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# REGULARITY OF SOLUTIONS OF OBSTACLE PROBLEMS FOR ELLIPTIC EQUATIONS WITH OBLIQUE BOUNDARY CONDITIONS

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**Much has been written about various obstacle problems in the context of variational inequalities. In particular, if the obstacle is smooth enough and if the coefficients of associated elliptic operator satisfy appropriate conditions, then the solution of the obstacle problem has continuous first derivatives. For a general class of obstacle problems, we show here that this regularity is attained under minimal smoothness hypotheses on the data and with a one-sided analog of the usual modulus of continuity assumption for the gradient of the obstacle. Our results apply to linear elliptic operators with Hölder continuous coefficients and, more generally, to a large class of fully nonlinear operators and boundary conditions.**

## Introduction.

For a smooth bounded domain $\Omega \subset \mathbb{R}^n$ with unit inner normal $\gamma$, we are concerned with generalizations of the simple obstacle problem of finding a function $u \in W^{1,2}(\Omega)$ which minimizes the functional $\mathcal{F}$ defined on $W^{1,2}$ by

$$\mathcal{F}(v) = \int_\Omega |Dv|^2\,dx + \int_{\partial\Omega} v^2\,d\sigma$$

over the set of all $v \in W^{1,2}$ with $v \geq \psi$ for a given function $\psi$. From standard results in the theory of variational inequalities and the arguments in [**12**], it follows that this minimizer has bounded second derivatives if $\psi$ has bounded second derivatives and satisfies the inequality $\partial\psi/\partial\gamma - \psi \geq 0$ on $\partial\Omega$, which is assumed sufficiently smooth. To state our generalization of this problem, we note that the minimizer $u$ will be superharmonic in $\Omega$ and harmonic on the set where $u > \psi$; in addition $\partial u/\partial\gamma - u = 0$ on $\partial\Omega$. It is this formulation of the minimization problem that we wish to generalize.

We write $\mathbb{S}^n$ for the set of all $n \times n$ symmetric matrices, and we set $\Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ and $\Gamma' = \partial\Omega \times \mathbb{R} \times \mathbb{R}^n$. For real-valued, differential functions $F$ and $G$ defined on $\Gamma$ and $\Gamma'$, respectively, we consider the problem

$$\min\{-F(x,u,Du,D^2u),\, u-\psi\} = 0 \text{ in } \Omega,\ G(x,u,Du) = 0 \text{ on } \partial\Omega. \tag{0.1}$$

(We justify this somewhat nonstandard way of writing the problem by pointing out that, in the special case that $F$ is the Laplace operator, our solution $u$ will be superharmonic with $u \geq \psi$ and $u$ is harmonic on the set $\{u > \psi\}$.) Using subscripts to denote partial derivatives with respect to the variables $z \in \mathbb{R}$, $p \in \mathbb{R}^n$ and $r \in \mathbb{S}^n$, we assume at least that the matrix $F_r(x, z, p, r)$ is positive definite for all $(x, z, p, r) \in \Gamma$ (so that the equation $F(x, u, Du, D^2u) = 0$ is elliptic) and that $G_p(x, z, p) \cdot \gamma(x) > 0$ for all $(x, z, p) \in \Gamma'$ for the unit inner normal $\gamma$ to $\partial\Omega$ (so the boundary condition is an oblique derivative condition). If $\psi \in C^1$, then the analog of the condition $\partial\psi/\partial\gamma - \psi \geq 0$ would be $G(x, \psi, D\psi) \geq 0$ on $\partial\Omega$ (compare with [**12**, (0.7)]); however, we shall assume a weaker condition than continuity of $D\psi$ which still implies the continuity of $Du$, so we shall modify this condition appropriately (see conditions (2.1) and (3.9) below). Under suitable regularity hypotheses on $F$, $G$, $\psi$, and $\Omega$, we shall show that a modulus of continuity for the first derivatives of $u$ can be estimated in terms of known data. In conjunction with known first derivative estimates, our results give a complete description of the regularity of solutions for several problems. As particular examples, we mention here the capillarity obstacle problem from [**12**] and the Bellman equation problem with linear boundary condition from [**23**] (strictly speaking, we refer to the problem which the authors of that paper defer to a sequel, listed there as reference [**27**], which has never appeared in print). In [**12**], $F$ has the special form

$$F(x, z, p, r) = (1 + |p|^2)^{-1/2} \left( \delta^{ij} - \frac{p_i p_j}{1 + |p|^2} \right) r_{ij} + a(x, z), \tag{0.2a}$$

for a suitable, Lipschitz function $a$ and $G$ has the form

$$G(x, z, p) = \frac{p}{(1 + |p|^2)^{1/2}} \cdot \gamma + \varphi(x, z) \tag{0.2b}$$

for a suitable, smooth function $\varphi$ such that $\sup |\varphi(x, z)| < 1$. Because of the bound on the gradient of the solution of (0.1) in [**12**], it follows that our estimates apply to this problem assuming that $a$ is Lipschitz and $\varphi$ is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$. In [**23**], $F$ has the form

$$F(x, z, p, r) = \inf_{k \in J} (a_k^{ij}(x) r_{ij} + b_k^i(x) p_i + c_k(x) z + f_k(x)), \tag{0.3a}$$

where $J$ is some index set (assumed to be countable in [**23**]) and there are uniform (with respect to $k$) bounds on the $C^2$ norms of the coefficients $a_k^{ij}$, $b_k^i$, $c_k$, and $f_k$ as well as a positive lower bound (independent of $k$) on the minimum eigenvalue of the matrix $[a_k^{ij}]$; $G$ has the form

$$G(x, z, p) = \beta(x) \cdot p + b(x) z + g(x) \tag{0.3b}$$

for some vector $\beta$ such that $\beta\cdot\gamma$ is bounded from below by a positive constant, and the $C^2$ norms of $\beta$, $b$ and $g$ are assumed to be bounded. Again, from the gradient bounds proved in [**23**], it can be shown that our results apply to such problems if we only assume bounds on the Hölder norms of $a_k^{ij}$, $b_k^i$, $c_k$, and $f_k$ (see Theorem 2.2) and on the Hölder norms of $\beta$, $b$ and $g$ (see Theorem 3.2); for second derivative bounds, we need to assume that $\beta$, $b$, and $g$ have Hölder continuous derivatives (see Theorem 3.3). Of course, the uniform lower bounds on the minimum eigenvalue of $[a_k^{ij}]$ and on $\beta\cdot\gamma$ cannot be relaxed for our techniques to work.

In addition to the one-sided condition on $\psi$, our hypotheses are weaker than those in [**11**, Section 2], [**12**], [**16**, Section 4], [**1**], and [**2**] because we relax the smoothness hypotheses on $F$, $G$, and $\Omega$.

A basic interpolation inequality appears in Section 1, which allows us to use a weak Harnack inequality rather than the usual Harnack inequality. An interior regularity result is proved in Section 2 using a modification of the technique pioneered by Caffarelli and Kinderlehrer [**4**]. Specifically, we show (via the weak Harnack inequality) that our one-sided condition on $\psi$ implies a two-sided integral bound for $u - L$ with $L$ a suitable linear function, and then the interpolation inequality from Section 1 gives a two-sided estimate on the first derivatives of $u$. The corresponding estimates at the boundary are proved in Section 3. Most of our work is to analyze the hypotheses on the obstacle; only some simple elements of the theory of differential equations enters into this analysis. Some similar results, with a Dirichlet boundary condition replacing the oblique derivative boundary condition, appear in a preprint by Jensen [**13**]. The analysis of the obstacle also provides a straightforward extension to the two obstacle problem, which we present in Section 4, and Section 5 discusses applications of our methods to some degenerate variational inequalities; in particular, problems with the $p$-Laplacian operators are considered. We close in Section 6 with an outline of the existence theory in a special case.

Our notation follows that in [**10**]. In addition, we write $F^{ij}$ for the components of the matrix $F_r$ and $F^i$ for the components of the vector $F_p$. Similarly, $G^i$ denotes the components of the vector $G_p$. We always assume here that $\psi$ is Lipschitz with

$$|\psi| + |D\psi| \le \Psi_1, \tag{0.4}$$

and we define

$$\Gamma_1' = \{(x, z, p) \in \Gamma' : |z| + |p| \le \Psi_1\}.$$

## 1. An interpolation lemma.

Our first lemma is an improvement of results on second derivative estimates in terms of estimates on lower order derivatives. For brevity, if $\Sigma \subset \overline{\Omega}$, we use $|u|^{(b)}_{a;\Sigma}$ to denote the norms weighted in terms of distance to $\partial\Sigma \cap \Omega$.

**Lemma 1.1.** *Let $\Omega$ be a bounded Lipschitz domain, let $\Sigma$ be a subset of $\overline{\Omega}$, and suppose $u \in C^{2+\alpha}(\Sigma)$ for some $\alpha \in (0,1)$. Suppose that there are positive constants $C_1$ and $C_2$ such that*

$$[D^2u]_{\alpha;\Sigma\cap B(R)} \le C_1 R^{-\alpha} |D^2u|_{0;\Sigma\cap B(2R)} + C_2 \tag{1.1}$$

*for any two concentric balls $B(R)$ and $B(2R)$, with radii $R$ and $2R$, respectively, such that the boundary of $\Sigma \cap B(2R)$ is disjoint from $\Omega \setminus \Sigma$. Then there is a constant $C$ determined only by $C_1$, $\alpha$, and $\Omega$ such that*

$$|u|^{(0)}_{2+\alpha;\Sigma} \le C\left(|u|_{0;\Sigma} + C_2(\operatorname{diam}\Sigma)^{2+\alpha}\right). \tag{1.2}$$

*In addition, if $\Sigma = \overline{\Omega} \cap B(2R)$ for some ball $B(2R)$, and if $\kappa > 0$, then*

$$|D^2u|_{0;\Sigma'} \le C(C_1, \alpha, \kappa, \Omega)\left(R^{-2-(n/\kappa)} \|u\|_{\kappa;\Sigma} + C_2 R^{\alpha}\right), \tag{1.3}$$

*where $\Sigma' = \overline{\Omega} \cap B(R)$.*

*Proof.* The proof of (1.2) is a simple combination of the interpolation inequality

$$|D^2u|^{(2)}_0 \le C\left([D^2u]^{(2+\alpha)}_\alpha + |u|_0\right)^{2/(\alpha+2)} (|u|_0)^{\alpha/(2+\alpha)}$$

and the observation that (1.1) implies that

$$[D^2u]^{(2+\alpha)}_\alpha \le C\left(C_1 |D^2u|^{(2)}_0 + C_2(\operatorname{diam}\Sigma)^{2+\alpha}\right).$$

To prove (1.3), we imitate the proof of [**19**, Lemma 4.5]. From (1.2), we infer that

$$\rho \sup_{S(\rho)} |Du| \le C\left(\sup_{S(2\rho)} |u| + C_2\rho^{2+\alpha}\right),$$

where $S(\rho) = \Sigma \cap B(\rho)$ and the boundary of $S(2\rho)$ is disjoint from $\Omega \setminus \Sigma$. It follows that there is a constant $C_0$ determined only by $C_1$, $\alpha$, $\kappa$, and $\Omega$ such that

$$\operatorname*{osc}_{S(\theta\rho)} u \le C_0\theta\left(\sup_{S(2\rho)} |u| + C_2\rho^{2+\alpha}\right)$$

for any $\theta \in (0,1)$. Now we take $x_1$ so that $d(x_1)^{n/\kappa}|u(x_1)| \ge \frac{1}{2}|u|^{(n/\kappa)}_0$ and we choose our balls to be centered at $x_1$ with $\rho = \frac{1}{4}d(x_1)$. Then

$$|u(x)| \ge |u(x_1)|[1 - C_0\theta 2^{n/\kappa}] - C_0\theta C_2\rho^{2+\alpha}$$

for $x \in S(\theta\rho)$. If we take $\theta$ so small that $C_0\theta 2^{n/\kappa} \le 1/2$ and $C_0\theta \le 1$, then rearranging the resulting inequality and integrating over $S(\rho)$ yields

$$\rho^n |u(x_1)|^\kappa \le C \left( \int_{S(\theta\rho)} |u|^\kappa \, dx + C_2^\kappa \rho^{(2+\alpha)\kappa+n} \right),$$

and therefore

$$[u]_0^{(n/\kappa)} \le C \left( \|u\|_\kappa + C_2 R^{2+\alpha+(n/\kappa)} \right).$$

