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ON COMPOSITION OPERATORS WHICH PRESERVE BMO

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Dedicated to Professor Kôzô Yabuta on his sixtieth birthday

We characterize the Lebesgue measurable maps between Euclidean spaces which preserve BMO.

1. Introduction.

For a subdomain D of $\mathbf{R}^n$, $n \geq 1$, let BMO(D) be the space of all locally integrable functions f on D satisfying

$$(1) \qquad \|f\|_* = \|f\|_{*.D} = \sup_{Q \subset D} |Q|^{-1} \int_Q |f - f_Q| dx < \infty,$$

where $|Q|$ is the n-dimensional Lebesgue measure of Q, $f_Q = |Q|^{-1} \int_Q f dx$, and the supremum is taken over all closed cubes $Q \subset D$ with sides parallel to the coordinate axes.

Let D and D' be subdomains of $\mathbf{R}^m$ and $\mathbf{R}^n$, m, $n \geq 1$, respectively. We say that a map $F : D \to D'$ is measurable if $F^{-1}(E)$ is measurable for each measurable subset E of D'. We say that a measurable map $F : D \to D'$ is a *BMO map* if i) for each null set $E \subset D'$, $F^{-1}(E)$ is also a null set, and furthermore, ii) for each BMO(D') function f, $C_F(f) = f \circ F$ belongs to BMO(D). The condition i) guarantees the uniqueness of the function $f \circ F$. From the closed graph theorem each BMO map F induces a bounded operator C_F between BMO spaces.

Various partial results are known for the characterization of BMO maps. It seems, however, that we do not know almost anything yet for non-continuous BMO maps. The main purpose of the present paper is to give a characterization of BMO maps $F : \mathbf{R}^m \to \mathbf{R}^n$, $m, n \geq 1$ (Theorem 3.1).

Our argument depends on the following two celebrated results for BMO; a growth estimation for BMO functions due to John-Nirenberg, and the existence of certain extremal BMO functions due to Uchiyama (Propositions 4.1 and 4.2).

The present paper is organized as follows. First, we give various examples of BMO maps in §2. The main results of the present paper are given in §3. The following §4 is devoted to their proofs. Finally, in §5 we give a remark on BMO maps which are homeomorphisms between intervals.

In the following, a cube implies a closed cube with sides parallel to the coordinate axes, tQ denotes the cube with the same center as Q and expanded by a constant factor $t > 0$, and we use the letter C to denote a positive constant which may vary from place to place unless stated otherwise, that is, $f \le 2C$ implies $f \le C$, on the other hand, $f \le 2C_2$ does not necessarily mean $f \le C_2$. Also we sometimes write "$F : D \to D'$" even if $F(D) \not\subset D'$ under the assumption that both $A = F(D) \setminus D'$ and $F^{-1}(A)$ are null sets. For instance, we may write $F : \mathbf{R} \to (0, \infty)$, $F(x) = |x|$, instead of $F : \mathbf{R} \to [0, \infty)$, $F(x) = |x|$.

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2. Examples.

In the present section we give various examples of BMO maps.

Example 2.1. a) Let $F : D_1 \times D_2 \to D_1$, $D_1 \subset \mathbf{R}^m$, $D_2 \subset \mathbf{R}^n$, be the canonical projection. Then F is a BMO map satisfying $\|C_F\| = 1$. In particular, if $D_2 = \mathbf{R}^n$, then $\|C_F(f)\|_* = \|f\|_*$ holds for each $f \in \mathrm{BMO}(D_1)$.

b) Let $F : D \to D'$ be the inclusion map. Then F is a BMO map satisfying $\|C_F\| = 1$.

Example 2.2. Let $F : D \to D'$ be a homeomorphism between subdomains of $\mathbf{R}^n$, $n \ge 2$. If F is quasiconformal, then F is a BMO map satisfying $\|C_F\| \le C(n, K_F)$, where K_F is the maximal dilatation of F. Conversely, if F is a BMO map satisfying i) for each null set E, $F^{-1}(E)$ is also a null set, ii) F is ACL, iii) F is differentiable a.e., then F is a quasiconformal map satisfying $K_F \le C(n, \|C_F\|)$ (Reimann [**13**]).

Example 2.3. Let F be a homeomorphism of $\mathbf{R}$. Then F is a BMO map if and only if we can take constants K, $\alpha > 0$ so that

$$\frac{|F^{-1}(E \cap I)|}{|F^{-1}(I)|} \le K \left(\frac{|E \cap I|}{|I|} \right)^\alpha \tag{2}$$

holds for each pair of a measurable subset E of $\mathbf{R}$ and an interval I (Jones [**11**]). Note that (2) holds if and only if F^{-1} is absolutely continuous and its derivative $(F^{-1})'$ (or $-(F^{-1})'$) is an A_∞ weight (cf. (4)). In this case F^{-1} also satisfies the same condition, and so F induces a bijection of $\mathrm{BMO}(\mathbf{R})$.

Jones gave no explicit relation between the constants K, α above and $\|C_F\|$. In §3 we show, however, that his argument implicitly gives the following estimations: If (2) holds, then $\|C_F\| \le CK/\alpha$ for some universal constant $C > 0$; conversely, if F is a BMO map, then we can take constants K, α so that $K = C_1$ and $\alpha = C_2/\|C_F\|$, where $C_k > 0$, $k = 1, 2$, are universal constants. Hence $\|C_F\|$ and $\inf(K/\alpha)$ are comparable with universal constant factors, where the infimum is taken over all pairs of K, α satisfying (2) (Theorem 5.3) (cf. Mayer-Zinsmeister [**12**]).

Fominykh [**3**] gave a sufficient condition for spherically continuous maps between (finite or infinite) open intervals to be BMO maps, which partially extends Jones' result.

Example 2.4. Let $F : D \to D'$ be a nonconstant holomorphic map between plane domains. Then F is a BMO map if and only if we can take an integer $p > 0$ so that for each disk B satisfying $2B \subset D$, F is p-valent on B. In particular, a holomorphic map $F : \mathbf{C} \to \mathbf{C}$ is a BMO map if and only if it is a polynomial (Gotoh [**8**]).

Thus, whether a given nonconstant holomorphic map $F : D \to D'$ between plane domains is a BMO map or not is independent of the choice of its target D'. The following example shows that this does not extend to hold for general maps.

Example 2.5. a) Let $D = \{x \in \mathbf{R}^2 \,|\, 1 < |x| < 2\}$, $I = \{(0, x_2) \in \mathbf{R}^2 \,|\, -2 < x_2 < -1\}$, and $D_0 = D \setminus I$. Let F satisfy $F(x) = x$ on D_0 and $F(x) \in D_0$ on I. Then $F : D \to D$ is a BMO map, and $F : D \to D_0$ is not a BMO map, because $\mathrm{BMO}(D) \neq \mathrm{BMO}(D_0)$.

b) Let $F : D \to D'$ be a BMO map. Let D'_0 be a subdomain of D' satisfying $F(D) \subset D'_0$. Assume that each $\mathrm{BMO}(D'_0)$ function is the restriction of some $\mathrm{BMO}(D')$ function. (Such domains D'_0 are characterized as relative uniform domains with respect to D' (Gotoh [**7**].) For instance, uniform domains D'_0 satisfy this condition (Proposition 3.4). In this case $F : D \to D'$ is a BMO map if and only $F : D \to D'_0$ is a BMO map.

