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Let $\widetilde{SL(2,\mathbb{F})}$ be the metaplectic two-fold cover of $SL(2,\mathbb{F})$, the special linear group in two variables over a local field $\mathbb{F}$ of characteristic 0. The inverse image $\widetilde{T}$ of a maximal torus T in $\widetilde{SL(2,\mathbb{F})}$ is an abelian extension of T by ± 1. We consider the question of whether this extension is trivial. More generally we find the minimal subgroup A of the circle for which the extension is split when considered with coefficients in A. We see that $|A| = 2, 4$ or 8 in the p-adic case. We also find an explicit splitting function for the cocycle.

Introduction

Let $\widetilde{SL(2,\mathbb{F})}$ be the metaplectic two-fold cover of $SL(2,\mathbb{F})$, the special linear group in two variables over a local field $\mathbb{F}$ of characteristic 0. The inverse image $\widetilde{T}$ of a maximal torus T in $\widetilde{SL(2,\mathbb{F})}$ is an abelian extension of T by ± 1. We consider the question of whether this extension is trivial. We exclude the case $\mathbb{F}=\mathbb{C}$, which is trivial.

More generally suppose A is a subgroup of the circle $\mathbb{T}$ containing ± 1. The inclusion of ± 1 in A induces a map on cohomology, and defines an extension

$$1 \to A \to T_A \to T \to 1.$$

We say A is a *splitting group* for $\widetilde{T}$ if the extension $T_A \to T$ splits. It is well-known that $\mathbb{T}$ is a splitting group. We say a splitting group A is a *minimal splitting group* if no proper subgroup of A is a splitting group. It is easy to see the order of a minimal splitting group is a power of 2, and hence unique, if it is finite.

Let $(\ ,\ )_{\mathbb{F}}$ be the Hilbert symbol of $\mathbb{F}$, and let μ_n be the n^{th} roots of unity in $\mathbb{C}$.

Theorem 1. *The minimal splitting group $A_{\min}$ for T is given by:*

(a) *Suppose $T \simeq \mathbb{F}^*$. Then*

$$A_{\min} = \begin{cases} \mu_2 & (-1,-1)_{\mathbb{F}} = 1 \\ \mu_4 & (-1,-1)_{\mathbb{F}} = -1. \end{cases}$$

(b) *Suppose $T \simeq \mathbb{E}^1$ for $\mathbb{E}$ a quadratic extension of $\mathbb{F}$. Then*

$$A_{\min} = \begin{cases} \mu_2 & (-1,-1)_{\mathbb{F}} = 1 \\ \mu_4 & (-1,-1)_{\mathbb{F}} = -1, \ \mathbb{F} \textit{ non-archimedean}, \ -1 \notin \mathbb{E}^{*2} \\ \mu_8 & (-1,-1)_{\mathbb{F}} = -1, \ \mathbb{F} \textit{ non-archimedean}, \ -1 \in \mathbb{E}^{*2} \\ \mathbb{T} & \mathbb{F} = \mathbb{R}. \end{cases}$$

Remark 2. It is well-known that $(-1,-1)_{\mathbb{F}} = 1$ unless $\mathbb{F} = \mathbb{R}$, $\mathbb{Q}_2$, or an extension of $\mathbb{Q}_2$ of odd degree.

Theorem 1 is proved in Sections 3, 4 and 5. Here is an alternative realization of $\widetilde{T}$. A character of $\widetilde{T}$ is said to be *genuine* if it does not factor to T.

Theorem 3. *Let $\tau(z) = z^2$ $(z \in \mathbb{C}^*)$. Let $\widetilde{\alpha}$ be a genuine character of $\widetilde{T}$. Then $\widetilde{\alpha}^2$ factors to a character α of T, and $\widetilde{T}$ is isomorphic to the pullback of τ via α. In other words $\widetilde{T}$ is isomorphic to the $\sqrt{\alpha}$-extension of G.*

From this we obtain an interpretation of the minimal splitting group of Theorem 1. Let $n(\widetilde{T})$ be the minimal order of a genuine character of $\widetilde{T}$. Set $\mu_\infty = \mathbb{T}$.

Corollary 4. *The minimal splitting group for $\widetilde{T}$ is $\mu_{n(\widetilde{T})}$.*

For the proofs of Theorem 3 and Corollary 4 see Lemma 1.4.

We also give an explicit splitting of this extension, i.e., a function $\zeta : T \to A_{min}$ whose coboundary is the cocycle defining $\widetilde{T}$ (see §3 and Theorem 5.7).

These questions arise from the theory of the oscillator representation and dual pairs. The splitting plays a role in this context, for example see [**11**]. The case of $\mathbb{F}^*$ is well-known ([**4**], p. 42, attributed to J. Klose), as is the existence of a $\mathbb{T}$-splitting in general [**2**]. General results about the splitting of the metaplectic cover over subgroups are due to Kudla [**7**], and a splitting of the extension of an elliptic torus is found in [**7**], Proposition 4.8 (in the non-archimedean case it is easy to see this can be taken to be a μ_8-splitting). This paper grew out of an effort to simplify Kudla's formula. In the case of a p-adic field of odd residual characteristic a formula for a μ_2-splitting in some cases may be deduced from [**6**], cf. ([**4**], p. 43).

Many of the arguments, especially those of Section 1 apply to other abelian extensions of abelian groups, for example a maximal torus in the two-fold cover of $Sp(2n, \mathbb{F})$. If $\widetilde{G}$ is a non-linear n-fold cover of the $\mathbb{F}$ points of an algebraic group G, then the inverse image $\widetilde{T}$ of a maximal torus in G is typically not abelian. However similar arguments apply to the center of $\widetilde{T}$.

Throughout $\mathbb{F}$ denotes a local field of characteristic zero, and $(x, y)_{\mathbb{F}} \in \mu_2$ is the Hilbert symbol. For $x \in \mathbb{F}^*$ and $\psi_{\mathbb{F}}$ a non-trivial additive character of

$\mathbb{F}$, $\gamma_{\mathbb{F}}(x, \psi_{\mathbb{F}}) \in \mu_4$ is the Weil index. We use basic properties of the Hilbert symbol and the Weil index without further comment, see ([10], Appendix) for details. We make repeated use of the identities

$$\gamma_{\mathbb{F}}(x, \psi_{\mathbb{F}})\gamma_{\mathbb{F}}(y, \psi_{\mathbb{F}}) = (x, y)_{\mathbb{F}}\gamma_{\mathbb{F}}(xy, \psi_{\mathbb{F}}) \tag{1}$$

$$\gamma_{\mathbb{F}}(x, \psi_{\mathbb{F}})^2 = (-1, x)_{\mathbb{F}}. \tag{2}$$

If $\mathbb{E}$ is a quadratic extension of $\mathbb{F}$ then

$$(x, z)_{\mathbb{E}} = (x, Nz)_{\mathbb{F}} \quad (x \in \mathbb{F}^*, z \in \mathbb{E}^*) \tag{3}$$

$$(x, y)_{\mathbb{E}} = 1 \quad (x, y \in \mathbb{F}^*) \tag{4}$$

$$\gamma_{\mathbb{E}}(x, \psi_{\mathbb{E}})\gamma_{\mathbb{E}}(y, \psi_{\mathbb{E}}) = \gamma_{\mathbb{E}}(xy, \psi_{\mathbb{E}}) \quad (x, y \in \mathbb{F}^*). \tag{5}$$

I would like to thank Steve Kudla and Jonathan Rosenberg for many useful discussions, and Jim Schafer for assistance with the arguments in Section 1.

