

Pacific Journal of Mathematics

ON IMAGINARY QUADRATIC NUMBER FIELDS WITH
2-CLASS GROUP OF RANK 4 AND INFINITE 2-CLASS
FIELD TOWER

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Let k be an imaginary quadratic number field with $C_{k,2}$, the 2-Sylow subgroup of its ideal class group C_k , of rank 4. We show that k has infinite 2-class field tower for particular families of fields k , according to the 4-rank of C_k , the Kronecker symbols of the primes dividing the discriminant Δ_k of k , and the number of negative prime discriminants dividing Δ_k . In particular we show that if the 4-rank of C_k is greater than or equal to 2 and exactly one negative prime discriminant divides Δ_k , then k has infinite 2-class field tower.

Introduction.

Let k denote an algebraic number field and $C_{k,2}$ denote its 2-class group, i.e., the 2-Sylow subgroup of the ideal class group C_k (in the wide sense) of k ; denote by k_1 the Hilbert 2-class field of k . Let k_n (for n a nonnegative integer) be defined inductively as $k_0 = k$ and $k_{n+1} = (k_n)_1$. Then $k_0 \subseteq k_1 \subseteq k_2 \subseteq \dots \subseteq k_n \subseteq \dots$ is called the 2-class field tower of k . If n is the minimal integer such that $k_n = k_{n+1}$, then n is called the length of the tower. If no such n exists, then the tower is said to be of infinite length.

In 1964, Golod and Shafarevich (cf. [4]) established for the first time the existence of infinite p -class field towers, for p prime. In the case $p = 2$, their criterion (as refined by Gaschütz and Vinberg [10]) can be stated in the following way, where $E_{k,2}$ denotes the unit group of k mod its squares, E_k/E_k^2 : If $\text{rank } C_{k,2} \geq 2 + 2(\text{rank } E_{k,2} + 1)^{1/2}$ then k has infinite 2-class field tower. We shall refer to the above inequality as the Golod-Shafarevich inequality. We immediately see that for k imaginary with $\text{rank } C_{k,2} \geq 5$, or k real with $\text{rank } C_{k,2} \geq 6$, the Golod-Shafarevich inequality is satisfied and k thereby has infinite 2-class field tower. It is well-known that for k imaginary with $\text{rank } C_{k,2} = 2$ or 3, the 2-class field tower of k may be finite or infinite, and that if $\text{rank } C_{k,2} = 1$ then the 2-class field tower of k is finite and has length 1 (cf. [3], [6], [14], [17], [20]). It has been conjectured that for k imaginary with $\text{rank } C_{k,2} = 4$, k has infinite 2-class field tower (cf. [17], [18]).

A partial result in this direction, as proved by Hajir, is that if k is an imaginary quadratic number field such that $C_{k,2}$ contains a subgroup isomorphic to $Z_4 \times Z_4 \times Z_4$, then k has infinite 2-class field tower (cf. [6], [7]). We extend this result to particular fields k with rank $C_{k,2} = 4$ and 4-rank of C_k less than 3. Denoting the discriminant of k by Δ_k , our fields k are classified according to the 4-rank of C_k , the Kronecker symbols (p/q) of the primes dividing Δ_k , and the number of negative prime discriminants dividing Δ_k . We demonstrate that if the 4-rank of C_k is greater than or equal to 2 and exactly one negative prime discriminant divides Δ_k , then k has infinite 2-class field tower.

Preliminaries.

Our initial results are directly related to the following inequality (cf. [6], [7]):

Proposition 1. *Let F be a totally real field of degree n , E a totally complex quadratic extension of F , and t the number of prime ideals of F which ramify in E . If $t \geq 3 + 2\sqrt{n} + 1$ then the 2-class field tower of E is infinite.*

We will also need to utilize the well-known ambiguous class number formula, where for a cyclic extension K/F an ambiguous ideal class is an ideal class of K that remains invariant under the action of $\text{Gal}(K/F)$. We denote the subgroup of ambiguous ideal classes by $\text{Am}(K/F)$ and its Sylow 2-subgroup by $\text{Am}_2(K/F)$. We state the following two propositions: (cf. [12], [15]).