Example 2.6. Let $D = \mathbf{R}^{n-1} \times (0, \infty)$ be the upper half space. Then for each $f \in \mathrm{BMO}(D)$, its symmetric extension g, $g(x, y) = f(x, y)$ on D and $g(x, y) = f(x, -y)$ on $\mathbf{R}^n \setminus D$, is a $\mathrm{BMO}(\mathbf{R}^n)$ function satisfying $\|f\|_{*,\mathbf{R}^n} \le C\|f\|_{*,D}$, where $C > 0$ is a universal constant, which is called a reflection principle for BMO. In other words, the two-sheeted folding map $F : \mathbf{R}^n \to D$, $F(x, y) = (x, |y|)$, is a BMO map satisfying $\|C_F\| \le C$ (cf. Reimann-Rychener [**14**]).

Example 2.7. Let D be a quasidisk, that is, D is the image of the upper half plane under a quasiconformal map of $\overline{\mathbf{R}^2} = \mathbf{R}^2 \cup \{\infty\}$. Let $\tau : \overline{\mathbf{R}^2} \to \overline{\mathbf{R}^2}$ be the quasiconformal reflection with respect to ∂D. Then from Examples 2.2 and 2.6 the two-sheeted folding map $F : \mathbf{R}^2 \to D$, $F(x) = x$ on D and $F(x) = \tau(x)$ on $\mathbf{R}^n \setminus D$, is a BMO map.

Example 2.8. a) Let $D = \{r < |x| < r'\} \subset \mathbf{R}^n$, $n \ge 2$. Let $a = r'/r$ and set $D_k = \{a^k r < |x| < a^{k+1} r\}$, $k \in \mathbf{Z}$. We define an infinite-sheeted folding map $F : \mathbf{R}^n \to D$ as follows: Set $F(x) = x/a^{2k}$ on D_{2k} and $F(x) = F(\tau_k(x))$ on D_{2k+1}, where τ_k is the reflection with respect to the sphere

$\{|x| = a^{2k+1}r\}$. Then F is a BMO map and $\|C_F\| \le \frac{aC}{a-1}$, where $C = C(n) > 0$. This is a consequence of the reflection principle, the removability of one point for BMO, and Proposition 5.2 below.

b) We define an infinite-sheeted folding map $F : \mathbf{R} \to (0,1)$ as follows: Set $F(x) = x$ on $[0,1]$, $F(x) = 2-x$ on $[1,2]$, and $F(x) = F(x-2k)$, $2k \le x \le 2k+2$, $k \in \mathbf{Z}$. Then F is a BMO map. On the other hand, $F \times id_{\mathbf{R}} : \mathbf{R} \times \mathbf{R} \to (0,1) \times \mathbf{R}$ is not a BMO map: Let $f(x_1, x_2) = x_2$. Then $C_{F\times id_{\mathbf{R}}}(f)(x_1,x_2) = x_2$. Thus $f \in \mathrm{BMO}((0,1)\times\mathbf{R})$ and $C_{F\times id_{\mathbf{R}}}(f) \notin \mathrm{BMO}(\mathbf{R}\times\mathbf{R})$.

There are essentially non-continuous BMO maps.

Example 2.9. a) Let τ be a Möbius transformation of $\overline{\mathbf{R}^n} = \mathbf{R}^n \cup \{\infty\}$, $n \ge 2$. Let D be an arbitrary subdomain of $\mathbf{R}^n$ and $D' = \tau(D) \setminus \{\infty\}$. Then $F = \tau|D : D \to D'$ is a BMO map. This is a consequence of Example 2.2 and the removability of one point for BMO. (cf. Reimann-Rychener [**14**]. Also see Lemma 5.1 below.) For instance, $x \mapsto x/|x|^2$, which is discontinuous at the origin under the Euclidean topology, induces a bijection between $\mathrm{BMO}(\{|x| < 1\})$ and $\mathrm{BMO}(\{|x| > 1\})$.

b) Let $F : \mathbf{R} \to (0,1)$ be the infinite-sheeted folding map in Example 2.8 b). Then $G : \mathbf{R} \to (0,1)$, $G(x) = F(1/x)$, is a BMO map which is discontinuous at the origin even under the spherical topology.

Moreover, there are BMO maps between plane domains with essential singularities.

Example 2.10. Let $F(z) = \mathcal{P}(1/z)$, where $\mathcal{P}$ is the Weierstrass $\mathcal{P}$-function. Then $F : \mathbf{C} \to \mathbf{C}$ is a BMO map having the origin as an essential singularity. Another example is given by the Blaschke product $F : \mathbf{C} \to \mathbf{C}$,

$$F(z) = \prod_{k=0}^{\infty} \frac{z - 2^{-k}i}{z + 2^{-k}i} \prod_{k=1}^{\infty} \frac{2^k i - z}{2^k i + z}.$$

(See the next example.) Moreover, for an arbitrary plane domain D and an arbitrary sequence $\{z_k\} \subset D$, $z_k \to \partial D$, there exists a BMO meromorphic map $F : D \to \mathbf{C}$ having $\{z_k\}$ as simple poles satisfying $\|C_F\| \le C$, where $C > 0$ is a universal constant (Gotoh [**6**], [**8**]).

Contrary to the case of holomorphic maps between plane domains (Example 2.4), it seems difficult to estimate the operator norms for rational maps $F : \mathbf{C} \to \mathbf{C}$. As to this we only know the following.

Example 2.11. Let F be a finite Blaschke product on the unit disk Δ. Let t_ζ, $\zeta \in \Delta$, denote the Carleson constant associated with the zeros of the Blaschke product $(F-\zeta)/(1-\bar{\zeta}F)$. Let $s_F = \sup_{\zeta\in\Delta} t_\zeta$. Then for the operator norm $\|C_F\|$ of the map $F : \mathbf{C} \to \mathbf{C}$, we have $\|C_F\| \le C_1(s_F)$ and $s_F \le C_2(\|C_F\|)$. In particular, we can show that there exists a sequence of rational maps $F_k : \mathbf{C} \to \mathbf{C}$, $\deg F_k = k$, satisfying $\|C_{F_k}\| \le C$, where $C > 0$ is a universal constant (Gotoh [6]).

For related topics, see Astala [1], Smith [15], and Mayer-Zinsmeister [12].

3. Main theorem.

We say that a domain $D \subset \mathbf{R}^n$ is *admissible* if D is an increasing limit of some sequence of cubes. For instance, $\mathbf{R}^n$, half spaces with sides parallel to the coordinate axes, and open cubes are admissible.

Theorem 3.1 (Main Theorem). *For a measurable map $F : D \to D'$, $D \subset \mathbf{R}^m$, $D' \subset \mathbf{R}^n$, we consider the following conditions:*

(a) *We can take constants K, $\alpha > 0$ so that for an arbitrary pair of measurable subsets E_1, E_2 of D' we have*

$$\sup_{Q \subset D} \min_{k=1,2} \frac{|F^{-1}(E_k) \cap Q|}{|Q|} \le K \left(\sup_{Q' \subset D'} \min_{k=1,2} \frac{|E_k \cap Q'|}{|Q'|} \right)^{\alpha}, \tag{3}$$

where the suprema are taken over all cubes $Q \subset D$ and $Q' \subset D'$ respectively;

(b) *We can take constants γ, $0 < \gamma < 1/4$, and $\lambda > 0$ so that for an arbitrary pair of measurable subsets E_1, E_2 of D' satisfying*

$$\sup_{Q' \subset D'} \min_{k=1,2} \frac{|E_k \cap Q'|}{|Q'|} < \lambda,$$

we have

$$\sup_{Q \subset D} \min_{k=1,2} \frac{|F^{-1}(E_k) \cap Q|}{|Q|} < \gamma,$$

where the suprema are taken over all cubes $Q \subset D$ and $Q' \subset D'$ respectively;

(c) *F is a* BMO *map.*

Then we have (a) $\Rightarrow$ (b) $\Rightarrow$ (c). Moreover, if we can take an admissible domain D'_0 satisfying $F(D) \subset D'_0 \subset D'$, then all these conditions are equivalent.

