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We find families of words W where W is a product of k pieces for k=2. For k=3,4,6, W arises in a small cancellation group with single defining relation W=1. We assume W involves generators but not their inverses and does not have a periodic cyclic permutation (like XY...XYX for nonempty base word XY). We prove W or W written backwards equals ABCD where ABC, CDA are periodic words with base words of different lengths. One family includes words of the form EFGG for periodic words G, E, F with the same base word and increasing lengths. Other W are found using *Mathematica*.

1. Introduction.

A small cancellation condition on a group's defining relations yields, for example, a solution to the conjugacy problem. See [**1**, **4**]. There are 3 types of such conditions. Each includes a condition $C(k)$ for $k = 3$, 4 or 6, depending on the type. For a group G with one defining relation $R = 1$, $C(k)$ involves the set $[R]$ of cyclic permutations of R and of R^{-1}. A *piece* is a nonempty, initial subword of 2 distinct members of $[R]$. $C(k)$ requires that no word in $[R]$ is a product of fewer than k pieces.

To study a "large cancellation" group G and avoid all small cancellation types, we can use the condition that R is a product of 2 pieces. What does such a word R look like? For simplicity, in this paper we consider words involving generators but not their inverses. In particular, we study 2-piece words R, meaning

(1.1)
R involves generators but not their inverses and R is a product of 2 pieces.

An attempt to classify these words led to the results in this paper. These results also lie in the field of combinatorics on words which is surveyed in [**3**].

2. Summary of results.

For convenient exposition, from now on, a word W is a finite sequence of letters taken from some alphabet; $|W|$ = length of W; the empty word=1; $|1| = 0$. Write $W \sim V$ if W, V are cyclic permutations. W is 2-*piece* if

$(\exists U, V, Y, Z \neq 1)$ $W = UV \sim UY \sim VZ$, $Z \neq U$, $Y \neq V$. (U, V) is a 2-*piece pair*. W is *periodic* if $(\exists Q \neq 1, P,\ k \geq 2)$ $W = P(QP)^k$. $X < W$ means $W = XY$, $Y \neq 1$. W is *biperiodic* if $(\exists P, Q, R, S, m, n)$ $W = UV$, $U = P(QP)^m$, $V = R(SR)^n$, $SR < U$, $QP < V$, $1 \neq QP \neq SR \neq 1$, $URS \neq SRU$ and $VPQ \neq QPV$; $m, n \geq 1$.

The main results are: If a 2-piece word W has no periodic cyclic permutation then W or W written backwards is biperiodic. Each biperiodic word is 2-piece. W is biperiodic and $|S| < |RS| < |PR| < PQ| < |RSR|$ is equivalent to $(\exists A, B, a, b, c, m, n)$ $W = A(BA)^b(A(BA)^c)^m(A(BA)^a)^n$ together with $AB \neq BA$, $1 < a < b < c \leq 2a$, $m, n \geq 1$. Such a word W is not periodic for $n \geq 2$. Two other similar equivalences are proved. Other as yet unclassified biperiodic words are found using *Mathematica*.

The title of the paper refers to the periodic subwords $P(QP)^{m+1}$, $R(SR)^{n+1}$ which begin word $W = P(QP)^m R(SR)^n$ and its cyclic permutation $R(SR)^n P(QP)^m$, respectively, whenever W is biperiodic and hence 2-piece.

3. Terminology.

Terminology in the previous section is augmented as follows. Let A, B be words over some alphabet. The concatenation of words A, B is written as a product AB. The product of k copies of A, written A^k, is a *power* of A if $k \geq 0$, with $A^0 = 1$, and a *proper power* if $k \geq 2$. Call W *simple* if W is not a proper power. Note the empty word $E = 1$ is not simple since $E = E^2$. If $W = XYZ$ then X, Y, Z are *factors* of W; write $X, Y, Z \subseteq W$. X, Z are *left* and *right* factors; write $X \leq W$, $W \geq Z$. If U is a factor of W then U is *major* if $2|U| \geq |W|$ and *proper* if $U \neq W$. Proper left and right factors of W are indicated by $X < W$, $W > Z$. Denote W written backwards by W^*, the *reverse* of W. As in [**4**, p. 153], the *period* $\pi(W)$ of a word W is the minimum length of the words admitting W as a factor of some of their powers. Equivalent definitions of *periodic* are in Theorem 4.13. Call W *plain* if W has no periodic cyclic permutation.

We restate a definition to enable later reference to its parts:

Definition 3.1. Word W is *biperiodic* using U, V, P, Q, R, S, m, n if $W = UV$ for words U, V such that:

(3.1a) $U = P(QP)^m$ for some words $Q \neq 1$, P and some integer $m \geq 1$.

(3.1b) $V = R(SR)^n$ for some words $S \neq 1$, R and some integer $n \geq 1$.

(3.1c) $SR < U$, $QP < V$.

(3.1d) $1 \neq QP \neq SR \neq 1$.

(3.1e) $URS \neq SRU$, $VPQ \neq QPV$.

4. Preliminaries.

Let $A, B, \ldots$ denote words and $a, b, \ldots$ denote integers in the following Lemmas. Lemmas 4.1–4.3 are found in Propositions 1.3.4, 8.1.1 and Theorem 8.1.2 in [**3**]. Lemmas 4.4, 4.5, 4.6, 4.8, 4.9, 4.10, 4.12 are in the following Propositions in [**2**]: 1.2, 1.3, 1.4, 1.4″, 1.8, 1.16, 1.23 . Prove Lemma 4.7 from Lemma 4.6 and Lemma 4.11 from Lemma 4.10 using reverse words. Use S simple if and only if S^* simple.

Lemma 4.1. *$Y, Z \neq 1$, $XZ = YX$ imply $(\exists n \geq 0, U, V)$ $Y = UV$, $Z = VU$, $X = U(VU)^n$.*

Lemma 4.2. *$\pi(W) = \mathrm{Min}\{|W| - |V|\}$ where $V < W > V$.*

Lemma 4.3. *$p = \pi(XY)$, $q = \pi(YZ)$, $d = gcd(|X|, |Y|)$, $|Y| \geq p + q - d$ imply $p = q = \pi(XYZ)$.*

Lemma 4.4. *If $XY = YX$ then $(\exists S$ simple, $a, b \geq 0)$ $X =^a$, $Y =^b$.*

Lemma 4.5. *If S, T simple, $S^a = T^b$, $a, b \geq 1$ then $S = T$.*

Lemma 4.6. *If S is a simple word and $PS \leq S^n$, $n \geq 1$ then P is a power of S.*

Lemma 4.7. *If S is a simple word and $S^n \geq SP$, $n \geq 1$ then P is a power of S.*

Lemma 4.8. *If $PBA \leq A(BA)^r$, $r \geq 1$, $P \neq 1$ with BA simple then $(\exists e \geq 0)$ $P = A(BA)^e$.*

Lemma 4.9. *A cyclic permutation of a simple word is a simple word.*

Lemma 4.10. *$X \leq Y^e X$, $e > 0$, $Y \neq 1$ imply $(\exists t \geq 0, E)$ t, E unique, $X = Y^t E$, $E < Y$.*

Lemma 4.11. *$XY^e \geq X$, $e > 0$, $Y \neq 1$ imply $(\exists t \geq 0, E)$ t, E unique, $X = EY^t$, $Y > E$.*

Lemma 4.12. *If $XYZ = ZYX$, $X \neq 1$, $Z \neq 1$ then $(\exists a, b, c \geq 0, U, V)$ $X = U(VU)^a$, $Y = V(UV)^b$, $Z = U(VU)^c$ and the word UV is simple.*

Lemma 4.13. *Equivalent conditions on a word W are:*

(4.13a) *W has a proper major left factor which is also a right factor.*

(4.13b) $W = YX = XZ$, $|X| \geq |Y| > 0$.