1. Abstract Groups.

In this section we ignore the topology on T and consider it as an abstract group. We recall some standard facts from group cohomology and establish some notation. For example see [1].

Suppose G is a group, A is an abelian group, and G acts trivially on A. The equivalence classes of central extensions of G by A are parametrized by the group cohomology $H^2(G, A)$. Given an extension $p : H \to G$ let $s : G \to H$ be a section, i.e., $p \circ s = 1$. The cohomology class of the extension is represented by the 2-cocycle $c_s(g, h) = s(gh)s(h)^{-1}s(g)^{-1}$. When there is no danger of confusion we do not distinguish between c_s and its image $\overline{c}_s$ in $H^2(G, A)$. Any other such splitting s' is given by $s'(g) = s(g)\zeta(g)$ for some map $\zeta : G \to A$, and then $c_{s'}(g, h) = c(g, h)\zeta(gh)\zeta(h)^{-1}\zeta(g)^{-1}$. Thus $c_{s'} = c_s d\zeta$, and $\overline{c}_s = \overline{c}_{s'}$.

Conversely given a cocycle c we define H to be equal to $G \times A$ as a set, with multiplication $(g, a)(g', a') = (gg, aa'c(g, g'))$. The cocyle c is trivial in cohomology if and only if

$$c(g, h) = \zeta(g)\zeta(h)\zeta(gh)^{-1} \tag{6}$$

for some ζ, i.e., $d\zeta = c$. We say ζ is a *splitting* of the cocycle. Equivalently the splitting map $s(g) = (g, \zeta^{-1}(g))$ is a homomorphism. Any other splitting is then of the form $\zeta' = \zeta\alpha$ with $\alpha : G \to A$ a homomorphism.

Suppose $A = \mu_2$, with cocycle c, and $A \subset \mu_{ab}$ with b odd. If $\zeta : G \to \mu_{ab}$ is a splitting of c, then ζ^b is a μ_a splitting. Therefore we will restrict consideration to μ_n with n a power of 2.

Now suppose G is abelian. The universal coefficient theorem for group cohomology gives an exact sequence:

$$1 \to \mathrm{Ext}(G, A) \to H^2(G, A) \xrightarrow{\phi} \mathrm{Hom}(\Lambda^2 G, A) \to 1.$$

Here G and A are considered as $\mathbb{Z}$-modules, $\mathrm{Hom} = \mathrm{Hom}_{\mathbb{Z}}$, $\mathrm{Ext} = \mathrm{Ext}_{\mathbb{Z}}$, and $\mathrm{Hom}(\Lambda^2 G, A)$ consists of alternating, bilinear maps $G \times G \to A$.

If c is a 2-cocycle, representing the class $\overline{c} \in H^2(G, A)$, then $\phi(\overline{c})(g, h) = c(g, h)c(h, g)^{-1}$. In terms of the group, suppose $p : H \to G$ is the corresponding extension. For $g, h \in G$ and any section s let $\{g, h\}$ be the commutator $s(g)s(h)s(g)^{-1}s(h)^{-1}$. This is contained in A, is independent of the choice of s, and $\phi(\overline{c})(g, h) = \{g, h\}$. In particular $\phi(\overline{c}) = 1$ if and only if H is abelian, so $\mathrm{Ext}\,(G, A) \subset H^2(G, A)$ parametrizes the abelian extensions of G by A.

Let $G^n = \{g^n | \, g \in G\}$ and ${}_nG = \{g \in G \,| g^n = 1\}$. The next result is presumably well-known to the experts.

Lemma 1.1. *For any positive integer n, inclusion $\iota : {}_nG \hookrightarrow G$ induces an isomorphism:*

$$\mathrm{Ext}\,(G, \mu_n) \simeq \mathrm{Ext}\,({}_nG, \mu_n).$$

Proof. Consider the maps

$$G \xrightarrow{\alpha} G^n \xrightarrow{\beta} G$$

where $\alpha(g) = g^n$ and β is inclusion. The induced map $\alpha^*\beta^* : \mathrm{Ext}\,(G, \mu_n) \to \mathrm{Ext}\,(G, \mu_n)$ is induced by the n^{th} power map $g \to g^n$ on G. This is the same map as that induced by the n^{th} power map on μ_n, and therefore $\alpha^*\beta^* = 0$.

Now the long exact cohomology sequence corresponding to $0 \to G^n \xrightarrow{\beta} G \to G/G^n \to 0$ has final two terms $\mathrm{Ext}\,(G, A) \xrightarrow{\beta^*} \mathrm{Ext}\,(G^n, A) \to 0$. Therefore β^* is surjective, which implies $\alpha^* = 0$. On the other hand the short exact sequence

$$0 \to {}_nG \xrightarrow{\iota} G \xrightarrow{\alpha} G^n \to 0$$

gives rise to the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}\,(G^n, A) \to \mathrm{Hom}\,(G, A) \to \mathrm{Hom}\,({}_nG, A) \to \\ \mathrm{Ext}\,(G^n, A) \xrightarrow{\alpha^*} \mathrm{Ext}\,(G, A) \xrightarrow{\iota^*} \mathrm{Ext}\,({}_nG, A) \to 0. \end{aligned} \tag{7}$$

Since $\alpha^* = 0$, ι^* is an isomorphism. □

Remark 1.2. In our setting ${}_2T = \pm 1$. For the μ_2 extension $\widetilde{T}$ to split it is necessary that it splits over ± 1. Perhaps surprisingly the converse holds as well by the Lemma.

For later use we note an explicit formula for a splitting of $\alpha^*\beta^* c$. We drop the assumption that H is abelian, so let $p : H \to G$ be an extension, with section s and corresponding cocycle c.

Lemma 1.3. *Let*

$$\begin{aligned} \tau(g) &= s(g^n)s(g)^{-n} \\ &= c(g, g)^{-1} c(g, g^2)^{-1} \ldots c(g, g^{n-1})^{-1} \in A. \end{aligned}$$

Then

$$c(g^n, h^n) = \tau(g)\tau(h)\tau(gh)^{-1}\{g,h\}^{n(n-1)/2}. \tag{8}$$

If H is abelian then $d\tau = \alpha^*\beta^* c$.

Note that $\{g,h\}^{n(n-1)/2} = \pm 1$, and is identically 1 if n is odd. Compare ([**3**], p. 130) and ([**5**], §4).

