Pacific Journal of Mathematics

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Volume 200 No. 2 October 2001

PACIFIC JOURNAL OF MATHEMATICS
Vol. 200, No. 2, 2001

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Using the representation formulas obtained earlier, new comparison theorems for elliptic boundary value problems are developed. Properties of support function of convex domain are applied for proofs and for obtaining nonexistence theorems for solutions of capillary problems in the absence of gravity.

Let D_0 and D_1 $(D_0 \subset D_1)$ be plane convex figures. Denote by A_i and p_i the area and the perimeter of figure D_i $(i = 0, 1)$. Let's suppose that the inequality

$$\frac{A_1}{p_1} > \frac{A_0}{p_0} \tag{1}$$

holds. In the present paper, using (1), we shall obtain the comparison theorems for some elliptic boundary value problems (Sections 1-3). Proofs of these theorems were based on representation formulas, obtained earlier. The general steps of proving one of these are given in the Appendix. Further (Section 4) we will formulate the sufficient condition for (1) in terms of mixed area and will use these results to prove some nonexistence theorems for solutions of capillary problems in the absence of gravity (Section 5).

1. The comparison theorem for solutions of second boundary value problem for Helmholtz equation.

Let D be a convex planar domain with $C^{2,\alpha}$ boundary Γ. Hereinafter we denote by n the outward normal to Γ. Let $u(x, y)$ be a solution to the following problem

$$\Delta u = ku \quad \text{in}\, D, \qquad u_n|_\Gamma = R > 0. \tag{2}$$

In [7] we have proved the following:

Theorem 1. *Let u be the solution to problem (2), and z be a solution of second boundary value problem for the Poisson equation*

$$\Delta z = \frac{Rp}{A} \quad in\, D, \qquad z_n|_\Gamma = R, \tag{3}$$

such that

$$\iint\limits_D z\,dxdy = 0.$$

Then the solution of problem (2) *can be represented as*

$$u = \frac{Rp}{kA} + z + \omega, \tag{4}$$

where ω *satisfies the inequality* $\max|\omega| < C|k|$ *in* D.

Let u_i be the solution to (2) in the domain D_i. From (4) we immediately obtain:

Theorem 2. *Let domains* D_0 *and* D_1 *be such that the conditions of Theorem* 1 *and inequality* (1) *hold. Then there exists a number* $k_0 > 0$ *such that for any positive number* $k < k_0$ *the inequality* $u_0 > u_1$ *holds in the domain* D_0.

The result of Theorem 2 we had anounced in [**6**].

2. The comparison theorem for solutions of the third boundary value problem for Poisson equation.

In the third boundary value problem it is required to find a solution of the equation

$$\Delta u = -1 \tag{5}$$

in domain D with the boundary conditions

$$u + \beta\frac{\partial u}{\partial n} = 0 \quad (\beta > 0). \tag{6}$$

The solution of this problem satisfies the theorem of representation ([**5**], [**8**]).

Theorem 3. *Let the boundary* Γ *of plane convex domain* D *belongs to the class* $C^{2,\alpha}$ *and its curvature is separated from zero. Then*

$$u = \frac{\beta A}{p} + u_\infty + \omega, \tag{7}$$

where u_∞ *is the solution of Equation* (5) *such that*

$$\frac{\partial u_\infty}{\partial n} = -\frac{A}{p}, \qquad \int\limits_\Gamma u_\infty\,ds = 0,$$

and function ω *satisfies the inequality* $\max|\omega| < C\beta^{-1}$ *in* D.

Let u_i be the solutions of the third boundary value problems in domains D_i. From representation (7) we immediately obtain:

Theorem 4. *Let the domains D_0 and D_1 be such that the conditions of Theorem 3 and inequality (1) hold. Then there exists $\beta_0 > 0$ such that for any $\beta > \beta_0$ the inequality $u_1 > u_0$ holds in D_0.*

3. The comparison theorem of capillary surfaces heights in case of small gravity.

It is well-known (see [**3**]) that the searching of the form of liquid free surface in cylindrical tube under capillary forces and force of gravity is equivalent to the following boundary value problem. It is required to find the solution of the equation

$$\mathrm{div} Tu = ku \tag{8}$$

in domain D with boundary condition

$$(Tu, n) = \cos\gamma, \qquad (Tu = \nabla u/\sqrt{1+|\nabla u|^2}). \tag{9}$$

In the absence of gravity, the equation of liquid free surface takes the form

$$\mathrm{div} Tu = \frac{p\cos\gamma}{A}, \tag{10}$$

but boundary condition remains in form (9).