(4.13c) $(\exists k \geq 2, U \neq 1)$ $W \leq U^k$, $|W| \geq 2|U|$.

(4.13d) *W is periodic, that is, $(\exists B \neq 1,\ A, m \geq 2)$ $W = A(BA)^m$.*

(4.13e) $|W| \geq 2\pi(W)$ *and* $W \neq 1$.

Proof. **(a) if and only if (b):** Use definitions.

- **(b) implies (c):** Deduce $|W| = |YX| \geq |YY| = 2|Y|$ and $W < YW$. By Lemma 4.10, $W = Y^t E < Y^{t+1}$ for some word E with $t \geq 1$ because $|E| < |Y| < |W|$.
- **(c) implies (d):** From (c), $(\exists t \geq 2)\ t|U| \leq |W| < (1+t)|U|$. So $(\exists B \neq 1, A)\ U = AB$, $W = U^t A$, proving (d).
- **(d) implies (b):** Use $X = A(BA)^{m-1}$, $Y = AB$, $Z = BA$.
- **(d) implies (e):** From (d), $W < (AB)^{m+1}$ and so $\pi(W) \leq |AB|$. It follows that $|W| = |A(BA)^m| \geq m|BA| \geq 2|AB| \geq 2\pi(W)$, yielding (e).
- **(e) implies (c):** In general, $\pi(W) \leq |W|$. From (e), $(\exists V \neq 1,\ k \geq 1)$ $|V| = \pi(W)$, $W \subseteq V^k$. $k \geq 2$ since $2|V| = 2\pi(W) \leq |W| \leq k|V|$. Then $(\exists U \sim V)\ W \leq U^k$.

□

Lemma 4.14. *If $W = XY^eZ$, $Z \leq Y \geq X$, $Y \neq 1$, $e \geq 1$ then $(\exists B \neq 1, A, p)\ 0 \leq p \leq 2$, $W = A(BA)^{e+p}$, $|AB| = |Y|$, $XZ = A(BA)^p$.*

Proof. $(\exists C, D)\ Y = ZC = DX$. Let $Y_1 = XD$, $X_1 = XZ$. Then $X_1 \leq W = (Y_1)^e X_1$. Apply Lemma 4.10 to $X_1 \leq (Y_1)^e X_1$. $(\exists p \geq 0, A)\ X_1 = (Y_1)^p A$, $A < Y_1$. So $(\exists B \neq 1)\ Y_1 = AB$. Hence $W = (AB)^e(AB)^p A = A(BA)^{e+p}$; $XZ = X_1 = (AB)^p A = A(BA)^p$. Also $p \leq 2$ since $|Y^p A| = |(Y_1)^p A| = |X_1| = |XZ| \leq |Y^2|$. □

Lemma 4.15. *If $AB \neq BA$ then $(\exists C, D \neq 1, a, b \geq 0)$ CD simple, $A = C(DC)^a$, $B = D(CD)^b$. For $t \geq 0$, $A(BA)^t = C(DC)^{p(t)}$, $(AB)^t = (CD)^{q(t)}$, $p(t) = (a+b+1)t + a$, $q(t) = (a+b+1)t$.*

Proof. $(\exists$ simple S, $e \geq 1)\ AB = S^e$. $(\exists a, b \geq 0, C, D)\ S = CD$, $A = C(DC)^a$, $B = D(CD)^b$. $C, D \neq 1$ else A, B are powers of the same word and $AB = BA$, a contradiction. □

Lemma 4.16. *Y, $Z \neq 1$, $XZ = YX$ imply $(\exists r \geq 0, s \geq 1, C, D)$ CD simple, $Y = (CD)^s$, $Z = (DC)^s$, $X = C(DC)^r$.*

Proof. By Lemma 4.1, $(\exists n \geq 0, U, V)\ Y = UV$, $Z = VU$, $X = U(VU)^n$. $(\exists$ simple $S)$ UV is a power of S. $(\exists i, j \geq 0, C, D)\ S = CD$, $U = C(DC)^i$, $V = D(CD)^j$. Use $r = i + n(i+j+1)$, $s = i+j+1$. □

5. General results.

Each 2-piece word is simple (Theorem 5.5). If $W = UV$ is 2-piece then U, V appear again as factors of cyclic permutations of W. If U, V appear at least twice in W then W is periodic (Theorem 5.6). If a 2-piece word W is plain then W or W^* is biperiodic (Theorem 5.8). Each biperiodic word is 2-piece (Theorem 5.9). We start with some easily verifiable remarks.

Remark 5.1. If $PQ \neq QP$ and $m \geq 2$ then the periodic word $W = P(QP)^m$ is 2-piece, by definition, using $U = P(QP)^{m-1}$, $V = QP$, $Y = PQ$, $Z = QPP(QP)^{m-2}$.

Remark 5.2. A word is simple, periodic, 2-piece or biperiodic if and only if its reverse has the same property.

Remark 5.3. If (U, V) is a 2-piece pair then so are (V, U) and (V^*, U^*).

Given a 2-piece word W, W^* inherits properties as follows:

Remark 5.4. If a 5-tuple (W, U, V, Y, Z) of words satisfies $W = UV \sim YU \sim ZV$, $1 \neq U \neq Z$, $1 \neq V \neq Y$ then so does $(W^*, V^*, U^*, Z^*, Y^*)$.

Theorem 5.5. *Each* 2*-piece word is a simple word.*

Proof. Let W be 2-piece word, $W = UV \sim UY \sim VZ$, $1 \neq U \neq Z$, $1 \neq V \neq Y$. $(\exists A, B)$ $W = AB$, $UY = BA$. Suppose W is not simple. Then $(\exists$ simple $X)$ $W = X^m$, $m \geq 2$.

If $|X| \leq |U|$ then $(\exists C)$ $U = XC$. So $AX \leq AUY \leq ABAB = X^{2m}$. By Lemma 4.6, A is a power of X, so is W and hence so is B. Thus $AB = BA$. Then $UV = AB = BA = UY$ implies $V = Y$, a contradiction.

If $|X| > |U|$ then $|X| \leq |V|$. By Remark 5.3, $W^* = V^*U^*$ is 2-piece and $W^* = (X^*)^m$. Get a contradiction for W^*, V^*, U^* as previously with W, U, V. □

Theorem 5.6. *If* $U, V \neq 1$ *each appear at least twice in* $W = UV$ *then* W *is periodic. In other words,* $W = UV = IUJ = KVL$ *and* $U, V, I, L \neq 1$ *imply* W *is periodic.*

Proof. Assume $W = UV = XUT = RVY$ and $U, V, X, Y \neq 1$. The change in letters allows a more pleasing factorization $W = RST$ in Case 1.

Case 1: $|X| < |U|$; $|Y| < |V|$. Then $(\exists F, G)$ $UGT = XUT = W = RFV = RVY$, $|U| > |X| = |G| > 0$, $|V| > |F| = |Y| > 0$. So $U = RF$, $V = GT$, $W = RFGT$. Let $S = FG$. Then $W = RST$. Hence $UGT = W = RST = W = RFV$ imply $UG = RS$, $ST = FG$. Apply Lemma 4.2 to $W_1 = RS$, $W_2 = ST$.