Proposition 2. *Let K/F be a cyclic extension of prime degree p . Then $|\text{Am}(K/F)| = h(F) \cdot p^{t-1} / (E : H)$ where t is the number of (finite or infinite) primes of F which are ramified in K/F , $E = E_F$ is the unit group of F , $H = E \cap N_{K/F}K^x$ is the subgroup of units which are norms of elements of K^x , and K^x is the multiplicative group of K .*

Proposition 3. *Let K/F be a quadratic extension of an algebraic number field where $h(F)$ is odd. Then $|\text{Am}_2(K/F)| = 2^e$ where e is the 2-rank of C_K .*

Results.

We begin by obtaining some conditions on the Kronecker symbols of the primes dividing Δ_k , directly related to Proposition 1, to insure that an imaginary quadratic number field k with rank $C_{k,2} = 4$ has infinite 2-class field tower.

Lemma 1. *Let k be an imaginary quadratic number field such that rank $C_{k,2} = 4$. If for some prime $p_j \equiv 1 \pmod{4}$, or $p_j = 2$ in which case we further assume that 8 is a fundamental discriminant dividing Δ_k , we have*

$\left(\frac{p_j}{p_k}\right) = \left(\frac{p_j}{p_l}\right) = \left(\frac{p_j}{p_m}\right) = 1$, p_j, p_k, p_l, p_m distinct primes, $p_j p_k p_l p_m | \Delta_k$, then k has infinite 2-class field tower.

Proof. We proceed in a similar way to Hajir in [6]. Let $F = Q(\sqrt{p_j})$, $E = k(\sqrt{p_j})$. We see that E is a CM field with maximal field subfield F such that either 7 or 8 primes ramify from F to E . Since $7 \geq 3 + 2\sqrt{2+1}$, it follows from Proposition 1 that E has infinite 2-class field tower. Since E is an unramified quadratic extension of k , k has infinite 2-class field tower as well. \square

We utilize the following notational convenience: If d_j is a negative prime discriminant we let p_j denote the prime dividing d_j if $d_j \neq -4$, and $p_j = 1$ if $d_j = -4$.

Lemma 2. *Let k be an imaginary quadratic number field such that $\text{rank } C_{k,2} = n$, $n \geq 1$. Let $L = Q(\sqrt{d_j})$ and $F = k(\sqrt{d_j})$, where d_j is a negative prime discriminant, $p_j | \Delta_k$. Then exactly $2n$ prime ideals in L are ramified in F if and only if $\left(\frac{-p_j}{p_i}\right) = 1$, $i \neq j$, for all primes $p_i | \Delta_k$, $1 \leq i \leq n+1$.*

Proof. Assume $\left(\frac{-p_j}{p_i}\right) = 1$, $i \neq j$, for all primes $p_i | \Delta_k$, $1 \leq i \leq n+1$. It follows that there are exactly n primes p_i dividing Δ_k that split in $L = Q(\sqrt{d_j})$. Since $F = k(\sqrt{d_j})$ is an unramified quadratic extension of k , these n primes p_i each have ramification index 2 in F . We therefore can conclude that each of these primes p_i must ramify from L to F . There are no other prime ideals that ramify from L to F , since if there were a prime ideal P_m in L that ramifies in F such that $P_m \cap Q = p_m \neq p_i$, $1 \leq i \leq n+1$, it would imply that p_m ramifies in F . But p_m does not divide Δ_k unless $p_m = p_j$, and F is an unramified quadratic extension of k . Since p_j has ramification index 2 in F , we therefore conclude that exactly $2n$ prime ideals in L are ramified in F . The converse is proved in a similar way and is left to the reader. \square

We note that in our proof of Lemma 1 we were able to utilize the full strength of Proposition 1 by requiring only 7 primes to ramify from F to E , whereas Hajir, in his original proof that if the 4-rank of C_k is greater than or equal to 3 then k has infinite 2-class field tower (cf. [6]), assumed that $p_i \equiv 1 \pmod{4}$, $\left(\frac{p_j}{p_i}\right) = 1$, $j \neq i$, and therefore 8 primes ramified from F to E .