Proof. This follows from the identity

$$[s(g)s(h)]^n = s(g)^n s(h)^n \{h,g\}^{n(n-1)/2}.$$

Using $s(g)s(h) = s(gh)c(g,h)$ and $s(g)^n = s(g^n)\tau(g)^{-1}$, the left hand side is equal to

$$s(gh)^n = s(g^n h^n)\tau(gh)^{-1}.$$

The right hand side is

$$\begin{aligned} &s(g^n)s(h^n)\tau(g)^{-1}\tau(h)^{-1}\{h,g\}^{n(n-1)/2} \\ &= s(g^n h^n)c(g^n,h^n)\tau(g)^{-1}\tau(h)^{-1}\{h,g\}^{n(n-1)/2} \end{aligned}$$

and the first assertion follows. Since $\alpha^*\beta^* c(g,h) = c(g^n, h^n)$, the second assertion is equivalent to

$$c(g^n, h^n) = \tau(g)\tau(h)\tau(gh)^{-1} \tag{9}$$

which is (8) for H abelian. □

For H an abelian extension (9) implies τ is a character when restricted to ${}_nG$. In terms of the exact sequence (7), $c \in \mathrm{Ext}\,(G, A)$, $\beta^* c \in \mathrm{Ext}\,(G^n, A)$, $\alpha^*\beta^* c = 0$, and $\beta^* c$ is the image of $\tau \in \mathrm{Hom}\,({}_nG, A)$. Thus $\beta^* c = 0$ if τ extends to an element of $\mathrm{Hom}\,(G, A)$.

More generally, we try to find a splitting subgroup A of $\beta^* c$, i.e., c restricted to G^n, together with an explicit formula. Note that (9) does not necessarily define such a splitting since the function $g^n \to \tau(g)$ is not necessarily well-defined. Let α be a character of G whose restriction to ${}_nG$ is equal to τ^{-1}. Then $\zeta_\alpha(g^n) := \tau(g)\alpha(g)$ is well-defined, and $d\zeta_\alpha = c$. The minimal splitting subgroup for $\beta^* c$ is thus the minimal subgroup A of $\mathbb{T}$, containing μ_n, such that τ restricted to ${}_nG$ can be extended to a character of G with values in A.

Characters and the $\sqrt{\alpha}$ extension.

For the remainder of this section let $p : \widetilde{G} \to G$ be a μ_2 extension of a group G. We do not assume that G or $\widetilde{G}$ is abelian. If α is a character of G, and $\tau(z) = z^2$ $(z \in \mathbb{C}^*)$ then the pullback of τ via α is a μ_2 extension of G, and may be realized as the subgroup of $G \times \mathbb{C}^*$ given by $\{(g,z) \mid \alpha(g) = \tau(z)\}$. Projection on the second factor is a genuine character $\widetilde{\alpha}$ of $\widetilde{G}$ satisfying $\widetilde{\alpha}^2 = \alpha \circ p$. This is sometimes denoted the $\sqrt{\alpha}$-extension of G. It may or may not be the trivial extension.

We see that $\widetilde{G}$ has a $\mathbb{T}$-splitting if and only if there is a genuine character of $\widetilde{G}$. More precisely:

Lemma 1.4. *Suppose there is a genuine character $\widetilde{\alpha}$ of $\widetilde{G}$. Then $\widetilde{\alpha}^2$ factors to a character α of G, and $\widetilde{G}$ is isomorphic to the $\sqrt{\alpha}$ extension of G. If* $\mathrm{Image}(\widetilde{\alpha}) \subset A \subset \mathbb{T}$ *then $G_A \simeq G \times A$. The minimal splitting group for G is $\mu_{n(\widetilde{G})}$ where $n(\widetilde{G})$ is the minimal order of a genuine character of $\widetilde{G}$.*

Conversely if $\widetilde{G}_A \simeq G \times A$ then there is a genuine character of $\widetilde{G}$ with values in A.

Note that there exists a genuine character $\widetilde{\alpha}$ of $\widetilde{G}$ if and only if $z \notin [\widetilde{G}, \widetilde{G}]$ where z is the non-trivial element in the inverse image of 1. In particular this holds if $\widetilde{G}$ is abelian, which proves the existence of a $\mathbb{T}$-splitting (cf. [**2**]).

Proof. The map $\phi : g \to (p(g), \widetilde{\alpha}(g)) \subset G \times \mathbb{T}$ is an isomorphism of $\widetilde{G}$ with the pullback of τ via α. This is a subgroup of $G \times A$, and ϕ extends to an isomorphism of $\widetilde{G}_A$ with $G \times A$. The final two assertions are immediate. $\square$

Remark 1.5. In the setting of the Lemma, suppose ζ is a $\mathbb{T}$-splitting of the cocycle defining $\widetilde{G}$ (with respect to a section s). Then $\alpha := \zeta^2$ is a character of G, and $\widetilde{G}$ is isomorphic to the $\sqrt{\alpha}$ cover of G.

Theorem 3 and Corollary 4 are immediate consequences of the Lemma. Theorem 1 also follows from the Lemma, from a computation of $n = n(\widetilde{T})$: n is the minimal power of 2 such that $z \notin \widetilde{T}^n$. We follow a different approach, by giving explicit formulas for the minimal splitting ζ in Sections 3-5.

2. Moore cohomology and $\widetilde{SL(2,\mathbb{F})}$.

Now suppose G and A are locally compact topological groups, A is abelian, and G acts continuously on A. In our applications A will either be μ_n or $\mathbb{T}$, with trivial G action. C. Moore has defined cohomology groups $H^n_{\mathrm{top}}(G, A)$ using measurable cochains [**8**]. In the case of a totally disconnected group it is equivalent to use continuous cochains. Viewing G and A as abstract groups, there is a natural homomorphism $H^2_{\mathrm{top}}(G, A) \to H^2(G, A)$. In general it is neither surjective nor injective.

Now let $G = GL(2, \mathbb{F})$. We recall the definition of the standard cocycle on G [**6**], cf. ([**4**], p. 41). For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ let $x(g) = c$ (resp. d) if $c \neq 0$ (resp. $c = 0$). Then

$$c(g, h) = (x(g)x(gh), x(h)x(gh))_{\mathbb{F}}(\det(g), x(g)x(gh))_{\mathbb{F}}. \tag{10}$$

This defines a μ_2-extension $\widetilde{GL(2,\mathbb{F})}$ of $GL(2,\mathbb{F})$, with distinguished section s. We write $\widetilde{GL(2,\mathbb{F})}$ in cocycle notation as usual. The restriction to $SL(2,\mathbb{F})$ is isomorphic to $\widetilde{SL(2,\mathbb{F})}$.

Up to conjugation $GL(2,\mathbb{F})$ contains one hyperbolic torus isomorphic to $\mathbb{F}^*\times\mathbb{F}^*$, and for each quadratic extension $\mathbb{E}$ of $\mathbb{F}$ one elliptic torus isomorphic to $\mathbb{E}^*$. The commutator of two elements z, w of $\mathbb{E}^*$ is given by ([**3**], p. 128)

$$\begin{aligned}\{z,w\} &= (z,\overline{w})_{\mathbb{E}} \\ &= (z,w)_{\mathbb{E}}(Nz,Nw)_{\mathbb{F}}.\end{aligned} \tag{11}$$

Here and elsewhere we suppress the map $\iota:\mathbb{E}^*\hookrightarrow GL(2,\mathbb{F})$ from the notation, and write $\{z,w\}:=\{\iota(z),\iota(w)\}$. The commutator is trivial when restricted to $\mathbb{E}^1=\mathbb{E}^*\cap SL(2,\mathbb{F})$.