Hence

$$|u|_{0;\Omega\cap B(3R/2)} \le C \left( R^{-n/\kappa} \|u\|_\kappa + C_2 R^{2+\alpha} \right)$$

for any ball $B(3R/2)$. The desired result follows from this one after applying (1.2) with $\Sigma$ replaced by $\Omega \cap B(3R/2)$. □

## 2. Interior derivative estimates.

Our main ingredient is a pointwise estimate of how fast $u$ moves away from the obstacle near a contact point. In this section, we prove this estimate at an interior point. The argument is a straightforward modification of that in [**4**], but because of Lemma 1.1, we only need to estimate the $L^\kappa$ norm of a function related to $u$; this estimate is proved quite simply. Our basic assumption on the obstacle $\psi$ is that there are functions $Y$ defined on $\Omega$ and $\zeta$ defined on $[0, \operatorname{diam} \Omega]$ with $\zeta$ continuous and increasing such that

$$\psi(x_1) \ge \psi(x_2) + Y(x_2) \cdot (x_1 - x_2) - \zeta(|x_1 - x_2|)|x_1 - x_2| \tag{2.1}$$

for all $x_1$ and $x_2$ in $\Omega$. We have not assumed that $\zeta(0) = 0$, even though this assumption is needed to conclude that $Du$ is actually continuous, because it does not affect the form of our estimates. Note that the usual assumption (from [**1**, **4**, **11**, **12**, **16**, **18**]) is that $|D\psi(x_1) - D\psi(x_2)| \le \zeta(|x_1 - x_2|)$, which is equivalent to the combination of (2.1) and the companion inequality

$$\psi(x_1) \le \psi(x_2) + Y(x_2) \cdot (x_1 - x_2) + \zeta(|x_1 - x_2|)|x_1 - x_2|.$$

Our condition includes functions which are not continuously differentiable even if $\zeta(0) = 0$. For example, if $(\psi_\alpha)_{\alpha\in I}$ is a family of functions (with arbitrary index set $I$) satisfying (2.1), then a simple calculation shows that $\psi$ defined by $\psi(x) = \sup_{\alpha\in I} \psi_\alpha(x)$ also satisfies this condition provided we have a uniform $L^\infty$ bound on $\psi_\alpha$ and $D\psi_\alpha$. In particular, condition (2.1) includes the obstacles studied by Troianiello in [**30**, **31**].

**Lemma 2.1.** *Suppose that $u \in W^{2,n}_{\mathrm{loc}}$ satisfies*

$$\min\{-F(x, u, Du, D^2u), u - \psi\} = 0 \text{ in } \Omega \tag{2.2}$$

*and that there are positive constants $\lambda$, $\Lambda$, and $\mu_0$ such that*

$$\lambda|\xi|^2 \le F^{ij}(x, u, Du, D^2u)\xi_i\xi_j \tag{2.3}$$

*for all* $x \in \Omega$ *and* $\xi \in \mathbb{R}^n$,

$$F(x, u, Du, 0) \geq -\mu_0 \lambda, \tag{2.4a}$$

$$|F_r(x, u, Du, tD^2 u)| \leq \Lambda, \tag{2.4b}$$

*for all* $x \in \Omega$ *and all* $t \in [0, 1]$. *Suppose also that* (2.1) *holds and that* $x_0$ *is a point such that* $u(x_0) = \psi(x_0)$. *Then there are constants* $\kappa$ *and* $C$ *determined only by* $n$ *and* $\Lambda/\lambda$ *such that the function* $\bar{u}$ *defined by*

$$\bar{u}(x) = u(x) - u(x_0) - Y(x_0) \cdot (x - x_0) \tag{2.5}$$

*satisfies the estimate*

$$\left( R^{-n} \int_{B(x_0, R/2)} |\bar{u}|^\kappa \, dx \right)^{1/\kappa} \leq C[\mu_0 R^2 + \zeta(R) R] \tag{2.6}$$

*for all* $R \leq d(x_0)/2$.

*Proof.* Since $u \geq \psi$ and $u(x_0) = \psi(x_0)$, it follows from (2.1) that $\bar{u} \geq -\zeta(R)R$ in $B(x_0, R)$. Next, we note that $a^{ij} D_{ij} u \leq -F(x, u, Du, 0)$ for

$$a^{ij}(x) = \int_0^1 F^{ij}(x, u(x), Du(x), tD^2 u(x)) \, dt.$$

It follows that $v = \bar{u} + \zeta(R)R$ satisfies the conditions $a^{ij} D_{ij} v \leq \lambda \mu_0$ and $v \geq 0$ in $B(x_0, R)$. Therefore [**10**, Theorem 9.22] and the obvious inequality $\inf_{B(R/2)} v \leq v(0) = \zeta(R)R$ yield

$$\left( R^{-n} \int_{B(R/2)} |v|^\kappa \, dx \right)^{1/\kappa} \leq C[\mu_0 R^2 + \zeta(R) R],$$

and the triangle inequality gives

$$\left( R^{-n} \int_{B(R/2)} |\bar{u}|^\kappa \, dx \right)^{1/\kappa} \leq C(\kappa) \left[ \left( R^{-n} \int_{B(R/2)} |v|^\kappa \, dx \right)^{1/\kappa} + \zeta(R) R \right].$$

We complete the proof by combining these last two inequalities. □

Note Lemma 2.1 continues to hold if we only assume that the minimum in (2.2) is nonnegative; however, our full regularity result will use that the minimum is zero.

The regularity of the derivatives of $u$ at an arbitrary point follows from this estimate and Lemma 1.1 by a simple variation of the argument in [**4**].

**Theorem 2.2.** *Suppose that* $u$, $\psi$, *and* $F$ *satisfy conditions* (2.1)-(2.4) *with* $\zeta$ *a continuous increasing function on* $[0, \operatorname{diam} \Omega]$ *satisfying*

$$\frac{\zeta(t_1)}{t_1} \geq \frac{\zeta(t_2)}{t_2} \text{ if } t_1 \leq t_2. \tag{2.7}$$

*Suppose also that there are constants* $\alpha \in (0,1)$ *and* $\mu_1$ *such that*

$$|F(x,z,p,r) - F(y,w,q,r)| \le (\mu_0 + \mu_1|r|)\lambda|x-y|^\alpha, \tag{2.8}$$

*and that* $F$ *is convex or concave with respect to* $r$*. Then there is a constant* $C$ *determined only by* $n$*,* $\alpha$*,* $\mu_1$*,* $\Lambda/\lambda$*,* $|u|_1$*,* $|\psi|_1$*, and* $\operatorname{diam}\Omega$ *such that*

$$|Du(x_1) - Du(x_2)| \le C\left[\zeta(|x_1-x_2|) + \left(\mu_0 + \frac{\sup|Du|}{d(x_1)}\right)|x_1-x_2|\right] \tag{2.9}$$

*for all* $x_1$ *and* $x_2$ *in* $\Omega$ *with* $|x_1 - x_2| \le \frac{1}{4}\min\{d(x_1), d(x_2)\}$.

*Proof.* Using $I$ to denote the contact set $I = \{x \in \Omega : \psi(x) = u(x)\}$, we consider three cases:

(i) both points are in $I$,
(ii) one point is in $I$,
(iii) neither point is in $I$.

In all cases, we set $\rho = |x_1 - x_2|$ and $Z = \zeta(\rho) + \mu_0\rho$.

In the first case, we use (2.6) twice, first with $R = \rho$ and $x_0 = x_1$ and then with $R = 2\rho$ and $x_0 = x_2$ to infer that

$$\left(\rho^{-n}\int_{B(x_1,\rho)} |u(x) - \psi(x_i) - Y(x_i)\cdot(x-x_i)|^\kappa\,dx\right)^{1/\kappa} \le CZ\rho$$

for $i = 1, 2$ because $B(x_1,\rho) \subset B(x_2, 2\rho) \subset \Omega$. Next, we use the observation that

$$\begin{aligned}&\psi(x_1) - \psi(x_2) - Y(x_2)\cdot(x_1-x_2) + [Y(x_1) - Y(x_2)]\cdot(x-x_1)\\ &\quad= [u(x) - \psi(x_1) - Y(x_1)\cdot(x-x_1)] - [u(x) - \psi(x_2) - Y(x_2)\cdot(x-x_2)]\end{aligned}$$

along with the triangle inequality to infer that

$$\begin{aligned}\Bigg(\rho^{-n}\int_{B(x_1,\rho)} |\psi(x_1) - \psi(x_2) - Y(x_2)\cdot(x_1-x_2)&\\ + V\cdot(x-x_1)|^\kappa\,dx\Bigg)^{1/\kappa} &\le CZ\rho,\end{aligned}$$

where $V = Y(x_1) - Y(x_2)$. In addition, (2.1) implies that

$$\psi(x_1) - \psi(x_2) - Y(x_2)\cdot(x_1-x_2) + V\cdot(x-x_1) \ge V\cdot(x-x_1) - \zeta(\rho)\rho$$

in $B(\rho)$. We therefore infer that

$$\left(\rho^{-n}\int_{B(x_1,\rho)} \left([V\cdot(x-x_1)]^+\right)^\kappa\,dx\right)^{1/\kappa} \le CZ\rho,$$

and this inequality easily gives $|V| \le CZ$. It follows that

$$|u(x_1) - u(x_2) - Y(x_2)\cdot(x_1 - x_2)| \le C[\zeta(|x_1 - x_2|) + \mu_0|x_1 - x_2|]|x_1 - x_2| \tag{2.10}$$

for any $x_1$ and $x_2$ in $I$, so $u$ is differentiable in the interior of $I$ with $Du = Y$ there.

In the second case, we may assume without loss of generality that $x_1 \in I$, and we write $\xi_2$ for the closest point to $x_2$ in $I$. Note that $u$ is a solution of the equation $F(x, u, Du, D^2u) = 0$ in $\Sigma_0 = B(x_2, |x_2 - \xi_2|)$, so (1.1) holds with $\Sigma$ any subset of $\Sigma_0$. (This estimate is proved in [**28**], but the precise form used here does not appear in that reference; see [**19**, Theorem 14.7] for a proof of the corresponding parabolic estimate.) If $x$ is on the line segment between $x_2$ and $\xi_2$, it follows from (1.3) with $R = 2|x - \xi_2|$ (applied to $\bar{u}$ defined with $\xi_2$ in place of $x_0$) that

$$|D\bar{u}(x)| \le C[R^{-1-(n/\kappa)}\|u\|_\kappa + \mu_0 R^{1+\alpha}]$$

and hence

$$|Y(\xi_2) - Du(x_2)| \le C|x_2 - \xi_2| \le CZ.$$

Since $|x_1 - \xi_2| \le 2\rho$, it follows from Case (i) that $|Y(x_1) - Y(\xi_2)| \le CZ$, and hence (2.10) holds if $x_1 \in I$ and $x_2 \notin I$. Therefore $u$ is also differentiable on $\partial I$ with $Du = Y$ there. Now that we know $Du = Y$ on $I$, our estimates imply (2.9) for $x_1 \in I$ and $x_2 \in \Omega$.

