Pacific Journal of Mathematics

CR EXTENSION FOR L^p CR FUNCTIONS ON A QUADRIC SUBMANIFOLD OF C^n

Albert Boggess

Volume 201 No. 1 November 2001

PACIFIC JOURNAL OF MATHEMATICS
Vol. 201, No. 1, 2001

CR EXTENSION FOR L^p CR FUNCTIONS ON A QUADRIC SUBMANIFOLD OF C^n

Albert Boggess

We consider the space, $\mathrm{CR}^p(M)$, consisting of CR functions which also lie in $L^p(M)$ on a quadric submanifold M of C^n of codimension at least one. For $1 \le p \le \infty$, we prove that each element in $\mathrm{CR}^p(M)$ extends uniquely to an H^p function on the interior of the convex hull of M. As part of the proof, we establish a semi-global version of the CR approximation theorem of Baouendi and Treves for submanifolds which are graphs and whose graphing functions have polynomial growth.

1. Definitions and main results.

We will be working in $C^n = C^m \times C^d$ with coordinates

$$(w = u + iv, z = x + iy) \in C^m \times C^d.$$

A bilinear form $q : C^m \times C^m \mapsto C^d$ is said to be a *quadric form* if

$$\overline{q(w_1, w_2)} = q(\overline{w_2}, \overline{w_1}) \quad \text{for } w_1, w_2 \in C^m.$$

Note that this requirement implies that $\overline{q(w, \overline{w})} = q(w, \overline{w}) \in R^d$ for all $w \in C^m$. A submanifold $M \subset C^n$ is said to be a *quadric submanifold* if there exists a quadric form q such that

$$M = \{(w, z) \in C^m \times C^d; \mathrm{Re}\, z = q(w, \overline{w})\}.$$

The closed convex hull of M, denoted $ch(M)$, can be identified with $M + \Gamma$ where

$$(1) \qquad \Gamma = \text{closed convex hull of } \{q(w, \overline{w});\ w \in C^m\} \subset R^d.$$

The set Γ can be identified with the convex hull of the image of the Levi form of M at the origin (see [**B1**] or [**BP**] for details). We are interested in the case where the interior of $ch(M)$ is nonempty.

We say that F belongs to $H^p(M + \Gamma)$ if F is holomorphic on the interior of $M + \Gamma$ and

$$||F||_{H^p(M+\Gamma)} = \sup_{x \in \text{interior}\{\Gamma\}} \left(\int_{m \in M} |F(m + x)|^p \, d\sigma(m) \right)^{1/p} \quad \text{is finite}$$

where $d\sigma(w, y) = \sqrt{1 + |\nabla q(w, \overline{w})|^2} d\lambda(w)\, dy$ is the usual surface measure for M (graphed over the y and w - variables and $d\lambda(w)$ is Lebesgue measure on

C^m). If $p = \infty$, then the integral on the right is replaced by $\sup_{m\in M} |F(m+x)|$. This definition can also be localized (yielding the space $H^p_{\text{loc}}(M+\Gamma)$) by restricting the domain of integration to a small open subset about any given point in M. This definition of H^p for $ch(M) = M + \Gamma$ is analogous to the usual defintion of H^p for an open set $D \subset C^n$ or a tube domain $B + iR^n$, where $B \subset R^n$. In our context, M plays the role of the boundary of D or iR^n in the tube case.

In [**BN**], it was shown that for $1 \le p \le \infty$, the space $H^p_{\text{loc}}(M+\Gamma)$ is isomorphic to the space of functions in $L^p_{\text{loc}}(M)$ that satisfy the tangential Cauchy-Riemann equations in the sense of distribution theory. In this paper, we globalize these results for quadrics. The key new ingredient here is a semi-global version of the CR approximation theorem (by entire functions) of Baouendi and Treves [**BT**]. A global version of this approximation theorem does *not* hold in general (see [**BT**], Examples 3.1 and 3.2). However a global version does hold for tube-like CR manifolds (see [**B2**]). A semi-global version for rigid submanifolds whose graphing function has polynomial growth will be established in Section 2. The proof of our global CR extension theorem will follow from this approximation theorem and by an analytic disc construction (Sections 3, 4 and 5).

To precisely state our theorem, we need the following definitions. For $1 \le p \le \infty$, we define $\text{CR}^p(M)$ to be the space of functions in $L^p(M)$ (with respect to surface measure on M) satisfying the tangential Cauchy-Riemann equations on M in the sense of distribution theory. We say that a subcone $\Gamma' \subset \Gamma \subset R^d$ is *smaller* than Γ (and write $\Gamma' < \Gamma$) if $\Gamma' \cap S \subset\subset \text{interior}\{\Gamma\} \cap S$ where S is the unit sphere in R^d. Our main result is the following.

Theorem 1. *Suppose M is a quadric submanifold of C^n and suppose the interior of the convex hull of the Leviform (i.e.,* interior$\{\Gamma\}$*) is nonempty. Let $1 \le p \le \infty$ and suppose $f \in \text{CR}^p(M)$. Then there exists a unique $F \in H^p(M+\Gamma)$ which extends f in the sense that if $\Gamma' < \Gamma$, then*

$$\lim_{x\in\Gamma',\ x\mapsto 0} \int_{m\in M} |F(m+x) - f(m)|^p \, d\sigma(m) = 0 \quad \text{if } 1 \le p < \infty. \tag{2}$$

If $p = \infty$, then

$$\lim_{x\in\Gamma',\ x\mapsto 0} F(m+x) = f(m) \quad \textit{for almost all } m \in M. \tag{3}$$

For $1 \le p \le \infty$

$$||F||_{H^p(M+\Gamma)} = ||f||_{L^p(M)}. \tag{4}$$

Conversely, if $F \in H^p(M+\Gamma)$, then there exists an $f \in \text{CR}^p(M)$ such that (2), (3) *and* (4) *hold.*

Since $H^p(M+\Gamma) \subset H^p_{\text{loc}}(M+\Gamma)$ all the local results in [**BN**] apply to $H^p(M+\Gamma)$. In particular, pointwise almost everywhere limits of F exist on

M within admissible approach regions of $M+\Gamma$. These approach regions lie within any smaller cone $\Gamma' < \Gamma$ and allow quadratically-tangential approach along the complex tangent directions and nontangential approach along the totally real tangent directions.

If $\Gamma = R^d$, then $M+\Gamma = C^n$ and so the extension, F, is an entire function with a uniform bound on $||F||_{L^p(M+x)}$ for all $x \in R^d$. If $1 \leq p < \infty$, then this F must be zero. If $p = \infty$, then F must be constant. Thus, we have the following corollary.

