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In this paper we consider, for $1 \leq m < p < 2$, the generalized KPZ equation $u_t = \triangle(u^m) - |\nabla u|^p$. For $m = 1$, we show existence and uniqueness of the so called very singular solution which is self-similar. A complete classification of self-similar solutions is also given. For $m > 1$, we establish the existence of very singular self-similar solution and prove that such a solution must have compact support. Moreover, we derive the interface relation. Recent experience with parallel equations where the gradient term $|\nabla u|^p$ is replaced by u^p indicates that the self-similar solutions are crucially important in study intermediate asymptotic behavior of general solutions.

1. Introduction.

In this paper, we consider the equation

$$u_t = \Delta u - |\nabla u|^p \quad \text{in} \quad R^n \times (0, +\infty), \quad 1 < p < 2 \tag{1.1}$$

and its porous media counterpart

$$u_t = \Delta(u^m) - |\nabla u|^p \quad \text{in} \quad R^n \times (0, +\infty), \quad 1 < m < p < 2. \tag{1.2}$$

Equation (1.1) is called generalized KPZ equation which arises from modelling of growth mechanism for surfaces through ballistic deposition, see [**15**], [**16**]. The model is derived from consideration that the growth mechanism is governed (approximating) by local rules. In such a model, $u(x,t)$ is the height above the underlying substrate which describes the interface profile, or the surface of the material. Experiments and numerical simulation show that $u(x,t)$ behaviours in a self-similar way. One important aspect in the study of such a model is then to find out the scaling exponents and functions which characterize the self-similarity of surface on a large space-time scale.

The actual physical model involving (1.1) is subject to random initial data and thorough analytical understanding is beyond our ability.

In this paper, we consider a simpler case, where the initial value $u(x,0) = u_0(x)$ is a deterministic function.

The main purpose is to study the existence and detailed characterization of special profiles of various type of self-similar solutions to (1.1) and (1.2), see Theorems 2.1 and 3.1 for more details.

Here by a **self-similar solution** we mean that u has the form

$$u(x,t) = t^{-\alpha} f(|x|t^{-\beta}). \tag{1.3}$$

For Equation (1.1),

$$\alpha := \frac{2-p}{2(p-1)}, \qquad \beta := \frac{1}{2}, \tag{1.4}$$

and f as a function of $r = |x|t^{-\beta}$, defined on $[0, +\infty)$, solves

$$f'' + \frac{n-1}{r} f' + \beta r f' + \alpha f - |f'|^p = 0 \quad \forall \ r > 0. \tag{1.5}$$

For Equation (1.2),

$$\begin{aligned} \alpha &:= \frac{2-p}{p(3-m)-2}, \\ \beta &:= \frac{p-m}{p(3-m)-2} \quad (1 < m < p < 2 \Rightarrow p(3-m)-2 > 0), \end{aligned} \tag{1.6}$$

and f as a function of $r = |x|t^{-\beta}$, defined on $[0, +\infty)$, solves

$$(f^m)'' + \frac{n-1}{r}(f^m)' + \beta r f' + \alpha f - |f'|^p = 0 \quad \forall \ r > 0. \tag{1.7}$$

In particular, we are able to show the existence of the so called very singular solutions for both (1.1) and (1.2).

By a **singular solution** we mean a nonnegative and nontrivial solution which is continuous in $R^n \times [0, +\infty) \backslash \{(0,0)\}$ and satisfies

$$\lim_{t \searrow 0} \sup_{|x| > \varepsilon} u(x,t) = 0 \quad \forall \ \varepsilon > 0. \tag{1.8}$$

A singular solution is called a **very singular solution** if

$$\lim_{t \searrow 0} \int_{|x| \le \varepsilon} u(x,t) dx = \infty \qquad \forall \ \varepsilon > 0. \tag{1.9}$$

Note that condition (1.9) is equivalent to, if u is given by (1.3),

$$\lim_{r \to \infty} r^{\alpha/\beta} f(r) = 0. \tag{1.10}$$

Furthermore, if $n\beta < \alpha$ and the solution f of (1.5) or (1.7) satisfies (1.10), then $u(x,t)$ given explicitly by (1.3) satisfies (1.8) and (1.9), i.e., it is a very singular self-similar solution of (1.1) or (1.2).

In recent years, several authors of [3]-[9], [11]-[14] and [17]-[20] studied the existence of very singular self-similar solutions and their role in describing the intermediate asymptotic behavior of general solutions to

$$u_t = \triangle(u^m) - u^p, \quad 0 < m < \infty, \ \ p > 1 \tag{1.11}$$

and

$$u_t = \mathrm{div}(|\nabla u|^{m-1}\nabla u) - u^p, \quad 0 < m < \infty, \ \ p > 1. \tag{1.12}$$

But, it can be seen from what follows that (1.1) and (1.2) have some peculiar properties very different from (1.11) and (1.12).

We mention that the Cauchy problem of (1.1) and related problems have been studied by Ben-Artzi and Koch [1], B. Gilding et al [10] and more recently by Benachou and Laurencot [2]. In particular, Benachou and Laurencot [2] proved the existence and uniqueness of very singular solution of (1.1), but their proof is different from ours.

The organization of this paper is as follows. In §2 we study (1.5) and give a complete classification of its solutions. In particular, the existence and uniqueness of very singular self-similar solution is proved. In §3, we study (1.7) and establish the existence of a very singular self-similar solution which has compact support. The interface relation is also shown.

2. Study of self-similar solutions to (1.1).

In this section we study (1.5) and give a complete classification of its solutions in relation to the initial value $f(0)$. In particular, we prove the existence and uniqueness of very singular self-similar solution. We consider the solution of (1.5) with initial value

$$f(0) = a > 0, \qquad f'(0) = 0. \tag{2.1}$$

For each $a > 0$, (1.5) and (2.1) has a unique solution $f(r;a)$, at least locally. If we multiply (1.5) by r^{n-1} and integrate from 0 to r, we get

$$f' r^{n-1} = -\beta r^n f(r) + \int_0^r s^{n-1}[(\beta n - \alpha) f + |f'|^p] \, ds.$$

Another integration on $[0, r]$ after dividing the above equation by r^{n-1} and simple calculation yield

$$f(r) = a - a\frac{\alpha}{2n} r^2 + o(r^2).$$

This shows how the solution behaviours as $r \to 0$.