We illustrate Hajir's method of proof of the above result in the case where a negative prime discriminant d_j divides Δ_k , $\left(\frac{-p_j}{p_i}\right) = 1$, $j \neq i$, as follows (cf. [7]):

Lemma 3. *Let k be an imaginary quadratic number field such that $\text{rank } C_{k,2} = 4$. Assume there exists a negative prime discriminant d_j , $d_j | \Delta_k$,*

such that $\left(\frac{-p_i}{p_i}\right) = 1$, $j \neq i$, for all primes $p_i | \Delta_k$, $1 \leq i \leq 5$. Then k has infinite 2-class field tower.

Proof. Let $L = Q(\sqrt{-p_j})$ and $F = k(\sqrt{-p_j})$. By Lemma 2 we see that exactly 8 prime ideals in L are ramified in F . By Proposition 2 and Proposition 3, it follows that $\text{rank } C_{F,2} \geq 6$. Since $\text{rank } E_{F,2} = 2$, we obtain the Golod-Shafarevich inequality: $\text{rank } C_{F,2} \geq 6 \geq 2 + 2\sqrt{2+1}$ and therefore F has infinite 2-class field tower. Since F is an unramified quadratic extension of k , k has infinite 2-class field tower as well. \square

We state the following corollaries of Lemma 3:

Corollary 1. *Let k be an imaginary quadratic number field such that $\text{rank } C_{k,2} = 4$, exactly one negative prime discriminant divides Δ_k , and $\Delta_k \equiv 4 \pmod{8}$. Then k has infinite 2-class field tower.*

Proof. Since k has exactly one negative prime discriminant and $\Delta_k \equiv 4 \pmod{8}$, all the odd primes dividing Δ_k are congruent to 1 mod 4. We therefore have $\left(\frac{-1}{p_i}\right) = 1$ for all odd primes $p_i | \Delta_k$ and our result follows immediately from Lemma 3. \square

Corollary 2. *Let k be an imaginary quadratic number field, $\text{rank } C_{k,2} = 4$, such that five negative prime discriminants divide Δ_k . Then the following fields k have infinite 2-class field tower, where q_i , $1 \leq i \leq 5$, is a prime congruent to 3 mod 4:*

$$Q(\sqrt{-q_1 q_2 q_3 q_4 q_5}), \quad \left(\frac{-q_i}{q_j}\right) = \left(\frac{-q_i}{q_k}\right) = \left(\frac{-q_i}{q_l}\right) = \left(\frac{-q_i}{q_m}\right) = 1, \\ \{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$$

$$Q(\sqrt{-q_1 q_2 q_3 q_4}), \quad \left(\frac{-q_i}{q_j}\right) = \left(\frac{-q_i}{q_k}\right) = \left(\frac{-q_i}{q_l}\right) = \left(\frac{-q_i}{2}\right) = 1, \\ \{i, j, k, l\} = \{1, 2, 3, 4\}.$$

Proof. It is immediate by applying Lemma 3 to each field k that we have infinite 2-class field tower. We note that these are all the possible fields satisfying the assumptions of our corollary for which we are able to apply Lemma 3. \square

For the cases when exactly one negative prime discriminant divides Δ_k where $\Delta_k \not\equiv 4 \pmod{8}$, and exactly three negative prime discriminants divide Δ_k , we utilize the following lemma:

Lemma 4. *Let k be an imaginary quadratic number field such that $\text{rank } C_{k,2} = 4$, at least two of the prime discriminants dividing Δ_k are positive, and $\left(\frac{p_1}{p_3}\right) = \left(\frac{p_2}{p_3}\right) = 1$ where p_1 and p_2 are distinct primes dividing positive*

prime discriminants dividing Δ_k , and p_3 is a prime dividing a positive or negative prime discriminant dividing Δ_k , $p_1 \neq p_3 \neq p_2$. Then k has infinite 2-class field tower.