3. The hyperbolic torus.

Let $\iota:\mathbb{F}^*\hookrightarrow SL(2,\mathbb{F})$, so $T=\iota(\mathbb{F}^*)$ is a hyperbolic torus. After conjugation we may assume $\iota(x)=\mathrm{diag}(x,x^{-1})$. We drop ι from the notation and identify x with $\iota(x)$. We prove Theorem 1 in this case, together with a formula for the minimal splitting. While these results are well-known they are not easy to find in the literature, and the calculation illustrates some of the ideas in the next section.

The restriction of $\widetilde{SL(2,\mathbb{F})}$ to T defines a μ_2 extension $\widetilde{T}$ of T with cocycle

$$c(x,y)=(x,y)_{\mathbb{F}}.$$

In particular $(-I,\epsilon)^2=(I,(-1,-1)_{\mathbb{F}})$, so the restriction of the extension to $\pm I$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ if $(-1,-1)_{\mathbb{F}}=1$, or $\mathbb{Z}/4\mathbb{Z}$ if $(-1,-1)_{\mathbb{F}}=-1$. By Lemma 1.1 the minimal splitting group for $\widetilde{T}$, considered as an abstract group, is μ_2 (resp. μ_4) if $(-1,-1)_{\mathbb{F}}=1$ (resp. -1). It remains to show this splitting is measurable. We do this by computing it explicitly.

Fix ψ and write $\gamma(x)=\gamma_{\mathbb{F}}(x,\psi)$. The key point is that properties (1) and (2) of the Weil index shows that $d\gamma=c$ and $\gamma^4=1$, so γ is a measurable μ_4 splitting of c. This completes the proof in case $(-1,-1)_{\mathbb{F}}=-1$, so assume $(-1,-1)_{\mathbb{F}}=1$. Let α be a character of $\mathbb{F}^*$ satisfying $\alpha(x^2)=(-1,x)_{\mathbb{F}}$. To see that such a character exists, define α restricted to $\mathbb{F}^{*2}$ by this formula; it is well-defined since $\alpha((-x)^2)=(-1,-x)_{\mathbb{F}}=(-1,-1)_{\mathbb{F}}\alpha(x^2)=\alpha(x^2)$. Extend arbitrarily from $\mathbb{F}^{*2}$ to $\mathbb{F}^*$. By (2) $\alpha(x)^2\gamma(x)^2=(-1,x)_{\mathbb{F}}^2=1$. Let $\zeta_\alpha=\gamma\alpha$, i.e.,

$$\zeta_\alpha(x)=\gamma(x,\psi_{\mathbb{F}})\alpha(x).$$

Then $d\zeta_\alpha=d\gamma=c$ and ζ_α is a μ_2-splitting of c.

Choose representatives $a_1,a_2,\dots,a_m\in\mathbb{F}^*$ of generators of $\mathbb{F}^*/\mathbb{F}^{*2}\simeq(\mathbb{Z}/2\mathbb{Z})^m$. (By [4], Lemma 0.3.2, $2^m=4/|2|_{\mathbb{F}}$.) Given any choice of signs

ϵ_i we may choose α so that $\zeta(a_i) = \epsilon_i$, and $\alpha(x^2) = (-1, x)_{\mathbb{F}}$ for all $x \in \mathbb{F}$. Then ζ extends uniquely to a splitting.

For example, if $-1 \in \mathbb{F}^{*2}$ then we may take $\alpha = 1$ and $\zeta(x) = \gamma_{\mathbb{F}}(x, \psi_{\mathbb{F}}) = \pm 1$. On the other hand, suppose $-1 \notin \mathbb{F}^{*2}$ and the residual characteristic of $\mathbb{F}$ is odd. We may take representatives $\pm 1, \pm\varpi$ for $\mathbb{F}^*/\mathbb{F}^{*2}$ (ϖ is a uniformizing parameter) and then choose ζ satisfying:

$$\zeta(\pm x^2) = (-1, x)_{\mathbb{F}}$$
$$\zeta(\pm\varpi x^2) = \pm(-1, x)_{\mathbb{F}}.$$

4. Elliptic tori.

Let T be an elliptic torus of $SL(2, \mathbb{F})$ as in §2. Thus $\mathbb{E}$ is a quadratic extension of $\mathbb{F}$, $\iota : \mathbb{E}^1 \hookrightarrow SL(2, \mathbb{F})$ is an embedding, and $T = \iota(\mathbb{E}^1)$. As in §3 we fix ι and drop it from the notation.

As in the case of the hyperbolic torus, $(-I, \epsilon)$ has order 2 or 4 depending on whether $(-1, -1)_{\mathbb{F}} = +1$ or -1. By Lemma 1.1 this proves μ_2 is a splitting group if and only if $(-1, -1)_{\mathbb{F}} = 1$, so assume $(-1, -1)_{\mathbb{F}} = -1$.

Let $\mathbb{F} = \mathbb{R}, \mathbb{E} = \mathbb{C}$. Since $T = T^2$, it is enough to find a splitting of $\beta^* c$ as in §2. Choose a character α of $\mathbb{E}^1$ such that $\alpha(-1) = -1$, i.e., $\alpha(z) = z^n$ for n odd. It is easy to see $c(-z, -z) = -c(z, z)$, and the discussion in §2 shows that

$$\zeta_\alpha(z^2) := c(z, z)\alpha(z) \tag{12}$$

is a well-defined (measurable) $\mathbb{T}$-splitting. Since ζ_α is surjective onto $\mathbb{T}$ for any α, this shows that there is no A-splitting for any proper subgroup A of $\mathbb{T}$.

We now assume $\mathbb{F}$ is non-archimedean. Since $(-1, -1)_{\mathbb{F}} = -1$, $\mathbb{F}$ is an extension of $\mathbb{Q}_2$ of odd degree, and $-1 \notin \mathbb{F}^{*2}$. If $-1 \notin \mathbb{E}^{*2}$ then ${}_4T = \{\pm 1\}$. Since $(-I, \zeta)^4 = (I, \pm\zeta^2)^2 = I$ for all $\zeta \in \mu_4$, the μ_4-extension splits over ${}_4T$.

Suppose $-1 = \delta^2$ ($\delta \in \mathbb{E}^*$). We claim ${}_4T = {}_8T = \{\pm 1, \pm\delta\}$. It is enough to show $\mathbb{E}^*$ does not contain a primitive eighth root of unity, or equivalently $\delta \notin \mathbb{E}^{*2}$. Since $\mathbb{F}$ is an extension of $\mathbb{Q}_2$ of odd degree, $2 \notin \mathbb{F}^{*2}$. But then $(a + b\delta)^2 = \delta$ implies $a^2 = \pm\frac{1}{2}$, which is a contradiction.

For any $\zeta \in \mu_8$ we compute $(\delta, \zeta)^8 = (-I, \pm\zeta^2)^4 = (I, -\zeta^4)^2 = I$, which implies the extension splits over ${}_8T$.