So $\pi(RS) \leq |RS| - |U|$, $\pi(ST) \leq |ST| - |V|$ since $RS = UG = XU$, $ST = FV = VY$. Then $|S| = |F| + |G| = |RS| - |U| + |ST| - |V| \geq \pi(RS) + \pi(ST)$. By Lemma 4.3, $\pi(RST) = \pi(RS) = \pi(ST)$. Since $|W| \geq |S| \geq \pi(RS) + \pi(ST) = 2\pi(W)$, W is periodic by (4.13e) in Lemma 4.13.

Case 2: $|U| \leq |X|$; $|V| \leq |Y|$. Then $|U| \leq |UT| \leq |V|$, $|V| \leq |RV| \leq |U|$. So $|U| = |V|$, $U = V$ and hence $W = UU$ is periodic.

Case 3: $|X| < |U|$; $|V| \leq |Y|$. Then $(\exists F)$ $UFT = XUT = W$ with $|U| > |X| = |F| > 0$. So $UF = XU$. By Lemma 4.16, $(\exists C, D, r \geq 0, s \geq 1)$ CD simple, $U = C(DC)^r$, $F = (DC)^s$, $X = (CD)^s$. $r \geq 1$

since $|U| > |X| \geq |CD|$. $DC \leq V$ since $V = FT$. Thus $|W| \geq |UF| \geq 2|CD|$. Also $(\exists P, Q)$ $U = PVQ$.

Then $PDC \leq PV \leq U = C(DC)^r$. Also $DC \sim CD$ implies DC simple by Lemma 4.9. By Lemma 4.8, $P = C(DC)^e$, $e \geq 0$ implying $V \leq (DC)^{r-e}$. So $W \leq C(DC)^{2r-e}$; hence $\pi(W) \leq |CD|$. Then $|W| \geq 2|CD| \geq 2\pi(W)$ implies W is periodic by (4.13e) in Lemma 4.13.

Case 4: $|U| \leq |X|$; $|Y| < |V|$. Then V^*, U^* each appear at least twice in $W^* = V^*U^*$. Apply Case 3 to W^*; get W^* periodic. By Remark 5.2, W is periodic.

□

Lemma 5.7. *Let $W = UV$ be a plain word. Assume cyclic W has 2nd occurrences U'', V'' of the words U, V, respectively. Then* (i) *U, U'' overlap and V, V'' overlap and* (ii) *U'' is a factor of one of the words UV, VU and V'' is a factor of the other.*

Proof. First prove results (1)-(7).

(1) Conclusions for U, V, U'', V'' apply to $V^*, U^*, (V'')^*, (U'')^*$. By Remarks 5.2, 5.3, 5.4, assumptions on W, U, V, U'', V'' apply to $W^*, V^*, U^*, (V'')^*, (U'')^*$, respectively.

(2) Neither UV nor VU has both factors U'', V''. Use Theorem 5.6.

(3) U, U'' overlap or V, V'' overlap. If not then U has factor V'', V has factor U''. Therefore $U = V$, $W = UU$, W is periodic, contradicting W is plain.

(4) U, U'' overlap implies ($U'' \subseteq UV$ or $U'' \subseteq VU$). Suppose U, U'' overlap and U'' is not a factor of UV or VU. Then $(\exists A, B, C \neq 1)$ $W = ABCV$, $U = ABC$, $U'' = CVA$. So $CV < ABCV > CV$, $AB < CVAB > AB$. Then $ABCV$ or $CVAB$ has a major left and right factor, namely, CV or AB, respectively. Thus $ABCV$ or $CVAB$ is periodic by Lemma 4.13, contradicting W is plain. Thus (4) is true.

(5) V, V'' overlap implies ($V'' \subseteq VU$ or $V'' \subseteq UV$). Use (1), (4). Get $V^*, (V'')^*$ overlap implies $((V'')^* \subseteq V^*U^*$ or $(V'')^* \subseteq U^*V^*)$. This implies (5).

(6) U, U'' overlap implies V, V'' overlap. If not then U, U'' overlap but V, V'' do not. So $V'' \subseteq U$. Also $U'' \subseteq UV$ or $U'' \subseteq VU$. Then $U, U'', V, V'' \subseteq UV$ (or VU), contradicting (2).

(7) V, V'' overlap implies U, U'' overlap. Use (1), (6). Therefore $V^*, (V'')^*$ overlap implies $U^*, (U'')^*$ overlap. This implies (7).

Now (i) follows from (3), (6), (7) and (ii) follows from (i), (2), (4), (5). □

Theorem 5.8. *Each plain, 2-piece word W is biperiodic or its reverse is biperiodic.*

Proof. By Remarks 5.2 and 5.3, W^* is plain, 2-piece. Since $W = UV$ satisfies the 2-piece condition, cyclic W has 2nd occurrences U'', V'' of the words U, V, respectively. By Lemma 5.7, there are 2 cases:

Case 1: $U'' \subseteq UV, V'' \subseteq VU$. Using the 2-piece property and Lemma 5.7 Part (i), it follows that $UV = W = UDB = CU''B$, $VU = VFG = EV''G$, $U = FG$, $V = DB$, $Y = BC$, $Z = GE$, $|U| > |C|$, $|V| > |E|$ for some words $B, C, D, E, F, G \neq 1$. Since $|U| > |C|$, $|V| > |E|$, we can apply Lemma 4.1 to $UD = CU$ and $VF = EV$.

$(\exists Q \neq 1, P, m \geq 1)$ $U = P(QP)^m$, $C = PQ$, $D = QP$ and $(\exists S \neq 1, R, n \geq 1)$ $V = R(SR)^n$, $E = RS$, $F = SR$. Thus $UV = UDB = UQPB$, $QP < V = R(SR)^n$ and $VU = VFG = VSRG$, $SR < U = P(QP)^m$, implying Conditions (3.1a), (3.1b) and (3.1c).

If $|PQ| = |RS|$ then $QP < V = R(SR)^n$ implies $QP = RS$. So $W = PX^{m+n}R$ for $X = QP$ and W is periodic by Lemma 4.14, a contradiction. So (3.1d) is true.

If $URS = SRU$ then $FU = SRU = URS = UE = FGE = FZ$ and $U = Z$, not true. If $QPV = VPQ$ then $DV = QPV = VPQ = DBC = DY$ and $V = Y$, not true. So $URS \neq SRU$, $QPV \neq VPQ$, (3.1e) is true, making W biperiodic.

Case 2: $V'' \subseteq UV$, $U'' \subseteq VU$. Then $(V'')^* \subseteq V^*U^*$, $(U'')^* \subseteq U^*V^*$. By Remark 5.4, Case 1 applies to W^* so W^* is biperiodic.