Corollary 1. *If M is a quadric submanifold of C^n with $\Gamma = R^d$, then there are no nonzero elements in* $\mathrm{CR}^p(M)$ *for $1 \leq p < \infty$. Any function belonging to* $\mathrm{CR}^\infty(M)$ *is constant.*

2. The approximation theorem.

In [**BT**], Baouendi and Treves proved that continuous CR functions can be locally approximated by entire functions. As already mentioned, a global version of this theorem does not hold in general. In this section, we prove a semi-global approximation theorem in the L^p - norm for rigid CR manifolds, whose graphing function has polynomial growth. This class of submanifolds includes the quadric submanifolds (whose graphing function grows quadratically) which is our main interest. The key idea in the proof uses Baouendi and Treves' technique of convolving the CR function against a kernel along a slice that passes through a variable point (in a Radon transform-like fashion) and then using Stokes Theorem to show that the slice can be fixed (independent of the point in question). The new ingredient here is the use of a different exponential kernel which takes into account the polynomial growth of the graphing function to M. Some additional technicalities are needed to handle the L^p - norm.

Theorem 2. *Suppose $M = \{(w = u+iv, z);\ \mathrm{Re}\, z = h(u,v)\}$ where $h : C^m \mapsto R^d$ is a smooth function with the following polynomial growth estimate:*

$$|(Dh)(u,v)| \leq C|w|^N \quad \text{for all } w \in C^m$$

where D is any first-order derivative and where C and N are uniform positive integers. Suppose f is an element of $\mathrm{CR}^p(M)$ *for $1 \leq p < \infty$. Then there exists a sequence of entire functions F_k such that for each compact set $K \subset M$, $F_k \mapsto f$ in $L^p(K)$ as $k \mapsto \infty$. If $p = \infty$, then F_k converges pointwise almost everywhere on M to f.*

For $1 \leq p \leq \infty$ and for each compact set $K \subset M$, there is a constant C_K such that $||F_k||_{L^p(K)} \leq C_K ||f||_{L^p(K)}$ for all $k = 1, 2, \ldots$.

Proof. M is parameterized by the following function $H : R^m \times R^m \times R^d \mapsto C^m \times C^d$

$$H(u,v,y) = (u+iv, h(u,v)+iy) \quad \text{for } u,\ v \in R^m\ y \in R^d.$$

For fixed $u \in R^m$, let

$$M_u = \{H(u,t,s); t \in R^m,\ s \in R^d\}.$$

For any point $(w,z) = (u+iv,z) \in M$, M_u is an $n = m+d$-dimensional real slice of M that passes through (w,z).

For $\zeta = (\zeta_1, \dots, \zeta_n) \in C^n$ and any positive integer q let

$$(\zeta)^q = \zeta_1^q + \cdots + \zeta_n^q.$$

Note that if $\zeta \in R^n$ and q is even then $(\zeta)^q \approx |\zeta|^q$. For $\eta \in C^d$ and $\zeta \in C^m$, define

$$E(\eta,\zeta) = \frac{1}{C_1} e^{(\zeta)^2 - (\eta)^{4N}}$$

where C_1 is a constant chosen so that

$$\int_{t \in R^m,\ s \in R^d} E(it, is)\, dt\, ds = 1. \tag{5}$$

We have the following lemma.

Lemma 1. *Suppose* $f \in L^p(M)$ *(not necessarily* CR*). For* $(w,z) = (u+iv, x+iy) \in M \subset C^n = C^{m+d}$ *and for* $\epsilon > 0$*, define*

$$G_\epsilon(f)(w,z) = \frac{1}{i^n \epsilon^n} \int_{(\eta,\zeta) \in M_u} f(\eta,\zeta) E\left(\frac{w-\eta}{\epsilon}, \frac{z-\zeta}{\epsilon}\right) d\eta \wedge d\zeta \tag{6}$$

where $d\eta = d\eta_1 \wedge \cdots \wedge d\eta_m$ *and* $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_d$*. Then,* $G_\epsilon(f)(w,z)$ *is well-defined for almost all* $(w,z) \in M$*. If* $1 \le p < \infty$*, then for any compact set* $K \subset M$*,* $||G_\epsilon(f) - f||_{L^p(K)} \mapsto 0$ *as* $\epsilon \mapsto 0$*.*

If $p = \infty$*, then a subsequence* G_{ϵ_k} *converges to* f *pointwise almost everywhere on* M*. If* $f \in \mathrm{CR}^p(M)$*,* $1 \le p \le \infty$*, and if* $K \subset M$ *is a compact set, then there is a constant* C_K *such that* $||G_\epsilon(f)||_{L^p(K)} \le C_K ||f||_{L^p(K)}$*.*

Proof of the Lemma. The principal term in the kernel of the operator G_ϵ looks like a convolution operator. Therefore, the proof of this lemma proceeds in three steps. We first use a change of variables, which is typical for analyzing convolution operators. Then we estimate the remainder terms and complete the proof.

Step 1. **Change of Variables.** For $(w = u+iv, z) \in M$ and $(\eta,\zeta) \in M_u$, we write

$$(w,z) = (u+iv, h(u,v)+iy) \quad \text{and} \quad (\eta,\zeta) = (u+it, h(u,t)+is).$$

So

$$w - \eta = i(v-t) \quad \text{and} \quad z - \zeta = (h(u,v) - h(u,t) + i(y-s)).$$

We obtain

$$G_\epsilon(f)(w,z) = \frac{1}{C_1\epsilon^n}\int_{t,s} f(H(u,t,s))e^{J_\epsilon(u,v,y,t,s)}\,dt\,ds \tag{7}$$

where

$$J_\epsilon(u,v,y,t,s) = \frac{1}{\epsilon^2}\left[h(u,v)-h(u,t)+i(y-s)\right]^2 - \frac{(v-t)^{4N}}{\epsilon^{4N}}.$$

We now make the change of variables:

$$t = v - \epsilon t' \qquad s = y - \epsilon s'$$

and obtain

$$\begin{aligned} G_\epsilon(f)(w,z) &= \frac{1}{C_1}\int_{t',s'} f(H(u,v-\epsilon t', y-\epsilon s'))e^{J_\epsilon(u,v,y,v-\epsilon t', y-\epsilon s')}\,dt' ds' \\ &= \frac{1}{C_1}\int_{t',s'} f(H(u,v-\epsilon t', y-\epsilon s')) \end{aligned}$$

$$\exp\left\{-\left(s' - i\left[\frac{h(u,v)-h(u,v-\epsilon t')}{\epsilon}\right]\right)^2 - (t')^{4N}\right\}\,dt'\,ds'. \tag{8}$$

Step 2. **Estimate of Terms.** The real part of the exponent of the integrand in (8) is

(9)

$$\operatorname{Re} J_\epsilon(u,v,y,v-\epsilon t', y-\epsilon s') = -|s'|^2 - (t')^{4N} + \left(\frac{h(u,v-\epsilon t')-h(u,v)}{\epsilon}\right)^2.$$

Since $|(Dh)(u,v)| \le C(|u|^N + |v|^N)$, we have (by Mean Value)

$$|h(u,t)-h(u,v)| \le C(|u|^N + |v|^N + |v-t|^N)|v-t|.$$

Therefore

$$\left|\frac{h(u,v-\epsilon t')-h(u,v)}{\epsilon}\right|^2 \le C(|u|^{2N}+|v|^{2N}+\epsilon^{2N}|t'|^{2N})|t'|^2.$$