If we denote by $(0, R(a))$ the maximal existence interval where $f > 0$, then $f' < 0$ in $(0, R(a))$ and either (i) $R(a) = \infty$ and $\lim_{r \nearrow \infty} f(r;a) = 0$, or (ii) $R(a) < \infty$ and $f(R(a);a) = 0$. The main results of this section read as follows.

Theorem 2.1. *Assume that $1 < p < 2$. For each $a > 0$, let $f(r;a)$ be the solution of* (1.5), (2.1). *Then the following conclusions hold:*

(i) *If $2\alpha \le n$, then $R(a) = \infty$ and $\liminf_{r\to\infty} r^{2\alpha} f(r;a) > 0$.*

(ii) *If $2\alpha > n$, then there exists $a^* > 0$ such that the following classification is valid:*

(a) *If $a \in (0, a^*)$, then $R(a) < \infty$ and $f' < 0$ in $(0, R(a)]$.*

(b) *If $a \in (a^*, \infty)$, then $R(a) = \infty$, $f(r;a)$ is strictly increasing and $f'(r;a)$ is strictly decreasing with respect to a. And for some $k(a) > 0$,*

$$f(r;a) = k(a) r^{-2\alpha} \left\{1 + 2\alpha(2 + 2\alpha - n - [2\alpha k(a)]^{p-1}) r^{-2} + o(r^{-2})\right\} \quad \text{as } r \to \infty. \tag{2.2}$$

(c) *If $a = a^*$, then $\lim_{r\to\infty} r^{2\alpha} f(r;a^*) = 0$, and for some $k(a^*) > 0$,*

$$f(r;a^*) = k(a^*) r^{2\alpha - n} e^{-r^2/4} \left\{1 - 2(2\alpha - n)(\alpha - 1) r^{-2} + o(r^{-2})\right\} \quad \text{as } r \to \infty. \tag{2.3}$$

This theorem shows that (1.1) has a very singular self-similar solution if and only if $1 < p < (2+n)/(1+n)$, and in case of existence the solution is unique.

We write (1.5) as

$$\begin{cases} f' = v, \\ v' = -\frac{n-1}{r} v - \frac{r}{2} v - \alpha f - |v|^{p-1} v. \end{cases} \tag{2.4}$$

Lemma 2.1. *Assume that $a > 0$ and $f = f(r;a)$ is the solution of* (1.5), (2.1), $(0, R(a))$ *is the maximal existence interval where $f > 0$. Then*

$$|f'(r)| \le (\alpha a)^{1/p} \quad \textit{for all} \quad 0 \le r \le R(a).$$

Proof. First, we consider the case where f'' is negative in an interval. If there exist two constants b and c such that $0 < b < c \le R(a)$ and $f''(r) \le 0$ in (b, c). Then by Equation (1.5) we have $|f'(r)|^p \le \alpha f(r) \le \alpha a$. Therefore, $|f'(r)| \le (\alpha a)^{1/p}$ for all $r \in [b, c]$.

On the other hand, if b_1 and c_1 are so given that $0 < b_1 < c_1 \le R(a)$, $f''(r) \ge 0$ in (b_1, c_1) and $f''(b_1) = 0$, then $f'(r) \ge f'(b_1)$, and hence $|f'(r)| \le |f'(b_1)|$ for all $r \in [b_1, c_1]$.

Since $f''(0) = -\alpha a/n < 0$, the above consideration show that the conclusion of Lemma 2.1 holds. □

For any given $\lambda > 0$, we denote $\mathcal{L}_\lambda = \{(f, v) : f > 0, \ -\lambda f < v < 0\}$.

Lemma 2.2. *For any given $\lambda > 0$ there exists an $r_\lambda := 2(\lambda + \alpha/\lambda)$ such that $\mathcal{L}_\lambda$ is positively invariant for $r > r_\lambda$. That is, if $(f(r_\lambda), v(r_\lambda)) \in \mathcal{L}_\lambda$, then the orbit $(f(r), v(r))$ of* (2.4) *remains in $\mathcal{L}_\lambda$ for all $r \ge r_\lambda$.*

Proof. Since the vector field points into $\mathcal{L}_\lambda$ from the positive f-axis, we need to show that it also points into $\mathcal{L}_\lambda$ from the ray

$$l_\lambda = \{(f,v) : f > 0, \quad v = -\lambda f\}.$$

We have on l_λ,

$$\frac{v'}{f'} = -\frac{n-1}{r} - \frac{r}{2} - \alpha\frac{f}{v} - |v|^{p-1} < -\frac{r}{2} + \frac{\alpha}{\lambda}.$$

Hence,

$$\frac{v'}{f'} < -\lambda \quad \text{on} \quad l_\lambda$$

if $r \geq r_\lambda := 2(\lambda + \alpha/\lambda)$. This completes the proof. $\square$

By using similar arguments as those of Lemmas 5-7 in [4], the following result can easily be shown to hold. But, for simplicity, we omit the details.

Lemma 2.3. *Suppose $f(r;a) > 0$ for all $r > 0$. Then*

(i) $(f(r;a), v(r;a)) \to (0,0)$ *as* $r \to \infty$.
(ii) $\lim_{r\to\infty} \frac{v(r;a)}{f(r;a)} = L_a$ *exists, and* $L_a = 0$ *or* $L_a = -\infty$.

We now prove Theorem 2.1(i), which gives the nonexistence results of very singular self-similar solutions when $2\alpha \leq n$.

Proof of Theorem 2.1(i). Suppose $2\alpha \leq n$. Multiplying (1.5) by $r^{2\alpha-1}$ we have, for $r \in (0, R(a))$,

$$\left(r^{2\alpha-1}f' + \frac{1}{2}r^{2\alpha}f\right)' = (2\alpha - n)r^{2\alpha-2}f' + r^{2\alpha-1}|f'|^p > 0.$$

The function $g(r) := r^{2\alpha-1}f' + \frac{1}{2}r^{2\alpha}f$ is strictly increasing in $(0, R(a))$. Note that $\lim_{r\searrow 0} g(r) = 0$, we get $g > 0$ in $(0, R(a))$. Since $f' < 0$, we conclude that $R(a) = \infty$ and $f \searrow 0$ as $r \nearrow \infty$. In addition, $r^{2\alpha}f(r;a) \geq 2g(r)$ and $g(r)$ is increasing, hence $\liminf_{r\to\infty} r^{2\alpha}f > 0$. This completes the proof. $\square$

In the sequel of this section we always assume that $2\alpha > n$. Let L_a be given as in Lemma 2.3, we define

$$\begin{aligned} \mathcal{A} &= \{a > 0 : R(a) < \infty\}, \\ \mathcal{B} &= \{a > 0 : R(a) = \infty \ \text{ and } \ L_a = 0\}, \\ \mathcal{C} &= \{a > 0 : R(a) = \infty \ \text{ and } \ L_a = -\infty\}. \end{aligned}$$

By Lemma 2.3 we know that $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = (0, \infty)$. It is obvious that these sets are disjoint.