Below we consider M. Miranda question ([**3**], Sec. 5.3, 5.4): Does a liquid in a “wide” capillary tube rise lower than in a “narrow” one. This question is equivalent to the following problem: Let u_0 and u_1 be solutions of Equation (8) in domains D_0 and D_1 $(D_0 \subset D_1)$ with boundary conditions (9) on boundaries Γ_0 and Γ_1. Is it right that $u_0 > u_1$ in D_0?

In [**3**] some conditions for an affirmative answer are given, and also an example for which the answer is negative.

D. Siegel has proved in [**13**] for plane domain with $C^{2,\alpha}$ boundary the following:

Theorem 5. *Let there exists a solution z to the problem (10)-(9). Then solution u of the problem (8)-(9) can be represented as*

$$u = \frac{p\cos\gamma}{kA} + z + \omega \tag{11}$$

while the function ω satisfies the inequality $\max|\omega| < C|k|$ in D.

L_2-estimate of ω was received in [**7**].

Now the comparison theorem is immediately following from representation (11).

Theorem 6. *Let $0 < \gamma < \pi/2$, the inequality (1) holds and there exist the solutions of the problem (10)-(9) in domains D_0 and D_1, then exists $k_0 > 0$ such that for any $0 < k < k_0$ the inequality $u_0 > u_1$ holds in D_0.*

We note, that for special cases of domains D_i (D_1 is a disk or D_0 is a disk of sufficiently small radius), the comparison Theorems 2, 4 and 6 have been obtained by other methods for arbitrary positive k and β in [**3**], [**12**], [**5**].

On the other hand, it is evident that if the inequality reverse (1) holds, then $u_0 < u_1$ in D_0.

4. Geometrical theorem.

Let us obtain now a sufficient condition under which the inequality (1) holds. We have proved the same implication in [**6**], where we assume sufficient smoothness of a boundaries. In present paper this result is reduced in Example 3.

Let A_{01} be the mixed area of figures D_0 and D_1.

Theorem 7. *Let figures D_0 and D_1 be such that*

$$(p_0 + p_1)A_1 \geq 2A_{01}p_1. \tag{12}$$

Then the inequality (1) *holds.*

Proof. We shall use the formulas from standard manuals ([**1**], [**11**]) on the geometry of convex figures.

Let $D_\theta = (1-\theta)D_0 + \theta D_1$ be the linear family of convex figures. It is well-known that the area A_θ of the figure D_θ is given by formula

$$A_\theta = (1-\theta)^2 A_0 + 2\theta(1-\theta)A_{01} + \theta^2 A_1, \tag{13}$$

and its perimeter p_θ is given by formula

$$p_\theta = (1-\theta)p_0 + \theta p_1. \tag{14}$$

We note that inequality (1) immediately follows from (12) and the Frobenius inequality

$$2A_{01} \geq \frac{A_0 p_1}{p_0} + \frac{A_1 p_0}{p_1}.$$

We shall give another proof whose details give additional information. Let us consider the function

$$f(\theta) = \frac{A_\theta}{p_\theta}.$$

We shall prove that this function is concave in the segment $[0,1]$ and its left derivative $f'(1)$ is positive because of (12). Hence we shall prove that the function $f(\theta)$ monotonically increases. Using formulas (13) and (14), we obtain

$$f(\theta) = \frac{(1-\theta)^2 A_0 + 2\theta(1-\theta)A_{01} + \theta^2 A_1}{(1-\theta)p_0 + \theta p_1}.$$

On the other hand

$$(1-\theta)f(0) + \theta f(1) = (1-\theta)\frac{A_0}{p_0} + \theta\frac{A_1}{p_1}.$$

After elementary algebraic transformations we see that the concavity condition for $f(\theta)$

$$f(\theta) \geq (1-\theta)f(0) + \theta f(1)$$

is equivalent to Frobenius inequality . If we calculate the left derivative

$$f'(1) = \lim_{\epsilon \to 0} \frac{f(1) - f(1-\epsilon)}{\epsilon},$$

using formulas (13) and (14), we obtain

$$\begin{aligned} f(1) - f(1-\epsilon) &= \frac{F_1}{p_1} - \frac{\epsilon^2 F_0 + 2\epsilon(1-\epsilon)F_{01} + (1-\epsilon)^2 F_1}{\epsilon p_0 + (1-\epsilon)p_1} \\ &= \epsilon \frac{(p_0+p_1)A_1 - 2A_{01}p_1}{p_1^2} + O(\epsilon^2). \end{aligned} \tag{15}$$

It is evident that the derivative $f'(1)$ (the coefficient of ϵ in (15)) is nonnegative because of (12).

Theorem 7 has been proved.