For a compact set K, there is a constant C_K such that $C(|u|^{2N}+|v|^{2N}) \le C_K$ for all (u,v) belonging to K, and so

$$\left|\frac{h(u,v-\epsilon t')-h(u,v)}{\epsilon}\right|^2 \le C_K|t'|^2 + C\epsilon^{2N}|t'|^{2N+2}. \tag{10}$$

From (9) and (10), we obtain

$$\operatorname{Re} J_\epsilon(u,v,y,v-\epsilon t', y-\epsilon s') \le -|s'|^2 - (t')^{4N} + C_K|t'|^2 + C\epsilon^{2N}|t'|^{2N+2}$$

for $(u,v)\in K$. There exists an $\epsilon_0 > 0$ and $R_K > 0$ (depending only on K) such that for $0 \le \epsilon \le \epsilon_0$ and for $t' \in R^m$ with $|t'| \ge R_K$,

$$-(t')^{4N} + C_K|t'|^2 + C\epsilon^{2N}|t'|^{2N+2} \le -(1/2)(t')^{4N}.$$

Therefore

$$\operatorname{Re} J_\epsilon(u,v,y,v-\epsilon t', y-\epsilon s') \le -(1/2)(t')^{4N} - |s'|^2$$

for $s' \in R^d$, $t' \in R^m$ with $|t'| \ge R_K$ and $(u,v) \in K$. This inequality together with (8) implies

$$||G_\epsilon(f)||_{L^p(K)} \le \widetilde{C}_K ||f||_{L^p(K)} \tag{11}$$

for some constant $\widetilde{C}_K$ depending only on K.

Step 3. **Proof that $G_\epsilon(f) \mapsto f$.** We return to $G_\epsilon(f)$ as given in (8). We first assume f is continuous with compact support. We have shown that for (w,z) in the compact set $K \subset M$, the integrand in (8) is dominated by

$$\widetilde{C}_K \left(\sup |f| e^{-(1/2)(t')^{4N} - |s'|^2} \right)$$

for t' large and ϵ small where $\widetilde{C}_K$ is a constant depending only on K. The right side is integrable in $(t', s') \in R^m \times R^d$. By the Dominated Convergence Theorem, we obtain

$$\lim_{\epsilon \mapsto 0} G_\epsilon(f)(w,z) = \frac{f(H(u,v,y))}{C_1} \int_{t',s'} e^{-(s' - iD_v h(u,v)\cdot t')^2 - (t')^{4N}} \, dt' ds' \tag{12}$$

uniformly for $(w,z) \in K \subset M$.

To evaluate the integral on the right, consider the following function defined on the set, $M_{m\times m}$, consisting of $m \times m$ complex-valued matricies:

$$I(Z) = \frac{1}{C_1} \int_{t',s'} e^{-(s'+Z\cdot t')^2 - (t')^{4N}} \, dt' ds' \quad \text{for} \quad Z \in M_{m\times m}.$$

Since the integrand is exponentially decreasing in t' and s', $I(Z)$ is an entire function of Z. When Z has real entries, $I(Z) = 1$, which can be seen by using the change of variables $\hat{s} = s' + Z \cdot t'$ and $\hat{t} = t'$ together with the choice of constant C_1 (see (5)). By the identity theorem for analytic functions, $I(Z) = 1$ for all $Z \in M_{m\times m}$. Therefore by (12),

$$\lim_{\epsilon \mapsto 0} G_\epsilon(f)(w,z) = f(H(u,v,y)) = f(w,z)$$

and this limit is uniform in $(w,z) = H(u,v,y) \in K \subset M$. This completes the proof of the lemma in the case where f is continuous with compact support.

If f belongs to $L^p(M)$ and $1 \le p < \infty$, then f can be approximated in $L^p(M)$ by a continuous, compactly supported $\widetilde{f}$. We then have

$$\begin{aligned} ||G_\epsilon(f) - f||_{L^p(K)} &\le ||G_\epsilon(f - \widetilde{f})||_{L^p(K)} + ||G_\epsilon(\widetilde{f}) - \widetilde{f}||_{L^p(K)} + ||\widetilde{f} - f||_{L^p(K)} \\ &\le (\widetilde{C}_K + 1)||\widetilde{f} - f||_{L^p(K)} + ||G_\epsilon(\widetilde{f}) - \widetilde{f}||_{L^p(K)} \quad \text{by (11).} \end{aligned}$$

The right side can be made as small as desired since we have already shown $G_\epsilon \widetilde{f}$ approximates a continuous $\widetilde{f}$.

If $f \in L^\infty$, then f also belongs to L^1_{loc} and the above arguments show that $G_\epsilon(f) \mapsto f$ in $L^1(K)$. Thus, there is a subsequence, $G_{\epsilon_k}(f)$, which converges to f pointwise almost everywhere. This completes the proof of the lemma.

The integrand defining $G_\epsilon(f)(w,z)$ is analytic in both z and w (see (6)). However, the domain of integration (M_u) depends on w (since $u = \mathrm{Re}(w)$) and therefore $G_\epsilon(f)(w,z)$ is not necessarily holomorphic in w. The next lemma states that if f is CR on M, then the domain of integration in $G_\epsilon(f)(w,z)$ can be chosen to be independent of (w,z) and thus $G_\epsilon(f)$ is an entire function. This next lemma will then complete the proof of the approximation theorem.

Lemma 2. *Suppose $f \in L^p(M)$ and suppose K, E and $G_\epsilon(f)$ are given as in Lemma 1. For any fixed $u_0 \in R^m$, define*

$$F_\epsilon(f)(w,z) = \frac{1}{i^n \epsilon^n} \int_{(\eta,\zeta)\in M_{u_0}} f(\eta,\zeta) E\left(\frac{w-\eta}{\epsilon}, \frac{z-\zeta}{\epsilon}\right) d\eta \wedge d\zeta$$

where $d\zeta = d\zeta_1 \wedge \cdots \wedge d\zeta_d$, $d\eta = d\eta_1 \wedge \cdots \wedge d\eta_m$. There exists $u^0 \in R^m$ such that $F_\epsilon(f)(w,z)$ is a well-defined, entire function of (w,z). If in addition, f satisfies the tangential Cauchy-Riemann equations on M, then for each ϵ, $F_\epsilon(f)(w,z) = G_\epsilon(f)(w,z)$ for almost all $(w,z) \in M$.

Proof. The proof involves two steps. First, we assume f is C^1 and CR. The proof in this case will follow from Stokes Theorem. When f is not C^1 some additional technicalities are involved.