Lemma 2.4. *The set $\mathcal{A}$ is nonempty and open.*

Proof. We choose $a = \epsilon \ll 1$ and let

$$w_\epsilon(r) = \varepsilon^{-1} f(r;\varepsilon).$$

Then w_ε satisfies

$$\begin{cases} w_\varepsilon'' + \frac{n-1}{r} w_\varepsilon' + \frac{r}{2} w_\varepsilon' + \alpha w_\varepsilon + \varepsilon^{p-1} |w_\varepsilon'|^{p-1} w_\varepsilon' = 0, \\ w_\varepsilon(0) = 1, \quad w_\varepsilon'(0) = 0. \end{cases} \tag{2.5}$$

Let $E(w_\varepsilon) = \alpha w_\varepsilon^2 + (w_\varepsilon')^2$, then $\frac{d}{dr} E(w_\varepsilon) \le 0$. Therefore, $E(w_\varepsilon) \le \alpha$ for all $\varepsilon > 0$. Consequently, both w_ε and w_ε' are uniformly bounded with respect to $r \ge 0$ and $\varepsilon > 0$. It follows by the standard continuity argument that

$$w_\varepsilon \to w \quad \text{ as } \varepsilon \to 0 \text{ in } C^2([0,R])$$

for any $R > 0$, where w is the solution of the reduced problem

$$\begin{cases} w'' + \frac{n-1}{r} w' + \frac{r}{2} w' + \alpha w = 0, \\ w(0) = 1, \quad w'(0) = 0. \end{cases} \tag{2.6}$$

We claim that w has a zero. Suppose on the contrary that $w(r) > 0$ for all $r > 0$. By (2.6) we have

$$\left(r^{n-1} w'(r) + \frac{1}{2} r^n w(r) \right)' = \left(\frac{n}{2} - \alpha \right) r^{n-1} w(r) < 0. \tag{2.7}$$

Therefore, $w'(r) + \frac{1}{2} r w(r) < 0$ for all $r > 0$. Thus we have $w(r) \le \exp\{-\frac{r^2}{2}\}$, and $\limsup_{r\to\infty} r^{n-1} w'(r) = 0$. But, an integration of (2.7) gives

$$r^{n-1} w'(r) + \frac{1}{2} r^n w(r) = \int_0^r \left(\frac{n}{2} - \alpha \right) s^{n-1} w(s)\, ds < -C, \qquad r \gg 1$$

for some constant $C > 0$. It is a contradiction.

Since $w' < 0$ at the first zero of w, it follows that for ε sufficiently small, w_ε has a zero as well. This shows that $\mathcal{A}$ is nonempty.

By the uniqueness and continuous dependence on the initial data of solution we see that $\mathcal{A}$ is open. □

Lemma 2.5. *The set $\mathcal{B}$ is nonempty and open.*

Proof. We first show that if initial data a is suitably large then the corresponding orbit must stay in $\mathcal{L}_1$ for all $r \ge 0$. This implies that $a \in \mathcal{B}$.

Let r_0 be the first value such that the orbit intersects with the boundary of $\mathcal{L}_1$. It is clear that $v(r_0) = -f(r_0)$. Consequently, using Lemma 2.1, we have

$$f(r_0) = -v(r_0) \le (\alpha a)^{1/p}, \tag{2.8}$$

and

$$f(r_0) = f(0) + \int_0^{r_0} f'(s)ds \tag{2.9}$$
$$\geq a + \int_0^{r_0} \{-(\alpha a)^{1/p}\}ds$$
$$= a - (\alpha a)^{1/p} r_0.$$

(2.8) combined with (2.9) yields

$$r_0 \geq \frac{a - (\alpha a)^{1/p}}{(\alpha a)^{1/p}} \to \infty \quad \text{ as } \ a \to \infty.$$

It contradicts to Lemma 2.2.

Now we prove that $\mathcal{B}$ is open. Suppose $a_0 \in \mathcal{B}$. Then by the definition of $\mathcal{B}$, there exists $r_0 > 0$ such that $(f(r_0; a_0), v(r_0; a_0)) \in \mathcal{L}_1$. Hence, by continuous dependence on initial data there exists a neighbourhood Σ of a_0 such that if $a \in \Sigma$, then $f(r; a) > 0$ for all $r \in [0, r_0]$ and $(f(r_0; a), v(r_0; a)) \in \mathcal{L}_1$. It follows from Lemmas 2.2 and 2.3 that if $a \in \Sigma$, the corresponding $L_a = 0$, so that $a \in \mathcal{B}$. □

Lemma 2.6. *Assume that* $f_1(0) = a_1 > 0, \quad f_2(0) = a_2 > 0$. *If* $a_2 > a_1$ *then*

$$f_2(r) > f_1(r), \qquad f_2'(r) < f_1'(r) \qquad \forall \ \ 0 < r < R(a_1).$$

Proof. Let $w = f_1 f_2' - f_1' f_2$, then w satisfies

$$w' + \left\{ \frac{n-1}{r} + \frac{r}{2} + |f_2'|^{p-1} \right\} w = -f_1' f_2 [|f_2'|^{p-1} - |f_1'|^{p-1}] \tag{2.10}$$
$$\triangleq F(r).$$

Because $f_i''(0) = -a_i/n$, $a_1 < a_2$ and $f_i'(0) = 0$, it follows that $f_2'(r) < f_1'(r)$, and consequently $F(r) > 0$ for $r \ll 1$. Denote

$$r_0 = \sup\{0 < r < R(a_1) : f_2'(s) < f_1'(s) \ \ \forall \ \ s \in (0, r)\}.$$

Then we have

$$w' + \left\{ \frac{n-1}{r} + \frac{r}{2} + |f_2'|^{p-1} \right\} w = F(r) > 0 \qquad \forall \ \ 0 < r < r_0.$$

Since $w(0) = 0$, it follows that

$$w(r) > 0, \quad \text{i.e.,} \quad (f_2/f_1)' > 0 \qquad \forall \ \ 0 < r < r_0.$$

Therefore, $f_2(r) > f_1(r)$ for all $0 \leq r \leq r_0$.