□

Theorem 5.9. *Each biperiodic word is a 2-piece word.*

Proof. Let $W = UV$ be biperiodic using P, Q, R, S, U, V, m, n. VU is biperiodic using R, S, P, Q, V, U, n, m. By symmetry and Remark 5.3, we may assume $|SR| < |PQ|$. By Conditions (3.1a), (3.1b) and (3.1c), $SR < PQ$ and $QP < R(SR)^n$. Define F, J by $SRF = PQ$, $QPJ = R(SR)^n$. We now check that W satisfies the definition of being 2-piece by using $Y = JPQ$, $Z = FP(QP)^{m-1}RS$.

$$\begin{aligned} UY &= P(QP)^m JPQ \sim P(QP)^{m+1} J = UQPJ = UV \\ VZ &= R(SR)^n FP(QP)^{m-1} RS \sim SRFP(QP)^{m-1} R(SR)^n \\ &= PQP(QP)^{m-1} V = UV. \end{aligned}$$

If $Y = V$ then $VPQ = QPJPQ = QPY = QPV$, a contradiction. If $Z = U$ then we get a contradiction from $SRU = SRZ = PQP(QP)^{m-1}RS = URS$. Thus W is 2-piece. □

6. Factoring some 2-piece words.

The 2-piece words to be factored are two types of biperiodic words. They are called *biperiodic*-1 and *biperiodic*-2 words. They are 2-piece by Theorem 5.9.

Their factorizations are called *binary*-1 and *binary*-2 words and have factors $A, B, AB \neq BA$. See Theorems 6.10 and 6.11. A 3rd type of biperiodic word, called a *biperiodic*-3 word, has a cyclic permutation possessing a 3rd type of factorization, a *binary*-3 word (Theorem 6.12). Each binary-3 word has a binary-2 cyclic permutation (Remark 6.4). Likewise, each biperiodic-3 word has a biperiodic-2 cyclic permutation, using Theorem 6.12, Remark 6.4 and Theorem 6.11.

Results in this section, together with Remark 5.3, will show that 2-piece words can be found using the above factorizations and their reverses. More precisely, we have:

Theorem 6.1. *Binary-*1 *and binary-*2 *words and their reverses are* 2*-piece words. Each binary-*3 *word has a* 2*-piece cyclic permutation and so does its reverse.*

The types of biperiodic words and factorizations are defined as follows:

Definition 6.2. A word W is *biperiodic*-1, *biperiodic*-2 or *biperiodic*-3 if W satisfies Definition 3.1 together with (6.2a), (6.2b) or (6.2c), respectively.

(6.2a) $$|S| < |RS| < |PR| < |PQ| < |RSR|.$$

(6.2b) $$P = 1, |SR| < |Q|.$$

(6.2c) $$|RS| \leq |P| < |PQ|.$$

Definition 6.3. A word W is *binary*-1, *binary*-2 or *binary*-3 if (6.3a), (6.3b) or (6.3c), respectively, with $AB \neq BA$. Call such W *binary*. Terminology for later use: W is binary-1 *using* $A, B : AB \neq BA$ and W is binary-1 *for* A, B, h, i, j. Similar terminology applies to binary-2 and binary-3.

(6.3a) $W = A(BA)^i(A(BA)^j)^m(A(BA)^h)^n$, $1 < h < i < j \leq 2h$, $m, n \geq 1$.

(6.3b) $W = (A(BA)^i)^m(AB)^j$, $1 \leq i, i+1 \leq j$, $m, n \geq 1$.

(6.3c) $W = (A(BA)^i)^m A(BA)^j$, $1 \leq i, i+2 \leq j$, $m, n \geq 1$.

Remark 6.4. Each binary-3 word W has a cyclic permutation V which is binary-2. In particular, if W satisfies (6.3c), use $V = (A(BA)^i)^{m+1}(AB)^{j-i}$.

The proof that the biperiodic-1 and binary-1 conditions are equivalent for a word W requires 3 lemmas involving closely related conditions defined as follows:

Definition 6.5. A word W is *biperiodic*-1* if W satisfies (3.1a)-(3.1d) with $m = n = 1$. Notice the omission of (3.1e). In other words, W satisfies:

(6.5a) $$(\exists P, Q, R, S)\ W = PQPRSR, \quad SR < PQ, \quad QP < RSR,$$
$$|S| < |RS| < |PR| < |PQ|.$$

Definition 6.6. W is *binary*-1* if (6.3a) with $m = n = 1$. ($AB \neq BA$ not required.)

Lemma 6.7. *Word W is biperiodic-1^* if and only if*

(6.7a)
$(\exists F, I, J, P, Q, R, S, T, U)$ *all words* $\neq 1$ *except possibly S and*
$W = PQPRSR$, $SRF = PQ$, $QP = RSI$, $R = IJ$, $P = ST$, $Q = RU$.

Proof. Assume (6.5a). Then $SR < PQ$, $QP < RSR$ imply $PQ = SRF$, $QP = RSI$, $R = IJ$ for some words $F, I, J \neq 1$. $|RS| < |PR| < |PQ|$ imply $|S| < |P|$, $|R| < |Q|$. Using these inequalities and $SR < PQ$, $QP < RSR$ we get $P = ST$, $Q = RU$ for some words $T, U \neq 1$. Thus (6.7a) is true. (6.5a) follows easily from (6.7a). □

Lemma 6.8. *If W is biperiodic-1^* then W is binary-1^* using $A, B : B \neq 1$.*

Proof. By Lemma 6.7, we can assume W satisfies (6.7a) from which we deduce:

(1)	$(\exists V \neq 1)\ F = VU$	since $STRU = PRU = PQ = SRF$ and so $F > U$.
(2)	$UP = SI$	since $RUPJ = QPJ = RSIJ$.
(3)	$\lvert T\rvert < \lvert I\rvert$	since $\lvert UST\rvert = \lvert UP\rvert = \lvert SI\rvert$, $\lvert UT\rvert = \lvert I\rvert$.
(4)	$RV = TR$	since (1) and $SRVU = SRF = PQ = STRU$.
(5)	$(\exists K \neq 1)\ I = TK$	since (3), (4) and $R = IJ$.
(6)	$Q = KJF$	since $PQ = SRF = SIJF$ $= STKJF = PKJF$.
(7)	$KJFST = TKJSTK$	by (5), (6) and $KJFST = QP$ $= IJSI = TKJSTK$.
(8)	$TK = KT$	since, from (7), $K \leq TK \geq T$.
(9)	$(\exists N \neq 1,\ r, s \geq 1)\ K = N^r$, $T = N^s$	using (8) and Lemma 4.4.
(10)	$SN^r = SK = US$, hence $SN^r > S$	since (8), (5), (2) and $P = ST$ imply $SKT = STK = SI = UP = UST$.
(11)	$JFS = TJSK$	since (7) and (8).
(12)	$JVUS = TJSK$	since (11) and (1).
(13)	$JV = TJ = N^sJ$, hence $J < N^sJ$	using (12), (10) and (9).
(14)	$(\exists t \geq 0, D)\ S = DN^t$, $N > D$	by applying Lemma 4.11 to (10) $SN^r > S$.
(15)	$(\exists u \geq 0, L)\ J = N^uL$, $L < N$	by applying Lemma 4.10 to (13) $J < N^sJ$.
(16)	$PR = DN^bL$, $b = r + 2s + t + u$	since $PR = STIJ$ $= DN^tN^sN^{r+s}N^uL$.

(17)	$UPR = DN^cL$, $c = 2r + 2s + t + u$	since $UPR = USTIJ$ $= SKTIJ = SKTKTJ$.
(18)	$SR = DN^aL$, $a = r + s + t + u$	since $SR = SIJ = STKJ$ $= DN^tN^{r+s}N^uL$.
(19)	$(\exists C, M)\ N = LM = CD$	using (14) and (15).