If coefficient of ϵ in (15) is negative then inequality opposite (12) holds. This means that between figures D_θ there exists the figure such that $f(\theta) > f(1)$.

On the other hand we shall obtain the condition for inequality opposite (1) if we calculate the right derivative of the function $f(\theta)$ in zero.

Theorem 8. *Let*

$$2A_{01} \leq \frac{p_0 + p_1}{p_0} A_0.$$

Then

$$\frac{A_1}{p_1} < \frac{A_0}{p_0}.$$

Let us consider three important special cases.

Example 1. Let D_1 be a disk with radius R_1. In this case we have $2A_{01} = R_1 p_0$. It is evident that $p_0 < 2\pi R_1$, hence

$$R_1 p_0 2\pi R_1 < (p_0 + 2\pi R_1)\pi R_1^2,$$

and inequality (12) holds.

Example 2. Let D_0 be a disk with radius

$$r < \frac{A_1}{p_1} \frac{1}{1 - \dfrac{2\pi A_1}{p_1^2}}.$$

We know that $2A_{01} = rp_1$. After algebraic transformations we obtain

$$r(p_1^2 - 2\pi A_1) < A_1 p_1,$$

or

$$rp_1p_1 < (2\pi r + p_1)A_1,$$

hence inequality (12) holds.

We note that from isoperimetric inequality

$$\frac{A_1}{p_1}\frac{1}{1-\dfrac{2\pi A_1}{p_1^2}} \le \frac{2A_1}{p_1}$$

(equality holds only if D_1 is a disk), hence, using Example 1, we can to improve the previous result: Let $D_0 \subset D_1$ and D_0 is contained in the disc of radius $R_0 < 2A_1/p_1$ then inequality (1) holds.

Example 3. Let domain D_1 has smooth boundary, whose curvature K_1 satisfies the inequality

$$0 < K_1 \le \frac{p_1}{A_1}. \tag{16}$$

Then inequality (12) holds.

Indeed, let $h_i(\phi)$ be the support function of the domain D_i. Then the following formulas are valid ([**1**])

$$2A_{01} = \int_0^{2\pi} (h_0 h_1 - h_0' h_1')d\phi,$$

$$\frac{1}{K_1} = h_1'' + h_1,$$

$$p = \int_0^{2\pi} h\, d\phi, \qquad 2A = \int_0^{2\pi} (h^2 - h'^2)\, d\phi.$$

Using the inequality (16) we obtain

$$\begin{aligned} 2A_{01} &= \int_0^{2\pi} h_0(h_1'' + h_1)d\phi \\ &= \int_0^{2\pi} (h_0 - h_1)(h_1'' + h_1)d\phi + \int_0^{2\pi} (h_1^2 - h_1'^2)d\phi \\ &\le 2A_1 - \frac{A_1}{p_1}(p_1 - p_0) \\ &= \frac{A_1(p_1+p_0)}{p_1}. \end{aligned}$$

Hence the inequality (12) holds.

Examples stated above show that condition (12) can be used for checking inequality (1).

5. The nonexistence theorems for solutions of capillary problem in the absence of gravity.

Let us return to the problem (10)-(9). If $\gamma = 0$ the important condition of the existence of solution for this problem in the domain D_1 is the following: Let D_0 be an arbitrary subdomain of D_1 then if the solution of the problem (9)-(10) exists then inequality (1) holds. ([**2**]).

Giusti has proved ([**4**]) that for convex domains the sufficient condition of the existence is the inequality (16). Moreover ([**3**]), if the solution of the problem (10)-(9) exists for $\gamma = 0$ then it exists for any $0 < \gamma \leq \pi/2$. Using this statements and our previous speculations we can reformulate the result [**2**] as sufficient condition of nonexistence for problem (10)-(9).

Theorem 9. *Let exists such convex subdomain D_0 of domain D_1 that the inequality opposite (12) holds. Then if $\gamma = 0$ then a solution of (10)-(9) does not exist.*

Proof. Really, if inequality opposite (12) holds then there exists a domain $\overline{D} \subset D_1$ in linear family $D_\theta = (1-\theta)D_0 + \theta D_1$ such that

$$\frac{\overline{A}}{\overline{p}} > \frac{A_1}{p_1}.$$

In particular, (10)-(9) has no solutions, if D_1 is a regular polygon. Indeed, we can put as D_0 the disk inscribed into D_1 .

Using results of Section 4, we can add the following simple condition of nonexistence of solutions for (10)-(9) in case of $\gamma = 0$.