In particular, all the conditions above are equivalent if D' is admissible. The implication (a) $\Rightarrow$ (b) is trivial. We show that (a) implies (c) with $\|C_F\| \le C(m,n)K/\alpha$ (Lemma 4.8). Furthermore, if we can take an admissible domain D'_0 satisfying $F(D) \subset D'_0 \subset D'$, then we show that (c) implies (a) with constants $K = K(m,n)$ and $\alpha = C(m,n)/\|C_F\|$ (Lemma 4.10). Thus we have:

Corollary 3.2. *If we can take an admissible domain D'_0 satisfying $F(D) \subset D'_0 \subset D'$, then the operator norm $\|C_F\|$ and* $\inf(K/\alpha)$ *are comparable with* *constant factors depending only on m and n, where the infimum is taken over all pairs of constants K, α satisfying the estimation* (3).

Let F be a homeomorphism of $\mathbf{R}$. Then the condition (2) in Example 2.3 implies (3) with the same constants K, α. And so we may regard the Main Theorem as an extension of Jones' result.

We say that a weight w is an A_∞ weight on D if we can take a constant α, $K > 0$ so that

$$\frac{\int_{E\cap Q} w dx}{\int_Q w dx} \le K \left(\frac{|E\cap Q|}{|Q|}\right)^\alpha \tag{4}$$

holds for each pair of a measurable set $E \subset D$ and a cube $Q \subset D$. A weight w is an A_∞ weight if and only if we can take constants ε, δ, $0 < \varepsilon, \delta < 1$, so that for each pair of E, Q satisfying $|E \cap Q|/|Q| < \delta$, we have $\int_{E\cap Q} w dx / \int_Q w dx < \varepsilon$. The equivalence of the conditions (a) and (b) of the Main Theorem implies the corresponding result holds for BMO maps.

Recall that for a weight w on D, $f = \log w$ belongs to $\mathrm{BMO}(D)$ if and only if w^γ is an A_∞ weight on D for some $\gamma > 0$. Hence,

Corollary 3.3. *We can add the condition*

(d) *For each A_∞ weight w on D', $w^\gamma \circ F$ is an A_∞ weight on D for some $\gamma > 0$.*

to the list of the Main Theorem in the sense that (c) $\Leftrightarrow$ (d) *holds.*

We say that a domain $D \subset \mathbf{R}^n$ is *uniform* if

$$k_D(x, y) \le C \log\left(\frac{d(x, \partial D) + d(y, \partial D) + |x - y|}{\min\{d(x, \partial D), d(y, \partial D)\}}\right), \qquad x, y \in D,$$

holds for some $C > 0$, where k_D is the quasihyperbolic metric on D. Uniform domains are invariant under quasiconformal maps on $\overline{\mathbf{R}^n}$. Half spaces are uniform domains. In the case of a simply connected plane domain D, D is uniform if and only if D is a quasidisk. The uniformness can be characterized by the BMO extension property.

Proposition 3.4 (Jones [**10**]). *A domain $D \subset \mathbf{R}^n$ is uniform if and only if each* $\mathrm{BMO}(D)$ *function is the restriction of some* $\mathrm{BMO}(\mathbf{R}^n)$ *function.*

In this case, for each $g \in \mathrm{BMO}(D)$ we can take $f \in \mathrm{BMO}(\mathbf{R}^n)$, $f|D = g$, so that $\|f\|_* \le C\|g\|_*$, where $C > 0$ is a constant depending only on n and the constant of uniformness.

Corollary 3.5. *Let D and D' be subdomains of $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. Assume that D' is uniform. Then for a measurable map $F : D \to D'$, the following conditions are equivalent:*

(a) *We can take constants K, $\alpha > 0$ so that for an arbitrary pair of measurable subsets E_1, E_2 of D' we have*

$$\sup_{Q\subset D}\min_{k=1,2}\frac{|F^{-1}(E_k)\cap Q|}{|Q|}\le K\left(\sup_{Q'\subset \mathbf{R}^n}\min_{k=1,2}\frac{|E_k\cap Q'|}{|Q'|}\right)^{\alpha}, \tag{5}$$

where the suprema are taken over all cubes $Q \subset D$ and $Q' \subset \mathbf{R}^n$ respectively;

(b) *We can take constants γ, $0 < \gamma < 1/4$, and $\lambda > 0$ so that for an arbitrary pair of measurable subsets E_1, E_2 of D' satisfying*

$$\sup_{Q'\subset \mathbf{R}^n}\min_{k=1,2}\frac{|E_k\cap Q'|}{|Q'|}<\lambda,$$

we have

$$\sup_{Q\subset D}\min_{k=1,2}\frac{|F^{-1}(E_k)\cap Q|}{|Q|}<\gamma,$$

where the suprema are taken over all cubes $Q \subset D$ and $Q' \subset \mathbf{R}^n$ respectively;

(c) *F is a* BMO *map;*

(d) *$G = i\circ F : D \to \mathbf{R}^n$ is a* BMO *map, where $i : D \to \mathbf{R}^n$ is the inclusion map.*

We cannot replace the condition "$Q' \subset \mathbf{R}^n$" in (a) (and in (b)) above with "$Q' \subset D'$" (Example 4.11).

Remark 3.6. We may replace the assertion (a) of the Main Theorem with

(a′) For each $N \ge 2$ we can take constants K, $\alpha > 0$ so that for arbitrary measurable subsets $E_1, \dots , E_N$ of D' we have

$$\sup_{Q\subset D}\min_{1\le k\le N}\frac{|F^{-1}(E_k)\cap Q|}{|Q|}\le K\left(\sup_{Q'\subset D'}\min_{1\le k\le N}\frac{|E_k\cap Q'|}{|Q'|}\right)^{\alpha}, \tag{6}$$

where the suprema are taken over all cubes $Q \subset D$ and $Q' \subset D'$ respectively,

or

(a″) The assertion (a′) holds for some $N \ge 2$.

Similarly, we may replace the assertion (b) of the Main Theorem with

(b′) For each $N \ge 2$ we can take constants γ, $0 < \gamma < 1/4$, and $\lambda > 0$ so that for arbitrary measurable subsets $E_1, ..., E_N$ of D' satisfying

$$\sup_{Q'\subset D'}\min_{1\le k\le N}\frac{|E_k\cap Q'|}{|Q'|}<\lambda,$$

we have

$$\sup_{Q\subset D}\min_{1\le k\le N}\frac{|F^{-1}(E_k)\cap Q|}{|Q|}<\gamma,$$

where the suprema are taken over all cubes $Q\subset D$ and $Q'\subset D'$ respectively,

or

(b″) The assertion (b′) holds for some $N\ge 2$.

The implications (a′) ⇒ (a) ⇒ (a″), (b′) ⇒ (b) ⇒ (b″), (a) ⇒ (b), (a′) ⇒ (b′), and (a″) ⇒ (b″) are trivial. Furthermore, we obtain (a″) ⇒ (a) and (b″) ⇒ (b) by setting $E_2=E_3=\cdots=E_N$. In the next section we show (b) ⇒ (c), and (c) ⇒ (a) (under the additive assumption). It is easy to check that we can show (c) ⇒ (a′) in the same way.

Note that we can also rewrite Corollaries 3.2 and 3.5 similarly.