In the third case, we set $d^*(x) = \operatorname{dist}(x, I)$ and $m_0 = \min\{d^*(x_1), d^*(x_2)\}$, and we consider three possibilities. If $2\rho \ge m_0$, then, with $\xi_i$ denoting the closest point to $x_i$ in $I$, we have

$$|Du(x_1) - Du(x_2)| \le |Du(x_1) - Du(\xi_1)| + |Du(\xi_1) - Du(\xi_2)| + |Du(\xi_2) - Du(x_2)|,$$

and the three terms on the right-hand side of this inequality are estimated either by Case (i) or Case (ii) along with the observation that

$$|x_1 - \xi_1| \le C\rho,\ |\xi_1 - \xi_2| \le C\rho,\ |x_2 - \xi_2| \le C\rho.$$

If $2\rho < m_0$ and $d(x_1) \le m_0$, then we can use Lemma 1.1 as in Case (ii) and (2.7) to infer that

$$|D^2u| \le C\left[\frac{\zeta(m_0)}{m_0} + \mu_0 m_0^\alpha\right] \le C\left[\frac{\zeta(\rho)}{\rho} + \mu_0\right]$$

on the line segment joining $x_1$ and $x_2$. An easy integration of this inequality yields (2.9) in this case as well. Finally, if $2\rho < m_0$ and $d(x_1) > m_0$, then (1.3) with $\kappa = \infty$ gives the desired result. □

Note that the hypothesis (2.7) really involves no loss of generality. Specifically if $\zeta$ is a continuous, increasing function, then the function $\zeta_1$, defined by

$$\zeta_1(t) = t \sup_{s \geq t} \frac{\zeta(s)}{s},$$

satisfies (2.7) and $\zeta_1 \geq \zeta$, so (2.1) holds with $\zeta_1$ in place of $\zeta$. In addition $\zeta_1$ is clearly continuous. To see that $\zeta_1$ is increasing, we let $t_1 < t_2$ and choose $s_i$ so that $\zeta_1(t_i) = (t_i/s_i)\zeta(s_i)$. If $\zeta(s_1) = \zeta(s_2)$, then $\zeta_1(t_1)/\zeta_1(t_2) = t_1/t_2 < 1$. If $\zeta(s_1) < \zeta(s_2)$, then $s_1 < t_2$, so $\zeta_1(t_1) = (t_1/s_1)\zeta(s_1) \leq (t_1/s_1)\zeta(t_2) \leq \zeta(t_2) \leq \zeta_1(t_2)$. Moreover, if $\zeta(0) = 0$, then $\zeta_1(t) \to 0$ as $t \to 0$, as we see by considering two cases. If $\zeta(s)/s$ is bounded as $s \to 0$, say by $S$, then $\zeta_1(t) \leq St \to 0$. If $\zeta(s)/s$ is unbounded as $s \to 0$, let $(s_j)$ be a sequence tending to zero with $\zeta(s_j)/s_j \geq j$ and $\zeta(s_j)/s_j \geq \zeta(s)/s$ if $s \geq s_j$. Then $\zeta_1(s_j) = \zeta(s_j)$ so $\zeta_1(s_j) \to 0$, and then $\zeta(t) \to 0$ as $t \to 0$ because $\zeta_1$ is increasing. In addition, we note (see [**24**, Section 3.5] for details) that the modulus of continuity for a function defined on an open set satisfies (2.7).

Condition (2.8) can be weakened, say to

$$|F(x,z,p,r) - F(y,w,q,r)| \leq (\mu_0 + \mu_1|r|)\lambda|x-y|^\alpha + \mu_0'\lambda|p-q|^\alpha,$$

since this condition is only used to infer the appropriate form of the Hölder for second derivatives of solutions of the equation $F(x,u,Du,D^2u) = 0$ (see [**28**]). In particular, our results apply to the operator $F$ defined by (0.3a) if we assume uniform Hölder estimates on the functions $a_k^{ij}$, $b_k^i$, $c_k$, and $f_k$ along with a uniform lower bound on the minimum eigenvalue of $[a_k^{ij}]$; this structure was considered in [**23**]. Moreover, we can infer condition (2.8) for more general classes of fully nonlinear, uniformly elliptic operators $F$ once we have a Hölder gradient estimate for $u$. Such an estimate follows by virtue of the following variant of Theorem 2.2, which is also important for our study of oblique derivative problems.

**Theorem 2.3.** *Suppose $u$, $\psi$, and $F$ satisfy conditions (2.1)-(2.4) with $\zeta$ a continuous, increasing function on $[0, \operatorname{diam}\Omega]$. Suppose also that $F$ is concave or convex with respect to $r$. Suppose finally that there are a positive constant $\nu_1$ and a continuous, increasing function $\zeta_1$ with $\zeta_1(0) = 0$ such that*

$$|F(x,z,p,r) - F(y,w,q,r)| \leq \mu_0\lambda + \lambda(\nu_1|p-q| + \zeta_1(|x-y|))|r|. \tag{2.11}$$

*Then there are positive constants $\alpha(n, \Lambda/\lambda, \nu_1)$ and $C(n, \zeta_1, \Lambda/\lambda, \nu_1, \operatorname{diam}\Omega)$ such that*

$$\frac{\zeta(t_1)}{t_1^\alpha} \geq \frac{\zeta(t_2)}{t_2^\alpha} \text{ for } t_1 \leq t_2 \tag{2.12}$$

*implies*

(2.13)
$$|Du(x_1) - Du(x_2)| \le C\left[\zeta(|x_1 - x_2|) + \left(\mu_0 + \frac{\sup|Du|}{d(x_1)^\alpha}\right)|x_1 - x_2|^\alpha\right]$$

*for all* $x_1$ *and* $x_2$ *in* $\Omega$ *with* $|x_1 - x_2| \le \frac{1}{4}\min\{d(x_1), d(x_2)\}$.

*Proof.* We basically follow the proof of Theorem 2.2. The main notational change is that we set $Z = \zeta(\rho) + \mu_0\rho^\alpha$. From the argument in [**19**, Lemma 12.13] (see also [**3**, Theorem 2] and [**32**]), we infer that

$$[Du]_{\alpha;B(R)} \le C[R^{-\alpha}|Du|_{0;B(2R)} + \mu_0]$$

if $B(2R) \subset \Omega$ and $F(x, u, Du, D^2u) = 0$ in $B(2R)$. The proof is completed by using this inequality in the obvious modification of Lemma 1.1. □

Note that if $\zeta$ satisfies (2.7), then $\zeta_2$ defined by $\zeta_2(t) = (\sup\zeta)^{1-\alpha}(\zeta(t))^\alpha$ satisfies (2.12), so Theorem 2.3 also does not restrict our choice of obstacles.

Condition (2.11) is certainly satisfied for quasilinear operators, that is, $F(x, z, p, r) = a^{ij}(x, z, p)r_{ij} + a(x, z, p)$ provided $[a^{ij}]$ is elliptic, continuous with respect to $x$ and $z$, and Lipschitz with respect to $p$. In particular (after using the gradient bound from [**12**]), this result applies when $F$ is given by (0.2a). Moreover, we can remove the hypothesis that $F$ be either concave or convex with respect to $r$ in Theorem 2.3 by considering viscosity solutions as in [**3**, **32**] and suitably modifying the arguments. Finally, as noted before, we can replace condition (2.11) by any condition which yields the Hölder gradient estimate

$$\operatorname*{osc}_{B(x_0,r)} Du \le C\left[\left(\frac{r}{R}\right)^\alpha \operatorname*{osc}_{B(x_0,R)} Du + \mu_0 r^\alpha\right].$$

See [**5**] for an alternative structure condition which provides such an estimate.

## 3. Estimates for the oblique derivative problem.

To prove a modulus of continuity estimate for the gradient up to the boundary for the oblique derivative problem, we use a slight variation of the ideas in the proof of Theorem 2.2. We begin with a preliminary estimate which is related to the boundary condition in which we write $v'$ for the first $n-1$ components of the vector $v$. The connection of this lemma to our original problem will be made clear in Theorem 3.2.

**Lemma 3.1.** *Let* $\omega_0$, $\omega_1$, *and* $r$ *be positive constants with* $\omega_0 > \omega_1$, *and define*

(3.1) $$K = \{x^n \ge \omega_0|x'|,\ |x| \le r\}, E = \{x^n \ge \omega_1|x'|,\ r/4 < |x| \le r\}.$$

*Let $\bar\psi$ be a Lipschitz function defined in $K$ and suppose that there are positive constants $z$ and $\kappa$ along with a vector-valued function $\bar Y$ such that*

$$\bar\psi(x) \ge \bar\psi(x_1) + \bar Y(x_1)\cdot(x - x_1) - z \tag{3.2}$$

*for all $x$ and $x_1$ in $K$ and*

$$\left(\int_E |\bar\psi|^\kappa \, dx\right)^{1/\kappa} \le z r^{n/\kappa}. \tag{3.3}$$

*Suppose also that there is a Lipschitz function $g$ defined on $\mathbb{R}^n$ with*

$$\left|\frac{\partial g}{\partial p'}\right| \le \mu_2\chi_0, \tag{3.4a}$$

$$\frac{\partial g}{\partial p_n} \ge \chi_0 \tag{3.4b}$$

*for some positive constants $\chi_0$ and $\mu_2$ with $\mu_2\omega_1 < 1$. Then*

$$g(0) \ge g(\bar Y(0)) - C(n,\kappa,\omega_0,\omega_1,\mu_2)\chi_0 z/r. \tag{3.5}$$

*Proof.* The first step is to prove a pointwise upper bound for $\bar\psi$ in

$$E' = \{x^n \ge |x'|/\mu_2, 3r/8 \le |x| \le 3r/4\}.$$

To prove this estimate, let $x_1$ be a point in $E'$ at which the maximum of $\bar\psi$ is attained and suppose that $\bar\psi(x_1) > 2z$. Then

$$\left(\int_{B(x_1,\rho)} |\bar\psi|^\kappa \, dx\right)^{1/\kappa} \le \left(\int_E |\bar\psi|^\kappa \, dx\right)^{1/\kappa} \le z r^{n/\kappa} \tag{3.6}$$

for any $\rho$ such that $B(x_1,\rho) \subset E$. In particular, we can take $\rho = C(\omega_1,\mu_2)r$. With this choice for $\rho$, we set

$$E^+ = \{x \in B(x_1,\rho) : \bar Y(x_1)\cdot(x - x_1) \ge 0\},$$

and note that $|E^+| \ge \frac{1}{2}|B(x_1,\rho)| \ge Cr^n$. In addition, for $x \in E^+$, we have

$$\bar\psi(x) \ge \bar\psi(x_1) + \bar Y(x_1)\cdot(x - x_1) - z \ge \bar\psi(x_1) - z,$$

and therefore

$$\begin{aligned}\left(\int_{B(x_1,\rho)} |\bar\psi(x)|^\kappa \, dx\right)^{1/\kappa} &\ge \left(\int_{E^+} \bar\psi(x)^\kappa \, dx\right)^{1/\kappa} \ge (|E^+|(\bar\psi(x_1) - z)^\kappa)^{1/\kappa} \\ &\ge C[\bar\psi(x_1) - z]r^{n/\kappa}.\end{aligned}$$

In conjunction with (3.6), this inequality implies that $\bar\psi \le Cz$ on $E'$.

Next, we note that there is a point $x_2$ with $|x_2| \in (7r/16, 9r/16)$ and $x_2^n > 2\mu_2|x_2'|$ such that $\bar\psi(x_2) \ge -Cz$. In addition, if $\bar Y(x_2) \ne 0$, then there is a positive constant $c_2$ such that $x_3 = x_2 + c_2 r\overline{Y}(x_2)/|\overline{Y}(x_2)| \in E'$. Therefore

$$Cz \ge \bar\psi(x_3) \ge \bar\psi(x_2) + \overline{Y}(x_2)\cdot(x_3 - x_2) - z = \bar\psi(x_2) - c_2 r|\overline{Y}(x_2)| - z.$$

It follows that $|\overline{Y}(x_2)| \le Cz/r$ and hence

$$\bar\psi(x) \ge \bar\psi(x_2) + \overline{Y}(x_2)\cdot(x - x_2) - z \ge -Cz \tag{3.7}$$

for any $x \in K$.