Step 1. **Suppose $f \in C^1 \cap \mathrm{CR}^p(M)$.** Fix any $u_0 \in R^m$. As shown in the proof of Lemma 1, the kernel $E(w-\eta, z-\zeta)$ is exponentially decreasing in $\mathrm{Im}\,(\eta)$ and $\mathrm{Im}\,(\zeta)$. Thus, $F_\epsilon(f)(w,z)$ is a well-defined, entire function of (w,z). If f satisfies the tangential Cauchy-Riemann equations, then we will show that $F_\epsilon(f) = G_\epsilon(f)$ on M by using the Stokes Theorem argument as in [**BT**] (see also [**B1**]). For any $(w,z) \in M$, we connect $u = \mathrm{Re}\,(w)$ to u_0 by a smooth path $\gamma_u(r)$, $0 \le r \le 1$ in R^m with $\gamma_u(0) = u_0$ and $\gamma_u(1) = u$. Define

$$\widetilde{M}_u = \{H(\gamma_u(r), v, y);\ y \in R^d,\ v \in R^m,\ \ 0 \le r \le 1\}.$$

$\widetilde{M}_u$ is an $m+d+1 = n+1$-dimensional manifold whose (manifold) boundary is M_u and M_{u_0}. By Stokes Theorem,

$$\begin{aligned} &F_\epsilon(f)(w,z) - G_\epsilon(f)(w,z) \\ &= \frac{1}{i^n\epsilon^n} \int_{(\eta,\zeta)\in\widetilde{M}_u} d\left\{ f(\eta,\zeta) E\left(\frac{w-\eta}{\epsilon}, \frac{z-\zeta}{\epsilon}\right) d\eta \wedge d\zeta \right\} \end{aligned}$$

for $(w,z) \in M$. Since the form on the right involves the $(n,0)$-form $d\eta \wedge d\zeta$, the d on the right is really just $\overline{\partial}$. If f satisfies the tangential Cauchy-Riemann equations then the form on the right is $\overline{\partial}$-closed because $E(\eta,\zeta)$ is holomorphic in both η and ζ. Thus, the right side is zero and so $F_\epsilon(f)(w,z) = G_\epsilon(f)(w,z)$ for every $(w,z) \in M$. This completes the proof in the case f is C^1.

Step 2. **Suppose that $f \in \mathrm{CR}^p(M)$ but not C^1.** Fix any $(w,z) \in M$. For $u \in R^m$, let

$$F(u) = \int_{(\eta,\zeta)\in M_u} f(\eta,\zeta) E\left(\frac{w-\eta}{\epsilon}, \frac{z-\zeta}{\epsilon}\right) d\eta \wedge d\zeta.$$

Since f belongs to $L^p(M)$ and $E((w-\eta)/\epsilon, (z-\zeta)/\epsilon)$ is exponentially decreasing as $(\eta,\zeta) \mapsto \infty$ along M_u, F is locally integrable on R^m. For almost every lower dimensional slice, the restriction of F is locally integrable along that slice. We are particularly interested in one-dimensional slices where F is integrable. In the following lemma, we shall show that along any such slice, F is almost everywhere constant provided f satisfies the tangential Cauchy-Riemann equations.

Sublemma 1. *Suppose f satisfies the tangential Cauchy-Riemann equations* (*in the sense of distribution theory*). *Let L be a one-dimensional slice of R^m such that $F|_L$ is locally integrable on L. If u^0 and u^1 belong to the Lebesgue set of $F|_L$, then $F(u^0) = F(u^1)$.*

Proof of Sublemma. Without loss of generality, suppose L is the u_1-axis, $u^1 = 0$ (the origin) and $u^0 = (u_1^0, 0, \ldots, 0)$ with $u_1^0 > 0$.

For $r > 0$, let ψ_r be an approximation to the identity with the following properties:

1. ψ_r is a smooth, even and nonnegative function defined on the real line.
2. $\int \psi_r(x)\, dx = 1$.
3. The support of $\psi_r \subset [-r, r]$.

These properties imply that ψ_r converges weakly (in the sense of distribution theory) to δ_0 (the delta function centered at the origin). Recall that if $g : R \mapsto R$ is locally integrable and if x^0 belongs to the Lebesgue set of g, then

$$\lim_{r\mapsto 0} \int_R f(x)\psi_r(x - x^0)\, dx = f(x^0)$$

(see [**SW**]).

Let ϕ_r satisfy

$$\phi_r'(x) = \psi_r(x) - \psi_r(x - u_1^0).$$

Graphs of $\psi_r(x) - \psi_r(x - u_1^0)$ and $\phi_r(x)$ are given in Figures 1 and 2, respectively. If 0 and u_1^0 belong to the Lebesgue set of $F(u_1, 0 \ldots 0)$, then the following limits hold:

$$(13)\quad \lim_{r\mapsto 0} \int \phi_r'(u_1) F(u_1, 0, \ldots, 0)\, du_1 \quad = \quad F(0, \ldots, 0) - F(u_1^0, 0, \ldots, 0).$$

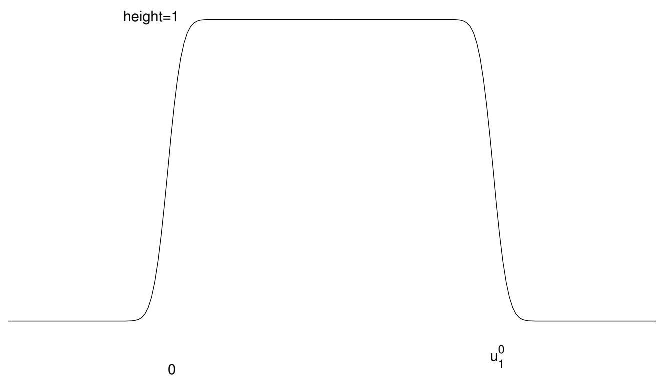

Figure 1. Graph of $\phi_r'(x) = \psi_r(x) - \psi_r(x - u_1^0)$.

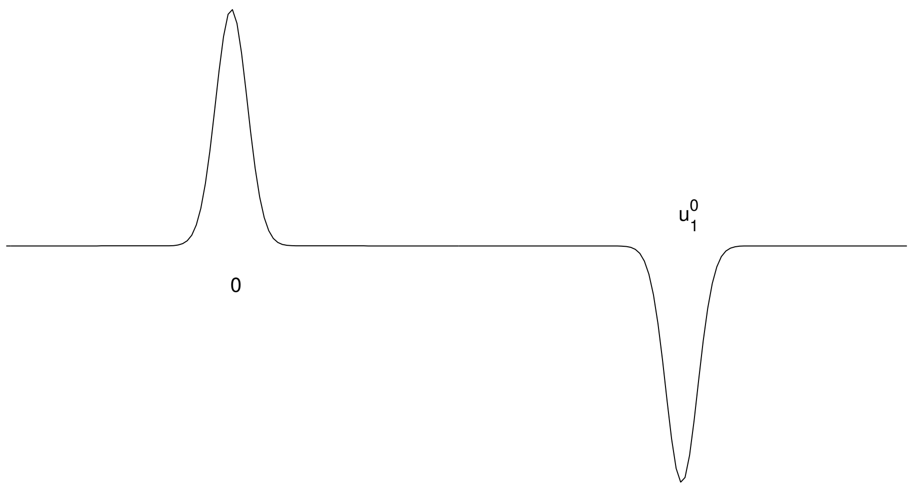

Figure 2. Graph of $\phi_r(x)$.