Proof. By the assumptions of our lemma, we can write $k = Q(\sqrt{-p_1 p_2 p_3 p_4 p_5})$ or $k = Q(\sqrt{-p_1 p_2 p_3 p_4})$ where $p_1 \equiv p_2 \equiv 1 \pmod{4}$, or $p_1 = 2$ and $p_2 \equiv 1 \pmod{4}$. By Martinet (cf. [17], Proposition 5) we see immediately that k has infinite 2-class field tower. \square

We now let $k = Q(\sqrt{-p_1 p_2 p_3 p_4 p_5})$ where $\Delta_k \not\equiv 4 \pmod{8}$ and exactly one negative prime discriminant divides Δ_k . We define a Kronecker symbol configuration of k to be a complete list of Kronecker symbols $\left(\frac{p_i}{p_j}\right)$, $i \leq j$, $l \leq i \leq 5$, $l \leq j \leq 5$. We denote a Kronecker symbol configuration by listing all the Kronecker symbols $\left(\frac{p_i}{p_j}\right)$ as above with $\left(\frac{p_i}{p_j}\right) = 1$ (respectively -1), where the remaining Kronecker symbols $\left(\frac{p_i}{p_j}\right)$, $i < j$, are assumed to be -1 (respectively 1).

In Table 1 we utilize the Rédei & Reichardt conditions [19] to list all possible Kronecker symbol configurations (without loss of generality) according to the 4-rank of C_k .

Table 1.

| 4-rank of C_k | possible Kronecker symbol configurations |
|-----------------|--|
| 4 | all Kronecker symbols equal 1 |
| 3 | $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_4}\right) = \left(\frac{p_1}{p_5}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = -1$ |
| 2 | $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_2}{p_3}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = \left(\frac{p_1}{p_5}\right) = \left(\frac{p_2}{p_5}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_4}\right) = \left(\frac{p_1}{p_5}\right) = \left(\frac{p_2}{p_5}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = -1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = -1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_2}{p_3}\right) = -1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_5}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_5}\right) = \left(\frac{p_2}{p_3}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_4}{p_5}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_5}\right) = \left(\frac{p_2}{p_3}\right) = 1$ |

| 4-rank of C_k | possible Kronecker symbol configurations |
|-----------------|--|
| 1 | $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_4}\right) = \left(\frac{p_2}{p_3}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_4}\right) = \left(\frac{p_2}{p_3}\right) = -1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_4}\right) = -1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_2}{p_4}\right) = -1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = 1$ |
| 0 | all Kronecker symbols equal -1 $\left(\frac{p_1}{p_2}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_2}{p_4}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = \left(\frac{p_1}{p_5}\right) = \left(\frac{p_2}{p_3}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = \left(\frac{p_1}{p_5}\right) = \left(\frac{p_2}{p_3}\right) = \left(\frac{p_4}{p_5}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_4}\right) = \left(\frac{p_2}{p_5}\right) = \left(\frac{p_3}{p_5}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = \left(\frac{p_1}{p_5}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_4}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_5}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_5}\right) = \left(\frac{p_2}{p_3}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = \left(\frac{p_1}{p_5}\right) = -1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = \left(\frac{p_1}{p_5}\right) = \left(\frac{p_2}{p_3}\right) = -1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_5}\right) = -1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = \left(\frac{p_1}{p_5}\right) = \left(\frac{p_2}{p_5}\right) = 1$ $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_3}{p_4}\right) = \left(\frac{p_1}{p_3}\right) = \left(\frac{p_1}{p_4}\right) = \left(\frac{p_1}{p_5}\right) = -1$ |