Therefore μ_4 (resp. μ_8) is a minimal splitting group for T if $\mathbb{F}$ is non-archimedean, $(-1, -1)_{\mathbb{F}} = -1$, and $-1 \notin \mathbb{E}^{*2}$ (resp. $-1 \in \mathbb{E}^{*2}$). It remains to show these splittings can be chosen to be measurable. In the next section we give explicit such splittings.

Remark 4.1. We have shown the cohomology class $\overline{c} \in H^2_{\text{top}}(G, A)$ has image 0 in $H^2(G, A)$ in the given cases. An argument due to Jonathan

Rosenberg shows that the map $H^2_{\mathrm{top}}(G, A) \to H^2(G, A)$ is injective in this situation. Since we are interested in explicit formulas for the splittings in any case, we do not pursue this approach. For G perfect (not at all the case here!) the injectivity of ϕ is known ([**9**], Theorem 2.3).

5. Explicit splittings for elliptic tori.

We continue with the notation of the previous section. The embedding $\iota : \mathbb{E}^1 \hookrightarrow SL(2, \mathbb{F})$ extends to an embedding $\iota : \mathbb{E}^* \hookrightarrow GL(2, \mathbb{F})$. We will make use of the non-abelian μ_2 extension of $\mathbb{E}^*$ obtained by restricting the extension $\widetilde{GL(2, \mathbb{F})}$ of $GL(2, \mathbb{F})$.

We proceed as follows. Since $\{z, w\} = (z, \overline{w})_{\mathbb{E}}$, the commutator is trivial on $\mathbb{E}^{*2}$. By the method of §1 we find a splitting of this extension. As in §3 we also find a splitting of the extension of $\mathbb{F}^* \subset \mathbb{E}^*$. The extension of $\mathbb{E}^{*2}\mathbb{F}^*$ is abelian, and by an explicit version of the Mayer-Vietoris sequence we obtain a splitting of this extension. Finally $\mathbb{E}^1$ is contained in $\mathbb{E}^{*2}\mathbb{F}^*$, and we restrict to obtain a splitting of the extension of $\mathbb{E}^1$.

Fix non-trivial additive characters $\psi_{\mathbb{F}}$ of $\mathbb{F}$ and $\psi_{\mathbb{E}}$ of $\mathbb{E}$. Recall (5) the restriction of $\gamma_{\mathbb{E}}(\cdot, \psi_{\mathbb{E}})$ to $\mathbb{F}^*$ is a quadratic character.

Lemma 5.1. *Suppose λ (respectively μ) is a splitting of the cocycle restricted to $\mathbb{E}^{*2}$ (respectively $\mathbb{F}^*$). Assume $\lambda(x) = \mu(x)$ for $x \in \mathbb{E}^{*2} \cap \mathbb{F}^*$. For $z \in \mathbb{E}^*, x \in \mathbb{F}^*$ let $\zeta_{\lambda,\mu}(z^2x) := \lambda(z^2)\mu(x)c(x, z^2)$. Then $\zeta_{\lambda,\mu}$ is a well-defined splitting of the cocycle restricted to $\mathbb{E}^{*2}\mathbb{F}^*$.*

Conversely if ζ is any splitting of the cocycle restricted to $\mathbb{E}^{*2}\mathbb{F}^*$ then $\zeta = \zeta_{\lambda,\mu}$ with $\lambda = \zeta|_{\mathbb{E}^{*2}}$ and $\mu = \zeta|_{\mathbb{F}^*}$.

Proof. Let ζ be any splitting of the cocycle. Then $\zeta(z^2x) = \zeta(z^2)\zeta(x)c(x, z^2)$ by (6) and the second assertion is immediate.

Given λ and μ, choose any splitting ζ. Then λ and $\zeta|_{\mathbb{E}^{*2}}$ both define splittings, so $\lambda(z^2) = \zeta(z^2)\alpha(z^2)$ for some character α of $\mathbb{E}^{*2}$. Similarly $\mu(x) = \zeta(x)\beta(x)$ for some character β of $\mathbb{F}^*$. Let τ be a character of $\mathbb{E}^*$ extending α and β; this exists since, for $x \in \mathbb{E}^{*2} \cap \mathbb{F}^*$, $\lambda(x) = \mu(x)$ implies $\alpha(x) = \beta(x)$. Then

$$\begin{aligned} \lambda(z^2)\mu(x)c(x, z^2) &= \zeta(z^2)\alpha(z^2)\zeta(x)\beta(x)c(x, z^2) \\ &= \zeta(z^2x)\alpha(z^2)\beta(x) \\ &= \zeta(z^2x)\tau(z^2x). \end{aligned}$$

This shows that $\zeta_{\lambda,\mu}$ is well-defined and is a splitting of the cocycle. □

Lemma 5.2. (1) *Choose a character α of $\mathbb{E}^*$ satisfying*

$$\alpha(-1) = (-1, -1)_{\mathbb{F}}\gamma_{\mathbb{E}}(-1, \psi_{\mathbb{E}}) \tag{13}$$

and let

$$\lambda_\alpha(z^2) = c(z, z)\gamma_{\mathbb{E}}(z, \psi_{\mathbb{E}})\gamma_{\mathbb{F}}(Nz, \psi_{\mathbb{F}})\alpha(z).$$

Then λ_α is a well-defined splitting of the cocycle restricted to $\mathbb{E}^{*2}$. Furthermore every splitting of the cocycle restricted to $\mathbb{E}^{*2}$ is equal to λ_α for some α satisfying (13).

(2) *Let β be a character of $\mathbb{F}^*$ and let*

$$\mu_\beta(x) = \gamma_{\mathbb{F}}(x, \psi_{\mathbb{F}})\beta(x).$$

Then μ_β is a splitting of the cocycle restricted to $\mathbb{F}^*$, and every splitting of the cocycle restricted to $\mathbb{F}^*$ is equal to μ_β for some β.

(3) *Suppose $\alpha \in \widehat{\mathbb{E}^*}, \beta \in \widehat{\mathbb{F}^*}$ satisfy*

$$\alpha(z) = \gamma_{\mathbb{F}}(z^2, \psi_{\mathbb{F}})\gamma_{\mathbb{F}}(Nz, \psi_{\mathbb{F}})\gamma_{\mathbb{E}}(z, \psi_{\mathbb{E}})c(z, z)\beta(z^2) \quad (z^2 \in \mathbb{F}^*). \tag{14}$$

In particular α satisfies (13). For $z \in \mathbb{E}^*, x \in \mathbb{F}^*$ define

$$\begin{aligned}\zeta_{\alpha,\beta}(z^2 x) &:= \lambda_\alpha(z^2)\mu_\beta(x)c(x, z^2) \\ &= \gamma_{\mathbb{E}}(z, \psi_{\mathbb{E}})\gamma_{\mathbb{F}}(Nz, \psi_{\mathbb{F}})\gamma_{\mathbb{F}}(x, \psi_{\mathbb{F}})\alpha(z)\beta(x)c(z, z)c(x, z^2).\end{aligned} \tag{15}$$

Then ζ is a well-defined splitting of the cocycle restricted to $\mathbb{E}^{*2}\mathbb{F}^*$. Furthermore every splitting of the cocycle restricted to $\mathbb{E}^{*2}\mathbb{F}^*$ is equal to $\zeta_{\alpha,\beta}$ for some α, β satisfying (14).