To continue, we define $\xi$ to be the unit vector in the direction of

$$\int_0^1 g_p(t\overline{Y}(0))\,dt,$$

so

$$g(\overline{Y}(0)) - g(0) = \int_0^1 g_p(t\overline{Y}(0))\cdot\overline{Y}(0)\,dt \le C\chi_0\xi\cdot\overline{Y}(0).$$

Now set $\rho = r/(2\xi^n)$. It is easy to see that $r/2 \le \rho \le Cr$, In addition, we infer from our estimate $\bar\psi \le Cz$ on $E'$ along with (3.2) and (3.7) that

$$Cz \ge \bar\psi(\rho\xi) \ge \bar\psi(0) + \overline{Y}(0)\cdot(\rho\xi) - z \ge -Cz + \rho\overline{Y}(0)\cdot\xi.$$

It follows that $\overline{Y}(0)\cdot\xi \le Cz/r$, which yields (3.5). □

To state our gradient estimate for the oblique derivative problem, we use $\Gamma_2'$ to denote the set of all $(x,z,p) \in \Gamma'$ with $|z| + |p| \le \max\{|u|_1, \Psi_1\}$. Because of the way that a Hölder gradient estimate is used to prove second derivative estimates for the oblique derivative problem without an obstacle, we first prove our estimate in a situation analogous to that in Theorem 2.3.

**Theorem 3.2.** *Let $u \in W^{2,n}_{\mathrm{loc}} \cap C^1(\overline{\Omega})$ solve (0.1) with $\partial\Omega \in C^{1,\alpha}$ for some $\alpha \in (0,1)$ and $F$ either convex or concave with respect to $r$. Suppose that there are positive constants $\lambda$, $\Lambda$, $\mu_0$, and $\nu_1$ along with a continuous, increasing function $\zeta_1$ with $\zeta_1(0) = 0$ such that conditions (2.3), (2.4), and (2.11) hold. Suppose also that there are positive constants $\chi_0$, $\mu_2$, and $\mu_3$ such that*

$$G_p(x,z,p)\cdot\gamma(x) \ge \chi_0, \tag{3.8a}$$

$$|G_p(x,z,p)\cdot\tau(x)| \le \mu_2\chi_0, \tag{3.8b}$$

$$|G(x,z,p) - G(y,w,p)| \le \mu_0\chi_0(|x-y| + |z-w|)^\alpha \tag{3.8c}$$

*for all $(x,z,p)$ and $(y,w,p)$ in $\Gamma_2'$ and any vector field $\tau(x)$ with $\tau\cdot\gamma = 0$. Suppose further that there is a continuous increasing function $\zeta$ on $[0,\operatorname{diam}\Omega]$ satisfying (2.12) such that $\psi$ satisfies (2.1) and*

$$G(x,\psi,Y) \ge 0 \tag{3.9}$$

*for all $x \in \partial\Omega$. Then there are constants $\alpha_0(n, \mu_2, \nu_1, \Lambda/\lambda)$ and $C$ determined only by $n$, $\alpha$, $\Lambda/\lambda$, $\mu_2$, $\Psi_1$, $\zeta_1$, and $\Omega$ such that $\alpha \le \alpha_0$ implies*

$$|Du(x_1) - Du(x_2)| \le C[\zeta(|x_1 - x_2|) + (\mu_0 + \sup |Du|)|x_1 - x_2|^\alpha] \tag{3.10}$$

*for all $x_1$ and $x_2$ in $\Omega$.*

*Proof.* We imitate the proof of Theorem 2.2. First, we show (as in Lemma 2.1) that, if $x_0 \in \Omega$ is a point at which $u(x_0) = \psi(x_0)$ and if $\bar{u}$ is defined by (2.5), then

$$\left(R^{-n} \int_{B(x_0,R/2)\cap\Omega} |\bar{u}|^\kappa \, dx\right)^{1/\kappa} \le C[(\mu_0 + \mu_2)R^{1+\alpha} + \zeta(R)R] \tag{3.11}$$

for any sufficiently small $R$ (that is, $R$ is smaller than a constant determined only by $\mu_2$ and $\Omega$). If $d(x_0) \ge R$, then this inequality is just (2.6). If $d(x_0) < R$, then we first prove an estimate for $G(x, u(x), Y(x_0))$ by appropriate application of Lemma 3.1.

Let $x^*$ be a closest point to $x_0$ in $\partial\Omega$. By rotation and translation, we may assume that $x^*$ is the origin and that $x_0$ is on the positive $x^n$-axis. Then $K \subset \Omega$ provided $\omega_0 > 1/\mu_2$ and $R$ is sufficiently small (determined only by $\Omega$ and $\mu_2$), and $g(p) = G(x^*, \psi(x^*), p + Y(x_0))$ satisfies (3.4). Next, we define $\bar{\psi}$ by $\bar{\psi}(x) = \psi(x) - \psi(x_0) - Y(x_0) \cdot (x - x_0)$ and we set $\bar{Y}(x) = Y(x) - Y(x_0)$. For $z = C[\mu_0 R^\alpha + \zeta(R)]R$ and $r = 2d(x_0)$, we have (3.2) directly from (2.1) because $r \le R$. Now, we note that using a chaining argument in the proof of [**10**, Theorem 9.22] allows us to replace $B(x_0, R/2)$ by $E$ and $R$ by $r$ in the proof of (2.6). Thus, we obtain

$$\left(r^{-n} \int_E |\bar{u}|^\kappa \, dx\right)^{1/\kappa} \le Cz,$$

which yields (3.3) because $\bar{u} \ge \bar{\psi} \ge -C\zeta(r)r$ in $E$. It then follows from Lemma 3.1 that

$$G(x^*, \psi(x^*), Y(x_0)) = g(0) \ge -Cz$$

because $g(\bar{Y}(0)) = G(x^*, u(x^*), Y(x^*)) \ge 0$. For $x \in B(x_0, R) \cap \partial\Omega$, we have

$$|\psi(x^*) - u(x)| \le |\psi(x^*) - \psi(x_0)| + |u(x) - u(x_0)| \le (\Psi_1 + |Du|_0)|R,$$

and therefore

$$G(x, u(x), Y(x_0)) \ge -\mu_0 \chi_0 (\Psi_1 + |Du|_0 + 1)^\alpha R^\alpha - C\chi_0 z \ge -Cz/R.$$

It follows that

$$\beta \cdot D\bar{u} \le C\chi_0[\zeta(R) + \mu_0 R^\alpha]$$

on $B(x_0, R) \cap \partial\Omega$ for

$$\beta(x) = \int_0^1 G_p(x, u, tDu + (1-t)Y(x_0)) \, dt.$$

We then infer (3.11) by arguing as in Lemma 2.1 but with [**20**, Theorem 4.2] in place of [**10**, Lemma 9.22].

To prove the modulus of continuity estimate for $Du$, we consider the three cases from Theorem 2.2 with $Z = \zeta(\rho) + \mu_0\rho^\alpha$. In addition, we set $\Omega[y,R] = B(y,R)\cap\Omega$. In Case (i), we note that there is a cone $Q$ with height $\rho$, opening angle $\theta$ (determined only by $\Omega$), and vertex 0 such that $x_i + Q \subset \Omega[x_i,\rho]$ for $i = 1,2$. It follows that

$$\left(\rho^{-n}\int_{\Omega[x_1,\rho]}\left([V\cdot(x-x_1)]^+\right)^\kappa\,dx\right)^{1/\kappa} \le CZ\rho,$$

and similar reasoning gives

$$\left(\rho^{-n}\int_{\Omega[x_2,\rho]}\left([V\cdot(x-x_2)]^-\right)^\kappa\,dx\right)^{1/\kappa} \le CZ\rho.$$

Combining these two estimates gives

$$\left(\rho^{-n}\int_{Q}|V\cdot x|^\kappa\,dx\right)^{1/\kappa} \le CZ\rho,$$

which again implies $|V| \le CZ$. For Cases (ii) and (iii), we proceed as in Theorem 2.3 with [**19**, Lemma 13.22] in place of [**19**, Lemma 12.13] to prove (3.10). □

The remarks from Section 2 show that this result applies to the examples from [**1**, **2**, **12**, **16**, **23**]. The function $G$ given by (0.2b) satisfies conditions (3.8) by virtue of the gradient bound in [**12**] and $G$ from (0.3b) clearly satisfies these conditions if $\beta$, $b$ and $g$ are Hölder continuous. Moreover, if $(\psi_\alpha)_{\alpha\in I}$ is a family of $C^1$ functions which satisfy conditions (2.1) and (3.9) with $Y = D\psi_\alpha(x)$, then it is immediate that there is a vector field $Y$ such that $\psi = \sup_{\alpha\in I}\psi_\alpha$ satisfies these conditions.

We note that this result is a purely local one. Hence if the hypotheses of the theorem are satisfied only in a neighborhood $N$ of some point $x^*$, then we obtain a modulus of continuity estimate for the first derivatives of $u$ in $N'\cap\Omega$ for any compact subset $N'$ of $N$. The corresponding local result was proved by B. Huisken [**11**] although she only considered quasilinear equations and her hypotheses are stronger than ours.

In addition, we have the following result which corresponds to Theorem 2.2.

**Theorem 3.3.** *Let $u \in W^{2,n}_{\mathrm{loc}}\cap C^1(\overline{\Omega})$ solve (0.1) with $\partial\Omega\in C^{1,\alpha}$ for some $\alpha > 0$. Suppose that there are positive constants $\lambda$, $\lambda_1$, $\Lambda$, $\mu_0$, and $\mu_1$ such that conditions (2.1), (2.3), (2.4), (2.7), (2.8), (3.8) are satisfied. Suppose also that $G\in C^{1,\alpha}(\Gamma_2')$ and that $\zeta(t)\le z_0t^\alpha$ for some $z_0$. Then there is* *a constant $C$ determined only by $n$, $z_0$, $\alpha$, $\Lambda/\lambda$, $\mu_0$, $\mu_1$, $\mu_2$, $\mu_3$, $\Psi_1$, $\zeta_1$, $\sup|Du|$, $|G|_{1,\alpha}$, and $\Omega$ such that*

$$|Du(x_1) - Du(x_2)| \le C(\zeta(|x_1 - x_2|) + |x_1 - x_2|). \tag{3.12}$$

*Proof.* We observe that our hypotheses imply a Hölder estimate for $Du$. With this estimate, we can follow the proof of Theorem 2.2 with [**29**] (as modified in [**19**, Theorem 14.22] to deal with nonlinear boundary conditions; this step uses the Hölder gradient estimate) in place of [**28**]. □

In particular, if $\zeta(t) = t$, Theorem 3.3 gives a bound on the second derivatives of $u$.

## 4. The double obstacle problem.

The crucial new element in our study of double obstacle problems is a Harnack-type inequality for the difference between the upper obstacle and the lower obstacle. The basic ideas for this inequality were used in Lemma 3.1, but, here, we shall use some precise information on how fast the ratio of the maximum of the difference to its minimum goes to one on a ball of shrinking radius, provided the obstacles are defined in a ball of fixed radius. Specifically, we have the following result.

**Lemma 4.1.** *Suppose $\psi_1$ and $\psi_2$ are two functions defined in $B(x_0, r)$ with $\psi_1 \le \psi_2$ and that there are two vector fields $Y_1$ and $Y_2$ such that*

$$\psi_1(x_1) \ge \psi_1(x_2) + Y_1(x_2)\cdot(x_1 - x_2) - \zeta(|x_1 - x_2|)|x_1 - x_2|, \tag{4.1a}$$

$$\psi_2(x_1) \le \psi_2(x_2) + Y_2(x_2)\cdot(x_1 - x_2) + \zeta(|x_1 - x_2|)|x_1 - x_2| \tag{4.1b}$$

*for all $x_1$ and $x_2$ in $B(x_0, r)$. Then for any $\varepsilon \in (0, 1)$, we have*

$$\sup_{B(x_0,\varepsilon r)} (\psi_2 - \psi_1) \le \frac{1+\varepsilon}{1-\varepsilon} \inf_{B(x_0,\varepsilon r)} (\psi_2 - \psi_1) + 4\varepsilon(\zeta(2\varepsilon r) + \zeta(r))r. \tag{4.2}$$

*Proof.* Set $\psi = \psi_2 - \psi_1$ and $I = \inf_{B(x_0,\varepsilon r)} \psi$. Then choose $x_2$ so that $|x_0 - x_2| \le \varepsilon r$ and $\psi(x_2) = I$. Our first step is to show that

$$|Y(x_2)| \le \frac{I}{(1-\varepsilon)r} + 2\zeta(r), \tag{4.3}$$

so let us assume that $Y(x_2) \ne 0$ and set $\xi = Y(x_2)/|Y(x_2)|$. Then for $R < (1-\varepsilon)r$, we have that $x_2 - R\xi \in B(x_0, r)$, so (4.1) implies that

$$\begin{aligned}\psi(x_2 - R\varepsilon) &\le \psi(x_2) - Y(x_2)\cdot(R\xi) + 2\zeta(R)R \\ &= I - R|Y(x_2)| + 2\zeta(R)R.\end{aligned}$$

We infer (4.3) from this inequality by sending $R \to (1-\varepsilon)r$ and noting that $\psi(x_2 - R\xi) \ge 0$.