4. Proofs of the Main Theorem and Corollary 3.5.

The following two results play fundamental roles in the proof of the Main Theorem. The latter one shows that the growth estimation of BMO functions given by the former one is remarkably precise.

Proposition 4.1 (John-Nirenberg [9]). *Let $f\in \mathrm{BMO}(D)$, $D\subset\mathbf{R}^n$, and $Q\subset D$ be a cube. Then*

$$|\{x\in Q\,|\,|f(x)-f_Q|\ge t\}|\le C_1|Q|\exp\left(-C_2\frac{t}{\|f\|_*}\right),\quad t\ge 0,$$

where C_1, $C_2>0$ are constants depending only on n.

Proposition 4.2 (Uchiyama [17], cf. Garnett-Jones [4]). *Let D be an admissible subdomain of $\mathbf{R}^n$. Let $N\ge 2$, $t>1$, and E_1,...,E_N be measurable subsets of D satisfying*

$$\sup_{Q\subset D}\min_{1\le k\le N}\frac{|E_k\cap Q|}{|Q|}\le 2^{-nt},$$

where the supremum is taken over all cubes $Q\subset D$. Then there exist $\mathrm{BMO}(D)$ *functions f_1, ... ,f_N satisfying $\sum_{k=1}^N f_k=1$ and*

$$0\le f_k\le 1,\qquad f_k=0\ \text{ on } E_k,\qquad \|f_k\|_*\le C/t,\qquad (0\le k\le N),$$

where $C=C(n,N)>0$.

Garnett-Jones showed the assertion when $D=\mathbf{R}^n$, $N=2$ and $E_1\subset Q$, $E_2=\mathbf{R}^n\setminus 2Q$. Uchiyama extended their result to the form above.

First, we give a variant of the John-Nirenberg Theorem.

Lemma 4.3. *Let $f \in \mathrm{BMO}(D)$, $D \subset \mathbf{R}^n$, and $Q \subset D$ be a cube. Then*

$$\begin{aligned} &\min\{|\{x \in Q \mid f(x) \geq t\}|, |\{x \in Q \mid f(x) \leq s\}|\} \\ &\leq C_1 |Q| \exp\left(-C_2 \frac{t-s}{\|f\|_*}\right), \qquad -\infty < s \leq t < \infty, \end{aligned}$$

where C_1, $C_2 > 0$ are constants depending only on n.

Proof. We may assume $f_Q \leq (s+t)/2$. Then from the John-Nirenberg theorem we have

$$\begin{aligned} |\{x \in Q \mid f(x) \geq t\}| &\leq \left| \left\{ x \in Q \mid |f(x) - f_Q| \geq \frac{t-s}{2} \right\} \right| \\ &\leq C|Q| \exp\left(-C \frac{t-s}{\|f\|_*}\right). \end{aligned}$$

□

Conversely,

Lemma 4.4. *Let f, $f(x) \neq \pm\infty$ (a.e.), be a measurable function on a domain $D \subset \mathbf{R}^n$. Assume that there exist constants C_1, $C_2 > 0$ such that for each cube $Q \subset D$ we have*

$$\begin{aligned} &\min\{|\{x \in Q \mid f(x) \geq t\}|, |\{x \in Q \mid f(x) \leq s\}|\} \\ &\leq C_1 |Q| e^{-C_2(t-s)}, \qquad -\infty < s \leq t < \infty. \end{aligned}$$

Then f is a $\mathrm{BMO}(D)$ function satisfying $\|f\|_* \leq 4(C_1+1)C_2^{-1} \exp(2C_2)$.

This is a direct consequence of the following.

Lemma 4.5. *Let $\lambda : \mathbf{R} \to [0,1]$ be a nonconstant, non-decreasing function. Assume that there exist constants C_1, $C_2 > 0$ such that*

$$\min(\lambda(s), 1 - \lambda(t)) \leq C_1 e^{-C_2(t-s)}, \qquad -\infty < s \leq t < \infty.$$

Then we can take $t_0 \in \mathbf{R}$ so that

$$\max(\lambda(t_0 - t), 1 - \lambda(t_0 + t)) \leq (C_1 + 1) e^{2C_2} e^{-C_2 t}, \qquad t \geq 0.$$

Proof. Since λ is nonconstant, $\lambda(t) \to 0$ $(t \to -\infty)$, and $\lambda(t) \to 1$ $(t \to \infty)$. Let $s_k = \sup\{t \mid \lambda(t) \leq 1 - \lambda(t+k)\}$, $k \geq 1$. Then s_k is non-increasing, $s_k + k$ is non-decreasing, and

$$\lambda(s_k - 1) \leq 1 - \lambda(s_k + k - 1), \qquad \lambda(s_k + 1) > 1 - \lambda(s_k + k + 1).$$

Set $t_0 = s_1$. First, assume $k \leq t < k+1$, $k \geq 2$. Then

$$\begin{aligned} 1 - \lambda(t_0 + t) \leq 1 - \lambda(s_{k-1} + k) \leq C_1 e^{-C_2(k-1)} \leq C_1 e^{2C_2} e^{-C_2 t}, \\ \lambda(t_0 - t) \leq \lambda(s_{k-1} - 1) \leq C_1 e^{-C_2(k-1)} \leq C_1 e^{2C_2} e^{-C_2 t}. \end{aligned}$$

Next, if $0 \leq t < 2$, then $\max\{\lambda(t_0 - t), 1 - \lambda(t_0 + t)\} \leq 1 \leq e^{2C_2} e^{-C_2 t}$. □

Proof of Lemma 4.4. Set $\lambda(t) = |\{x \in Q \mid f(x) \le t\}|/|Q|$. Then λ satisfies the assumption of Lemma 4.5 with the same constants C_1, C_2. Thus we can take t_0 so that

$$\max\{\lambda(t_0 - t), 1 - \lambda(t_0 + t)\} \le (C_1 + 1)e^{2C_2}e^{-C_2 t}, \quad t \ge 0.$$

And so

$$\mu(t) := |\{x \in Q \mid |f(x) - t_0| \ge t\}| \le 2(C_1 + 1)|Q|e^{2C_2}e^{-C_2 t}, \quad t \ge 0.$$

Hence

$$\int_Q |f - f_Q|dx \le 2\int_Q |f - t_0|dx = 2\int_0^\infty \mu(t)dt \le 4(C_1 + 1)C_2^{-1}e^{2C_2}|Q|.$$

□

Lemma 4.6. *Let $F : D \to D'$ satisfy the condition* (a) *of the Main Theorem. Then $K \ge 1$ and $\alpha \le 1$.*

Proof. We obtain $K \ge 1$ by setting $E_1 = E_2 = D'$.