Let

$$\widetilde{M}_1 = \{H(u_1, 0, \dots 0, v, y);\ u_1 \in R,\ v \in R^m,\ y \in R^d\}.$$

$\widetilde{M}_1$ is a $d + m + 1 = n + 1$ real-dimensional slice of M and $\widetilde{M}_1$ contains M_0 and M_{u^0}. Since f is CR on M, we have

$$d\ [f(\eta, \zeta)\, d\eta \wedge d\zeta] = 0$$

as currents on M. Since $f|_{\widetilde{M}_1}$ is locally integrable, this equation also holds as currents on $\widetilde{M}_1$. Using the notation $\langle , \rangle$ for the pairing between currents and forms on $\widetilde{M}_1$, we have

$$0 = \left\langle f, d\left[\phi_r(\eta,\zeta) E\left(\frac{w-\eta}{\epsilon}, \frac{z-\zeta}{\epsilon}\right) d\eta \wedge d\zeta\right]\right\rangle.$$

Since E is holomorphic in both η and ζ, this equation becomes

$$0 = \int_{\widetilde{M}_1} \phi_r'(u_1) f(\eta,\zeta) E\left(\frac{w-\eta}{\epsilon}, \frac{z-\zeta}{\epsilon}\right) du_1 \wedge d\eta \wedge d\zeta.$$

In view of Equation (13), the right side converges to

$$\int_{M_0} f(\eta,\zeta) E\left(\frac{w-\eta}{\epsilon}, \frac{z-\zeta}{\epsilon}\right) d\eta \wedge d\zeta \\ - \int_{M_{u^0}} f(\eta,\zeta) E\left(\frac{w-\eta}{\epsilon}, \frac{z-\zeta}{\epsilon}\right) d\eta \wedge d\zeta = F(0) - F(u^0).$$

Thus, $F(u^0) = F(0)$, as claimed, and the sublemma is now proved.

As mentioned earlier, F is locally integrable along almost every one dimensional subspace of R^m. Along any such subspace, L, the Lebesgue set of $F|_L$ is a set of full measure within L. Therefore, we can find a set $U \subset R^m$, whose complement is a set of measure zero with the following property:

- If $(u_1^0, \dots, u_m^0)$ belongs to U, then for each $1 \le i \le m$, the function
$$u_i \mapsto F(u_1^0, \dots, u_{i-1}^0, u_i, u_{i+1}^0, \dots, u_m^0)$$
is integrable *and* u_i^0 belongs to the Lebesgue set of this function.

By throwing out a set of measure zero, we can assume that the intersection of U with any one-dimensional line that is parallel with one of the coordinate axis is either empty or a set of full one-dimensional measure.

By a translation, assume that the origin, 0, belongs to U. We claim that $F(0) = F(u)$ for almost every $u \in R^m$. Indeed, Sublemma 1 implies

$$F(0, \dots, 0) = F(u_1, 0, \dots, 0)$$

for every $(u_1, 0, \dots, 0)$ belonging to U (i.e., for almost every u_1). For each $(u_1, 0, \dots, 0) \in U$, Sublemma 1 implies

$$F(u_1, 0, \dots, 0) = F(u_1, u_2, 0, \dots, 0)$$

for each $(u_1, u_2, 0, \dots, 0)$ which belongs to U. Continuing in this way, we conclude that $F(0) = F(u)$ on the set

$$\widetilde{U} = \{(u_1, \dots, u_m) \in R^m;\ (u_1, \dots, u_i, 0, \dots, 0) \\ \text{belongs to } U \text{ for each } 1 \le i \le m\}.$$

The complement of $\widetilde{U}$ is a set of measure zero in R^m. Therefore, we conclude that if $u^0 = 0$, then $F_\epsilon(f)(z,w) = G_\epsilon(f)(z,w)$ for almost all $(z, w = u+iv) \in$

M_u and for almost all slices $M_u \subset M$. The proof of Lemma 2 is now complete.

By taking an increasing sequence of compact sets $K_k \subset M$, $k = 1, 2, \dots,$ and using Lemma 1, we can find a subsequence $F_k = F_{\epsilon_k}$ which converges to f as $k \mapsto \infty$, in $L^p(K)$, for any compact set $K \subset M$. The proof of Theorem 2 is now complete.

3. Analytic discs for quadrics.

Now we return to the class of quadric submanifolds $M = \{(w, z) \in C^m \times C^d;\ \mathrm{Re}\,(z) = q(w, \overline{w})\}$ where $q : C^m \times C^m \mapsto C^d$ is a quadric form. In this section, we summarize the results of analytic discs in [**BP**] and a related subaveraging estimate given in [**BN**].

Lemma 3. *Suppose $\alpha_0, \alpha_1, \dots \in C^m$ are given with $\sum_{j=0}^{\infty} |\alpha_j| < \infty$ and suppose $y \in R^m$ is given. Let $W : \{|\zeta| \le 1\} \mapsto C^m$ and $G : \{|\zeta| \le 1\} \mapsto C^d$ be analytic discs defined by*

$$\begin{aligned} W(\zeta) &= \sum_{j=0}^{\infty} \alpha_j \zeta^j \\ G(\zeta) &= \sum_{k=0}^{\infty} q(\alpha_k, \overline{\alpha_k}) + 2 \sum_{j>k\ge 0} q(\alpha_j, \overline{\alpha_k}) \zeta^{j-k} + iy. \end{aligned}$$

Then $A(\zeta) = (W(\zeta), G(\zeta))$ is an analytic disc with values in $C^n = C^m \times C^d$ whose boundary lies in M. The center of this disc is the point

$$A(\zeta = 0) = \left(\alpha_0, \sum_{k=0}^{\infty} q(\alpha_k, \overline{\alpha_k}) + iy \right).$$

The set of disc-centers

$$\left\{ A(\alpha_0,\, \alpha_1, \dots, y)(\zeta = 0);\ \alpha_j \in C^m,\ \text{with } \sum_{j=0}^{\infty} |\alpha_j| < \infty,\ y \in R^d \right\}$$

is the closed convex hull of M.