Now let $x \in B(x_0, \varepsilon r)$ and use (4.1) to infer that

$$\begin{aligned}\psi(x) &\le \psi(x_2) + Y(x_2)\cdot(x-x_2) + 2(|x-x_2|)|x-x_2| \\ &\le I + 2\varepsilon r|Y(x_2)| + 4\zeta(2\varepsilon r)\varepsilon r.\end{aligned}$$

Simple algebra then completes the proof. □

We also shall use the following simple variant of (4.2).

**Corollary 4.2.** *In addition to the hypotheses of Lemma 4.1, suppose that $\psi_1(x_0) = 0$ and $Y_1(x_0) = 0$. Then*

$$\sup_{B(x_0,\varepsilon r)} \psi_2 \le \frac{1+\varepsilon}{1-\varepsilon} \inf_{B(x_0,\varepsilon r)} \psi_2^+ + \frac{4\varepsilon}{1-\varepsilon}(\zeta(2\varepsilon r) + \zeta(r))r. \tag{4.4}$$

*If also $\psi_2 \ge 0$ on $B(x_0, \varepsilon r)$, then*

$$\sup_{B(x_0,\varepsilon r)} \psi_1 \le \frac{4\varepsilon}{1-\varepsilon^2} \inf_{B(x_0,\varepsilon r)} \psi_2 + \frac{6\varepsilon}{1-\varepsilon^2}(\zeta(2\varepsilon r) + \zeta(r))r. \tag{4.5}$$

*Proof.* Because $\psi_2 \ge -\zeta(r)r$, we can follow the proof of Lemma 4.1 with $\psi$ replaced by $\psi_2 + \zeta(r)r$ to infer that

$$\begin{aligned}\sup_{B(x_0,\varepsilon r)} (\psi_2 + \zeta(r)r) &\le \frac{1+\varepsilon}{1-\varepsilon} \inf_{B(x_0,\varepsilon r)} (\psi_2 + \zeta(r)r) + 2\varepsilon(\zeta(2\varepsilon r) + \zeta(r))r \\ &\le \frac{1+\varepsilon}{1-\varepsilon} \inf_{B(x_0,\varepsilon r)} (\psi_2^+ + \zeta(r)r) + 2\varepsilon(\zeta(2\varepsilon r) + \zeta(r))r,\end{aligned}$$

so

$$\sup_{B(x_0,\varepsilon r)} \psi_2 \le \frac{1+\varepsilon}{1-\varepsilon} \inf_{B(x_0,\varepsilon r)} \psi_2^+ + \left(\frac{1+\varepsilon}{1-\varepsilon} - 1\right)\zeta(r)r + 2\varepsilon(\zeta(2\varepsilon r) + \zeta(r))r.$$

Since $(1+\varepsilon)/(1-\varepsilon) - 1 = 2\varepsilon/(1-\varepsilon)$ and $1 < 1/(1-\varepsilon)$, this inequality gives (4.4).

Next, we set $\psi = \psi_2 - \psi_1$, $I = \inf_{B(x_0,\varepsilon r)} \psi$ and $I_2 = \inf_{B(x_0,\varepsilon r)} \psi_2$ to see that

$$I_2 \le \psi_2(x_0) = \psi(x_0) \le \sup_{B(x_0,\varepsilon r)} \psi \le \frac{1+\varepsilon}{1-\varepsilon} I + 4\varepsilon(\zeta(2\varepsilon r) + \zeta(r))r$$

and hence

$$\begin{aligned}\sup_{B(x_0,\varepsilon r)} \psi_1 &\le \sup_{B(x_0,\varepsilon r)} \psi_2 - I \\ &\le \left(\frac{1+\varepsilon}{1-\varepsilon} - \frac{1-\varepsilon}{1+\varepsilon}\right) I_2 + \left(\frac{4\varepsilon}{1+\varepsilon} + \frac{2\varepsilon}{1-\varepsilon}\right)(\zeta(2\varepsilon r) + \zeta(r))r.\end{aligned}$$

The desired inequality follows from this one by simple algebra. □

These lemmata allow us to imitate the argument in [**18**, Lemma 1.1] to prove an analog of Lemma 2.1 when $u$ is a solution of the double obstacle problem:

$$\psi_1 \le u \le \psi_2 \text{ in } \Omega, \tag{4.6a}$$

$$\min\{-F(x,u,Du,D^2u), u-\psi_1\} = 0 \text{ if } u < \psi_2, \tag{4.6b}$$

$$\min\{F(x,u,Du,D^2u), \psi_2 - u\} = 0 \text{ if } u > \psi_1. \tag{4.6c}$$

**Lemma 4.3.** *Suppose $u$, $\psi_1$, and $\psi_2$ are as above, and suppose that there are positive constants $\lambda$, $\Lambda$, and $\mu_0$ such that conditions (2.3), (2.4b), and*

$$|F(x,u,Du,0)| \le \mu_0\lambda \tag{4.7}$$

*are satisfied. If $x_0$ is a point such that $u(x_0) = \psi_1(x_0)$, then there are positive constants $C$, $\delta$, and $\kappa$ determined only by $n$ and $\Lambda/\lambda$ such that $\bar{u}$, defined by*

$$\bar{u}(x) = u(x) - u(x_0) - Y_1(x_0)\cdot(x-x_0), \tag{4.8}$$

*satisfies the estimate*

$$\left(R^{-n}\int_{B(x_0,\delta R)} |\bar{u}|^\kappa\,dx\right)^{1/\kappa} \le C[\mu_0R^2 + \zeta(R)R] \tag{4.9}$$

*for all $R \le d(x_0)$.*

*Proof.* We first note that the hypotheses of this lemma are unchanged if we subtract the same linear function from $u$, $\psi_1$ and $\psi_2$, so we may assume that $\psi_1(x_0) = 0$ and $Y_1(x_0) = 0$. Then, for $\varepsilon \in (0,1)$ to be chosen, we set $I_2 = \inf_{B(x_0,\varepsilon R)} \psi_2$.

If $I_2 \le 12\zeta(R)R + \mu_0R^2$, then (4.4) implies that $\psi_2 \le C(\varepsilon)[\mu_0R^2 + \zeta(R)R]$ in $B(x_0,\varepsilon R)$, and (4.9) follows for any $\delta \le \varepsilon$.

On the other hand, if $I_2 > 12\zeta(R)R + \mu_0R^2$, we set

$$M = (1-\varepsilon)I_2,$$
$$M_1 = \frac{5\varepsilon}{1-\varepsilon^2}I_2 + 2\zeta(R)R,$$
$$U = \min\{u, M\} + M_1,$$

and we note that $U \ge 0$. Now, for $\eta > 0$, define $f_\eta$ by

$$f_\eta(t) = (\max\{t,0\}^3 + \eta^3)^{1/3} - \eta,$$

and set

$$U_\eta = M + M_1 - f_\eta(M-u).$$

Then $U_\eta \to U$ uniformly as $\eta \to 0$ and $U_\eta \ge 0$ in $B(x_0,\varepsilon R)$. Moreover, because $a^{ij}D_{ij}u \le \lambda\mu_0$ wherever $u \le M$ and $f_\eta$ is $C^2$ with $f_\eta'' \ge 0$, it follows that $a^{ij}D_{ij}U_\eta \le \lambda\mu_0$. It follows from the weak Harnack inequality [**10**, Theorem 9.22] that

$$\left((\varepsilon R)^{-n}\int_{B(x_0,\varepsilon R/2)}|U_\eta|^\kappa\,dx\right)^{1/\kappa} \le C_1\left[\inf_{B(\varepsilon R/2)} U_\eta + \mu_0(\varepsilon R)^2 + \zeta(\varepsilon R)(\varepsilon R)\right]$$

for some $C_1(n,\Lambda/\lambda)$ and $\kappa(n,\Lambda/\lambda)$. Sending $\eta\to 0$, we infer that

$$\left((\varepsilon R/2)^{-n}\int_{B(x_0,\varepsilon R/2)}|U|^\kappa\,dx\right)^{1/\kappa} \le C_1[M_1+\varepsilon\mu_0 R^2+\varepsilon\zeta(R)R] \tag{4.10}$$

because $U(x_0)=M_1$.

Next, we set $M_2=\sup_{B(x_0,\varepsilon R)}\psi_1+\zeta(R)R$ and $V=\max\{u,M_2\}$. By a similar approximation argument, we infer from the local maximum principle [**10**, Theorem 9.20] that there is a constant $C_2(n,\Lambda/\lambda)$ so that

$$\begin{aligned}\sup_{B(x_0,\varepsilon R/4)} V &\\ &\le C_2\left[\left(\left(\frac{\varepsilon R}{2}\right)^{-n}\int_{B(x_0,\varepsilon R/2)}V^\kappa\,dx\right)^{1/\kappa}+\mu_0(\varepsilon R)^2+\zeta(\varepsilon R)\varepsilon R\right].\end{aligned}$$

Now we note that $u\le U$ (because $M_1\ge 0$ and $M+M_1\ge\sup_{B(x_0,\varepsilon R)}\psi_2$) and $M_2\le M_1-\zeta(R)R$ (because $M_2\le 5I_2\varepsilon/(1-\varepsilon^2)$), so $U_2\le U_1$ provided $\varepsilon\le 1/2$. It follows that

$$\begin{aligned}\sup_{B(x_0,\varepsilon R/4)} u &\le C_2(C_1[M_1+\mu_0\varepsilon R^2+\varepsilon\zeta(R)R]+\mu_0\varepsilon R^2+\varepsilon\zeta(R)R)\\ &\le C_1C_2\frac{2\varepsilon}{1-\varepsilon}\psi_2(x_2)+\left(\frac{5C_1}{1-\varepsilon^2}+C_1+1\right)C_2\varepsilon I_2.\end{aligned}$$

By taking $\varepsilon$ sufficiently small, we conclude that $u<\psi_2$ on $B(x_0,\varepsilon R/4)$. Therefore, $u+\zeta(R)R$ is a positive supersolution on $B(x_0,\varepsilon R/4)$ and we can use the weak Harnack inequality directly to infer (4.9) with $\delta=\varepsilon/8$. □

Note that the arguments of Lemmata 3.1 and 4.3 can be combined to prove pointwise decay of $\bar u$ near a contact point. Specifically, suppose $u$ satisfies (2.2) with $F$ satisfying (2.3), (2.4b), and (4.7). If $u(x_0)=\psi(x_0)$, then (2.6) and the proof of $\bar\psi\le Cz$ in $E'$ (from Lemma 3.1) give a constant $c_1$ such that $\bar\psi\le c_1[\zeta(R)+\mu_0 R]R$ in $B(R)$. The local maximum principle applied to $\max\{\bar u,(1+c_1)[\zeta(R)+\mu_0 R]R\}$ then yields $\bar u\le C[\zeta(R)+\mu_0 R]R$ in $B(R/4)$. With this pointwise estimate in hand, we can imitate the proof of [**4**, Theorem 2.3] to obtain a modulus of continuity estimate for solutions of obstacle problems with linear equations when $\zeta$ does not necessarily satisfy condition (2.7).

For our purposes, the next important step is to obtain a modulus of continuity for $Du$.