We assert that $r_0 = R(a_1)$. Suppose to the contrary that $r_0 < R(a_1)$, then $f_2'(r_0) = f_1'(r_0)$. On the other hand, by (1.5) we have

$$\begin{aligned} f_2''(r_0) &= -\tfrac{n-1}{r_0} f_2'(r_0) - \tfrac{r_0}{2} f_2'(r_0) + |f_2'(r_0)|^p - \alpha f_2(r_0) \\ &= -\tfrac{n-1}{r_0} f_1'(r_0) - \tfrac{r_0}{2} f_1'(r_0) + |f_1'(r_0)|^p - \alpha f_2(r_0) \\ &= f_1''(r_0) + \alpha(f_1(r_0) - f_2(r_0)) < f_1''(r_0). \end{aligned}$$

It contradicts to the definition of r_0. This lemma is proved. □

Proof of Theorem 2.1(ii). By Lemmas 2.4-2.6 and the proofs of Lemmas 2.4 and 2.5 we know that there exist $a_i : 0 < a_1 \le a_2 < \infty$ such that $\mathcal{A} = (0, a_1)$, $\mathcal{B} = (a_2, \infty)$, $\mathcal{C} = [a_1, a_2]$.

For any $a \in \mathcal{C}$, the corresponding orbit satisfies

$$\lim_{r\to\infty} \frac{f'(r)}{f(r)} = -\infty.$$

To prove (2.3), we define $E(r) = rv(r) + \frac{1}{2} r^2 f(r)$, $G(r) = r^2 E(r) - (2\alpha - n) r^2 f(r)$. Similar to the proofs of Lemmas 13-15 in [4] we can prove that

$$\lim_{r\to\infty} \frac{v(r)}{rf(r)} = -\frac{1}{2}, \quad \lim_{r\to\infty} \frac{E(r)}{f(r)} = 2\alpha - n, \quad \lim_{r\to\infty} \frac{G(r)}{f(r)} = 4(\alpha - 1)(2\alpha - n).$$

Using the same argument as in the proof of Theorem 2 in [4] it follows that (2.3) holds.

For any $a \in \mathcal{B}$, the corresponding orbit satisfies

$$\lim_{r\to\infty} \frac{f'(r)}{f(r)} = 0.$$

Similar to the proof of Theorem 4 in [4] we know that (2.2) holds.

We prove $a_1 = a_2$. If $a_1 < a_2$, by Lemma 2.6 and (2.10) we have that $w' + b(r)w = F(r) > 0$ for all $r > 0$, where $w = f_1 f_2' - f_1' f_2$ and $b(r) := \frac{n-1}{r} + \frac{r}{2} + |f_2'|^{p-1}$. Therefore,

$$w(R) \exp\left\{ \int_{r_0}^{R} b(s)ds \right\} > w(r) \exp\left\{ \int_{r_0}^{r} b(s)ds \right\}.$$

Because

$$f_i(r) \approx k(a_i) r^{2\alpha - n} \exp\left\{ -\frac{r^2}{4} \right\}, \quad f_i'(r) \approx k(a_i) r^{2\alpha+1-n} \exp\left\{ -\frac{r^2}{4} \right\}, \quad r \gg 1,$$

it follows that

$$w(r) \exp\left\{ \int_{r_0}^{r} b(s)ds \right\} < w(R) \exp\left\{ \int_{r_0}^{R} b(s)ds \right\} \to 0 \quad \text{as} \quad R \to \infty.$$

It is a contradiction. Therefore, $a_1 = a_2 := a^*$.

Lemma 2.6 shows that $f(r; a)$ is strictly increasing and $f'(r; a)$ is strictly decreasing with respect to a. The proof of Theorem 2.1 is complete. □

3. Existence of very singular self-similar solution of (1.2).

In this section we prove the existence and uniqueness of very singular self-similar solution to (1.2). Similar to §2, we consider (1.7) with initial data

$$f(0) = b > 0, \qquad f'(0) = 0. \tag{3.1}$$

For each $b > 0$, (1.7), (3.1) has a unique solution $f(r; b)$. If we denote by $(0, R(b))$ the maximal existence interval where $f > 0$, then $f' < 0$ in $(0, R(b))$. The main results of this section read as follows.

Theorem 3.1. *Assume that $1 < m < p < 2$. Then the following conclusions hold:*

(i) *If $\alpha \le n\beta$, then $R(b) = \infty$ and $\liminf_{r\to\infty} r^{\alpha/\beta} f(r; b) > 0$.*

(ii) *If $\alpha > n\beta$, then there exist one closed set $\mathcal{B}$ and two open sets $\mathcal{A}$ and $\mathcal{C}$ of $(0, \infty)$ satisfying*

$$\mathcal{A}\cup\mathcal{B}\cup\mathcal{C} = (0,\infty), \quad (b,\infty)\subset\mathcal{A} \ \text{ if } \ b\gg 1, \ \text{ and } \ (0,b)\subset\mathcal{C} \ \text{ if } \ 0<b\ll 1,$$

such that the following classification is valid:

(a) *If $b \in \mathcal{C}$, then $R(b) < \infty$ and $(f^m)'(R(b)) < 0$.*

(b) *If $b \in \mathcal{A}$, then $R(b) = \infty$, $\lim_{r\to\infty}(f(r; b), f'(r; b)) = (0, 0)$. And for some $\varphi(b) > 0$,*

$$\lim_{r\to\infty} r^{\alpha/\beta} f(r; b) = \varphi(b).$$

(c) *If $b \in \mathcal{B}$, then $R(b) < \infty$ and $f'(R(b)) = 0$. That is, the solution $f(r; b)$ has compact support, and*

$$f(r) > 0 \ \text{ for } \ 0 \le r < R(b), \qquad f(r) = 0 \ \text{ for } \ r \ge R(b).$$

Moreover,

$$\lim_{r\nearrow R(b)} (f^{m-1}(r))' = -\frac{(m-1)\beta}{m} R(b). \tag{3.2}$$

Where α and β are given in (1.6).

This theorem shows that (1.2) has a very singular self-similar solution if and only if $\alpha > n\beta$.

Remark. (3.2) is the important profile relation which we believe should give the optimal regularity of general solutions with compact support.