Next, assume $\alpha > 1$. Let $Q_0 = [p, q] \times P_0 \subset \mathbf{R} \times \mathbf{R}^{n-1} = \mathbf{R}^n$ be a cube in D'. Let $l = q - p$. Let $I_1 = [p, p + l/4]$, $I_2 = [q - l/4, q]$. We decompose I into 2^s subintervals $J_k = [p + 2^{-s}(k-1)l, p + 2^{-s}kl]$, $1 \le k \le 2^s$, where s is a sufficiently large integer. Let $E_k = (J_k \times \mathbf{R}^{n-1}) \cap D'$. Let Q be a cube in D. Let k_0 be the integer k which maximizes $|F^{-1}(E_k) \cap Q|$, $1 \le k \le 2^s$. Then

$$\sup_{Q' \subset D'} \min\left\{\frac{|E_k \cap Q'|}{|Q'|}, \frac{|E_{k_0} \cap Q'|}{|Q'|}\right\} \le \frac{4}{2^s},$$

holds for each $k \in \Sigma_1$ or for each $k \in \Sigma_2$, where $\Sigma_1 = \{k \mid J_k \subset I_1\}$ and $\Sigma_2 = \{k \mid J_k \subset I_2\}$. Thus from the assumption we have

$$\frac{|F^{-1}(E_k) \cap Q|}{|Q|} \le K\left(\frac{4}{2^s}\right)^\alpha,$$

for each $k \in \Sigma_1$ or for each $k \in \Sigma_2$. It follows from $\sharp\Sigma_1 = \sharp\Sigma_2 = 2^{s-2}$ that

$$\min_{j=1,2} |F^{-1}((I_j \times \mathbf{R}^{n-1}) \cap D') \cap Q| \le 2^{s-2}K\left(\frac{4}{2^s}\right)^\alpha |Q| \to 0, \quad s \to \infty.$$

Therefore, the measure $\mu(S) = |F^{-1}((S \times \mathbf{R}^{n-1}) \cap D') \cap Q|$ on $[p, q]$ is absolutely continuous and satisfies $\mu([a, a + t]) = 0$ or $\mu([a + 3t, a + 4t]) = 0$ for each a, t with $p \le a < a + 4t \le q$. Thus $\mu([p, q]) = 0$, and so $|F^{-1}(Q_0) \cap Q| = 0$. Since Q and Q_0 are arbitrary, we have $|F^{-1}(D')| = 0$, which is a contradiction. □

Lemma 4.7. *Let $F : D \to D'$, $D \subset \mathbf{R}^m$, $D' \subset \mathbf{R}^n$, be a* BMO *map. Then $\|C_F\| \ge 1$.*

Proof. Note that if E is a measurable subset of D satisfying $|E| > 0$, $|D\setminus E| > 0$, then the characteristic function f of E satisfies $\|f\|_* = \|f\|_{*,D} = 1/2$.

Let F be a BMO map. Fix a cube $Q \subset D$ and set $\lambda(E) = |F^{-1}(E) \cap Q|$. Then λ is an absolutely continuous finite measure on D', thus we can take $t_0 \in \mathbf{R}$ so that $\lambda(E_0) = \lambda(D' \setminus E_0) = |Q|/2$, where $E_0 = ((-\infty, t_0] \times \mathbf{R}^{n-1}) \cap D'$. Let f be the characteristic function of E_0. Then $f \circ F$ is the characteristic function of $F^{-1}(E_0)$, and so from the first paragraph we have $\|f\|_* = \|f \circ F\|_* = 1/2$, which implies the assertion. □

The following lemma shows (a) ⇒ (c) of the Main Theorem with an estimation of the operator norm.

Lemma 4.8. *Let $F : D \to D'$, $D \subset \mathbf{R}^m$, $D' \subset \mathbf{R}^n$, be a measurable map. Assume that there exist constants K, $\alpha > 0$ such that for each pair of measurable subsets E_1, E_2 of D', we have*

$$\sup_{Q\subset D} \min_{k=1,2} \frac{|F^{-1}(E_k) \cap Q|}{|Q|} \leq K \left(\sup_{Q'\subset D'} \min_{k=1,2} \frac{|E_k \cap Q'|}{|Q'|} \right)^{\alpha}.$$

Then F is a BMO *map satisfying $\|C_F\| \leq CK/\alpha$, where $C = C(m, n) > 0$.*

Proof. Assume that F satisfies the assumption of the lemma. Then the inverse image of a null set is trivially a null set.

Let $f \in \mathrm{BMO}(D')$, $E_1 = \{x \in D' \,|\, f(x) \leq s\}$, $E_2 = \{x \in D' \,|\, f(x) \geq t\}$, $-\infty < s \leq t < \infty$, and $Q' \subset D'$. From Lemma 4.3 we have

$$\min\{|E_1 \cap Q'|, |E_2 \cap Q'|\} \leq C|Q'| \exp\left(-\frac{C(t-s)}{\|f\|_*}\right).$$

It follows from the assumption and the fact $\alpha \leq 1$ (Lemma 4.6) that for an arbitrary cube $Q \subset D$ we have

$$\min\{|F^{-1}(E_1) \cap Q|, |F^{-1}(E_2) \cap Q|\} \leq CK|Q| \exp\left(-\frac{C\alpha(t-s)}{\|f\|_*}\right).$$

Now, $F^{-1}(E_1) = \{x \in D \,|\, f(F(x)) \leq s\}$, $F^{-1}(E_2) = \{x \in D \,|\, f(F(x)) \geq t\}$, and so from Lemma 4.4 and the fact $K \geq 1$ (Lemma 4.6) we have $F \circ f \in \mathrm{BMO}(D)$ and

$$\|f \circ F\|_* \leq \frac{CK}{\alpha} \|f\|_* \exp\left(\frac{C\alpha}{\|f\|_*}\right).$$

Finally, applying this estimation to tf, $t > 0$, and letting $t \to \infty$, we obtain $\|f \circ F\|_* \leq \frac{CK}{\alpha} \|f\|_*$. □

Next, to show (b) ⇒ (c) of the Main Theorem, we need the following.

Proposition 4.9 (Strömberg [**16**]). *Let f be a measurable function on $D \subset \mathbf{R}^n$. Assume that we can take constants γ, $0 < \gamma < 1/2$, and $\lambda > 0$ so that for each cube $Q \subset D$ we have*

$$\inf_{c \in \mathbf{R}} |\{x \in Q \,|\, |f(x) - c| \geq \lambda\}| \leq \gamma |Q|.$$

Then f is a BMO(D) *function satisfying $\|f\|_* \leq C\lambda$, where $C = C(n, \gamma) > 0$.*

Proof of Main Theorem (b) $\Rightarrow$ (c). Assume that F satisfies the condition (b) of the Main Theorem. Then the inverse image of a null set is trivially a null set.

Let $f \in \mathrm{BMO}(D')$. We may assume $\|f\|_* = 1$. Let $-\infty < s < t < \infty$, $E_1 = \{x \in D' \,|\, f(x) \leq s\}$, $E_2 = \{x \in D' \,|\, f(x) \geq t\}$, and $g = C_F(f)$. Then from Lemma 4.3 there exists $C_1 > 0$ such that if $t - s \geq C_1$, then

$$\sup_{Q' \subset D'} \min_{k=1,2} \frac{|E_k \cap Q'|}{|Q'|} \leq C e^{-C(t-s)} < \lambda,$$

and so

$$\sup_{Q \subset D} \min_{k=1,2} \frac{|F^{-1}(E_k) \cap Q|}{|Q|} < \gamma.$$

For $Q \subset D$ we set

$$s_Q = \sup \left\{ s \in \mathbf{R} \,\middle|\, |\{x \in Q \,|\, g(x) \leq s\}| \leq |\{x \in Q \,|\, g(x) \geq s + C_1\}| \right\}.$$

Since $g \neq \pm\infty$ (a.e.), we have $s_Q \neq \pm\infty$. Thus

$$|\{x \in Q \,|\, g(x) \leq s_Q - 1\}| < \gamma, \qquad |\{x \in Q \,|\, g(x) \geq s_Q + C_1 + 1\}| < \gamma,$$

and so if we set $c_Q = s_Q + C_1/2$ and $\delta = 1 + C_1/2$, then

$$|\{x \in Q \,|\, |g(x) - c_Q| > \delta\}| \leq 2\gamma \ \ (< 1/2).$$

Hence $g \in \mathrm{BMO}(D)$ by Proposition 4.9. □

Finally, we show the remaining implication (c) $\Rightarrow$ (a) of the Main Theorem.