We are now able to state the following theorem:

Theorem 1. *Let k be an imaginary quadratic number field with $\text{rank } C_{k,2} = 4$, $\Delta_k \not\equiv 4 \pmod{8}$, and exactly one negative prime discriminant dividing Δ_k . Then the following fields k have infinite 2-class field tower, where $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$:*

(A) 4-rank of C_k equal to 2, 3 or 4

- (B) 4-rank of C_k equal to 1 and Kronecker symbol configuration of k not $(\frac{p_i}{p_j}) = (\frac{p_i}{p_k}) = 1$ where either p_j or p_k is the prime dividing the negative prime discriminant dividing d_k
- (C) 4-rank of C_k equal to 0 (i.e., $C_{k,2}$ elementary) and Kronecker symbol configuration of k not one of the following types:
- all Kronecker symbols equal -1
 - $(\frac{p_i}{p_j}) = 1$
 - $(\frac{p_i}{p_j}) = (\frac{p_k}{p_l}) = 1$
 - $(\frac{p_i}{p_j}) = (\frac{p_k}{p_l}) = (\frac{p_i}{p_m}) = 1$ where either p_j or p_m is the prime dividing the negative prime discriminant dividing d_k .

Proof. For case (A) with 4-rank of C_k equal to 3 or 4 the result has been established by Hajir (cf. [6], [7]). For case (A) with 4-rank of C_k equal to 2, and cases (B) and (C), we apply Lemma 4 to our fields listed in Table 1 to establish our result. \square

From Table 1 we see that there are 32 possible Kronecker symbol configurations when exactly one negative prime discriminant divides Δ_k , $\Delta_k \not\equiv 4 \pmod{8}$. From Theorem 1 we find that for 27 of these Kronecker symbol configurations, k has infinite 2-class field tower. The unknown cases can be summarized by means of the 4-rank of C_k as follows:

Table 2.

| 4-rank of C_k | number of Kronecker symbol configurations where 2-class field tower of k may be finite |
|-----------------|--|
| 4 | 0 |
| 3 | 0 |
| 2 | 0 |
| 1 | 1 |
| 0 | 4 |

Remark 1. For the case when exactly three negative prime discriminants divide Δ_k , one can again utilize the Rédei & Reichardt conditions and Lemma 4 to obtain fields with infinite 2-class field tower. We note that in this case the Kronecker symbol configuration $(\frac{p_1}{p_2}) = (\frac{p_1}{p_3}) = (\frac{p_1}{p_4}) = 1$ may not satisfy the requirements of Lemma 4; however, Lemma 1 may be used when $p_1 \not\equiv 1 \pmod{4}$, or when $p_1 = 2$ if 8 is a fundamental discriminant dividing Δ_k . For the case when five negative prime discriminants divide Δ_k , one can utilize the Rédei & Reichardt conditions and Corollary 2 to obtain

fields with infinite 2-class field tower. In a follow-up paper we will demonstrate that our techniques allow us to conclude that k has infinite 2-class field tower for imaginary quadratic number fields k when $\text{rank } C_{k,2} = 4$, C_k has 4-rank equal to 2, and either five negative prime discriminants divide Δ_k or $\Delta_k \not\equiv 4 \pmod{8}$ (cf. [1]).

Examples.