**Theorem 4.4.** *Suppose that $u$, $\psi_1$, $\psi_2$, and $F$ satisfy conditions (4.6), (4.7), (2.3), and (2.4b) with $\zeta$ a continuous increasing function on $[0, \operatorname{diam}\Omega]$. Suppose also that there are constants $\alpha$, $\nu_0$ and $\nu_1$ along with a continuous increasing function $\zeta_1$ with $\zeta_1(0) = 0$ such that conditions (2.11) and (2.12) hold. If $F$ is convex or concave with respect to $r$, then there are constants $\alpha_0(n, \Lambda/\lambda, \nu_1)$ and $C$ determined only by $n$, $\alpha$, $\nu_1$, $\Lambda/\lambda$, $|u|_1$, and $\operatorname{diam}\Omega$ such that $\alpha \le \alpha_0$ implies (3.12) for all $x_1$ and $x_2$ in $\Omega$ with $|x_1 - x_2| \le \frac{1}{4}\min\{d(x_1), d(x_2)\}$.*

Of course, the double obstacle analog of Theorem 2.2 holds with $\alpha = 1$ if condition (2.11) is replaced by (2.8).

We can use the same ideas for oblique derivative problems, but the proofs are more complicated. In place of Lemma 4.1, we have a similar, but more subtle, inequality. To state our results more simply, we let $\omega$ be a $C^1$ function in some $(n-1)$-dimensional ball $B(0, R)$ with $R > 0$ and $\omega(0) = 0$, and we set $\omega_0 = \sup|D\omega|$. We also define

$$K[r] = \{x \in \mathbb{R}^n : x^n < r - (\omega_0 + 1)|x'|, x^n > \omega(x')\},$$
$$\Sigma[r] = \{x \in \mathbb{R}^n : x^n < r - (\omega_0 + 1)|x'|, x^n = \omega(x')\}$$

for $r \in (0, R)$, where here and below we abbreviate $x' = (x^1, \dots, x^{n-1})$.

**Lemma 4.5.** *Let $\psi_1$ and $\psi_2$ be two functions defined in $K[r] \cup \Sigma[r]$ for some $r \in (0, R)$ with $\psi_1 \le \psi_2$ there. Suppose that there are vector fields $Y_1$ and $Y_2$ such that conditions (4.1) hold for all $x_1$ and $x_2$ in $K[r] \cup \Sigma[r]$. Suppose also that there are positive constants $\alpha \le 1$, $\mu_3 < 1/\omega_0$, $\mu_3$, and $\chi_0$ along with a function $G$ such that*

$$\left|\frac{\partial G}{\partial p'}(x, z, p)\right| \le \mu_3\chi_0, \tag{4.11a}$$

$$G^n(x, z, p) \ge \chi_0, \tag{4.11b}$$

$$|G(x, z, p) - G(x, w, p)| \le \mu_0\chi_0|w - z|^\alpha \tag{4.11c}$$

*for any $(x, z, p) \in \Sigma[r] \times \mathbb{R} \times \mathbb{R}^n$ and any $w \in \mathbb{R}$, and set*

$$g_0 = \frac{1}{\chi_0}\inf_{K[\eta r]}(G(x, \psi_2(x), Y_2(x)) - G(x, \psi_1(x), Y_1(x)))^+. \tag{4.12}$$

*Then, for any $\varepsilon \in (0, 1)$, there is a constant $\eta(\varepsilon, \mu_3, \omega_0)$ such that*

(4.13)
$$\sup_{K[\eta r]}(\psi_2 - \psi_1) \le (1 + \varepsilon)\inf_{K[\eta r]}(\psi_2 - \psi_1) + 3\varepsilon\zeta(r)r + C_1\varepsilon r^{2/(2-\alpha)} + \varepsilon g_0 r$$

*for $C_1 = 3(\mu_0 \sup[\psi_2 - \psi_1]^{\alpha/2})^{2/(2-\alpha)}$.*

*Proof.* To simplify the notation, we shall set $\psi = \psi_2 - \psi_1$, $Y = Y_2 - Y_1$, and $K = K[\eta r]$. In addition, we write $I$ for the infimum of $\psi$ over $K$ and we let $x_2$ be a point in the closure of $K$ at which $\psi(x_2) = I$. We now consider several cases.

Suppose first that $x_2 \in \Sigma[r]$. Then we infer from (4.11) and (4.12) that

$$g_0\chi_0 \ge -\mu_0\chi_0|\psi(x_2)|^\alpha + v \cdot Y(x_2) \tag{4.14}$$

for some vector $v$ with $|v'| \le \mu_3 v^n$ and $v^n \ge \chi_0$. Now we set $\xi = v/|v|$ and we set

$$x_1 = x_2 + \frac{\varepsilon}{8}r\xi,\ x_3 = x_2 + \frac{1}{2}r\xi.$$

If $\eta < 1/2$, it follows that $x_1$ and $x_3$ are in $\Omega[r]$. Setting $I_1 = \psi(x_1)$, we see that

$$\begin{aligned} I_1 &\le I + Y(x_2)\cdot(x_1 - x_2) + \frac{\varepsilon}{8}\zeta(r)r = I + \frac{\varepsilon r}{8}Y(x_2)\cdot\xi + \frac{\varepsilon}{8}\zeta(r)r \\ &\le I + \frac{\mu_0\varepsilon}{8}I^\alpha r + \frac{\varepsilon}{8}\zeta(r)r + \frac{\varepsilon}{8}g_0 r \\ &\le \left(1 + \frac{\varepsilon}{2}\right)I + \frac{C_1}{3}\varepsilon r^{2/(2-\alpha)} + \frac{\varepsilon}{8}\zeta(r)r + \frac{\varepsilon}{8}g_0 r \end{aligned}$$

by virtue of (4.14) and Young's inequality. Now we obtain two estimates for $Y(x_1)$. First, there is a constant $k(\omega_0, \mu_3)$ such that $B(x_1, k\varepsilon r) \subset K[r]$ and then the proof of (4.3) with $\varepsilon = 1/2$ shows that

$$|Y(x_1)| \le \frac{2I_1}{k\varepsilon r} + 4\zeta(r) \le \frac{3}{2\varepsilon kr}I + \frac{2C_1}{3k}r^{\alpha/(2-\alpha)} + \left(\frac{2}{k} + 4\right)\zeta(r) + \frac{g_0}{4k}.$$

Moreover,

$$0 \le \psi(x_3) \le I_1 + \left(\frac{1}{2} - \frac{\varepsilon}{8}\right)rY(x_1)\cdot\xi + \zeta(r)r$$

because $x_3 - x_1 = (1/2 - \varepsilon/8)r\xi$ and hence

$$-rY(x_1)\cdot\xi \le \frac{1 + \varepsilon/2}{1/2 - \varepsilon/8}I + 8\zeta(r)r + \frac{4}{3}C_1\varepsilon r^{2/(2-\alpha)} + \frac{\varepsilon}{2}g_0 r.$$

Now we note that, for any $x \in K$, we have $|x| < 2\eta r$, and hence

$$\psi(x) \le \psi(x_1) + Y(x_1)\cdot(x - x_1) + 4\eta\zeta(4\eta r)r.$$

To analyze the right hand side of this inequality, we first observe that $x - x_1 = (x - x_2) + (x_2 - x_1)$ and that $|x - x_2| \le 4\eta r$. It follows that

$$\begin{aligned} &Y(x_1)\cdot(x-x_1)\\ &\le -\frac{\varepsilon}{8} rY(x_1)\cdot\xi + 4\eta r|Y(x_1)|\\ &\le \left((1+\varepsilon/2)\frac{\varepsilon/8}{1/2-\varepsilon/8} + \frac{6\eta}{2\varepsilon k}\right) I + \left(\left(\frac{2}{k}+4\right)2\eta + \varepsilon\right)\zeta(r)r\\ &+ C_1\left(\frac{\varepsilon^2}{3} + \frac{8\eta}{3k}\right) r^{2/(2-\alpha)} + \left(\frac{\varepsilon}{2}+\frac{2\eta}{2k}\right) g_0 r, \end{aligned}$$

and

$$\begin{aligned} \psi(x) \le \left((1+\varepsilon/2)\frac{1/2}{1/2-\varepsilon/8} + \frac{6\eta}{k\varepsilon}\right) I + \frac{C_1}{3}\left(2\varepsilon + \frac{8\eta}{k}\right) r^{2/(2-\alpha)}\\ + \left(2\varepsilon + \left(\frac{2}{k}+6\right)2\eta\right)\zeta(r)r + \left(\frac{\varepsilon}{2}+\frac{2\eta}{2k}\right) g_0 r \end{aligned}$$

provided $4\eta \le 1$. By simple calculation, $(1+\varepsilon/2)(1/2)/((1/2)-\varepsilon/8) < 1+\varepsilon$, so we can take $\eta$ sufficiently small to infer (4.13) in this case.

If $x_2 \notin \Sigma[r]$, then $Y^n(x_2) \ge 0$ and we can imitate the calculations of the preceding case with $\xi = (0, \dots, 0, 1)$, to see that

$$\psi(x) \le \left(\frac{1/2}{1/2-\varepsilon/8} + \frac{4\eta}{k\varepsilon}\right) I + \left[\left(\frac{1}{2k}+8\right)2\eta + \left(\frac{5/8}{1/2-\varepsilon/8}+1\right)\varepsilon\right]\zeta(r)r,$$

which implies (4.13) if $\eta$ is sufficiently small. □

Our next step is to prove a corresponding estimate for our general geometric situation.

**Lemma 4.6.** *Let $\psi_1$ and $\psi_2$ be two functions defined in $\overline{\Omega}$ with $\psi_1 \le \psi_2$. Suppose conditions (4.1) and (3.8) are satisfied. Let $x_0 \in \Omega$ and set*

$$g_1(r) = \frac{1}{\chi_0} \sup_{\partial\Omega[r]} (G(x, \psi_2(x), Y_2(x)) - G(x, \psi_1(x), Y_1(x)))^+. \tag{4.15}$$

*If $\partial\Omega \in C^1$, then for any $\varepsilon > 0$, there are constants $R(\mu_2, \Omega)$, $\delta(\varepsilon, \mu_2, \Omega)$ and $C(\mu_2, \mu_0, \alpha, \sup(\psi_2 - \psi_1))$ such that*

$$\sup_{\Omega[\delta r]}(\psi_2-\psi_1) \le (1+\varepsilon)\inf_{\Omega[\delta r]}(\psi_2-\psi_1) + C\varepsilon[\zeta(r)r + r^{2/(2-\alpha)}] + \varepsilon g_1(r) r \tag{4.16}$$

*for any $x_0 \in \Omega$ and $r \in (0, R)$.*

*Proof.* Let $x_1$ be a closest point to $x_0$ in $\partial\Omega$, which we can take to be the origin, and rotate axes so that $x_0' = 0$ and $x_0^n > 0$. Then there is a constant $R_1$ determined only by $\mu_2$ and $\Omega$ so that there is a function $\omega$ with $\mu_2 \sup|D\omega| < 1/2$ such that

$$\Omega[R] = \{x \in \mathbb{R}^n : |x - x_0| < R, x^n > \omega(x')\}$$

and $\omega(0) = 0$ and $D\omega(0) = 0$. By choosing $R < R_1$ sufficiently small, we can also arrange that $|G^n - G_p \cdot \gamma| \le \frac{1}{2}\chi_0$ and $|D\omega| < 1/2$. It follows that conditions (4.11a,b) hold with $\mu_3 = 2\mu_2$.

Now take $\eta$ to be the constant from Lemma 4.5 and note that there is a constant $\eta_1$ such that $d(x_0) \le \eta_1 r$ implies that $\Omega[\eta_1 r] \subset K[\eta r]$. Therefore (4.16) holds in this case with any $\delta \le \eta_1$. On the other hand, if $d(x_0) > \eta_1 r$, then Lemma 4.1 (with $\eta_1 r$ in place of $r$) implies (4.16) in this case with $\delta = \eta_1\varepsilon/3$ because $(1+\varepsilon/3)/(1-\varepsilon/3) \le 1+\varepsilon$. Combining this two cases yields the desired result with $\delta = \eta_1\varepsilon/2$. □

As before, we then have the following estimates.