Lemma 4.10. *Let $F : D \to D'$, $D \subset \mathbf{R}^m$, $D' \subset \mathbf{R}^n$, be a* BMO *map. Assume that there exists an admissible domain D_0' satisfying $F(D) \subset D_0' \subset D'$. Then there exist constants K, $\beta > 0$ depending only on m and n such that for an arbitrary pair of measurable subsets E_1, E_2 of D' we have*

$$\sup_{Q \subset D} \min_{k=1,2} \frac{|F^{-1}(E_k) \cap Q|}{|Q|} \leq K \left(\sup_{Q' \subset D'} \min_{k=1,2} \frac{|E_k \cap Q'|}{|Q'|} \right)^{\beta/\|C_F\|}.$$

Proof. Let $G : D \to D_0'$, $G(x) = F(x)$. Let $g \in \mathrm{BMO}(D_0')$. Since admissible domains are uniform domains with uniformly bounded constants of uniformness, from Proposition 3.4 we can take $f \in \mathrm{BMO}(D')$ so that $f|D_0' = g$ and $\|f\|_* \le C\|g\|_*$. Thus $\|g \circ G\|_* = \|f \circ F\|_* \le C\|f\|_*$. Therefore, G is a BMO map satisfying $\|C_G\| \le C\|C_F\|$.

Let E_1, E_2 be an arbitrary pair of measurable subsets of D'. Let $E_k' = E_k \cap D_0'$. Then

$$2^{-nt} := \sup_{Q' \subset D_0'} \min_{k=1,2} \frac{|E_k' \cap Q'|}{|Q'|} \le \sup_{Q' \subset D'} \min_{k=1,2} \frac{|E_k \cap Q'|}{|Q'|}.$$

Because of Lemma 4.7, we may assume $t > 1$. From the Uchiyama theorem there exist $\mathrm{BMO}(D_0')$ functions f_1, f_2 satisfying $f_1 + f_2 = 1$ and

$$0 \le f_k \le 1, \qquad f_k = 0 \ \text{ on } E_k', \qquad \|f_k\|_* \le C/t, \qquad (k = 1, 2).$$

Let $g_k = f_k \circ G$. Then $g_1 + g_2 = 1$ and

$$0 \le g_k \le 1, \qquad g_k = 0 \ \text{ on } G^{-1}(E_k'), \qquad \|g_k\|_* \le C\|C_G\|/t, \qquad (k = 1, 2).$$

Let Q be an arbitrary cube in D. Since $(g_1)_Q + (g_2)_Q = 1$, we may assume $(g_1)_Q \ge 1/2$. Then from the John-Nirenberg theorem we have

$$\begin{aligned} |G^{-1}(E_1') \cap Q| &\le |\{x \in Q \,|\, |g_1(x) - (g_1)_Q| \ge 1/2\}| \\ &\le C|Q| \exp\left(-\frac{C}{\|g_1\|_*}\right) \le C|Q| \exp\left(-\frac{Ct}{\|C_G\|}\right), \end{aligned}$$

and so we obtain

$$\begin{aligned} \sup_{Q \subset D} \min_{k=1,2} \frac{|F^{-1}(E_k) \cap Q|}{|Q|} &= \sup_{Q \subset D} \min_{k=1,2} \frac{|G^{-1}(E_k') \cap Q|}{|Q|} \\ &\le C \left(\sup_{Q' \subset D_0'} \min_{k=1,2} \frac{|E_k' \cap Q'|}{|Q'|} \right)^{\frac{C}{\|C_G\|}} \\ &\le C \left(\sup_{Q' \subset D'} \min_{k=1,2} \frac{|E_k \cap Q'|}{|Q'|} \right)^{\frac{C}{\|C_F\|}}. \end{aligned}$$

□

Proof of Corollary 3.5. (a) ⇒ (b) is trivial. (c) ⇔ (d) follows from Proposition 3.4. (d) ⇒ (a) is a consequence of the Main Theorem.

Finally, assume that (b) holds. Let E_1 and E_2 be measurable subsets of $\mathbf{R}^n$ satisfying

$$\sup_{Q' \subset \mathbf{R}^n} \min_{k=1,2} \frac{|E_k \cap Q'|}{|Q'|} < \lambda.$$

Then $E'_k = E_k \cap D'$ satisfies the assumption. Thus

$$\sup_{Q\subset D} \min_{k=1,2} \frac{|G^{-1}(E_k)\cap Q|}{|Q|} = \sup_{Q\subset D} \min_{k=1,2} \frac{|F^{-1}(E'_k)\cap Q|}{|Q|} < \gamma,$$

and so G satisfies the condition (b) of the Main Theorem. Hence (d) follows. □

Example 4.11. Let $D' = \{x \in \mathbf{R}^2 \,|\, |x_1| < 4,\ |x_2| < 5\} \setminus \{x \in \mathbf{R}^2 \,|\, x_1 \geq 0,\ |x_2| \leq 1\}$, where $x = (x_1, x_2)$. Then D' is a uniform domain. Let $a_1 = (2, 3)$, $a_2 = (2, -3)$, $E_1 = \{|x - a_1| \leq 1\}$, and $E_2 = \{|x - a_2| \leq 1\}$. Let $D = \{x \in \mathbf{R}^2 \,|\, |x_1| < 1,\ |x_2| < 1\}$. Let $F : D \to D'$ be a conformal map. Then F is a BMO map. On the other hand, $\min\{|E_1 \cap Q'|, |E_2 \cap Q'|\} = 0$ holds for each cube $Q' \subset D'$. Thus we cannot replace the condition "$Q' \subset \mathbf{R}^n$" in Corollary 3.5 with $Q' \subset D'$.

5. Homeomorphisms between intervals.

The Main Theorem gives a characterization of BMO maps between (finite or infinite) open intervals. Jones result (Example 2.3) implies that in the case of homeomorphisms of $\mathbf{R}$ we can reduce the condition to a much simpler form. The purpose of the present section is to show that his argument really characterizes BMO maps which are homeomorphisms between general open intervals with an explicit estimation on operator norms.

Recall that the space BMO($\mathbf{R}$) is invariant under Möbius transformations of $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ (cf. Riemann-Rychener [**14**]). More generally:

Lemma 5.1. *Let I_1 and I_2 be open intervals on $\mathbf{R}$. Let τ be a Möbius transformation of $\overline{\mathbf{R}}$ satisfying $\tau(I_1) = I_2$. Then $F = \tau|I_1 : I_1 \to I_2$ is a* BMO *map satisfying $C^{-1} \leq \|C_F(f)\|_*/\|f\|_* \leq C$, $f \in$ BMO(I_2), where $C > 0$ is a universal constant.*

Proof. Let $f \in$ BMO(I_2). Let I be an interval satisfying $2I \subset I_1$. Then $\max_I |F'| \leq C \min_I |F'|$, thus

$$|I|^{-1} \int_I |f \circ F - f_{F(I)}| dx \leq C|F(I)|^{-1} \int_{F(I)} |f - f_{F(I)}| dx \leq C\|f\|_*,$$

and so $\|C_F(f)\| \leq C\|f\|_*$ from the proposition below. □

Proposition 5.2 (cf. Reimann-Rychener [**14**]). *Let $f \in L^1_{\mathrm{loc}}(D)$, $D \subset \mathbf{R}^n$, and $t \geq 1$. Assume that*

$$\sup |Q|^{-1} \int_Q |f - f_Q| dx \leq \lambda$$

holds for each Q satisfying $tQ \subset D$. Then $f \in$ BMO(D) and $\|f\|_* \leq Ct\lambda$, where $C = C(n) > 0$.