Using (18) and (19) with Lemma 4.14 and (16) and (19) with Lemma 4.14, we have:

$$(\exists B \neq 1,\ A, p \geq 0)\ SR = A(BA)^{a+p},\ |AB| = |N|,$$
$$DL = A(BA)^p,\ |A| < |N| \text{ and hence } AB = DC.$$

$$(\exists G \neq 1,\ H, q \geq 0)\ PR = G(HG)^{b+q},\ |GH| = |N|,$$
$$DL = G(HG)^q,\ |G| < |N| \text{ and hence } GH = DC.$$

So $p|N| + |A| = |DL| = q|N| + |G|$, implying $p = q$, $|A| = |G|$. Then $A = G$, $B = H$ since $AB = DC = GH$. So $PR = A(BA)^{b+p}$. Similarly from (17) and (19) with Lemma 4.14, $UPR = A(BA)^{c+p}$. From (16), (17) and (18), $1 < a < b < c \leq 2a$. So $1 < a+p < b+p < c+p \leq 2a+p \leq 2a+2p = 2(a+p)$. Let $h = a + p$, $i = b + p$, $j = c + p$.

Then $W = PQPRSR = (PR)(UPR)(SR) = A(BA)^iA(BA)^jA(BA)^h$ satisfies (6.3a) for $m = n = 1$. So W is binary-1*. □

Lemma 6.9. *If W is binary-1* using $A, B : A \neq 1$ then W is biperiodic-1*.*

Proof. Assume word W satisfies (6.3a) with $A \neq 1$, $m = n = 1$. Since $i - h \leq i + h - j$, there exists an integer r with $0 < i - h \leq r \leq i + h - j$. Therefore $0 \leq h - i + r$, $j + r \leq i + h$, $0 < j - h \leq i - r$, $j - i + r \leq h$. Also

$$(*) \qquad (AB)^{j-i+r} < A(BA)^h$$

since $A \neq 1$.

Define $P = (AB)^r$, $Q = A(BA)^{i-r}(AB)^{j-i}$, $R = A(BA)^{i-r}$, $S = (AB)^{h-i+r}$. Note that $P, Q, R > 1$, $S \geq 1$, $W = PQPRSR$.

W is biperiodic-1* with (6.5a) true because:

$$SR < PQ \text{ since } SR = A(BA)^h < A(BA)^i(AB)^{j-i} = PQ,\ h < i.$$

$$QP < RSR \text{ since } QP = A(BA)^{i-r}(AB)^{j-i+r}$$
$$< A(BA)^{i-r}A(BA)^h = RSR \text{ using } (*).$$

$$|RS| < |PR| \text{ since } |S| = |(AB)^{h-i+r}| < |(AB)^r| = |P|,\ AB \neq 1,\ h < i.$$

$$|PR| < |PQ| \text{ since } |R| = |A(BA)^{i-r}| < |A(BA)^{i-r}(AB)^{j-i}| = |Q|.$$

□

Theorem 6.10. *A word is biperiodic-1 if and only if it is binary-1.*

Proof. Assume that W is biperiodic-1 with P, Q, R, S as in (3.1a)-(3.1e) and (6.2a). Word $W_1 = PQPRSR$ is biperiodic-1*. By Lemma 6.7, $(\exists F, I, J, T, U)$ satisfying Condition (6.7a). By Lemma 6.8, W_1 is binary-1* and satisfies (6.3a) for $m = n = 1$ and some $B \neq 1$. As in the proof of Lemma 6.8:

$$W_1 = PQPRSR = PRUPRSR,$$

$$PR = A(BA)^i,\ UPR = A(BA)^j,\ SR = A(BA)^h.$$

So $W = P(QP)^m R(SR)^n = P(RUP)^m R(SR)^n = PR(UPR)^m (SR)^n$ implying (6.3a).

If $AB = BA$ then by Lemma 4.4, A, B (and hence W) are powers of the same word. So W is a proper power, not simple. By Theorems 5.9 and 5.5, W is 2-piece and simple, a contradiction. Thus $AB \neq BA$ and W is a binary-1.

Now assume W is binary-1. Define P, Q, R, S as in proof of Lemma 6.9 so that:

$$1 < SR < PQ,\ QP < RSR. \tag{6.10a}$$

$$(PQ)^m = A(BA)^i (A(BA)^j)^{m-1} (AB)^{j-i}. \tag{6.10b}$$

$$(PQ)^m PR = A(BA)^i (A(BA)^j)^{m-1} A(BA)^j = A(BA)^i (A(BA)^j)^m. \tag{6.10c}$$

$$(PQ)^m PR(SR)^n = A(BA)^i (A(BA)^j)^m (A(BA)^h)^n = W. \tag{6.10d}$$

$W = UV$ for $U = (PQ)^m P$, $V = R(SR)^n$. Then (6.10a) implies (3.1a)-(3.1d) are true.

If $URS = SRU$ then $A(BA)^{h+1} \leq A(BA)^i < PQ < URS$ and $A(BA)^h AB \leq SRP < SRU$ imply $A(BA)^{h+1} = A(BA)^h AB$. So $BA = AB$, a contradiction. Thus $URS \neq SRU$. If $VPQ = QPV$ then $VPQ > Q > (AB)^{j-i} > AB$, $QPV > V > SR > BA$ imply $AB = BA$, a contradiction. So $VPQ \neq QPV$, (3.1e) is true and W is biperiodic-1. $\square$

Theorem 6.11. *A word is biperiodic-2 if and only if it is binary-2.*

Proof. Assume W is biperiodic-2. Then $U = Q^m$, $1 < SR < Q < R(SR)^n = V$, implying $RS = SR$. By Lemma 4.4, $R = X^r$, $S = X^s$, $X \neq 1$, $r, s \geq 1$. Since $SR < Q < R(SR)^n$, $(\exists A \neq 1, B, t \geq 0)$ $Q = A(BA)^{r+s+t}$, $X = AB$. Then $W = (A(BA)^i)^m (AB)^j$ using $i = r + s + t$, $j = r + n(r + s)$. Also $X^i < X^i A = Q < R(SR)^n = X^j$ implies $i < j$, $i+1 \leq j$. Thus (6.3b) is true. If $AB = BA$ then by Lemma 4.4 W is a proper power, not simple. But W is 2-piece, simple by Theorems 5.9 and 5.5, a contradiction. So $AB \neq BA$ and W is binary-2.

Now assume W is binary-2. Use $Q = A(BA)^i$, $P = R = 1$, $S = AB$, $n = j$, $U = Q^m$, $V = R(SR)^n$. Then (3.1a)-(3.1d) and (6.2b) $P = 1$, $|SR| < |Q|$ are true. Suppose $URS = SRU$. Then

$$(A(BA)^i)^m AB = URS = SRU = AB(A(BA)^i)^m.$$

Hence, $AB = BA$, a contradiction. Suppose $VPQ = QPV$. Then

$$(AB)^n A(BA)^i = VPQ = QPV = A(BA)^i(AB)^n$$

hence, $AB = BA$, a contradiction. Thus (3.1e) is true and W is biperiodic-2. $\square$

Theorem 6.12. *Each biperiodic-3 word has a binary-3 cyclic permutation. Each binary-3 word has a biperiodic-3 cyclic permutation.*

Proof. Assume W is binary-3. Then $W \sim W_1 = BA(A(BA)^i)^m A(BA)^{j-1}$. Use $P = BA, Q = A(BA)^{i-1}$, $R = A$, $S = B$, $n = j$. Thus (3.1a)-(3.1d) and (6.2c) are true. If $URS = SRU$ then $URS > AB$, $SRU > P = BA$ imply $AB = BA$, a contradiction. If $VPQ = QPV$ then $A(BA)^i AB = QPRS < QPV$ and $A(BA)^{i+1} < A(BA)^j = V < VPQ$ imply $AB = BA$, a contradiction. So (3.1e) is true and W has a biperiodic-3 cyclic permutation W_1.