**Corollary 4.7.** *In addition to the hypotheses of Lemma 4.6, suppose that $\psi_1(x_0) = 0$ and $Y_1(x_0) = 0$, and set*

(4.17)
$$g_2(r) = \frac{1}{\chi_0} \sup_{\partial\Omega[r]} (G(x, \psi_2(x), Y_2(x)) - \min\{G(x,0,0), G(x, \psi_1(x), Y_1(x))\})^+.$$

*Then*

(4.18)
$$\sup_{B(x_0,\delta r)} \psi_2 \le (1+\varepsilon) \inf_{\Omega[\delta r]} \psi_2^+ + C\varepsilon[\zeta(r) + r^{\alpha/(2-\alpha)} + g_2(r) + (\zeta(r)r)^\alpha]r.$$

*If also $\psi_2 \ge 0$ in $\Omega[\delta r]$, then*

$$\sup_{B(x_0,\delta r)} \psi_1 \le 2\varepsilon \inf_{\Omega[\delta r]} \psi_2 + C\varepsilon[\zeta(r) + r^{\alpha/(2-\alpha)} + g_2(r) + (\zeta(r)r)^\alpha]r. \tag{4.19}$$

*Proof.* We follow the proof of Corollary 4.2, noting that

$$G(x, -\zeta(r)r, 0) \ge G(x,0,0) - \mu_0(\zeta(r)r)^\alpha$$

and that $(1+\varepsilon) - 1/(1+\varepsilon) \le 2\varepsilon$. □

The estimate on the modulus of continuity for the gradient of the solution of the double obstacle problem follows easily.

**Theorem 4.8.** *Let $\partial\Omega \in C^{1,\alpha}$ for some $\alpha \in (0,1)$, and let $\psi_1$ and $\psi_2$ be two functions satisfying condition (4.1) in $\Omega$ for some continuous increasing function $\zeta$. Suppose also that $\psi_1 \le \psi_2$ in $\Omega$. Let $u \in W^{2,n}_{\mathrm{loc}} \cap C^1(\overline{\Omega})$ satisfy (4.6) and $G(x,u,Du) = 0$ on $\partial\Omega$. Suppose there are constants $\lambda$, $\mu_0$, $\mu_2$, and $\nu_1$, along with a continuous increasing function $\zeta_1$ with $\zeta_1(0) = 0$ such* *that conditions* (2.11), (2.12), (2.3), (2.4b), (4.7), *and* (3.8) *hold. Suppose finally that*

$$G(x, \psi_1(x), Y_1(x)) \ge 0,\ G(x, \psi_2(x), Y_2(x)) \le 0, \tag{4.20}$$

*for all* $x \in \partial\Omega$. *If* $F$ *is concave or convex with respect to* $r$, *then there are constants* $\alpha_0(n, \mu_2, \Lambda/\lambda, \nu_1)$ *and* $C(n, \alpha, \Lambda/\lambda, \mu_1, \mu_2, |D\psi_1|_1, |D\psi_2|_1, \zeta_1, \Omega)$ *such that if* $\alpha \le \alpha_0$, *then* (3.10) *holds for all* $x_1$ *and* $x_2$ *in* $\Omega$.

*Proof.* We proceed by combining the proof of Theorem 3.2 with that of Theorem 4.4, taking Corollary 4.7 into account. ☐

We omit the obvious two-obstacle analog of Theorem 3.3.

## 5. Variational inequalities.

Our methods also apply to certain types of variational inequalities. In particular, let $H$ be a convex, $C^2$ function defined on $[0, \infty)$ with $H(0) = 0$ and suppose that $h = H'$ satisfies the conditions

$$\delta \le \frac{th'(t)}{h(t)} \le g_0 \tag{5.1}$$

for some positive constants $\delta$ and $g_0$, and all $t > 0$; we also assume that $H(1) = 1$ for simplicity. The model such function is $H(t) = t^m$ with $m > 1$. Let $W^{1,H}$ denote the set of all functions $v \in W^{1,1}$ with $H(|Dv|) \in L^1(\Omega)$, and write $K$ for a convex subset of $W^{1,H}$ such that $v \ge \psi$ for all $v \in K$. (For example if $H(t) = t^m$, then $W^{1,H} = W^{1,m}$ and we can take $K$ to be the set of all $v \in W^{1,m}$ with $v \ge \psi$.) We then consider the problem of finding a function $u \in K$ such that

$$\int_\Omega [A(x, u, Du) \cdot D(u - v) - B(x, u, Du)(u - v)]\, dx \le 0 \tag{5.2}$$

for all $v \in K$, where $A$ is a vector-valued function (for example $A(x, z, p) = h(|p|)p/|p|$) and $B$ is a scalar-valued function, which we shall assume to be bounded. Such problems have a long history for various choices of $h$ provided $A$ and $B$ satisfy suitable structure conditions; see, for example, [**14**, Section III.4], [**18**], [**6**], [**8**], [**9**], [**22**], [**25**], [**27**]. We note, however, that all of these works assume that $\psi$ has Hölder continuous gradient when trying to prove a modulus of continuity estimate for the gradient of $u$.

We first observe that, when $h(t)/t$ is bounded from above and below by positive constants and $A$ and $B$ are sufficiently smooth, smooth solutions of this variational inequality are also solutions of (2.2) with

$$F(x, z, p, r) = \frac{\partial A^i}{\partial p_j}(x, z, p) r_{ij} + \frac{\partial A^i}{\partial z}(x, z, p) p_i + \frac{\partial A^i}{\partial x^i}(x, z, p) + B(x, z, p).$$

More generally, we assume that $A$ is differentiable with respect to $p$ and that there are nonnegative constants $\alpha$, $\Lambda$ and $\Lambda_1$ with $\alpha \in (0,1)$ such that

$$\frac{\partial A^i}{\partial p_j}\xi_i\xi_j \geq \frac{h(|p|)}{|p|}|\xi|^2, \tag{5.3a}$$

$$|A_p| \leq \Lambda h(|p|)/|p|, \tag{5.3b}$$

$$|B| \leq \Lambda_1, \tag{5.3c}$$

$$|A(x,z,p) - A(y,w,p)| \leq \Lambda_1(|x-y| + |w-z|)^\alpha. \tag{5.3d}$$

These conditions were studied extensively in [**17**]. In fact, we have simplified the conditions there somewhat by assuming a known bound for the gradient of $u$.

We begin by proving an estimate like (2.6). As in [**18**], we first prove the estimate for a simpler problem.

**Lemma 5.1.** *Let $A$ be a vector valued function defined on $\mathbb{R}^n$ and suppose that there are positive constants $\delta$, $g_0$, and $\Lambda$ along with a function $h$ such that conditions* (5.1) *and* (5.3a,b) *are satisfied. Let $u$ and $\psi$ be in $C^{0,1}(B(x_0,r))$ for some ball $B(x_0,r)$ with $u \geq \psi$, and let $K$ be the set of all $v$ with $v - u \in W_0^{1,H}(B(x_0,r))$ and $v \geq \psi$ in $B(x_0,r)$. Then there is a unique solution $U$ of the variational inequality*

$$\int_{B(x_0,r)} A(DU)\cdot D(U-v)\,dx \leq 0 \quad \text{for all } v \in K, \tag{5.4}$$

*and there are constants $C_1(n,\delta,g_0,|Du|_0,|D\psi|_0,\Lambda)$, $C_2(\Lambda,n)$, $\theta(\Lambda,n,\delta,g_0)$, and $\kappa(\Lambda,n)$ such that*

$$[U]_{\theta;B(x_0,r)} \leq C_1 r^{1-\theta} \tag{5.5}$$

*and*

$$\left(r^{-n}\int_{B(x_0,r/2)} |U-L|^\kappa\,dx\right)^{1/\kappa} \leq C_2 \inf_{B(x_0,r/2)}(U-L) \tag{5.6}$$

*for any linear function $L$ such that $U - L \geq 0$ in $B(x_0,r)$.*

*Proof.* The standard theory of variational inequalities gives the existence and uniqueness of $U$. In addition, (5.5) follows from the arguments in [**18**, Lemma 1.3].

To prove (5.6), we proceed by approximation. First, we fix $\alpha \in (0,1)$ and note (from the proof of [**17**, Lemma 5.2]) that there is a sequence of $C^{1,\alpha}$ functions $(A_k)$ which converge uniformly to $A$ on compact subsets of $B(x_0,r)$ and which satisfy

$$\frac{\partial A_k^i}{\partial p_j}\xi_i\xi_j \geq \frac{h_k(|p|)}{|p|}|\xi|^2$$

and

$$\left|\frac{\partial A_k^i}{\partial p_j}\right| \le 2\Lambda h_k(|p|)/|p|$$

for functions $h_k$ satisfying (5.1). For $\varepsilon \in (0,1)$, define $\beta_\varepsilon$ by $\beta_\varepsilon(t) = (\min\{t,0\})^2/\varepsilon$, and let $U_k$ solve $\operatorname{div} A_k(DU_k) + \beta_{1/k}(U_k - \psi) = 0$ in $B(x_0,r)$ and $U_k = u$ on $\partial B(x_0,r)$. The existence of a unique solution to this problem is straightforward, and classical regularity theory implies that $U_k \in C^2(B(x_0,r))$. Thus, the weak Harnack inequality [**10**, Theorem 9.22] implies that

$$\left(r^{-n}\int_{B(x_0,r/2)} |U_k - L|^\kappa\,dx\right)^{1/\kappa} \le C(\Lambda,n)\inf_{B(x_0,r/2)}(U_k - L)$$

for any linear function $L$ with $U_k - L \ge 0$ in $B(x_0,r)$. It is not hard to show that $U_k$ converges uniformly to $U$ as $k \to \infty$ (see, for example, [**14**, Theorem IV.5.2]), so the desired result follows immediately. □

From this lemma and a suitable choice for the linear function $L$, we infer a version of (2.6).

**Lemma 5.2.** *Under the hypotheses given before Lemma 5.1, suppose that $\psi$ satisfies condition (2.1) for some continuous increasing function $\zeta$. Suppose $u(x_0) = \psi(x_0)$ and define $\bar u$ by (2.5). If $\kappa$ and $\theta$ are the constants from Lemma 5.1 and if $r \le 1$, then there is a constant $C$ determined only by $\Lambda$, $n$, $\Lambda_1$, $|Du|_0$ such that*

$$\left(r^{-n}\int_{B(x_0,r/2)} |\bar u|^\kappa\,dx\right)^{1/\kappa} \le C[r^{\alpha/(2n+2\theta)} + \zeta(r)]r. \tag{5.7}$$

*Proof.* Let $U$ be the solution of (5.4) given by Lemma 5.1 with $A(p) = A(x_0,u(x_0),p)$ and set $w = u - U$. Then we can use $v = U$ in (5.2) and $v = u$ in (5.4) to see from [**17**, (5.8) and Lemma 2.2] that

$$\int_{B(x_0,r)} H(|w|/r)\,dx \le C\int_{B(x_0,r)} H(|Dw|)\,dx \le Cr^{n+\alpha/2}$$

because $H$ is convex. Then Jensen's inequality gives

$$\int_{B(x_0,r)} |w|\,dx \le Cr^{n+1+\alpha/2} \tag{5.8}$$

because [**17**, Lemma 1.1(c)] says that $H(r^{\alpha/2})/H(1) \le r^{\alpha/2}/1$.

To continue, we use a variation of the argument in Lemma 1.1. Choose $x_1$ so that $d(x_1)|w(x_1)| \ge (1/2)|w|_0^{(n)}$ and set $\rho = \varepsilon d(x_1)$ with $\varepsilon \in (0,1/2)$. We have from (5.5) that

$$|w(x)| \ge |w(x_1)| - |w(x) - w(x_1)| \ge \frac{1}{2}d(x_1)^{-n}|w|_0^{(n)} - cr^{1-\theta}\rho^\theta$$

for $|x - x_1| \le \rho$, so

$$\int_{B(x_0,r)} |w|\,dx \ge \int_{B(x_1,\rho)} |w|\,dx \ge \frac{\omega_n}{2}\varepsilon^n |w|_0^{(n)} - c\omega_n r^{1-\theta}\rho^{n+\theta},$$

where $\omega_n$ is the measure of the $n$-dimensional unit ball. Therefore

$$|w|_0^{(n)} \le C\varepsilon^{-n}\int_{B(x_0,r)} w\,dx + C\varepsilon^\theta d^{n+\theta} r^{1-\theta} \le Cr^{n+1}(\varepsilon^{-n}r^{\alpha/2} + \varepsilon^\theta)$$

from (5.8). Now we take $\varepsilon = r^{\alpha/(2n+2\theta)}/2$ to conclude that

$$\sup_{B(x_0,r/2)} |w| \le cr^{-n}|w|_0^{(n)} \le Cr^{1+\alpha/(2n+2\theta)}.$$

Thus we can take $L(x) = \psi(x_0) - Y(x_0)\cdot(x - x_0) - \zeta(r)r - Cr^{1+\alpha/(2n+2\theta)}$ in Lemma 5.2, and hence (5.7) holds. □

The interior gradient modulus of continuity estimate for such problems follows by using the argument of Theorem 2.3 and the Hölder gradient estimates for weak solutions of divergence structure equations from [**17**, Section 5]. The correct form of this estimate is an easy consequence of the last inequality on page 346 of [**17**].