Proof. Part (1) is an extension of (8) to the case of a non-abelian group. Thus by (8) and (11),

$$\begin{aligned}c(z^2, w^2) &= c(z, z)c(w, w)c(zw, zw)\{z, w\} \\ &= c(z, z)c(w, w)c(zw, zw)(z, w)_{\mathbb{E}}(Nz, Nw)_{\mathbb{F}}.\end{aligned}$$

Replacing $(z, w)_{\mathbb{E}}$ by $\gamma_{\mathbb{E}}(z, \psi_{\mathbb{E}})\gamma_{\mathbb{E}}(w, \psi_{\mathbb{E}})\gamma_{\mathbb{E}}(wz, \psi_{\mathbb{E}})^{-1}$, and similarly $(Nz, Nw)_{\mathbb{F}}$ gives

$$c(z^2, w^2) = \tau(z)\tau(w)\tau(zw)^{-1}$$

with $\tau(z) = c(z, z)\gamma_{\mathbb{E}}(z, \psi_{\mathbb{E}})\gamma_{\mathbb{F}}(Nz, \psi_{\mathbb{F}})$. The same relation holds with $\tau(z)$ replaced by $\tau(z)\alpha(z)$.

We check the condition that $\lambda_\alpha(z^2) := \tau(z)\alpha(z)$ be well-defined:

$$\begin{aligned}&\tau(-z)\alpha(-z) \\ &= c(-z, -z)\gamma_{\mathbb{E}}(-z, \psi_{\mathbb{E}})\gamma_{\mathbb{F}}(N(-z), \psi_{\mathbb{F}})\alpha(-z) \\ &= c(-z, -z)\gamma_{\mathbb{E}}(-1, \psi_{\mathbb{E}})\gamma_{\mathbb{E}}(z, \psi_{\mathbb{E}})(-1, z)_{\mathbb{E}}\gamma_{\mathbb{F}}(Nz, \psi_{\mathbb{F}})\alpha(z)\alpha(-1).\end{aligned}$$

A simple calculation using (10) gives

$$c(-z, -z) = (-1, -1)_{\mathbb{F}}(-1, Nz)_{\mathbb{F}}c(z, z) \tag{16}$$

and inserting this gives

$$\begin{aligned}\tau(-z)\alpha(-z) &= (-1, -1)_{\mathbb{F}}\gamma_{\mathbb{E}}(-1, \psi_{\mathbb{E}})\alpha(-1)\tau(z)\alpha(z) \\ &= \tau(z)\alpha(z) \quad \text{by (13)}.\end{aligned}$$

Fix α satisfying (13). If λ is any splitting of the cocycle restricted to $\mathbb{E}^{*2}$ then $\lambda = \lambda_\alpha \delta$ for some character δ of $\mathbb{E}^{*2}$. Extend δ to a character δ^* of $\mathbb{E}^*$. Then $\lambda_\alpha \delta = \lambda_{\alpha\delta^{*2}}$, and $\alpha\delta^{*2}$ satisfies (13). This proves (1).

By (10), $c(\mathrm{diag}(x,x), \mathrm{diag}(y,y)) = (x,y)_{\mathbb{F}}$, and (2) follows as in Section 3.

For (3) apply Lemma 5.1. Inserting $\lambda_\alpha, \mu_\beta$ in the condition of the Lemma gives (14). The final assertion follows as in the proof of (1). This completes the proof. □

Let ζ be any splitting of the cocycle restricted to $\mathbb{E}^{*2}\mathbb{F}^*$. Then $\zeta = \zeta_{\alpha,\beta}$ for some α, β satisfying (14). This implies:

Lemma 5.3. *The map* $\alpha(z) := \gamma_{\mathbb{F}}(z^2, \psi_{\mathbb{F}})\gamma_{\mathbb{F}}(Nz, \psi_{\mathbb{F}})\gamma_{\mathbb{E}}(z, \psi_{\mathbb{E}})c(z,z)$ *is a character of* $\{z \in \mathbb{E}^* \mid z^2 \in \mathbb{F}^*\}$.

For completeness we also prove this directly:

$$\begin{aligned}\alpha(zw) &= \gamma_{\mathbb{F}}(z^2w^2, \psi_{\mathbb{F}})\gamma_{\mathbb{F}}(N(zw), \psi_{\mathbb{F}})\gamma_{\mathbb{E}}(zw, \psi_{\mathbb{E}})c(zw, zw) \\ &= \gamma_{\mathbb{F}}(z^2, \psi_{\mathbb{F}})\gamma_{\mathbb{F}}(w^2, \psi_{\mathbb{F}})(z^2, w^2)_{\mathbb{F}}\gamma_{\mathbb{F}}(Nz, \psi_{\mathbb{F}})\gamma_{\mathbb{F}}(Nw, \psi_{\mathbb{F}})(Nz, Nw)_{\mathbb{F}} \\ &\quad\ \gamma_{\mathbb{E}}(z, \psi_{\mathbb{E}})\gamma_{\mathbb{E}}(w, \psi_{\mathbb{E}})(z, w)_{\mathbb{E}}c(zw, zw) \\ &= \alpha(z)\alpha(w)c(z,z)c(w,w)c(z^2, w^2)\{z, w\}c(zw, zw) \quad \text{by (11)} \\ &= \alpha(z)\alpha(w) \quad \text{by (8)}.\end{aligned}$$

Remark 5.4. Given $\alpha \in \widehat{\mathbb{E}^*}$ there exists $\beta \in \widehat{\mathbb{F}^*}$ satisfying (14) if and only if (13) holds.

This follows by an argument as in the proof of Lemma 5.2 (1): Define $\beta(z^2) = \alpha(z)\gamma_{\mathbb{F}}(z^2, \psi_{\mathbb{F}})^{-1}\gamma_{\mathbb{F}}(Nz, \psi_{\mathbb{F}})^{-1}\gamma_{\mathbb{E}}(z, \psi_{\mathbb{E}})^{-1}c(z,z)$; this is well-defined if (13) holds, and extends to a character of $\mathbb{F}^*$.

Suppose $z \in \mathbb{E}^1$. By Hilbert's Theorem 90, $z = w/\overline{w}$ for some $w \in \mathbb{E}^*$. Then $z = w^2/N(w) \in \mathbb{E}^{*2}\mathbb{F}^*$, so ζ restricts to a splitting of the extension of $\mathbb{E}^1$. We now make explicit choices such that ζ is a $\mathbb{T}$ or μ_n splitting as in Theorem 1.