Now assume W is biperiodic-3. Then (3.1a)-(3.1c), $|RS| \leq |P| < |PQ|$ imply $SR \leq P$, $RS < QP$, $QP < R(SR)^n$. Thus $(\exists I, J \neq 1, T)$ $QP = RSI$, $QPJ = R(SR)^n$, $P = SRT$.

(1)	$QSRT = QP = RSI$, $IJ = R(SR)^{n-1}$	using $R(SR)^n = QPJ = RSIJ$.
(2)	$I = VT$ for some V, $\|V\| = \|Q\|$	from (1).
(3)	$QSRTJ = VTJSR$	since $QSRTJ = QPJ = RSIJ$ $= R(SR)^n = IJSR = VTJSR$.
(4)	$Q = V$	from (3).
(5)	$SRTJ = TJSR$	from (3).
(6)	$QSRTJ = RSIJ$ $= RSVTJ = RSQTJ$	using (1), (2) and (4).
(7)	$RSQ = QSR$	from (6).
(8)	$(\exists a, b, c \geq 0, C, D)$ $R = C(DC)^a$, $S = D(CD)^b$, $Q = C(DC)^c$, CD simple	from (7) and Lemma 4.11.
(9)	DC is simple; $SR = (DC)^{a+b+1}$	by Lemma 4.9 and (8).
(10)	$(\exists t \geq 1)$ $TJ = (DC)^t$	from (9), (5) and Lemmas 4.4 and 4.5.
(11)	$(\exists d \geq 0, F \neq 1, G)$ $T = (FG)^d F$, $DC = FG$	from (10).
(12)	$R(SR)^n = C(DC)^a((DC)^{a+b+1})^n$ $= C(DC)^r$ with $r = a + n(a+b+1) \geq n \geq 2$	from (8).
(13)	$\|C(DC)^{c+1}\| \leq \|C(DC)^{a+b+c+1}\|$ $= \|QSR\| \leq \|QP\|$	from (8) and $\|RS\| \leq \|P\|$.

(14)	$\vert QP\vert < \vert R(SR)^n\vert$ $= \vert C(DC)^{n(a+b+1)+a}\vert$	from $QP < R(SR)^n$ and (12).
(15)	$c+1 < n(a+b+1)+a = r$	from (13) and (14).
(16)	$QP = C(DC)^e F$, $e = a+b+c+d+2$	by (1) $QP = QSRT$, (8) and (11).
(17)	$(\exists B \neq 1, A, p \geq 0)$ $QP = A(BA)^{e+p}$, $AB = CD$. Let $i = e+p$ so that $i \geq 2$.	by (16), $QP = X_1(Y_1)^e Z_1$, $X_1 = C$, $Y_1 = DC = FG$, $Z_1 = F$; apply Lemma 4.14 to $W_1 = X_1(Y_1)^e Z_1$.
(18)	$R(SR)^n P = C(DC)^f F$, $f = r+a+b+d+1$	by $P = SRT$, (8) and (11).
(19)	$(\exists M \neq 1, L, q \geq 0)$ $R(SR)^n P = L(ML)^{f+q}$, $LM = CD$. Let $j = f+q$.	by (17), $R(SR)^n P = X_2(Y_2)^e Z_2$, $X_2 = C$, $Y_2 = DC = FG$, $Z_2 = F$; apply Lemma 4.14 to $W_2 = X_2(Y_2)^f Z_2$.
(20)	$\vert A\vert \equiv \vert CF\vert$ Modulo $\vert DC\vert$; $\vert A\vert \leq \vert DC\vert$	from (16), (17) and $\vert BA\vert = \vert DC\vert$.
(21)	$\vert L\vert \equiv \vert CF\vert$ Modulo $\vert DC\vert$; $\vert L\vert \leq \vert DC\vert$	from (18), (19) and $\vert ML\vert = \vert DC\vert$.
(22)	$\vert A\vert = \vert L\vert$, $A = L$, $B = M$, $L(ML)^j = A(BA)^j$	by (20) and (21).
(23)	$(j-i)\vert DC\vert = (j-i)\vert AB\vert$ $= \vert A(BA)^j\vert - \vert A(BA)^i\vert$ $= \vert R(SR)^n P\vert - \vert QP\vert$ $= \vert R(SR)^n\vert - \vert Q\vert$ $= \vert C(DC)^r\vert - \vert C(DC)^c\vert$ $= (r-c)\vert DC\vert$	since (17) $AB = CD$. from (19), (22) and (17). from (12) and (8).
(24)	$j - i = r - c > 1$ and hence $i+2 \leq j$	from (23) and (15).
(25)	W has cyclic permutation $W_3 = (QP)^m R(SR)^n P$ $= (A(BA)^i)^m A(BA)^j$	from (17), (19) and (22).
(26)	W_3 satisfies (6.3c)	from (17), (24) and (25).

If $AB = BA$, it follows that W_3, W are proper powers, not simple by (25) and Lemma 4.4. However, W is 2-piece, simple by Theorems 5.9 and 5.5, a contradiction. So $AB \neq BA$ and W_3 is binary-3. $\square$

7. Some nonperiodic binary words.

We prove that binary words, with some restrictions on their exponents, are not periodic. Details are in Theorems 7.5, 7.6 and 7.7. In the proofs, AB simple, $A, B \neq 1$ can be assumed in (6.3a)-(6.3c) instead of $AB \neq BA$ because of the following lemma.

Lemma 7.1. *For $k = 1, 2$ or 3, equivalent properties for a word W are:*

(i) *binary-k using $A, B : AB \neq BA$;*

(ii) *binary-k using $C, D : C, D \neq 1$, CD simple.*

Proof. Assume (ii) for W. Then $CD \neq DC$ by Lemma 4.4. Use $A = C$, $B = D$ to get (i). Now assume (i) for W. By Lemma 4.14, $(\exists C, D \neq 1, a, b \geq 0)$ CD simple, $A = C(DC)^a$, $B = D(CD)^b$. Define $p(t) = (1 + a + b)t + a$, $q(t) = (1 + a + b)t$ for $t \geq 0$.

Assume $k = 1$. Then $p(h) < p(i) < p(j)$ since $p(t)$ is strictly increasing. Since $j \leq 2h$, $p(j) \leq p(2h) = (1+a+b)2h+a \leq (1+a+b)2h+2a = 2p(h)$. Also $1 < p(h)$ since $1 < h \leq p(h)$. So W is binary-1 for $C, D, p(h), p(i), p(j)$.

Assume $k = 2$. Then $p(i)+1 = (1+a+b)i+a+1 \leq (1+a+b)(j-1)+a+1 = q(j) - b \leq q(j)$ since $i \leq j-1$. $1 \leq p(i)$ since $1 \leq i \leq p(i)$. So W is binary-2 for $C, D, p(i), q(j)$.