By virtue of the proposition above, repeating the argument of Jones, we can easily extend his result as follows.

Theorem 5.3 (cf. Jones [**11**]). *Let $F : I_1 \to I_2$ be a homeomorphism between open intervals.*

(a) *If $I_1 = \mathbf{R}$ and $I_2 \neq \mathbf{R}$, then F is not a* BMO *map.*
(b) *If $I_1 \neq \mathbf{R}$ or $I_2 = \mathbf{R}$, then the following conditions are equivalent:*
 (i) *F is a* BMO *map;*
 (ii) *There exist constant K_0, $\alpha_0 > 0$ such that for each measurable subset E of I_2 and each subinterval I of I_2 satisfying $2F^{-1}(I) \subset I_1$, we have*

$$\frac{|F^{-1}(E \cap I)|}{|F^{-1}(I)|} \leq K_0 \left(\frac{|E \cap I|}{|I|}\right)^{\alpha_0}. \tag{7}$$

Moreover, if (7) holds, then $\|C_F\| \leq CK_0/\alpha_0$ for some universal constant $C > 0$, and conversely, if F is a BMO *map, then we can take constants K_0, α_0 so that $K_0 = C_1$ and $\alpha_0 = C_2/\|C_F\|$, where $C_k > 0$, $k = 1, 2$, are universal constants.*

In particular, $\|C_F\|$ and $\inf(K_0/\alpha_0)$ are comparable with universal constant factors, where the infimum is taken over all pairs of K_0, α_0 satisfying (7).

Lemma 5.4. *Let I_0 be an interval. Let J_1 and J_2 be mutually disjoint subintervals of I_0. Then*

$$\sup_{I \subset I_0} \min_{k=1,2} \frac{|J_k \cap I|}{|I|} = \frac{s}{d(J_1, J_2) + 2s},$$

where $s = \min_{k=1,2} |J_k|$.

Proof of Theorem 5.3. First, assume $I_1 = \mathbf{R}$ and $I_2 \neq \mathbf{R}$. From the Möbius invariance of BMO we may assume $I_2 = (0, \infty)$ and F is sense preserving. Let $f(x) = \log x$. Then $f \in \mathrm{BMO}(I_2)$. On the other hand, since $g = f \circ F$ is an increasing function satisfying $\lim_{x\to\infty} g(x) = \infty$, $\lim_{x\to-\infty} g(x) = -\infty$, if we set $J_k = [-k, k]$ and $E_k = [-k, -k/2] \cup [k/2, k]$, then

$$|J_k|^{-1} \int_{J_k} |g - g_{J_k}| dx \geq (2k)^{-1} \int_{E_k} \geq 4^{-1} \left(g(k/2) - g(-k/2)\right) \to \infty,$$

as $k \to \infty$. Thus F is not a BMO map.

Next, assume that F satisfies the condition (ii) of (b). Then $K_0 \geq 1$ and $\alpha_0 \leq 1$. Let I be a subinterval of I_2 satisfying $2F^{-1}(I) \subset I_1$. Let $f \in \mathrm{BMO}(I_2)$, $I' = F^{-1}(I)$, $g = f \circ F$, and $E_t = \{x \in I \,|\, |f(x) - f_I| \geq t\}$, $t \geq 0$. Then from the John-Nirenberg theorem we have $|E_t| \leq C|I| \exp(-Ct/\|f\|_*)$, thus

$$\mu(t) := |\{x \in I' \mid |g(x) - f_I| \geq t\}| = |F^{-1}(E_t)| \\ \leq CK_0|I'| \exp\left(-\frac{C\alpha_0 t}{\|f\|_*}\right)$$

and so

$$|I'|^{-1} \int_{I'} |g - f_I| dx = \int_0^\infty \mu(t) dt \leq \frac{CK_0}{\alpha_0} \|f\|_*.$$

Hence, from Proposition 5.2 $\|g\|_* \leq CK_0 \alpha_0^{-1} \|f\|_*$.

Finally, assume that F satisfies the condition (i) of (b). We may assume that F is sense preserving. Let $I_k = (p_k, q_k)$, $k = 1, 2$. Let E be a measurable subset of I_2. Let $I = [a, b]$ be an interval satisfying $2F^{-1}(I) \subset I_1$. Let $l = |I|$ and $l' = |F^{-1}(I)|$. Let $E_1 = E \cap [a, a + l/2]$ and $E_2 = [b, q_2)$. Then $s := \min_{k=1,2} |F^{-1}(E_k)| \geq |F^{-1}(E_1)|/2$, thus from Lemma 5.4

$$\sup_{I' \subset I_2} \min_{k=1,2} \frac{|E_k \cap I'|}{|I'|} \leq \frac{\min_{k=1,2} |E_k|}{l/2 + 2\min_{k=1,2} |E_k|} \leq \frac{2|E_1|}{l} \leq \frac{2|E \cap I|}{l},$$

$$\sup_{I' \subset I_1} \min_{k=1,2} \frac{|F^{-1}(E_k) \cap I'|}{|I'|} \geq \frac{s}{(l' - |F^{-1}(E_1)|) + 2s} \geq \frac{|F^{-1}(E_1)|}{2l'}.$$

And so from the Main Theorem we have

$$\frac{|F^{-1}(E_1)|}{l'} \leq C \left(\frac{|E \cap I|}{l}\right)^{C/\|C_F\|}.$$

Since the same estimation holds for $E_1' = E \cap [a + l/2, b]$, we obtain (7). □

Contrary to the case of homeomorphisms of $\mathbf{R}$, no homeomorphism $F : \mathbf{R} \to (0, \infty)$ is a BMO map even if F^{-1} is a BMO map. Moreover, there exists a homeomorphism $F : (0, \infty) \to (0, \infty)$ such that F is a BMO map and F^{-1} is not a BMO map. One such example is given by $F(x) = \log(1 + x)$ (cf. Corollary 5.9). This example also shows that we can not drop the condition $2F^{-1}(I) \subset I_1$ in the statement above: Assume that the estimation (7) holds for $I = [\log 2, \log(a + 1)]$ and $E = [\log(a/2 + 1), \log(a + 1)]$, $a > 1$. Then

$$\frac{a/2}{a - 1} \leq K_0 \left(\frac{\log((a + 1)/(a/2 + 1))}{\log((a + 1)/2)}\right)^{\alpha_0}, \quad a > 1,$$

which is a contradiction. Another such example is given by $F : (0, \infty) \to (0, \infty)$, $F(x) = 1/x$.

Applying (7) to subintervals $I = [a, b]$ and $E = [x, y]$ of I_2, $a \leq x \leq y \leq b$, we obtain:

Corollary 5.5. *Let $F : I_1 \to I_2$ be a homeomorphism between open intervals. Assume that F is a* BMO *map. Then F^{-1} is locally a Hölder continuous function of order $C/\|C_F\|$, where $C > 0$ is a universal constant.*

The homeomorphism $F(x) = |x|^p \mathrm{sgn} x$, $p \geq 1$, of $\mathbf{R}$ shows that the estimation above is best possible. (See Example 5.8 below.)

Recall that for a homeomorphisms $G : J_1 \to J_2$ between (finite) closed intervals, G is absolutely continuous and G' is an A_∞ weight on J_1 if and only if G^{-1} is absolutely continuous and $(G^{-1})'$ is an A_∞ weight on J_2 (cf. Coifman-Fefferman [**2**]). Thus, under the assumption of the corollary above, F' (or $-F'$) satisfies the A_∞ condition uniformly on each I satisfying $2I \subset I_1$. In particular, $F' \in L^p_{\mathrm{loc}}(I_1)$ holds for some $p > 1$, and so F is also locally Hölder continuous. We do not know, however, whether the similar estimation holds or not for the order of F'. Moreover, from Proposition 5.2 we have:

Corollary 5.6. *Let $F : I_1 \to I_2$ be a homeomorphism between open intervals. Assume that F is a* BMO *map. Then we have $\| \log |F'| \|_* \leq C$, where $C = C(\|C_F\|) > 0$.*

It is easy to see that the corresponding result does not hold for $\log |(F^{-1})'|$.