**Theorem 5.3.** *Let $A$ and $B$ be, respectively, a vector-valued function and a scalar-valued function on $\Omega\times\mathbb{R}\times\mathbb{R}^n$, and let $H$ be a convex, $C^2$ function on $[0,\infty)$ with $H(0) = 0$, and suppose $h = H'$ satisfies (5.1). Suppose also that conditions (5.3a–d) are satisfied. Let $\psi$ satisfy (2.1), let $u \in C^{0,1}(\overline{\Omega})$ and suppose $u \ge \psi$ in $\Omega$. If $u$ is a solution of (5.2) with $K$ the set of all $v \in W^{1,H}$ with $v - u \in W_0^{1,H}$ and $v \ge \psi$, then there are constants $\sigma_0(\Lambda,\delta,g_0)$ and $C(n,\Lambda,\delta,g_0,\sup|Du|,\Psi_1,\Lambda_1,\alpha,\operatorname{diam}\Omega)$ such that*

$$|Du(x_1) - Du(x_2)| \le C\left[\zeta(|x_1 - x_2|) + \left(1 + \frac{\sup|Du|}{d(x_1)}\right)|x_1 - x_2|^\sigma\right] \tag{5.9}$$

*for all $x_1$ and $x_2$ in $\Omega$ with $|x_1 - x_2| \le \frac{1}{4}d(x_1)$, where $\sigma = \min\{\sigma_0, \alpha/(2n + 2\theta)\}$ and $\theta$ is the constant from Lemma 5.1.*

The corresponding boundary regularity result is similar, and the proof is similar.

**Theorem 5.4.** *Let $\partial\Omega \in C^{1,\alpha}$ for some $\alpha \in (0,1)$, let $A$ and $B$ be, respectively, a vector-valued function and a scalar-valued function on $\Omega\times\mathbb{R}\times\mathbb{R}^n$, let $a_0$ be a scalar valued function on $\partial\Omega\times\mathbb{R}$, and let $H$ be a convex, $C^2$ function on $[0,\infty)$ with $H(0) = 0$, and suppose $h = H'$ satisfies (5.1). Suppose also that conditions (5.3a–d) are satisfied and that*

$$|a_0(x,z) - a_0(y,w)| \le \Lambda_2(|x - y| + |z - w|)^\alpha \tag{5.10}$$

*for all* $(x,z)$ *and* $(y,w)$ *in* $\partial\Omega \times \mathbb{R}$. *Let* $\psi$ *satisfy* (2.1) *and*

$$A(x,\psi,Y)\cdot\gamma + a_0(x,\psi) \geq 0 \tag{5.11}$$

*on* $\partial\Omega$. *If* $u$ *is a solution of*

$$\int_\Omega [A(x,u,Du)\cdot D(u-v) - B(x,u,Du)(u-v)]\,dx \leq \int_{\partial\Omega} a_0(x,u)(u-v)\,d\sigma \tag{5.12}$$

*with* $K$ *the set of all* $v \in W^{1,H}$ *with* $v \geq \psi$, *then there are constants* $\sigma_0(\Lambda,\delta,g_0)$ *and* $C(n,\Lambda,\delta,g_0,\sup|Du|,\Psi_1,\Lambda_1,\alpha,\Omega)$ *such that*

$$|Du(x_1) - Du(x_2)| \leq C[\zeta(|x_1-x_2|) + |x_1-x_2|^\sigma] \tag{5.13}$$

*for all* $x_1$ *and* $x_2$ *in* $\Omega$, *where* $\sigma = \min\{\sigma_0, \alpha/(2n+2\theta)\}$.

*Proof.* To prove the analog of (5.7), we let $U$ solve the variational inequality

$$\int_\Omega A(x_0,u(x_0),DU)\cdot D(U-v)\,dx \leq \int_{\partial\Omega} [a_0(x_0,u(x_0)) + C_0R]\,d\sigma,$$

where $K$ is the set of all $v \in W^{1,H}$ with $v \geq \psi$ in $\Omega$ and $v = u$ on $\Omega \setminus \Omega[R]$; the constant $C_0$, which is independent of $x_0$ and $R$, is chosen so that

$$A(x_0,\psi(x_0),Y(x))\cdot\gamma(x) + a_0(x_0,\psi(x_0)) + C_0R \geq 0$$

for all $x \in \partial\Omega[R]$. The appropriate Hölder gradient estimate was proved in [**15**, Section 4] for the special case that $A$ depends only on $p$, $a_0$ is constant, and $B$ is identically zero, and the general estimate follows from the perturbation argument in [**17**, Section 5]. □

We leave the straightforward modifications of these results for double obstacle problems to the reader. We do observe that the previous results for double obstacle problems (specifically [**7**, **18**, **26**]) all assume that the obstacle has Hölder continuous first derivatives. Thus, we have improved these results by considering general moduli of continuity and also suitable one-sided conditions.

## 6. Existence of solutions.

A suitable existence theory for our obstacle problem is based on known *a priori* estimates and the penalization method of Lions (see [**14**, Section IV.5]). We assume first that $\partial\Omega \in C^3$ (although this assumption can be relaxed by the remarks at the end of [**21**, Section 3]), and we assume that $\psi$ satisfies (2.1) with $\zeta(t) = z_0 t$. In addition, we assume that (3.9) holds. For $\rho$ a $C^2(\overline{\Omega})$ function such that $D\rho = \gamma$ on $\partial\Omega$ (which always exists), we suppose that there are nonnegative constants $M_0$ and $M_1$ such that

$$zF(x,z,-M_1D\rho,-M_1D^2\rho) < 0 \text{ in } \Omega, \tag{6.1a}$$

$$zG(x,z,-M_1\gamma) < 0 \text{ on } \partial\Omega \tag{6.1b}$$

for $z \geq M_0$. Next, we assume that there are increasing functions $\mu$ and $\mu_0$ such that

$$\lambda(x,z,p,r)|\xi|^2 \leq F^{ij}(x,z,p,r)\xi_i\xi_j \leq \Lambda(x,z,p,r)|\xi|^2, \tag{6.2a}$$

$$\Lambda(x,z,p,r) \leq \mu(|z|)\lambda(x,z,p,r) \tag{6.2b}$$

$$|F(x,z,p,0)| \leq \mu_0(|z|)\lambda(x,z,p,r)[1+|p|^2] \tag{6.2c}$$

for all $(x,z,p,r) \in \Gamma$ and

$$|G(x,z,p')| \leq \mu_0(|z|)G_p(x,z,p)\cdot\gamma[1+|p'|] \tag{6.2d}$$

for all $(x,z,p) \in \Gamma'$, where $p' = p - (\gamma\cdot p)\gamma$. We also assume that there is an increasing function $\mu_1$ such that

$$(1+|p|)|F_p| + |F_z| + |F_x| \leq \mu_1(|z|)\lambda[1+|p|^2+|r|] \tag{6.3}$$

on $\Gamma$ and

$$(1+|p|)|G_p| + |G_z| + |F_x| \leq \mu_1(|z|)G_p\cdot\gamma[1+|p|] \tag{6.4}$$

on $\Gamma'$. Finally we assume that $F$ is concave (or concave) with respect to $r$ and that $\lambda$ is uniformly bounded above and uniformly positive on bounded sets of $\Gamma$, and we assume that $G_p(x,z,p)\cdot\gamma$ is uniformly bounded and uniformly positive on bounded subsets of $\Gamma'$. Note that [**21**, Lemma 7.1] implies the upper bound $u \leq M_1 \sup\rho$ while the obstacle condition imply that $u \geq \min\psi$. Hence, we may assume that conditions (6.2)–(6.4) hold with $\mu$, $\mu_0$, and $\mu_1$ independent of $z$ by redefining $F$ and $G$ for large $|z|$ as needed. In particular, we may assume that $F$ and $G$ are independent of $z$ for $z \leq \psi(x)$.

Now for $\varepsilon \in (0,1)$, we define $\beta_\varepsilon$ by

$$\beta_\varepsilon(t) = (\min\{t,0\})^2/\varepsilon.$$

It then follows from [**21**, Lemma 7.1, Theorems 3.3, 4.1, and 7.8] along with [**29**, Theorem 3.3] (see also [**19**, Theorems 14.22 and 14.23]) that the problem

$$\begin{aligned} F(x,u_\varepsilon,Du_\varepsilon,D^2u_\varepsilon) + \beta_\varepsilon(u_\varepsilon-\psi) &= 0 \quad \text{in } \Omega,\\ G(x,u_\varepsilon,Du_\varepsilon) + \varepsilon &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

has a $C^{2,\theta}(\overline{\Omega})$ solution for any $\varepsilon \in (0,1)$ and some $\theta \in (0,1)$ upon recalling our previous observations that we may take $\mu$, $\mu_0$, and $\mu_1$ independent of $z$. As previously remarked, [**21**, Lemma 7.1] implies that $(u_\varepsilon)$ is uniformly bounded, independent of $\varepsilon$.

Now we estimate $\beta_\varepsilon(u_\varepsilon - \psi)$. If the minimum of $u_\varepsilon - \psi$ is nonnegative, then $\beta_\varepsilon = 0$. In addition, at a boundary minimum,

$$-\varepsilon = G(x,u_\varepsilon,Du_\varepsilon) \geq G(x,u_\varepsilon,Y),$$

so if the minimum of $u_\varepsilon - \psi$ is negative, it must occur at some $x_0 \in \Omega$. In this case, $Du_\varepsilon(x_0) = Y(x_0)$ and $D^2u_\varepsilon \geq -2z_0I$, where $I$ denotes the $n \times n$ identity matrix, so

$$\beta_\varepsilon(u_\varepsilon - \psi)(x_0) = -F(x_0, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) \leq F(x_0, \psi(x_0), Y(x_0), -2z_0I).$$

It follows that $\beta_\varepsilon(u_\varepsilon - \psi_\varepsilon) \leq c_1$ for some nonnegative constant $c_1$ independent of $\varepsilon$. We can then use [**21**, Theorem 3.3] to infer a global gradient bound for $u_\varepsilon$, which is uniform with respect to $\varepsilon$. We can then apply [**3**, Theorem 2] (see [**19**, Lemma 13.21, and Theorems 14.14 and 14.20] for a discussion of the extension to the oblique derivative boundary condition) to infer that $[Du_\varepsilon]_\alpha \leq c_2$ for constants $\alpha \in (0, 1)$ and $c_2$ independent of $\varepsilon$. Finally, [**3**, Theorem 1] shows that $(D^2u_\varepsilon)$ is bounded in $L^p_{\text{loc}}(\Omega)$ for any $p < \infty$. From these estimates and the argument on pages 44 and 45 of [**1**], we infer that there is a sequence $(\varepsilon(j))$ such that $(u_{\varepsilon(j)})$ converges to a function $u \in W^{2,n}_{\text{loc}}(\Omega) \cap C^{1,\alpha}$ and that $u$ solves (0.1). Theorem 2.3 then implies that $u \in C^{1,1}(\overline{\Omega})$.

Note that a more thorough existence theory can be derived via approximation of the obstacle; however, the convergence of the approximating solutions to a function in $W^{2,n}_{\text{loc}}(\Omega)$ requires at least that the obstacle be a supremum of $W^{2,n}_{\text{loc}}(\Omega)$ functions. On the other hand, the extension to two-obstacle problems, which we leave to the reader, is very simple.

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