The proof of Theorem 3.1 (i) is similar to that of Theorem 2.1 (i), and we omit the details.

In the sequel we assume that $\alpha > n\beta$.

Let $z = f^m$, $\ a = b^m$, we deal with the reduced problem

$$\begin{cases} z'' + \frac{n-1}{r} z' + \beta r (z^{1/m})' + \alpha z^{1/m} - m^{-p} z^{p(1-m)/m} |z'|^p = 0 & r > 0, \\ z(0) = a > 0, \quad z'(0) = 0, \quad z(r) \ge 0. \end{cases} \tag{3.3}$$

Here, (1.10) becomes

$$\lim_{r\to\infty} r^{\alpha/\beta} z^{1/m}(r) = 0. \tag{3.4}$$

Let $z' = v$ then we have

$$\begin{cases} z' = v, \\ v' = -\frac{n-1}{r}v - \frac{\beta}{m} r z^{\frac{1}{m}-1} v - \alpha z^{\frac{1}{m}} - m^{-p} z^{p(1-m)/m} |v|^{p-1} v. \end{cases} \tag{3.5}$$

Lemma 3.1. *Assume that $a > 0$ and $z = z(r; a)$ is the solution of (3.3), $(0, R(a))$ is the maximal existence interval where $z > 0$. Then $z'(r) < 0$ for all $0 < r < R(a)$, and*

$$|z'(r)| \le m\alpha^{1/p} a^{(1+mp-p)/(mp)} \quad \text{for all } \ 0 \le r \le R(a). \tag{3.6}$$

Proof. It's proof is similar to that of Lemma 2.1 and we omit the details. □

Because $1 < m < p < 2$, it is clear that there exists θ such that

$$\max\left\{\frac{1}{m},\ \frac{1+mp-p}{mp}\right\} < \theta < \min\left\{1,\ \frac{2p-1}{mp},\ \frac{m+p-mp}{m(2-p)}\right\}. \tag{3.7}$$

For any given $\lambda,\quad \eta > 0$, we define

$$\mathcal{S}_{\lambda,\eta} := \{(z, v) : 0 < z \le \eta, \quad -\lambda z^\theta < v < 0\}.$$

Lemma 3.2. *For any given $\lambda,\quad \eta > 0$, there exists an $r_{\lambda,\eta} := \frac{m\alpha}{\beta\lambda}\eta^{1-\theta} + \frac{m\theta\lambda}{\beta}\eta^{\theta-1/m}$ such that $\mathcal{S}_{\lambda,\eta}$ is positively invariant for $r > r_{\lambda,\eta}$. That is, if $(z(r_{\lambda,\eta}), v(r_{\lambda,\eta})) \in \mathcal{S}_{\lambda,\eta}$, then the orbit $(z(r), v(r))$ of (3.5) remains in $\mathcal{S}_{\lambda,\eta}$ for all $r \ge r_{\lambda,\eta}$.*

Proof. Similar to the proof of Lemma 2.2, we need only to show that the orbit points into $\mathcal{S}_{\lambda,\eta}$ from the parabola

$$l_{\lambda,\eta} := \{(z, v) : 0 < z \le \eta, \quad v = -\lambda z^\theta\}.$$

On $l_{\lambda,\eta}$, by (3.5), we have

$$\begin{aligned} \frac{v'}{(z^\theta)'} &= -\frac{n-1}{\theta r} z^{1-\theta} - \frac{\beta}{m\theta} r z^{\frac{1}{m}-\theta} + \frac{\alpha}{\theta\lambda} z^{1-2\theta+\frac{1}{m}} \\ &\quad - \frac{1}{\theta} m^{-p} z^{1-\theta+p(1-m)/m} |v|^{p-1} \\ &< -\frac{\beta}{m\theta} r z^{\frac{1}{m}-\theta} + \frac{\alpha}{\theta\lambda} z^{1-2\theta+\frac{1}{m}} \le -\lambda \end{aligned} \tag{3.8}$$

if

$$-\frac{\beta}{m\theta} r + \frac{\alpha}{\theta\lambda} z^{1-\theta} \le -\lambda z^{\theta-\frac{1}{m}}.$$

Since $1/m < \theta < 1$ and $0 < z \le \eta$, it is clear that (3.8) holds if

$$r \ge r_{\lambda,\eta} := \frac{m\alpha}{\beta\lambda}\eta^{1-\theta} + \frac{m\theta\lambda}{\beta}\eta^{\theta-1/m}.$$

This completes the proof. $\square$

Define $\mathcal{S}_\lambda := \{(z,v) : z > 0, \quad -\lambda z^\theta < v < 0\}$. The orbit $(z(r), v(r))$ of (3.5) starting from $(a, 0)$ enters $\mathcal{S}_\lambda$ eventually means that there exists an $r_a : 0 < r_a < R(a)$ such that $(z(r), v(r)) \in \mathcal{S}_\lambda$ for all $r_a \leq r < R(a)$. Set

$$\mathcal{A} = \{a > 0 : \text{the orbit } (z,v) \text{ starting from } (a,0) \text{ enters } \mathcal{S}_1 \text{ eventually}\},$$
$$\mathcal{B} = \{a > 0 : R(a) < \infty, \quad z'(R(a)) = 0\},$$
$$\mathcal{C} = \{a > 0 : R(a) < \infty, \quad z'(R(a)) < 0\}.$$

Remark. For any $a \in \mathcal{A}$, the corresponding solution $z(r; a)$ satisfies $z' + z^\theta > 0$ when $r < R(a)$ and close to $R(a)$. This implies $R(a) = \infty$. Therefore, $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = (0, \infty)$. It is obvious that $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ do not intersect with each other.