In the rest of the present section, we give a remark on the global behavior of BMO maps. For an open interval $I = (a, b)$ $(\neq \mathbf{R})$, the hyperbolic metric ds_h is defined by

$$ds_h(x) = \frac{(b-a)dx}{(b-x)(x-a)}.$$

The hyperbolic metric is invariant under Möbius transformations of $\overline{\mathbf{R}}$ and comparable with the quasihyperbolic metric $dx/d(x, \partial I)$. Let $d_h(x, y)$ denote the hyperbolic distance between x and y.

Lemma 5.7. *Let $F : I_1 \to I_2$ $(I_1,\ I_2 \neq \mathbf{R})$ be a homeomorphism between open intervals. Assume that F is a* BMO *map. Then*

$$d_h(F(x), F(y)) \leq C\|C_F\|(d_h(x, y) + 1), \qquad x, y \in I_1, \tag{8}$$

where $C > 0$ is a universal constant.

Proof. Since both the quasihyperbolic metric and BMO are invariant under Möbius transformations, we may assume that $I_1 = I_2 = (0, \infty)$, F is sense preserving, $x = 1 = F(1)$, and $y = a > 1$. Applying the Main Theorem with $E_1 = (0, 1]$ and $E_2 = [F(a), \infty)$, and utilizing Lemma 5.4, we get

$$\frac{1}{a+1} \leq C \left(\frac{1}{F(a)+1} \right)^{C/\|C_F\|},$$

hence $F(a) \leq C a^{C\|C_F\|}$, which implies the assertion. □

Each hyperbolically Lipschitz continuous function is a BMO function and F preserves the space of all hyperbolically Lipschitz continuous functions if and only if F is hyperbolically Lipschitz continuous. The lemma above shows that the similar result holds for BMO. Note that quasiconformal maps satisfy the corresponding estimation with respect to the quasihyperbolic metric. The following example shows that the estimation (8) is best possible.

Example 5.8. Let $F : (0,\infty) \to (0,\infty)$, $F(x) = x^p$, $p \geq 1$. F is the hyperbolic dilation centered at the point 1: $d_h(F(x), F(y)) = pd_h(x,y)$, $x, y \in (0,\infty)$. A simple calculation shows F satisfies (7) with $K_0 = 1$ and $\alpha_0 = 1/p$. On the other hand, if we set $f(x) = \log x$, then $C_F(f) = pf$. Hence, $\|C_F\|$ and p are comparable with universal constant factors. Note that as to the antisymmetric extension $F_1 : \mathbf{R} \to \mathbf{R}$, $F_1(x) = |x|^p \mathrm{sgn} x$, of F, $\|C_{F_1}\|$ and p are comparable with universal constant factors similarly.

Corollary 5.9. *Let $F : I_1 \to I_2$, I_1, $I_2 \neq \mathbf{R}$, be a homeomorphism between open intervals. Assume that F is a* BMO *map. Then C_F is a bijection between* $\mathrm{BMO}(I_2)$ *and* $\mathrm{BMO}(I_1)$ *if and only if we can take a constant $C > 0$ so that for each interval $I \subset I_1$ satisfying $d(I, \partial I_1) = |I|$, we have $d(F(I), \partial I_2) \leq C|F(I_2)|$.*

Note that we can take such a constant $C > 0$ if and only if

$$C^{-1} \leq \frac{d_h(F(x), F(y)) + 1}{d_h(x,y) + 1} \leq C, \qquad x, y \in I_1,$$

holds for some $C \geq 1$.

Proof. Assume that we can take such a constant $C > 0$. Let $f \in \mathrm{BMO}(I_1)$. Then for each $I \subset I_1$ satisfying $d(I, \partial I_1) = |I|$, $\|f \circ F^{-1}\|_{*,F(I)} \leq C$ holds, thus from Proposition 5.2 we have $f \circ F^{-1} \in \mathrm{BMO}(I_2)$, and so F^{-1} is a BMO map. The converse assertion easily follows from Lemma 5.7. □

References

[1] K. Astala, *Remarks on quasiconformal mappings and* BMO*-functions*, Michigan Math. J., **30** (1983), 209-212, MR 85h:30022.

[2] R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math., **51** (1974), 241-250, MR 50 #10670, Zbl 291.44007.

[3] M.A. Fominykh, *Admissible changes of variables in the class of* BMO *functions*, Math. Notes., **43** (1988), 366-371, MR 89k:42021, Zbl 726.42014.

[4] J.B. Garnett and P.W. Jones, *The distance in* BMO *to* L^∞, Ann. of Math., **108** (1978), 373-393, MR 80h:46037, Zbl 383.26010.

[5] Y. Gotoh, *On the composition of functions of bounded mean oscillation with multivalent analytic functions*, J. Math. Kyoto Univ., **29** (1989), 309-315, MR 92c:30028, Zbl 701.30033.

[6] ———, *On the composition of functions of bounded mean oscillation with meromorphic functions*, J. Math. Kyoto Univ., **31** (1991), 635-642, MR 93a:30040, Zbl 754.30028.

[7] ———, BMO *extension theorem for relative uniform domains*, J. Math. Kyoto Univ., **33** (1993), 171-193, MR 94d:26016, Zbl 783.42013.

[8] ———, *On holomorphic maps between Riemann surfaces which preserve* BMO, J. Math. Kyoto Univ., **35** (1995), 299-324, MR 96m:30051, Zbl 857.30031.

[9] J. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14** (1961), 415-426, MR 24 #A1348, Zbl 102.04302.

[10] P.W. Jones, *Extension theorems for* BMO, Indiana Univ. Math. J., **29** (1980), 41-66, MR 81b:42047, Zbl 432.42017.

[11] ———, *Homeomorphisms of the line which preserve* BMO, Ark. Mat., **21** (1983), 229-231, MR 86a:42028, Zbl 527.42007.

[12] V. Mayer and M. Zinsmeister, *Groupes d'homéomorphismes de la droite et du cercle laissant invariant l'espace* BMO, Bull. London Math. Soc., **28** (1996), 24-32, MR 96j:30035, Zbl 840.30007.

[13] H.M. Reimann, *Functions of bounded mean oscillation and quasiconformal mappings*, Comm. Math. Helv., **49** (1974), 260-276, MR 50 #13513, Zbl 289.30027.

[14] H.M. Reimann and T. Rychener, *Funktionen beschränkter mittelerer Oszillation*, Lecture Notes in Math., **487**, Springer, 1975, MR 58 #23564, Zbl 324.46030.

[15] W. Smith, *Compactness of composition operators on* BMOA, Proc. Amer. Math. Soc., **127** (1999), 2715-2725, MR 99m:47040, Zbl 921.47025.

[16] J.-O. Strömberg, *Bounded mean oscillation with Orlicz norms and duality of Hardy spaces*, Indiana Univ. Math. J., **28** (1979), 511-544, MR 81f:42021, Zbl 429.46016.

[17] A. Uchiyama, *The construction of certain* BMO *functions and the corona problem*, Pacific J. Math., **99** (1982), 183-204, MR 84d:42022, Zbl 498.42009.

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