Sketch of Proof. This lemma appears in [**BP**]. The boundary of A is contained in M because

$$\begin{aligned} q(W(\zeta), \overline{W(\zeta)}) &= q\left(\sum_{j=0}^{\infty} \alpha_j \zeta^j, \overline{\sum_{k=0}^{\infty} \alpha_k \zeta^k} \right) \\ &= \sum_{k=0}^{\infty} q(\alpha_k, \overline{\alpha_k}) + 2\mathrm{Re} \sum_{j>k\ge 0} q(\alpha_j, \overline{\alpha_k}) \zeta^{j-k} \\ &= \mathrm{Re}\,(G(\zeta)). \end{aligned}$$

The center of A is

$$A(\zeta = 0) = \left(\alpha_0, \sum_{k=0}^{\infty} q(\alpha_k, \overline{\alpha_k}) + iy\right).$$

Letting $\alpha_0 = w \in C^m$, we obtain

$$A(\zeta = 0) = (w, q(w, \overline{w}) + iy) + \left(0, \sum_{k=1}^{\infty} q(\alpha_k, \overline{\alpha_k})\right).$$

The first term on the right parameterizes M (as (w, y) range over $C^m \times R^d$). The second term on the right parameterizes Γ, the closed convex hull of the map $w \in C^m \mapsto q(w, \overline{w})$ as in (1). Since the convex hull of M equals $M + \Gamma$, the proof of the lemma is complete.

Lemma 4. *Suppose $\Gamma' < \Gamma$ is the convex hull of*

$$q(a_1, \overline{a_1}), \ldots, q(a_N, \overline{a_N})$$

for some choice of vectors $a_1, \ldots, a_N \in C^m$. Let $s_j \geq 0$ for $1 \leq j \leq N$ and let $y \in R^d$ and $w \in C^m$ be given. Let

$$W(\zeta) = w + \sum_{j=1}^{N} s_j a_j \zeta^j \tag{14}$$

$$G(\zeta) = q(w, \overline{w}) + 2\sum_{j=1}^{N} s_j q(a_j, \overline{w})\zeta^j + \sum_{k=1}^{N} s_k^2 q(a_k, \overline{a_k}) + 2\sum_{N \geq j > k \geq 1} s_j s_k q(a^j, \overline{a^k})\zeta^{j-k} + iy. \tag{15}$$

Then $A = (G, W)$ has boundary contained in M and

$$A(\zeta = 0) = (w, q(w, \overline{w}) + iy) + \left(0, \sum_{k=1}^{N} s_k^2 q(a_k, \overline{a_k})\right) \tag{16}$$

which parameterizes $M + \Gamma'$ as y, w and s_j vary.

This lemma follows from Lemma 3 by letting $\alpha_0 = w$, $\alpha_j = s_j a_j$ for $1 \leq j \leq N$ and $\alpha_j = 0$ for $j > N$.

For a point $m \in M$ and $\delta > 0$, let $B(m, \delta)$ be the nonisotropic ball in M centered at m of radius δ. $B(m, \delta)$ is an ellipsoid of Euclidean length $\delta > 0$ in the m-complex tangent directions to M at m and of Euclidean length δ^2 in the d-totally real tangent directions. Details of the construction of these balls are given in [**NSW**]. Now we summarize the basic subaveraging estimate in [**BN**] (see Lemma 5.5).

Lemma 5. *Suppose $\Gamma' < \Gamma$. There exist constants C_1 and C_2 such that for any $(w^0, z^0) \in M + \Gamma'$, there exists a neighborhood V of (w^0, z^0) in C^n such that if F is an entire function then*

$$\sup_V |F| \leq \frac{C_1}{|B(m^0, C_2\sqrt{\delta})|} \int_{B(m^0, C_2\sqrt{\delta})} |F| d\sigma \tag{17}$$

where δ is the Euclidean distance from (w^0, z^0) to the point $m^0 = (w^0, q(w^0, \overline{w^0}) + iy^0) \in M$.

Sketch of Proof. The point of this lemma is that the values of F at points $(w, z) \in M + \Gamma'$ can be controlled by the L^1-norm of F over a nonisotropic ball in M whose radius is roughly equal to the square root of the distance from (w, z) to M. This lemma holds more generally for nonnegative plurisubharmonic functions in $M + \Gamma$ that are continuous up to M. This lemma is proved by first estimating the value of $F(w^0, z^0)$ by subaveraging over a small ball $V' \subset M + \text{interior}\{\Gamma\}$ centered at (w_0, z_0). Each point in V' can be realized as the center of an analytic disc A with boundary in M. From (16), we see that the Euclidean distance from the center of the analytic disc A to $(w, q(w, \overline{w}) + iy) \in M$ is proportional to $|s|^2$ where $s = (s_1, \ldots, s_N)$. In order to hit points a distance of δ away from M by the center of such a disc, we must have $|s| \approx \sqrt{\delta}$. From (14) and (15), it can be shown that the boundary of A lies in an ellipsoid of Euclidean radius $|s|$ in the complex-tangent directions and of Euclidean radius $|s|^2$ in the totally real directions. By subaveraging along these discs, the L^1-norm of F over V' can be dominated by the L^1-norm of F on the union of the boundaries of these discs, which in turn can be estimated by the L^1-norm over $B(m, C\sqrt{\delta})$ for an appropriate constant C. We refer the reader to Lemma 5.5 in [**BN**] for details.

4. Proof of the main theorem.

If f belongs to $\text{CR}^p(M)$, $1 \leq p \leq \infty$, Theorem 2 produces a sequence of entire functions, F_k, with $F_k \mapsto f$ in $L^p(K)$ for each compact subset $K \subset M$ (or pointwise almost everywhere if $p = \infty$). In particular, F_k converges to f in L^1 on each compact subset $K \subset M$. Applying Lemma 5 to $F_k - F_j$, we see that the sequence F_j is uniformly Cauchy on a neighborhood of each point in $M + \text{interior}\{\Gamma\}$. Therefore this sequence converges uniformly to an analytic function F on the compact subsets of the interior of $M + \Gamma$.

To prove the Theorem 1 for $1 \leq p \leq \infty$, we need to show the following two facts:

A. For any $x \in \text{interior}\{\Gamma\}$,

$$\int_{m\in M} |F(m+x)|^p \, d\sigma(m) \quad \leq \quad ||f||^p_{L^p(M)} \quad \text{if } 1 \leq p < \infty \tag{18}$$

$$|F(m+x)| \quad \leq \quad ||f||_\infty \quad \text{a.e. } m \text{ if } p = \infty. \tag{19}$$

B. Suppose $\Gamma' < \Gamma$,

$$\lim_{x\mapsto 0,\ x\in\Gamma'} \int_{m\in M} |f(m) - F(m+x)|^p \, d\sigma(m) = 0 \quad \text{if } 1 \leq p < \infty \tag{20}$$

$$\lim_{x\mapsto 0,\ x\in\Gamma'} |f(m) - F(m+x)| = 0 \quad \text{a.e. } m \text{ if } p = \infty. \tag{21}$$

These estimates all follow a similar pattern. The point $m+x$ is expressed as the center of an analytic disc. Then the approximating sequence, F_k, is subaveraged along the boundary of this disc which is contained in M. The estimate then follows by taking the limit as $k \mapsto \infty$. The details when $1 \leq p < \infty$ differ from $p = \infty$, so we isolate both cases.

The Case $1 \leq p < \infty$. We must show (18) and (20).