From Lemma 1, Corollary 2, and Theorem 1 we immediately obtain that the following fields k have infinite 2-class field tower. We list our fields according to the 4-rank of C_k .

$$\begin{aligned}
 C_{k,2} \text{ elementary: } k &= Q\left(\sqrt{-61, 620}\right) = Q\left(\sqrt{-3 \cdot 5 \cdot 13 \cdot 79}\right) \\
 k &= Q\left(\sqrt{-120, 180}\right) = Q\left(\sqrt{-5 \cdot 13 \cdot 17 \cdot 29}\right) \\
 k &= Q\left(\sqrt{-122, 655}\right) = Q\left(\sqrt{-3 \cdot 5 \cdot 13 \cdot 17 \cdot 37}\right) \\
 k &= Q\left(\sqrt{-212, 135}\right) = Q\left(\sqrt{-5 \cdot 7 \cdot 11 \cdot 19 \cdot 29}\right) \\
 k &= Q\left(\sqrt{-256, 360}\right) = Q\left(\sqrt{-2 \cdot 5 \cdot 13 \cdot 17 \cdot 29}\right) \\
 k &= Q\left(\sqrt{-430, 360}\right) = Q\left(\sqrt{-2 \cdot 5 \cdot 7 \cdot 29 \cdot 53}\right) \\
 k &= Q\left(\sqrt{-440, 115}\right) = Q\left(\sqrt{-3 \cdot 5 \cdot 13 \cdot 37 \cdot 61}\right) \\
 k &= Q\left(\sqrt{-850, 135}\right) = Q\left(\sqrt{-5 \cdot 11 \cdot 13 \cdot 29 \cdot 41}\right) \\
 k &= Q\left(\sqrt{-2, 035, 240}\right) = Q\left(\sqrt{-2 \cdot 5 \cdot 17 \cdot 41 \cdot 73}\right) \\
 k &= Q\left(\sqrt{-5, 863, 655}\right) = Q\left(\sqrt{-5 \cdot 7 \cdot 29 \cdot 53 \cdot 109}\right)
 \end{aligned}$$

$$\begin{aligned}
 4 - \text{rank of } C_k = 1: k &= Q\left(\sqrt{-184, 008}\right) = Q\left(\sqrt{-2 \cdot 3 \cdot 11 \cdot 17 \cdot 41}\right) \\
 k &= Q\left(\sqrt{-531, 867}\right) = Q\left(\sqrt{-3 \cdot 7 \cdot 19 \cdot 31 \cdot 43}\right) \\
 k &= Q\left(\sqrt{-2, 657, 415}\right) = Q\left(\sqrt{-3 \cdot 5 \cdot 29 \cdot 41 \cdot 149}\right) \\
 k &= Q\left(\sqrt{-6, 425, 679}\right) = Q\left(\sqrt{-3 \cdot 13 \cdot 37 \cdot 61 \cdot 73}\right)
 \end{aligned}$$

$$4 - \text{rank of } C_k = 2: k = Q\left(\sqrt{-3, 989, 095}\right) = Q\left(\sqrt{-5 \cdot 11 \cdot 29 \cdot 41 \cdot 61}\right)$$

Remark 2. Since the fields satisfying the conditions of Lemmas 1, 3, and 4 possess an unramified quadratic extension which satisfy the Golod-Shafarevich inequality, (cf. [19] in regard to Lemmas 1 and 4) it follows from Theorem 6 of Hajir (cf. [5]) that the rank of the 2-class groups of these fields tend to infinity.

In conclusion, we see that the conjecture concerning the 2-class field tower of k being infinite holds in a number of particular fields k when the 4-rank of C_k is equal to 0, 1, or 2, and always holds when the 4-rank of C_k is greater than or equal to 3. Our techniques allow us to obtain families of fields k with 4-rank of C_k equal to 0, 1, or 2 and k having infinite 2-class field tower, as well as the rank of the 2-class groups of the fields in the tower of k tending to infinity. However, the complete resolution of the conjecture concerning all imaginary quadratic number fields k with rank $C_{k,2} = 4$ is still a very open question.

Acknowledgement. I would like to express my gratitude to Chip Snyder for a number of helpful suggestions throughout this paper.

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Received December 9, 1999 and revised April 19, 2000.

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