Lemma 5.5. *We may choose* $\psi_{\mathbb{E}}$ *so that*

$$\gamma_{\mathbb{E}}(x, \psi_{\mathbb{E}}) = 1 \quad (x \in \mathbb{F}^*). \tag{17}$$

Proof. Since $\gamma_{\mathbb{E}}(\cdot, \psi_{\mathbb{E}})$ is a quadratic character of $\mathbb{F}^*$, $\gamma_{\mathbb{E}}(x, \psi_{\mathbb{E}}) = (x, y)_{\mathbb{F}}$ for some $y \in \mathbb{F}^*$. Suppose $\mathbb{E} = \mathbb{F}(\sqrt{\Delta})$. Then $\gamma_{\mathbb{E}}(\Delta, \psi_{\mathbb{E}}) = 1$ since Δ is a square in $\mathbb{E}^*$. Therefore $(\Delta, y)_{\mathbb{F}} = 1$, so $y = Nw$ for some $w \in \mathbb{E}^*$. Replacing $\psi_{\mathbb{E}}$ by $w\psi_{\mathbb{E}}$ gives (cf. [**10**], Appendix)

$$\begin{aligned}\gamma_{\mathbb{E}}(x, w\psi_{\mathbb{E}}) &= (x, w)_{\mathbb{E}}\gamma_{\mathbb{E}}(x, \psi_{\mathbb{E}}) \\ &= (x, Nw)_{\mathbb{F}}(x, y)_{\mathbb{F}} = 1.\end{aligned}$$

This completes the proof. □

Lemma 5.6. *Fix $\psi_{\mathbb{E}}$ satisfying Lemma 5.5. Choose α, β satisfying (14) and let $\zeta = \zeta_{\alpha,\beta}$. Then*

$$\zeta(z)^2 = \alpha(z) \quad (z \in \mathbb{E}^1).$$

Proof. Writing $z = w/\overline{w} = w^2/N(w)$ and applying the definition (15) gives

$$\zeta(w/\overline{w}) = (-1, Nw)_{\mathbb{F}} \gamma_{\mathbb{E}}(w, \psi_{\mathbb{E}}) \alpha(w) \beta(Nw^{-1}) c(w, w) c(Nw^{-1}, w^2) \tag{18}$$

and

(19)

$$\begin{aligned}
\zeta(w/\overline{w})^2 &= (-1, w)_{\mathbb{E}} \alpha(w^2) \beta(Nw^{-2}) \\
&= (-1, Nw)_{\mathbb{F}} \alpha(w^2) \alpha(Nw^{-2})(-1, Nw)_{\mathbb{F}} \gamma_{\mathbb{E}}(Nw^{-1}, \psi_{\mathbb{E}}) \quad \text{by (14)} \\
&= \alpha(w/\overline{w}) \gamma_{\mathbb{E}}(Nw, \psi_{\mathbb{E}}) \\
&= \alpha(w/\overline{w}) \quad \text{by (17)}.
\end{aligned}$$

□

We see that $\zeta_{\alpha,\beta}$ is a μ_{2n} splitting if and only if $\alpha(z)^n = 1$ for all $z \in \mathbb{E}^1$. We now complete the proof of Theorem 1.

Proof of Theorem 1. By (16) and (17) we have

$$\alpha(-1) = (-1, -1)_{\mathbb{F}}.$$

Therefore we may choose $\alpha = 1$ if $(-1, -1)_{\mathbb{F}} = 1$. Assume $(-1, -1)_{\mathbb{F}} = -1$. If $\mathbb{F} = \mathbb{R}$ then $\alpha(z) = z^n$ for n odd as in Section 4, so assume $\mathbb{F}$ is non-archimedean. If $-1 \notin \mathbb{E}^{*2}$ we may choose $\alpha^2 = 1$. If $-1 \in \mathbb{E}^{*2}$ then $-1 \notin \mathbb{E}^{*4}$ (cf. §4) and we may choose $\alpha^4 = 1$. □

We make some explicit choices and summarize the preceding discussion.

If $(-1, -1)_{\mathbb{F}} = -1$ and $-1 \notin \mathbb{E}^{*2}$ choose $z_1 \in \mathbb{E}^*$ with $(z_1, -1)_{\mathbb{E}} = -1$.

If $(-1, -1)_{\mathbb{F}} = -1$ and $-1 \in \mathbb{E}^{*2}$, i.e., $\mathbb{E} = \mathbb{F}(\sqrt{-1})$, then the norm residue symbol $(w, z)_{\mathbb{E},4}$ is defined. In particular the map $z \rightarrow (w, z)_{E,4}$ is a character of $\mathbb{E}^*$ of order 4. Choose $z_2 \in \mathbb{E}^*$ satisfying $(z_2, -1)_{\mathbb{E},4} = -1$.

Theorem 5.7. *Choose a non-trivial character $\psi_{\mathbb{E}}$ of $\mathbb{E}$ such that $\gamma_{\mathbb{E}}(x, \psi_{\mathbb{E}}) = 1$ for all $x \in \mathbb{F}^*$ (Lemma 5.5). For $z \in \mathbb{E}^*$ let*

$$\alpha(z) := \begin{cases} z & \mathbb{F} = \mathbb{R} \\ 1 & (-1, -1)_{\mathbb{F}} = 1 \\ (z_1, z)_{\mathbb{E}} & (-1, -1)_{\mathbb{F}} = -1, -1 \notin \mathbb{E}^{*2} \\ (z_2, z)_{\mathbb{E},4} & (-1, -1)_{\mathbb{F}} = -1, -1 \in \mathbb{E}^{*2}. \end{cases}$$

Then α is a character of $\mathbb{E}^*$ of order $\infty, 1, 2$ or 4 respectively, satisfying (13). Choose a character β of $\mathbb{F}^*$ satisfying

$$\beta(z^2) = \gamma_{\mathbb{F}}(z^2, \psi_F)^{-1} \gamma_{\mathbb{F}}(Nz, \psi_{\mathbb{F}})^{-1} \gamma_{\mathbb{E}}(z, \psi_{\mathbb{E}})^{-1} c(z, z) \alpha(z) \quad (z^2 \in \mathbb{F}^*) \tag{20}$$

(*cf. Remark* 5.4). *In particular*

$$\beta(x^2) = (-1, x)_{\mathbb{F}}\alpha(x) \quad (x \in \mathbb{F}^*).$$

Let

$$\zeta(w/\overline{w}) = \gamma_{\mathbb{E}}(w, \psi_{\mathbb{E}})\alpha(w)\beta(Nw^{-1})c(w, w)c(Nw^{-1}, w^2). \tag{21}$$

Then ζ *is a splitting of the cocycle restricted to* $\mathbb{E}^1$. *Furthermore for* $z \in \mathbb{E}^1$ $\zeta(x)^2 = \alpha(z)$, *and for* $\mathbb{F}$ *non-archimedean this gives* $\zeta(z)^n = \alpha(z)^{n/2} = 1$ *with*

$$n = \begin{cases} 2 & (-1,-1)_{\mathbb{F}} = 1 \\ 4 & (-1,-1)_{\mathbb{F}} = -1, -1 \notin \mathbb{E}^{*2} \\ 8 & (-1,-1)_{\mathbb{F}} = -1, -1 \in \mathbb{E}^{*2}. \end{cases}$$

Remark 5.8. With α, β as in the Theorem,

$$\zeta(w/\overline{w}) = \zeta_{\alpha,\beta}(w/\overline{w})(-1, Nw)_{\mathbb{F}}.$$

We have dropped the term $(-1, Nw)$, which is allowed since $w/\overline{w} \to (-1, Nw)$ is a quadratic character of $\mathbb{E}^1$.