Lemma 3.3. *The set $\mathcal{C}$ is non-empty and open. Moreover, $(0, b) \subset \mathcal{C}$ if $0 < b \ll 1$.*

Proof. We choose $a = \epsilon > 0$ small and let $w_\epsilon(t) = \varepsilon^{-1} z(r; \varepsilon)$, $t = r\varepsilon^{(1-m)/(2m)}$. For simplicity we replace t by r. Then w_ε satisfies

$$\begin{cases} w_\varepsilon'' + \frac{n-1}{r} w_\varepsilon' + \beta r (w_\varepsilon^{1/m})' + \alpha w_\varepsilon^{1/m} \\ \qquad + m^{-p} \varepsilon^{(3p-mp-2)/(2m)} w_\varepsilon^{p(1-m)/m} |w_\varepsilon'|^{p-1} w_\varepsilon' = 0, \\ w_\varepsilon(0) = 1, \quad w_\varepsilon'(0) = 0. \end{cases} \tag{3.9}$$

Let $E(w_\varepsilon) = \frac{2\alpha}{1+1/m} w_\varepsilon^{(1+m)/m} + (w_\varepsilon')^2$, then $\frac{d}{dr} E(w_\varepsilon) \leq 0$. Therefore, $E(w_\varepsilon) \leq 2m\alpha/(1+m)$ for all $\varepsilon > 0$, both w_ε and w_ε' are uniformly bounded with respect to $r \geq 0$ and $\varepsilon > 0$. Denote by $(0, R_\varepsilon)$ the maximal existence interval where $w_\varepsilon > 0$, then $w_\varepsilon'(r) < 0$ in $(0, R_\varepsilon)$.

We first consider the reduced problem ($\varepsilon = 0$)

$$\begin{cases} w'' + \frac{n-1}{r} w' + \beta r (w^{1/m})' + \alpha w^{1/m} = 0, \\ w(0) = 1, \quad w'(0) = 0. \end{cases}$$

It is easy to show that there exists an $r_0 : 0 < r_0 < \infty$ such that $w(r_0) = 0$, $w'(r_0) < 0$ and $w(r) > 0$ for all $0 \leq r < r_0$.

We will prove that when ε is small then the solution w_ε of (3.9) has the same properties as w. To this aim, let $\eta_0 > 0$ be such that

$$\eta_0 + \frac{(m-1)}{8m} r_0 w'(r_0) < 0, \tag{3.10}$$

$$\frac{n-1}{r_0} \eta_0 + \beta r_0 \eta_0^{1/m} + \frac{m^{1-p}}{m+p-mp} \left(\frac{2m\alpha}{1+m} \right)^{(p-1)/2} \eta_0^{(m+p-mp)/m} < -\frac{w'(r_0)}{4}. \tag{3.11}$$

Choose $r_1 : 0 < r_0 - r_1 \ll 1$ such that $w(r_1) < \eta_0$ and $w'(r_1) < w'(r_0)/2$. By the continuous dependence of solution on the parameter ε we have that if $\varepsilon \ll 1$, then the solution w_ε of (3.9) satisfies

$$R_\varepsilon > r_1, \quad w_\varepsilon(r_1) := \eta < \eta_0, \quad w'_\varepsilon(r_1) < w'(r_0)/2. \tag{3.12}$$

Since $|w'_\varepsilon(r)| \le [2m\alpha/(1+m)]^{1/2}$, it follows that for $r > r_1$,

$$\begin{aligned} w''_\varepsilon \le & -\frac{n-1}{r} w'_\varepsilon - \beta r (w_\varepsilon^{1/m})' - \alpha w_\varepsilon^{1/m} \\ & - m^{-p} \varepsilon^{(3p-mp-2)/(2m)} \left(\frac{2m\alpha}{1+m}\right)^{(p-1)/2} w_\varepsilon^{p(1-m)/m} w'_\varepsilon. \end{aligned}$$

Integrating this inequality from r_1 to r and using (3.11), (3.12) we have,

$$\begin{aligned} w'_\varepsilon + \beta r w_\varepsilon^{1/m} \le & \; w'_\varepsilon(r_1) + \frac{n-1}{r_1}\eta + \beta r_1 \eta^{1/m} + \beta \eta^{1/m}(r - r_1) \\ & + \frac{m^{1-p}}{m+p-mp} \left(\frac{2m\alpha}{1+m}\right)^{(p-1)/2} \varepsilon^{(3p-mp-2)/(2m)} \eta^{(m+p-mp)/m} \\ \le & \; \frac{1}{2} w'(r_0) + \frac{n-1}{r_0}\eta_0 + \beta r_0 \eta_0^{1/m} + \beta \eta^{1/m}(r - r_1) \\ & + \frac{m^{1-p}}{m+p-mp} \left(\frac{2m\alpha}{1+m}\right)^{(p-1)/2} \eta_0^{(m+p-mp)/m} \\ < & \; \frac{1}{4} w'(r_0) + \beta \eta^{1/m}(r - r_1) \le 0 \end{aligned} \tag{3.13}$$

if

$$r \le r_2 := r_1 - \frac{1}{4\beta} \eta^{-1/m} w'(r_0).$$

Integrating (3.13) from r_1 to r_2 and by (3.10) we get

$$\begin{aligned} w_\varepsilon^{(m-1)/m}(r_2) & < \eta^{(m-1)/m} - \frac{\beta(m-1)}{2m}(r_2^2 - r_1^2) \\ & < \eta^{-1/m} \left[\eta + \frac{(m-1)}{8m} r_0 w'(r_0) \right] \\ & < \eta^{-1/m} \left[\eta_0 + \frac{(m-1)}{8m} r_0 w'(r_0) \right] < 0. \end{aligned}$$

This shows that $R_\varepsilon < r_2$ and $w'_\varepsilon + \beta r w_\varepsilon^{1/m} < 0$ for all $r_1 < r \le R_\varepsilon$. Therefore, $w'_\varepsilon(R_\varepsilon) < 0$. And consequently, $a = \varepsilon \in \mathcal{C}$.

It can be seen from the above proof that for a solution z to (3.3), if there exists $r_1 > 0$ such that $z(r_1) \ll 1$ and $|z'(r_1)|$ is not too small, then the solution will reach zero at a finite $r = R$ and $z'(R) < 0$. Hence $\mathcal{C}$ is open. Lemma 3.3 is proved. $\square$

Lemma 3.4. *For any given $a > 0$. If the corresponding $R(a) = \infty$, then there exists a limit $\lim_{r\to\infty} r^{\alpha/\beta} z^{1/m}(r; a) = k(a)$ and $k(a) > 0$.*

Proof.