Theorem 10. *Let we can inscribe into D_1 the disk of radius*

$$r > \frac{p_1 A_1}{p_1^2 - 2\pi A_1}. \tag{17}$$

Then in case of $\gamma = 0$ the solution of the problem (10)-(9) does not exist.

Proof. We immediately obtain from (17)

$$r p_1 > \frac{2\pi r + p_1}{p_1} A_1. \tag{18}$$

We can take a disk of radius r as the domain D_0. It is evident that the inequality (18) is the inequality opposite (12), hence we can apply Theorem 9.

Let us obtain now the generalization of Theorem 10.

Theorem 11. *Let we can inscribe in domain D_1 the disc of radius r such that inequality (17) holds. Then problem (10)-(9) has no solutions for any contact angle γ satisfying the inequality*

$$\cos\gamma > \frac{A_1}{r p_1}\left(1 + \sqrt{\frac{4\pi(p_1 r - \pi r^2 - A_1)}{p_1^2 - 4\pi A_1}}\right). \tag{19}$$

Proof. We remind the general idea of nonexistence proofs: If we can find subdomain $\overline{D} \subset D_1$ such that

$$\frac{A_1}{p_1} < \frac{\overline{A}\cos\gamma}{\overline{p}}, \tag{20}$$

then problem (10)-(9) has no solution ([**3**]).

We shall find the subdomain $\overline{D}$ in a certain linear family D_θ. We can reformulate the nonexistence condition in the following form: Let there exist a subdomain $D_0 \subset D_1$ and number $\overline{\theta} \in (0,1)$ such that

$$f(1) < f(\overline{\theta}), \tag{21}$$

then the problem (10)-(9) has no solutions.

Let us construct the corresponding linear family.

Let the convex domain $D_0 \subset D_1$ be such that

$$2A_{01}p_0 > (p_0+p_1)A_0, \qquad 2A_{01}p_1 > (p_0+p_1)A_1. \tag{22}$$

It follows from Theorems 7 and 8, that function $f(\theta)$ reaches its maximum value in the interval $(0,1)$. Let us find this value. We represent the function $f(\theta)$ in the form

$$f(\theta) = -\frac{E\theta}{\Delta p} + \frac{S}{(\Delta p)^2} - \frac{G}{(\Delta p)^2(\theta\Delta p + p_0)},$$

where

$$E = 2A_{01} - A_0 - A_1, \qquad S = 2p_1(A_{01} - A_0) + p_0(A_0 - A_1),$$
$$G = 2p_0p_1A_{01} - p_1^2A_0 - p_0^2A_1, \qquad \Delta p = p_1 - p_0.$$

It follows from the Frobenius inequality that $G > 0$, and inequalities (22) shows that $E > 0$.

After calculations we see that function $f(\theta)$ reaches its maximum value in the point

$$\overline{\theta} = \frac{1}{\Delta p}\left(\sqrt{\frac{G}{E}} - p_0\right),$$

and this value is equal to

$$f(\overline{\theta}) = \frac{2}{(\Delta p)^2}\Big((p_0+p_1)A_{01} - p_1A_0 - p_0A_1 - \sqrt{GE}\Big). \tag{23}$$

Using inequality (21), we can reformulate the sufficient condition of nonexistence of solution to the problem (10)-(9): Let there exist a convex subdomain D_0 of the domain D_1 such that inequalities (22) holds, then for any contact angle γ, satisfying inequality

$$\cos\gamma > \frac{A_1(\Delta p)^2}{2p_1((p_0+p_1)A_{01} - p_1A_0 - p_0A_1 - \sqrt{GE})}, \tag{24}$$

problem (10)-(9) has no solutions.

Let D_0 be the the disk, which radius r satisfies the inequality (17). Then

$$A_0 = \pi r^2, \qquad p_0 = 2\pi r, \qquad 2A_{01} = rp_1,$$
$$G = \pi r^2(p_1^2 - 4\pi A_1), \qquad E = rp_1 - \pi r^2 - A_1.$$

Note that the second of inequalities (22) holds because of (17) and the first one holds automatically. Substituting last formulas in inequality (24) we obtain Theorem 11 after algebraic transformations.

Example. Let domain D_1 be the regular n -polygon circumscribed around a circle of radius r. Then the inequality (19) takes the form

$$\cos\gamma > \frac{1}{2}\left(1 + \sqrt{\frac{\pi}{n\tan\frac{\pi}{n}}}\right).$$

For large n we can write more simple formula

$$\gamma < \frac{\pi}{n\sqrt{6}}.$$

We see that this estimate is weaker than the exact one ([**3**], Th. 6.2): $\gamma < \pi/n$, but it holds the same form in case of smoothed angles.

Appendix.