Henceforth write $\mathbb{E} = \mathbb{F}(\delta)$ with $\Delta := \delta^2 \in \mathbb{F}$.

Remark 5.9. Condition (20) is equivalent to

$$\begin{aligned} \beta(x^2) &= (-1, x)_{\mathbb{F}}\alpha(x) \quad (x \in \mathbb{F}^*) \\ \beta(\Delta) &= \gamma_{\mathbb{F}}(-1, \psi_{\mathbb{F}})^{-1}\gamma_{\mathbb{E}}(\delta, \psi_{\mathbb{E}})^{-1}c(\delta, \delta)\alpha(\delta). \end{aligned}$$

The splitting has a simple formula on T^2:

$$\zeta(z^2) = c(z, z)\alpha(z) \quad (z \in \mathbb{E}^1),$$

which is independent of $\beta, \psi_{\mathbb{F}}$ and $\psi_{\mathbb{E}}$. Note that any two μ_2 splittings of T have the same restriction to T^2 since they differ by a quadratic character.

The map $w/\overline{w} \to Nw$ induces an isomorphism $T/T^2 \simeq N\mathbb{E}^*/\mathbb{F}^{*2}$. Choose representatives $a_1, \ldots, a_n$ of generators of $N\mathbb{E}^*/\mathbb{F}^{*2}$, with corresponding elements $z_1, \ldots, z_n \in T$. Given α there are two choices of each $\beta(a_i)$ differing by sign, and these signs may may be chosen arbitrarily. The following result follows easily.

Corollary 5.10. *Choose representatives* $z_1 \ldots, z_m$ *of generators of* $T/T^2 \simeq N\mathbb{E}^*/\mathbb{F}^{*2}$. *Choose* α *as in Theorem* 5.7. *Define*

$$\zeta(z^2) = c(z, z)\alpha(z) \quad (z \in \mathbb{E}^1)$$

and for $1 \leq i \leq m$ *let* $\zeta(z_i)$ *be either square root of* $\alpha(z_i)$. *Then* ζ *extends uniquely to a splitting of the cocycle as in Theorem* 5.7.

In the non-archimedean case by ([4], Lemma 0.3.2) $|T/T^2| = 2/|2|_{\mathbb{F}}$, which equals 2 if the residual characteristic of $\mathbb{F}$ is odd.

We conclude with a few remarks about the definition of ζ.

From the definition we have for $w \in \mathbb{E}^*$:

$$c(w,w) = (-Nw, x(w)x(w^2))_{\mathbb{F}}$$

and

$$c(Nw^{-1}, w^2) = (Nw, x(w^2))_{\mathbb{F}}.$$

Note that for $\lambda \in \mathbb{F}^*, w \in \mathbb{E}^*$ we have

$$c(\lambda, w) = (\lambda, x(w))_{\mathbb{F}}.$$

Fix $u \in \mathbb{E}^*$ with $\mathrm{trace}(u) = 0$, and define $\mathrm{Tr}_u : \mathbb{E}^* \to \mathbb{F}^*$ by

$$\mathrm{Tr}_u(z) = \begin{cases} \mathrm{trace}(z) & \mathrm{trace}(z) \neq 0 \\ uz & \mathrm{trace}(z) = 0. \end{cases}$$

Up to conjugation by $SL(2, \mathbb{F})$ we may assume $\iota(x + y\delta) = \begin{pmatrix} x & y\Delta/a \\ ya & x \end{pmatrix}$ for some $a \in \mathbb{F}^*$. If $w = x + y\delta$ with $xy \neq 0$ we have $x(w)x(w^2) = 2xy^2a^2$. Considering the cases with $xy = 0$ separately gives

$$c(w,w) = (-Nw, \mathrm{Tr}_{a\delta}(w))_{\mathbb{F}} \quad (z \in \mathbb{E}^*)$$

and

$$\zeta(z^2) = (-1, \mathrm{Tr}_{a\delta}(z))_{\mathbb{F}}\alpha(z) \quad (z \in \mathbb{E}^1).$$

For example if p is odd and $\mathbb{E} \neq \mathbb{F}(\sqrt{-1})$ then $m = 1$ (cf. Corollary 5.10), we may take $z_1 = -1$, $\alpha = 1$ and $\zeta(-1) = 1$ which gives

$$\zeta(\epsilon z^2) = c(z,z)c(\epsilon, z^2) \quad (\epsilon = \pm 1, z \in \mathbb{E}^1).$$

For example let $\mathbb{F} = \mathbb{R}$ and define $\iota(x+iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$. We take $\alpha(z) = z$ and $\beta(x) = \pm\sqrt{|x|}$. Then for $z \in \mathbb{E}^1$,

$$\zeta(z^2) = (-1, \mathrm{Tr}_{-i}(z))_{\mathbb{R}} z = \mathrm{sgn}(\mathrm{Tr}_{-i}(z))z$$

(independent of β). Note that $a = -1$ and $\mathrm{Tr}_{-i}(iy) = y$ ($y \in \mathbb{R}^*$).

References

[1] K.S. Brown, *Cohomology of Groups*, Number 87 in Graduate Texts in Mathematics, Springer-Verlag, Berlin-Heidelberg-New York, 1982, MR 83k:20002, Zbl 584.20036.

[2] L. Calabi, *Sur les extensions des groupes topologiques*, Ann. Mat. Pura Appl., **32**(**4**) (1951), 295-370, MR 14,245d, Zbl 054.01302.

[3] Y. Flicker, *Automorphic forms on covering groups of* GL(2), Invent. Math., **57** (1980), 119-182, MR 81m:10057, Zbl 431.10014.

[4] D.A. Kazhdan and S.J. Patterson, *Metaplectic forms*, Inst. Hautes Études Sci. Publ. Math., **59** (1984), 35-142, MR 87h:22024, Zbl 559.10026.

[5] ———, *Towards a generalized Shimura correspondence*, Adv. in Math., **60**(**2**) (1986), 161-234, MR 87m:22050, Zbl 616.10023.

[6] T. Kubota, *Automorphic forms and the reciprocity law in a number field*, preprint, Kyoto University, 1969.

[7] S. Kudla, *Splitting metaplectic covers of dual reductive pairs*, Israel J. Math., **87** (1994), 361-401, MR 95h:22019, Zbl 840.22029.

[8] C. Moore, *Extensions and low-dimensional cohomology theory of locally compact groups.* II, Trans. Amer. Math. Soc., **113** (1964), 64-86, MR 30 #2106, Zbl 131.26902.

[9] ———, *Group extensions of p-adic and adelic linear groups*, Inst. Hautes Études Sci. Publ. Math., **35** (1968), 157-222, MR 39 #5575, Zbl 159.03203.

[10] R. Ranga Rao, *On some explicit formulas in the theory of the Weil representation*, Pacific J. Math., **157** (1993), 335-371, MR 94a:22037, Zbl 794.58017.

[11] J. Rogawski and S. Gelbart, *L-functions and Fourier-Jacobi coefficients for the unitary group* U(3), Invent. Math., **105** (1991), 445-472, MR 93b:11059, Zbl 742.11030.

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