To prove (18), suppose

$$x = \sum_{j=1}^{N} q(a^j, \overline{a^j}) \in \text{interior}\{\Gamma\}$$

for some choice of $a_1, \ldots, a_N \in C^m$. Let $A(w,y)(\zeta)$ be the analytic disc given in Lemma 3, with $\alpha_0 = w$, $\alpha_j = a_j$ for $1 \leq j \leq N$ and $\alpha_j = 0$ for $j > N$. Here, w and y are treated as parameters. By Lemma 3, the center of this disc is

$$A(w,y)(\zeta = 0) = (w, q(w, \overline{w}) + x + iy). \tag{22}$$

Fix any $R > 0$. By subaveraging over this disc, and then integrating y and w over $\{|y|, |w| \leq R\}$, we have

$$\begin{aligned} &\int_{|w|,|y|\leq R} |F_k(w, q(w,\overline{w}) + x + iy)|^p \, d\sigma(w,y) \\ &\leq \int_{|w|,|y|\leq R} \int_0^1 |F_k(A(w,y)(e^{2\pi i t}))|^p \, dt \, d\sigma(w,y). \end{aligned} \tag{23}$$

The set

$$\{A(w,y)(e^{2\pi i t});\ |w|,\ |y| \leq R,\ 0 \leq t \leq 1\}$$

is contained in a compact set in M. Using the Approximation Theorem (Theorem 2), we let $k \mapsto \infty$ and replace F_k by f on the right side of (23). This, in turn, is dominated by $||f||^p_{L^p(M)}$. We therefore obtain

$$\int_{|w|,|y|\leq R} |F(w, q(w,\overline{w}) + x + iy)|^p \, d\sigma(w,y) \leq ||f||^p_{L^p(M)}.$$

Letting $R \mapsto \infty$ yields (18).

It suffices to prove (20) for a Γ' of the form given in Lemma 4. We use the analytic disc $A(y,w,s)(\zeta) = (W(\zeta), G(\zeta))$ given in Lemma 4 with $A(y,w,s)(\zeta=0) = (w, q(w,\overline{w}) + x + iy)$. Fix any $R > 0$. As in the proof of (18), we subaverage around the boundary of these discs, then integrate w, y to obtain

$$\int_{|w|,|y|\le R} |F_k(w,q(w,\overline{w})+iy) - F_k(w,q(w,\overline{w})+x+iy)|^p \, d\sigma(w,y)$$
$$\le \int_0^1 \int_{|w|,|y|\le R} |F_k(w,q(w,\overline{w})+iy) - F_k(A(y,w,s)(e^{2\pi it}))|^p \, dt \, d\sigma(w,y).$$

We then let $k \mapsto 0$ (using Theorem 2) and then let $R \mapsto \infty$ (in that order) to obtain

$$\int_{y,w} |f(w,q(w,\overline{w})+iy) - F(w,q(w,\overline{w})+x+iy)|^p \, d\sigma(w,y) \tag{24}$$
$$\le \int_0^1 \int_{w,y} |f(w,q(w,\overline{w})+iy) - f(A(s,y,w)(e^{2\pi it}))|^p \, dt \, d\sigma(w,y).$$

We must show the right side converges to zero as $s \mapsto 0$ (i.e., as $x \in \Gamma' \mapsto 0$).

Using (14) and (15), we can rewrite A as

$$A(y,w,s)(\zeta) = (w, q(w,\overline{w}) + iy) + \left(0, 2\sum_{j=1}^N s_j q(a_j,\overline{w})\zeta^j\right)$$
$$+ (W_0(s)(\zeta),\ G_0(s)(\zeta))$$

where

$$|W_0(s)(\zeta)| \le C|s| \quad \text{and} \quad |G_0(s)(\zeta)| \le C|s|^2 \quad \text{for } |\zeta| \le 1 \tag{25}$$

where C is a uniform constant independent of y and w.

The Jacobian determinant of the change of variables

$$\hat{w} = w \quad \text{and} \quad \hat{y} = y + 2\mathrm{Im}\left\{\sum_{j=1}^N s_j q(a_j,\overline{w})\zeta^j\right\}$$

is 1. After this change of variables, the right side of (24) only involves

$$\hat{A}(y,w,s)(e^{2\pi it}) = (w, q(w,\overline{w}) + iy) + (W_0(s)(e^{2\pi it}),\ G_0(s)(e^{2\pi it})).$$

In view of (25), the right side of (24) converges to zero as $s \mapsto 0$, because f belongs to $L^p(M)$. The proof of (20) is complete.

The Case $p = \infty$. For (19), fix any $w_0 \in C^m$, $y_0 \in R^d$ and $x_0 \in \Gamma'$. As in the case when $p < \infty$, we use Lemma 3 to write $(w, q(w,\overline{w}) + x_0 + iy) = A(w,y)(\zeta = 0)$ for each w and y in a ball $B(r) \subset C^m \times R^d$ centered at (w_0, y_0)

of radius $r > 0$. Since f is locally integrable on M, we can subaverage over the boundary of this disc and then integrate (w, y) over $B(r)$ to obtain

$$\begin{aligned}
&\frac{1}{|B(r)|}\int_{B(r)} |F_k(w, q(w,\overline{w}) + x + iy)|\, dy\, d\lambda(w)\\
&= \frac{1}{|B(r)|}\int_{B(r)} |F_k(A(w,y)(\zeta = 0))|\, dy\, d\lambda(w)\\
&\leq \frac{1}{|B(r)|}\int_{B(r)}\int_0^1 |F_k(A(w,y)(e^{2\pi it}))|\, dt\, dy\, d\lambda(w)
\end{aligned}$$

where $|B(r)|$ is the measure of $B(r)$. Since $|F_k|$ is dominated on the compact set

$$\{A(w,y)(e^{2\pi it});\ 0 \leq t \leq 1,\ (w,y) \in B(r)\}$$

by $C_K||f||_\infty$ (Theorem 2) and since $F_k|_M$ converges pointwise (a.e.) to f, we obtain

$$\begin{aligned}
&\frac{1}{|B(r)|}\int_{B(r)} |F(w, q(w,\overline{w}) + x + iy)|\, dy\, d\lambda(w)\\
&\leq \frac{1}{|B(r)|}\int_{B(r)}\int_0^1 |f(A(w,y)(e^{2\pi it}))|\, dt\, dy\, d\lambda(w)\\
&\leq ||f||_\infty.
\end{aligned}$$

The estimate in (19) with $m = (w_0, q(w_0, \overline{w_0}) + iy_0) \in M$ now follows by letting $r \mapsto 0$ (so $B(r)$ shrinks to the point (w_0, y_0)) in the above estimate.

To establish (21), fix any $m^0 \in M$ and apply Lemma 5 to the entire function $\widetilde{F}_k(w,z) := f(m^0) - F_k(w,z)$, Letting $k \mapsto \infty$, we obtain

$$\begin{aligned}
&|f(m^0) - F(m^0 + x)|\\
&\leq \frac{C_1}{|B(m^0, C_2\sqrt{|x|})|}\int_{B(m^0, C_2\sqrt{|x|})} |f(m^0) - f(m)| d\sigma(m)
\end{aligned}$$

for $x \in \Gamma'$. As $x \mapsto 0$, the right side converges to zero for almost all m^0 by the Maximal Function Theorem applied to the class of nonisotropic balls (see [**NSW**]). This completes the proof of the first part of Theorem 1 (i.e., extending f to F).