Assume $k = 3$. Then $0 < p(i)$ since $0 < i \leq p(i)$. Since $i \leq j - 2$,

$$p(i) + 2 = (1 + a + b)i + a + 2 \leq (1 + a + b)(j - 2) + a + 2$$
$$= q(j) - a - 2b \leq q(j).$$

So W is binary-3 for $C, D, p(i), p(j)$. Thus (ii) is true for W for $k = 1, 2, 3$. $\square$

By Lemma 7.1, binary-1 and binary-3 words are products of words X_k defined below. Results about such products appear in the next two lemmas.

Definition 7.2. For fixed words $A, B \neq 1$, AB simple, define $X_k = A(BA)^k$, $k \geq 0$.

Lemma 7.3. *Let $G = X_{a_1} \ldots m X_{a_m}$, $H = X_{b_1} \ldots m X_{b_n}$, $1 \leq a_i$, $1 \leq b_j$, $1 \leq i \leq m$, $1 \leq j \leq n$ with $2 \leq m, n$. Assume $G = H$. Then $m = n$, $a_i = b_i$ for $1 \leq i \leq m$.*

Proof. If not, pick least integer $k \geq 1$ with $a_k \neq b_k$. Assume $a_k < b_k$ so that $X_{a_k}BA \leq X_{b_k}$. Therefore $k < m$, $TX_{a_k}AB < TX_{a_k}X_{a_{k+1}} \leq G$, $TX_{a_k}BA < TX_{b_k} \leq H$ for $T = X_{a_1} \ldots m X_{a_{k-1}}$. So $AB = BA$; hence AB not simple by Lemma 4.4, a contradiction. $\square$

Lemma 7.4. *Let $W = X_{a_1} \ldots m X_{a_m}$, $1 \leq a_i$, $1 \leq i \leq m$, $m \geq 2$. Assume $(\exists F)$ $F \leq W \geq F$.*

(7.4a)
If $|X_{a_1}| < |F|$ then $(\exists s, b)$ $F = X_{a_1} \ldots m X_{a_s} X_b$, $1 \leq s < m$, $1 \leq b \leq a_{s+1}$.

(7.4b)
If $|X_{a_m}| < |F|$ then $(\exists t, c)$ $F = X_c X_{a_t} \ldots m X_{a_m}$, $1 < t \leq m$, $1 \leq c \leq a_{t-1}$.

Proof. Assume $|X_{a_1}| < |F|$. Using $F < W$, $W = X_{a_1} \ldots m X_{a_m}$ induces a factorization $F = Y_1 \ldots m Y_r Z$ where $Y_k = A$ or $Y_k = B$ for $1 \leq k \leq r$ and

Z equals P or Q for some words P, Q satisfying $1 \le P < A$, $1 \le Q < B$. Also $r \ge 3$ since $|F| > |X_{a_1}|$, $a_1 \ge 1$. Since AAA, BB do not appear in $W = X_{a_1} \dots m X_{a_m}$, $Y = Y_{r-2}Y_{r-1}Y_r$ equals BAA, AAB, BAB or ABA. To prove (7.4a) it suffices to prove that $Y = ABA$ and $Z = 1$.

The five cases for YZ are $BAAQ$, $AABP$, $BABP$, $ABAP$, $ABAQ$. As shown below, only Case 4 with $P = 1$ and Case 5 with $Q = 1$ can occur. So indeed $Y = ABA$, $Z = 1$.

Case 1 $BAAQ$**:** $W > F$, $W > BABA$ imply $BABA > BAAQ$. By Lemma 4.7, AQ is a power of BA but $|AQ| < |BA|$. So $AQ = 1$, contradicting $A \ne 1$.

Case 2 $AABP$**:** $W > F$, $W > ABA$ imply $ABA > ABP$. $ABA = RABP$ for some $R \ne 1$ with $|RP| = |A|$. So $ABAB = RABPB$, $RAB < ABAB$, AB simple. By Lemma 4.6, R is a power of AB but $|R| \le |A| < |AB|$. So $R = 1$, a contradiction.

Case 3 $BABP$**:** $W > F$, $W > ABA$ imply $ABA > ABP$ as in Case 2.

Case 4 $ABAP$**:** $W > F$, $W > BABA$ imply $BABA > BAP$. By Lemma 4.7, P is a power of BA but $|P| < |BA|$. So $P = 1$.

Case 5 $ABAQ$**:** $W > F$, $W > BABA$ imply $BABA > BAQ$. By Lemma 4.7, Q is a power of BA but $|Q| < |BA|$. So $Q = 1$.

Now assume $|X_{a_m}| < |F|$. So $W^* = (X_{a_m})^* \dots m(X_{a_1})^*$ and $(X_{a_t})^* = A^*(B^*A^*)^t$, $t \ge 0$. AB simple implies BA simple by Lemma 4.9. $(BA)^*$ is simple by Remark 5.2. A^*B^* is simple since $A^*B^* = (BA)^*$. Note that $F^* \le W^* \ge F^*$. Apply (7.4a) to W^*, F^*. So $(\exists t, c)$ $F^* = (X_{a_m})^* \dots m(X_{a_t})(X_c)^*$, $1 < t \le m$, $1 \le c \le a_{t-1}$. Take reverses to get (7.4b). ☐

Theorem 7.5. *Each binary-*1 *word* W *with* $n \ge 2$ *is not periodic.*

Proof. By Lemma 7.1, Definition 7.2, $W = X_i(X_j)^m(X_h)^n$, $1 < h < i < j \le 2h$ for some $A, B \ne 1$, AB simple. By (4.13a), it suffices to prove that each major left factor of W which is also a right factor is equal to W. Suppose $F \le W \ge F$, $2|F| \ge |W|$. Then $|F| > |X_i|$, $|F| > |X_h|$ since $|X_i| < |X_j| > |X_h|$. By Lemma 7.4, $F = X_iGHX_h$ where G is a product of one or more X_j and H is a product of one or more X_h. It also follows from Lemma 7.4 that:

(7.5a) $\quad X_iGH$ and the start of W have the same X_k factors.

(7.5b) $\quad GHX_h$ and the end of W have the same X_k factors.

$|F| \le |W|$ implies $|GH| \le |(X_j)^m(X_h)^{n-1}|$. It follows that:

(7.5c) $\quad GH$ and the start of $(X_j)^m(X_h)^{n-1}$ have the same X_k factors.

(7.5d) $\quad GH$ and the end of $(X_j)^m(X_h)^{n-1}$ have the same X_k factors.

(7.5c) implies $G = (X_j)^m$. (7.5d) implies $H = (X_h)^{n-1}$. Thus $F = W$ as required. ☐

Theorem 7.6. *Each binary-2 word W with $m(i+1) \leq j$, $3 \leq j$ is not periodic.*

Proof. By Lemma 7.1 and Definition 7.2, $W = (X_i)^m(AB)^j$, $1 \leq i$, $i+1 \leq j$, $m \geq 1$ for some $A, B \neq 1$, AB simple. By (4.13a) it suffices to show that W has no proper major left factor which is also a right factor. Suppose F is such a factor, $2|F| \geq |W|$, $F < W > F$. We show this implies $AB = BA$, a contradiction.

Since $m(i+1) \leq j$, $|(X_i)^m| = |A^m(BA)^{mi}| < |(AB)^{m(i+1)}| \leq |(AB)^j|$. Then $2|F| \geq |W|$ and $|(X_i)^m| < |(AB)^j|$ imply $|(X_i)^m| < |F|$. Using $F < W$, F has one of the forms:

$$(X_i)^m(AB)^rP,\ (X_i)^m(AB)^sAQ,\ (X_i)^mR$$

where $1 \leq r < j$, $0 \leq s < j$, $1 \leq P < A$, $1 \leq Q < B$, $1 < R < A$.