Let us consider now the general steps for proof Theorem 1. Hereafter we denote by C (with subscripts or without them) the constants depending on geometrical characteristics of domain D.

It is evident that in convex domain D Poincaré inequality holds

$$\iint\limits_D u^2\,dxdy \le \frac{1}{A}\left(\iint\limits_D u\,dxdy\right)^2 + \mu \iint\limits_D |\nabla u|^2\,dxdy.$$

Let

$$v = u - \frac{Rp}{kA}.$$

Function v satisfies the equation

$$\Delta v = kv + \frac{Rp}{A} \tag{25}$$

and $v_n = R$ on Γ. Let us integrate (2) over D . It is easy to see that

$$\iint\limits_D v\,dx\,dy = 0.$$

We subtract Equation (3) from Equation (25). We obtain

$$\Delta(v - z) = k(v - z) + kz.$$

Denote $\omega = v - z$. Then

$$\Delta\omega = k\omega + kz. \tag{26}$$

We multiply (26) on ω and integrate over D. We obtain

$$\iint_D \Delta\omega\omega\, dx\, dy = \iint_D k\omega^2\, dx\, dy + \iint_D z\omega\, dx\, dy. \tag{27}$$

We transform the left side of (27) by well-known formulas, taking into account that $\omega_n = 0$. We obtain

$$-\iint_D |\nabla\omega|^2\, dx\, dy = \iint_D k\omega^2\, dx\, dy + \iint_D kz\omega\, dx\, dy.$$

Taking into account that

$$\iint_D \omega\, dx\, dy = 0 \tag{28}$$

we use the Poincaré inequality

$$\begin{aligned}\left(k + \frac{1}{\mu}\right)\|\omega\|^2_{L_2} &\le k\|\omega\|^2_{L_2} + \iint_D |\nabla\omega|^2\, dx\, dy \\ &\le k\iint_D \omega^2\, dx\, dy + \iint_D |\nabla\omega|^2\, dx\, dy,\end{aligned}$$

and the Cauchy-Schvarz-Bunyakovskii inequality

$$-k\iint_D \omega z\, dx\, dy \le |k|\left(\iint_D z^2\, dx\, dy\right)^{1/2}\left(\iint_D \omega^2\, dx\, dy\right)^{1/2}.$$

We obtain after algebraic transformations

$$\|\omega\|_{L_2} \le \frac{\mu|k|}{1+\mu k}\|z\|_{L_2}. \tag{29}$$

By S.L. Sobolev embedding theorem:

$$\left(\max_{\overline{D}} \omega\right)^2 \le C_1\|\nabla_2\omega\|^2_{L_2} + C_2\|\omega\|^2_{L_2}. \tag{30}$$

L_2 - norm of second derivitives of ω in plane convex domain is estimated from L_2 - norm of operator $\Delta\omega$. The detailed proof of this estimate for solution of the first boundary value problem is given in [**10**]. The same proof yields the same estimate for solutions of second boundary value problem as well. Indeed, let $g = \omega_{xx}\omega_{yy} - \omega^2_{xy}$. It is evident that

$$\omega^2_{xx} + 2\omega^2_{xy} + \omega^2_{yy} = (\Delta\omega)^2 - 2g.$$

Using the identity

$$2\iint\limits_D g\,dxdy = \int\limits_\Gamma (\omega_x\omega_{xy} - \omega_y\omega_{xx})dx - (\omega_x\omega_{yy} - \omega_y\omega_{xy})dy,$$

we obtain because of boundary condition of problem (2) and convexity of domain D

$$2\iint\limits_D g\,dxdy = \int\limits_\Gamma K|\nabla\omega|^2\,ds > 0$$

and the estimate is proved.

Furthermore, we obtain from (26)

$$\|\nabla_2\omega\|_{L_2}^2 \le \|\Delta\omega\|_{L_2}^2 \le 2k^2(\|\omega\|_{L_2}^2 + \|z\|_{L_2}^2).$$

The statement of Theorem 1 follows from the substitution of the latter estimate in (30) using (29).

Remark. Analyzing the proof of Theorem 1, we can see that requirement to convexity of domain D is excessive. Indeed, we can require only the realizability of the Poincaré inequality, S.L. Sobolev embedding theorem and the possibility to estimate $||\nabla_2 u||_{L_2}$ by means of $||\Delta u||_{L_2}$. These conditions are contained in [**10**].

Acknowledgments. We would like to thank Professor R. Finn for his comments and suggestions. Professor R. Finn has played a critical role in the design and the completion of this paper. We are grateful to Doctor G.N. Krichevets for helpful discussion.

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Received March 19, 1999 and revised June 19, 2000.

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