The second part of Theorem 1 (constructing f from $F \in H^p(M + \Gamma)$) follows from Theorem 5.3 in [**BN**]. In that theorem, $f(m)$ is locally constructed as the $L^p_{\mathrm{loc}}(M+\Gamma)$-limit of $F(m+x)$ as $x \mapsto 0$. If $F \in H^p(M+\Gamma)$, then it follows that f belongs to $L^p(M)$ and satisfies the tangential Cauchy Riemann equations in the sense of distribution theory. The proof of Theorem 1 is now complete.

5. Uniqueness.

The requirement that $||F||_{H^p(M+\Gamma)} = ||f||_{L^p(M)}$ implies that the extension, F, of f in Theorem 1 is unique. However, we can prove uniqueness without this estimate. This is the content of the following theorem.

Theorem 3. *Suppose M is a quadric submanifold of C^n and suppose* interior $\{\Gamma\}$ *is not empty. Let $1 \le p < \infty$ and suppose $F \in H^p(M+\Gamma)$ with*

$$\lim_{x\in\Gamma',\ x\mapsto 0} \int_{m\in M} |F(m+x)|^p \, d\sigma(m) = 0 \tag{26}$$

for each $\Gamma' < \Gamma$. Then $F = 0$ on $M +$ interior $\{\Gamma\}$.

Proof. Pick any $x_0 \in \text{interior}\{\Gamma\}$ and let $y_0 \in R^d$ and $w_0 \in C^m$ be given. Choose a family of analytic discs $A(w,y)(\cdot)$, with boundary in M, for w and y near w_0 and y_0 as in Lemma 3. We have

$$A(w,y)(\zeta = 0) = (0, x_0) + (w, q(w, \overline{w}) + iy) \in M + \text{interior}\{\Gamma\}.$$

We need the following lemma. Let $Q : U \mapsto M$, $Q(w,y) = (w, q(w,\overline{w}) + iy)$ be the graphing function for M.

Lemma 6. *There exists a cone $\Gamma' < \Gamma$ and a neighborhood, $U \subset C^m \times R^d$, of (w_0, y_0) such that $A(w,y)(\zeta)$ belongs to $Q\{U\} + \Gamma' \subset M + \Gamma'$ for all $(w,y) \in U$.*

Assume the lemma for the moment. Since F is holomorphic in $M +$ interior$\{\Gamma\}$ and the image of the analytic disc $A(w,y)(\cdot)$ is contained in $M + \text{interior}\{\Gamma\}$, $|F(A(w,y)(\zeta))|^p$ is a subharmonic function of ζ for $|\zeta| \le 1$ and $(w,y) \in U$. Therefore, the following expression

$$F_r = \int\limits_{(w,y)\in U} \int_0^1 |F(A(w,y)(re^{2\pi it}))|^p \, dt \, d\sigma(w,y)$$

is monotonically increasing in r for $0 \le r < 1$. Since the image of $A(w,y)(\cdot)$ is contained in $M + \Gamma'$ (in view of Lemma 6), (26) implies that $\lim_{r\mapsto 1^-} F_r = 0$. Therefore, $F_r = 0$ for $0 \le r < 1$. In particular,

$$F\left((0,x_0) + (w_0, q(w_0, \overline{w_0}) + iy_0)\right) = F(A(w_0,y_0)(\zeta = 0)) = F_0 = 0$$

as desired.

Proof of Lemma 6. Let S be the set of all linear functionals on R^d which define the convex cone Γ (i.e., x belongs to $\overline{\Gamma}$ if and only if $\ell(x) \ge 0$ for all

$\ell \in S$). Write

$$\begin{aligned} A(\zeta) &= (W(\zeta),\ G(\zeta)) \\ &= \Big(0,\ \ \mathrm{Re}\{G(\zeta)\} - q(W(\zeta), \overline{W(\zeta)})\Big) \\ &\quad + \Big(W(\zeta),\ q(W(\zeta),\ \overline{W(\zeta)}) + i\mathrm{Im}\{G(\zeta)\}\Big). \end{aligned}$$

The second term on the right belongs to M. Therefore, it suffices to show that there is a constant $\eta > 0$ such that

$$-\ell\Big(\mathrm{Re}\{G(\zeta)\} - q(W(\zeta), \overline{W(\zeta)})\Big) \leq -\eta(1 - |\zeta|) \tag{27}$$

for all $|\zeta| \leq 1$ and all $\ell \in S$ of unit norm.

Using the bilinearity of q and the definition of S, it is easy to show that the left side of (27) is a subharmonic function of ζ. In addition, the left side of (27) is zero on $|\zeta| = 1$ (since the boundary of A is contained in M) and is strictly negative when $\zeta = 0$ (since $\mathrm{Re}G(\zeta = 0) - q(w, \overline{w}) = x_0$ belongs to the interior of Γ). Therefore, the estimate in (27) follows from the Maximum Principle and the Hopf Lemma for subharmonic functions.

References

[BT] M.S. Baouendi and F. Treves, *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields*, Ann. Math., **113** (1981), 387-421, MR 82f:35057, Zbl 491.35036.

[B1] A. Boggess, CR *Manifolds and the Tangential Cauchy-Riemann Complex*, CRC Press, 1991, MR 94e:32035, Zbl 760.32001.

[B2] ______, *The holomorphic extension of* H^p - CR *functions on tube submanifolds*, Proc. Am. Math. Soc., **127**(**5**) (1999), 1427-1435, MR 99h:32012, Zbl 926.32045.

[BN] A. Boggess and A. Nagel, *Local reproducing kernels on wedge-like domains with type* 2 *edges*, Pacific J. Math, **167**(**1**) (1995), 1-47, MR 96g:32008, Zbl 824.32002.

[BP] A. Boggess and J.C. Polking, *Holomorphic extension of* CR *functions*, Duke Mathematical Journal, **49** (1982), 757-784, MR 84j:32018, Zbl 506.32003.

[BD] A. Boivin and R. Dwilewicz, *Extension and approximation of* CR *functions on tube manifolds*, Trans. Am. Math. Soc., **350** (1998), 1945-1956, MR 98h:32011, Zbl 903.32005.

[NSW] A. Nagel, E.M. Stein and S. Wainger, *Balls and metrics defined by vector fields,* I: *Basic properties*, Acta. Math., **155** (1985), 105-147, MR 86k:46049, Zbl 578.32044.

[SW] E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971, MR 46 #4102, Zbl 232.42007.

Received November 16, 1999 and revised July 18, 2000.

Department of Mathematics
Texas A&M University
College Station, TX 77843-3368
E-mail address: boggess@math.tamu.edu