Step 1. First, as in the case of semilinear case, it can be shown following the argument of Lemmas 5-7 in [**4**] that $z(r) \to 0$ as $r \to \infty$. Since $z(r) > 0$, $z'(r) < 0$ for all $r > 0$ and $z(r) \to 0$ as $r \to \infty$, it is impossible that $z''(r) < 0$ for all $r \gg 1$. Differentiating Equation (3.3), it is easy to see that $z''(r) > 0$ for $r \gg 1$, in consequence, by (3.3),

$$\alpha z^{1/m} + \beta r(z^{1/m})' + \frac{m(n-1)}{r} z^{(m-1)/m}(z^{1/m})' - |(z^{1/m})'|^p < 0, \quad \forall \ r \gg 1.$$

For any given $\varepsilon > 0$, since $p > m > 1$ and $(z, z') \to (0, 0)$ as $r \to \infty$, it follows that

$$\alpha z^{1/m} + (\beta + \varepsilon) r(z^{1/m})' < 0, \quad \forall \ r \gg 1,$$

and consequently,

$$z(r) \le C r^{-m\alpha/(\beta+\varepsilon)}, \quad \forall \ r \gg 1. \tag{3.14}$$

Step 2. Chosen $\mu > m\alpha/\beta$ and define $h(r) = \mu z(r; a) + r z'(r; a)$. We claim that $h(r)$ does not change signs for $r \gg 1$. In fact, if $h(r_0) = 0$, then by using (3.3) we have

$$h'(r_0) = -\frac{\mu(\mu + 2 - n)}{r_0} z + (\mu\beta/m - \alpha) r_0 z^{1/m} + r_0^{1-p} m^{-p} \mu^p z^{p/m} > 0$$

provided that r_0 satisfies:

$$z^{(m-1)/m}(r_0; a) < \frac{\mu\beta/m - \alpha}{\mu(\mu + 2 - n)} r_0^2 \quad (\alpha > n\beta \Rightarrow \mu > n).$$

Hence, $h(r) > 0$ for $r > r_0$. Therefore,

$$h(r) < 0 \quad \text{for all } \ r \gg 1; \quad \text{or} \quad h(r) > 0 \quad \text{for all } \ r \gg 1.$$

If $h(r) = \mu z + r z' < 0$ for all $r \gg 1$, then $-rz'/z > \mu$, and consequently

$$\begin{aligned} z'' &= -\frac{n-1}{r} z' + z^{1/m}(-\alpha - \beta r z'/(mz)) + m^{-p} z^{p(1-m)/m} |z'|^p \\ &> z^{1/m}(\beta\mu/m - \alpha) \triangleq \delta z^{1/m}, \quad \delta > 0, \quad r \gg 1. \end{aligned} \tag{3.15}$$

Multiplying (3.15) by z' and integrating the results from r to ∞ we have

$$(z')^2 \ge [m\delta/(1+m)] z^{(1+m)/m}, \quad \text{i.e.,} \quad -z' \ge C z^{(1+m)/(2m)}, \quad r \gg 1.$$

Since $m > 1$, an integration of the last inequality yields that $z(r_0) = 0$ for some $r_0 < \infty$. It is a contradiction. Therefore, $h(r) > 0$ for all $r \gg 1$.

By using (3.14) we have

$$|z'| = -z' \le \mu r^{-1} z = O(r^{-1-m\alpha/(\beta+\varepsilon)}). \tag{3.16}$$

Multiplying (3.3) by $r^{(\alpha-\beta)/\beta}$ and an integration from 0 to r yields

$$\begin{aligned} (3.17) \qquad & z' r^{(\alpha-\beta)/\beta} + \beta r^{\alpha/\beta} z^{1/m} \\ & = \left(\frac{\alpha}{\beta} - n\right) \int_0^r z' s^{\frac{\alpha}{\beta}-2} ds + m^{-p} \int_0^r z^{\frac{p(1-m)}{m}} |z'|^p s^{\frac{\alpha}{\beta}-1} ds. \end{aligned}$$

By (3.16) it follows that two integrands of the right hand side of (3.17) converge and $\lim_{r\to\infty} z' r^{(\alpha-\beta)/\beta} = 0$. Consequently, the limit

$$\lim_{r\to\infty} r^{\alpha/\beta} z^{1/m} = k(a)$$

exists. It is obvious that $k(a) \geq 0$.

Now we show tht $k(a) > 0$. Assuming on the contrary that $k(a) = 0$. Multiplying (3.3) by $r^{(\alpha-\beta)/\beta}$ and an integration from r to ∞ yields, in view of (3.16),

$$\begin{aligned} & z' r^{(\alpha-\beta)/\beta} + \beta r^{\alpha/\beta} z^{1/m} \\ & = \left(n - \frac{\alpha}{\beta}\right) \int_r^\infty z' s^{\frac{\alpha}{\beta}-2} ds - m^{-p} \int_r^\infty z^{\frac{p(1-m)}{m}} |z'|^p s^{\frac{\alpha}{\beta}-1} ds. \end{aligned}$$

Consequently, $zr^M \to 0$ as $r \to \infty$ for any $M > 0$.

On the other hand, since $h(r) > 0$ for $r \gg 1$, it follows that $z(r) \geq Cr^{-\mu}$ for $r \gg 1$. We get a contradiction. Therefore, $k(a) > 0$. □

Lemma 3.5. *The set $\mathcal{A}$ is non-empty and open. Moreover, for any $a \in \mathcal{A}$, there exists a limit $\lim_{r\to\infty} r^{\alpha/\beta} z^{1/m}(r; a) = k(a)$ and $k(a) > 0$.*

Proof. Using (3.6) and the special choice of θ (see (3.7)), by following the proof of Lemma 2.5 we can show that if a is large, it is in $\mathcal{A}$. Since $1/m < \theta < 1$, the number $r_{1,\eta} := \frac{m\alpha}{\beta}\eta^{1-\theta} + \frac{m\theta}{\beta}\eta^{\theta-1/m} \to 0$ as $\eta \searrow 0$. Lemma 3.2 shows that $\mathcal{A}$ is open.

The last conclusion is a corollary of Lemma 3.4. □

Lemma 3.6. *The set $\mathcal{B}$ is non-empty. For any $a \in \mathcal{B}$, the corresponding solution $z(r; a)$ satisfies the following interface relation:*

$$\lim_{r \nearrow R(a)} \{z'/z^{1/m}\} = -\beta R(a).$$

Proof. From Lemmas 3.3 and 3.5 we know that $\mathcal{B}$ is non-empty.