- **Case 1.** $F = (X_i)^m(AB)^rP$**:** $W = (X_i)^m(AB)^j > F$, $r < j$ imply $(AB)^j > (AB)^rP$. By Lemma 4.7, $P = 1$. Then $F > BA$, $W > AB$ imply $AB = BA$.
- **Case 2.** $F = (X_i)^m(AB)^sAQ$**:** $W > F$ implies $(AB)^j > (AB)^sAQ$ and $(AB)^j > AQ$. By Lemma 4.7, AQ is a power of AB. Since $|AQ| < |AB|$, we have $AQ = 1$, $A = 1$, contradicting $AB \neq BA$. So this case cannot occur.
- **Case 3.** $F = (X_i)^mR$**:** $W > F$, $3 \leq j$, $1 \leq i$ imply $(AB)^3 > ABAR$. By Lemma 4.7, $AR = (AB)^t$ for some $t \geq 0$. Here $t \leq 1$ since $0 < |R| < |A|$. If $t = 0$ then $A = 1$, a contradiction. If $t = 1$ then $R = B$, $F > BA$, $W > AB$ so that $AB = BA$.

□

Theorem 7.7. *Each binary-3 word W is not periodic.*

Proof. By Lemma 7.1 and Definition 7.2, $W = (X_i)^mX_j$, $1 \leq i$, $i+2 \leq j$, $m \geq 1$ for some $A, B \neq 1$, AB simple. By (4.13a) it suffices to show that W has no proper major left factor which is also a right factor. Suppose F is such a factor, $2|F| \geq |W|$, $F < W > F$. We show this implies $AB = BA$, a contradiction.

Since $|W| \geq |X_iX_j| > |X_iX_i| = 2|X_i|$ we have $2|F| \geq |W| > 2|X_i|$. By (7.4a), F has one of the forms: $(X_i)^r$, $(X_i)^sX_b$, $(X_i)^mX_c$ where $1 \leq r \leq m$, $1 \leq s < m$, $1 \leq b < i$, $1 \leq c < j$. Thus F has a right factor X_iX_a for some a, $1 \leq a < j$ and hence $F > BAX_a$. Also $W > ABX_a$. Therefore $AB = BA$. □

8. Computing possibly biperiodic words.

Let $W = UV$, $SR < P(QP)^m = U$, $QP < R(SR)^n = V$, $0 < |RS| < |PQ| < |U|$, m, $n \geq 1$. W may be biperiodic. $p = |PQ|$, $q = |SR|$, $d = |U| - p$, $e = |V| - q$ satisfy $0 < d$, $0 < e$, $q < p < q + e$. Function g (see below) with inputs p, q, d, e, generates such a W as a list of integers ≥ 1. Format for g comes from *Mathematica* software, version 3.0.

$g[p\_, q\_, d\_, e\_] := \text{Module}[\{i, k, n = p + q + d + e, w\},$
$\text{If } [!((0 < d)\&\&(0 < e)\&\&(q < p)\&\&(p < q + e)),$
$\quad \text{Return}[\text{“Invalid Input”}]];$
$w = \text{Join } [\text{ Range}[q], \text{Range}[n - 2q], \text{Range}[q]];$
$\text{For}[k = n - q, k \geq p + d + 1, k - -, w[[k]] = w[[k + q]]];$
$\text{For}[k = p + d, k \geq q + 1, k - -, w[[k]] = w[[k + p]]];$
$\text{For}[k = q, k \geq 1, k - -, i = w[[k + p]]; w = w/.(k- > i)]; w].$

The observed output from g is (unpredictably) either biperiodic or a proper power.

In *Mathematica*, Range$[q]$ is the list of positive integers from 1 to q. Join$[a,\ b,\ c]$ concatenates lists a, b, c. The code k- - indicates integer k is decreased by 1 after each stage of a loop. $w[[i]] = i$-th element of the list w. The code $w\ =\ w\ /.\ (k\ -{>}\ \ i)$ rewrites list w by replacing each instance of the current value of k in list w by the current value of i.

9. Examples.

We give 2 sets of examples of biperiodic words $W = PQPRSR$ over the alphabet $\{a, b\}$. In Example (9.1), $|PR| < |RS|$, $P \neq 1$. In Example (9.2), $|PQ| < |PR|$, $P \neq 1$. Therefore (6.2a)-(6.2c) are not true. These examples include $g[24, 18, 3, 10]$ and $g[24, 18, 15, 14]$ for the function g defined in the previous section.

Example 9.1. $W = (CCDDCD)^2D(ab)^{j-i}$, $C = a(ba)^i$, $D = a(ba)^j$, $0 < i < j \leq 2i$.

Let $P = C$, $Q = CDDCD$, $U = PQP = CCDDCDC$, $R = CD(ab)^{j-i}$, $S = CC(ab)^{j-i}$, $V = RSR = CDDCDD(ab)^{j-i}$. W is biperiodic because:

$SR < PQ$	since $SR = CCDD(ab)^{j-i}$,
	$PQ = CCDDCD = CCDD(ab)^iaD$, $j \leq 2i$.
$QP < RSR$	since $C < D$, $QP = CDDCDC$,
	$RSR = CDDCDD(ab)^{j-i}$.
$\lvert PR\rvert < \lvert RS\rvert$	since $\lvert PR\rvert = \lvert CCD(ab)^{j-i}\rvert < \lvert CCDDC(ab)^{j-i}\rvert = \lvert RS\rvert$.
$URS \neq SRU$	else $URCCD = URSC = SRUC = SRCCDDCDCC$,
	$CCD = DCC$, not true.
$QPV \neq VPQ$	else $QPCDDCDDD = QPVC = VPQC$
	$= VPCDDCDC$, $DD > DC$, not true.

Using $i = 1$, $j = 2$ and shorthand $2 = ab$, $3 = aba$, $5 = ababa$, $W = (335535)(335535)52$. Rewrite W by replacing the letters a, b with the symbols 1, 2, respectively. The resulting word, written as a list, is equal to $g[24, 18, 3, 10]$.

Example 9.2. $W = X(XZY)^2XXY$, $X = (abb)^h ab$, $Y = (abb)^i ab$, $Z = (abb)^{2h+1}ab$, $0 < h < i$.

Let $P = XXX$, $Q = bX(abb)^{i-h}$, $PQ = XXZ(abb)^{i-h}$, $QP = bXYXX$, $U = PQP = XXZYXX$, $R = bXY$, $S = (abb)^h a$, $V = RSR = bXYXXY$. W is biperiodic because:

$SR < PQ$, $QP < RSR$	since $SR = XXY$.
$\lvert PQ\rvert < \lvert PR\rvert$	since $i - h < i$ implies $Q < R$.
$URS \neq SRU$	since $URS > S > ba$, $SRU > X > ab$ and $ab \neq ba$.
$VPQ \neq QPV$	since $VPQ > Q > bb$, $QPV > Y > ab$ and $bb \neq ab$.

Using $h = 1$, $i = 2$ and shorthand $5. = X$, $8. = Y$, $11. = Z$, $W = 5.5.11.8.5.11.8.5.5.8$. Rewrite W by replacing the letters a, b with the symbols 1, 2, respectively. The resulting word, written as a list, is equal to $g[24, 18, 15, 14]$.

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