For simplicity we denote $R = R(a)$. Putting Equation (3.3) into divergence form and integrating the results from r to R, we get

$$\begin{aligned} (3.18) \qquad r^{n-1} z' = & -\beta r^n z^{1/m} + (\alpha - n\beta) \int_r^R z^{1/m} s^{n-1}\, ds \\ & - m^{-p} \int_r^R z^{p(1-m)/m} |z'|^p s^{n-1}\, ds. \end{aligned}$$

Dividing (3.18) by $z^{1/m}$ and putting $r \to R$ yields

(3.19)
$$R^{n-1} \lim_{r\to R} \frac{z'}{z^{1/m}} = -\beta R^n - m^{-p} \lim_{r\to R} z^{-1/m} \int_r^R z^{p(1-m)/m} |z'|^p s^{n-1}\, ds.$$

By using L'Hospital's rule we get

(3.20)
$$\lim_{r\to R} z^{-1/m} \int_r^R z^{p(1-m)/m} |z'|^p s^{n-1} = m R^{n-1} \lim_{r\to R} (|z'|/z^{(m-1)/m})^{p-1}\, ds$$

if the limit of the right hand side of (3.20) exists.

In the following we will prove that

$$\lim_{r\to R} \{|z'|/z^{(m-1)/m}\} = 0. \tag{3.21}$$

Dividing (3.18) by $z^{(m-1)/m}$ and putting $r \to R$, note that $1/m > (m-1)/m$, we have

$$R^{n-1} \lim_{r\to R} \frac{z'}{z^{(m-1)/m}} = -\frac{m^{1-p}}{m-1} R^{n-1} \lim_{r\to R} z^{(p+1-mp)/m} |z'|^{p-1} \tag{3.22}$$

if the limit of the right hand side of (3.22) exists.

Choose $\sigma_1 = (p+m-mp)/m$, then $\sigma_1 < 1/m$. Similar to (3.22) we have

$$R^{n-1} \lim_{r\to R} \frac{z'}{z^{\sigma_1}} = -\frac{m^{-p}}{\sigma_1} R^{n-1} \lim_{r\to R} |z'|^{p-1} = 0. \tag{3.23}$$

Case 1. If $p+1 \geq mp$, then $\sigma_1 \geq (m-1)/m$, and consequently,

$$\lim_{r\to R} \left\{ |z'|/z^{(m-1)/m} \right\} \leq \lim_{r\to R} \{|z'|/z^{\sigma_1}\} = 0,$$

i.e., (3.21) holds.

Case 2. If $p+1 < mp$. We write

$$z^{(p+1-mp)/m} |z'|^{p-1} = \left(\frac{|z'|}{z^{(mp-p-1)/(mp-m)}} \right)^{p-1}, \tag{3.24}$$

and define a sequence

$$\sigma_{l+1} = (p-1)\sigma_l + 1 - p + p/m, \qquad l = 1, 2, \cdots. \tag{3.25}$$

We first discuss the properties of σ_l. If $\sigma_l < (mp-p-1)/(mp-m)$ for some l, then

$$\sigma_l < \frac{m+p-mp}{m(2-p)}, \qquad \text{and} \qquad \frac{mp-p-1}{mp-m} < \frac{m+p-mp}{m(2-p)} \tag{3.26}$$

because $m < 2$. The first inequality of (3.26) implies $\sigma_{l+1} > \sigma_l$. We assert that there exists l such that $\sigma_l \geq (mp-p-1)/(mp-m)$. Otherwise, $\sigma_l \to \sigma_0$ as $l \to \infty$ for some $\sigma_0 > 0$ and $\sigma_0 \leq (mp-p-1)/(mp-m)$. Letting $l \to \infty$

in (3.25) yields $\sigma_0 = (m+p-mp)/(2m-mp)$. It contradicts with the second inequality of (3.26).

We assume that $l_0 \geq 1$ is the first one such that $\sigma_{l_0} \geq (mp-p-1)/(mp-m)$. If $l_0 = 1$, then $\lim_{r\to R} z^{(p+1-mp)/m}|z'|^{p-1} \leq \lim_{r\to R}(|z'|/z^{\sigma_1})^{p-1} = 0$ by (3.23). Hence, (3.21) holds in view of (3.22). If $l_0 > 1$, we have

$$\begin{aligned}\sigma_{l+1} &= (p-1)\sigma_l + 1 - p + p/m < (mp-p-1)/m + 1 - p + p/m \\ &= 1 - 1/m < 1/m, \qquad \forall \ \ 1 \leq l < l_0\end{aligned}$$

because $\sigma_l < (mp-p-1)/(mp-m)$. Moreover, $\sigma_{l+1} > \sigma_l > \cdots > \sigma_1 = (m+p-mp)/m$ for all $1 \leq l \leq \sigma_{l_0} - 1$. Dividing (3.18) by $z^{\sigma_{l+1}}, \ \ 1 \leq l < l_0$, we have

$$\begin{aligned}(3.27)\ \ & R^{n-1}\lim_{r\to R}\frac{z'}{z^{\sigma_{l+1}}} \\ &= -m^{-p}\lim_{r\to R} z^{-\sigma_{l+1}}\int_r^R z^{p(1-m)/m}|z'|^p s^{n-1}\,ds, \quad l = 1, 2, \ldots, l_0 - 1,\end{aligned}$$

and

$$\begin{aligned}(3.28)\qquad & \lim_{r\to R} z^{-\sigma_{l+1}}\int_r^R z^{p(1-m)/m}|z'|^p s^{n-1}\,ds \\ &= \frac{R^{n-1}}{\sigma_{l+1}}\lim_{r\to R}(|z'|/z^{\sigma_l})^{p-1}, \quad l = 1, 2, , \ldots, l_0 - 1\end{aligned}$$

provided that the limit of the right hand side of (3.28) exists. Using (3.23) we have that the limit of the left hand side of (3.28) equals zero for $l = 1$. Step by step, by repeatedly using (3.27) and (3.28) it follows that the conclusion holds for $l = l_0 - 1$. That is,

$$R^{n-1}\lim_{r\to R}\frac{z'}{z^{\sigma_{l_0}}} = -m^{-p}\lim_{r\to R} z^{-\sigma_{l_0}}\int_r^R z^{p(1-m)/m}|z'|^p s^{n-1}\,ds = 0.$$

Because $\sigma_{l_0} \geq (mp-p-1)/(mp-m)$, by use of (3.22) it follows that (3.21) holds.

From (3.19)-(3.21) we know that Lemma 3.6 holds. □

Proof of Theorem 3.1 (ii). Denote $\varphi(b) = k(b^m)$. By Lemmas 3.3, 3.5 and 3.6 we know that Theorem 3.1 (ii) holds. □

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