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It is shown that if (M, ϕ, α) is a W*-dynamical system with M a type I von Neumann algebra then the entropy of α w.r.t. ϕ equals the entropy of the restriction of α to the center of M. If furthermore (N, ψ, β) is a W*-dynamical system with N injective then $h_{\phi\otimes\psi}(\alpha \otimes \beta) = h_\phi(\alpha) + h_\psi(\beta)$.

1. Introduction.

In the theory of non-commutative entropy the attention has almost exclusively been concentrated on non-type I algebras. We shall in the present paper remedy this situation by proving the basic facts on entropy of automorphisms of type I C*- and von Neumann-algebras. The results are as nice as one can hope. The CNT-entropy of an automorphism of a von Neumann algebra of type I with respect to an invariant normal state is the classical entropy of the restriction of the automorphism to the center of the algebra. If one factor of a tensor product of two von Neumann algebras is of type I and the other injective, then the entropy of a tensor product automorphism with respect to an invariant product state is the sum of the entropies. The results have obvious corollaries to type I C*-algebras. The main idea behind our proofs is the use of conditional expectations of finite index, as employed in [**GN**].

We shall use the notation $h_\phi(\alpha)$ for the CNT-entropy of a C*-dynamical system as defined by Connes, Narnhofer and Thirring in [**CNT**], and $h'_\phi(\alpha)$ for the ST-entropy defined by Sauvageot and Thouvenot in [**ST**].

2. Main results.

We first prove a general result for the Sauvageot-Thouvenot entropy for the restriction of an automorphism to a globally invariant C*-subalgebra of finite index.

Proposition 1. *Let (A, ϕ, α) be a unital C*-dynamical system. Let $B \subset A$ be an α-invariant C*-subalgebra (with $1 \in B$). Suppose there exists a conditional expectation $E\colon A \to B$ such that $E \circ \alpha = \alpha \circ E$, $\phi \circ E = \phi$ and $E(x) \geq cx$ for all $x \in A^+$ for some $c > 0$. Then $h'_\phi(\alpha) = h'_\phi(\alpha|_B)$.*

Proof. Let (C, μ, β) be a C*-dynamical system with C abelian. Using E we can lift any stationary coupling on $B \otimes C$ to a stationary coupling on $A \otimes C$. This, together with the property of monotonicity of relative entropy, shows that $h'_\phi(\alpha) \geq h'_\phi(\alpha|_B)$.

Conversely, suppose λ is a stationary coupling of (A, ϕ, α) with (C, μ, β), and P is a finite-dimensional subalgebra of C with atoms $p_1, \dots, p_n$. Let

$$\phi_i(a) = \frac{\lambda(a \otimes p_i)}{\mu(p_i)} \quad \text{for} \quad a \in A.$$

Then in the notations of [**ST**]

$$h'_\phi(\alpha) = \sup \left\{ H_\mu(P|P^-) - H_\mu(P) + \sum_{i=1}^n \mu(p_i) S(\phi, \phi_i) \;\middle|\; (C, \mu, \beta, \lambda, P) \right\}.$$

Since $\phi_i \leq \frac{1}{\mu(p_i)}\phi$, ϕ_i is normal in the GNS-representation of ϕ. Since E is ϕ-invariant, it extends to a normal conditional expectation of the closure of A in the GNS-representation onto the closure of B. Thus we can apply [**OP**, Theorem 5.15] to ϕ and ϕ_i, and (as in the proof of Lemma 1.5 in [**GN**]) get

$$\begin{aligned}\sum_{i=1}^n \mu(p_i) S(\phi, \phi_i) &= \sum_{i=1}^n \mu(p_i)(S(\phi|_B, \phi_i|_B) + S(\phi_i \circ E, \phi_i)) \\ &\leq \sum_{i=1}^n \mu(p_i) S(\phi|_B, \phi_i|_B) - \log c.\end{aligned}$$

It follows that $h'_\phi(\alpha) \leq h'_\phi(\alpha|_B) - \log c$. Then for each $m \in \mathbb{N}$

$$h'_\phi(\alpha) = \frac{1}{m} h'_\phi(\alpha^m) \leq \frac{1}{m} h'_\phi(\alpha^m|_B) - \frac{1}{m}\log c = h'_\phi(\alpha|_B) - \frac{1}{m}\log c.$$

Thus $h'_\phi(\alpha) \leq h'_\phi(\alpha|_B)$. □

By [**ST**, Proposition 4.1] the Sauvageot-Thouvenot entropy coincides with the CNT-entropy for nuclear C*-algebras. In fact, what is really necessary for the coincidence of the entropies, is the existence of a net of unital completely positive mappings γ_i of finite-dimensional C*-algebras into A such that $S(\phi, \psi) = \lim_i S(\phi \circ \gamma_i, \psi \circ \gamma_i)$ for any positive linear functional ψ on A, $\psi \leq \phi$. We therefore have:

Corollary 2. *If in the above proposition A and B are injective von Neumann algebras and ϕ is normal then $h_\phi(\alpha) = h_\phi(\alpha|_B)$.*

To prove our main result we need also two simple lemmas. The first lemma is more or less well-known.

Lemma 3. *Let (M, ϕ, α) be a W*-dynamical system. Then*

(i) *if p is an α-invariant projection in M such that* $\operatorname{supp}\phi \leq p$, *then* $h_\phi(\alpha) = h_\phi(\alpha|_{M_p})$;

(ii) *if* $\{p_i\}_{i\in I}$ *is a set of mutually orthogonal* α*-invariant central projections in* M, $\sum_i p_i = 1$, *then*

$$h_\phi(\alpha) = \sum_i \phi(p_i) h_{\phi_i}(\alpha_i),$$

where $\phi_i = \frac{1}{\phi(p_i)}\phi$ *is the normalized restriction of* ϕ *to* Mp_i, *and* $\alpha_i = \alpha|_{Mp_i}$.

Proof.
(i) easily follows from the definitions; (ii) follows from [**CNT**, VII.5(iii)], (i) and [**SV**, Lemma 3.3] applied to the subalgebras $M(p_{i_1} + \cdots + p_{i_n}) + \mathbb{C}(1 - p_{i_1} - \cdots - p_{i_n})$. ◻

The proof of the following lemma is left to the reader.

Lemma 4. *Let* T *be an automorphism of a probability space* (X, μ), $f \in L^\infty(X, \mu)$ *a* T*-invariant function such that* $f \geq 0$ *and* $\int_X f\, d\mu = 1$. *Let* μ_f *be the measure on* X *such that* $d\mu_f/d\mu = f$. *Then* $h_{\mu_f}(T) \leq ||f||_\infty h_\mu(T)$.

Theorem 5. *Let* (M, ϕ, α) *be a W*-dynamical system with* M *a von Neumann algebra of type I. Let* Z *denote the center of* M. *Then* $h_\phi(\alpha) = h_\phi(\alpha|_Z)$.

Proof. By Lemma 3(i) we may suppose that ϕ is faithful. Then M is a direct sum of homogeneous algebras of type I_n, $n \in \mathbb{N} \cup \{\infty\}$. By Lemma 3(ii) we may assume that M is homogeneous of type I_n. We first assume that $n \in \mathbb{N}$. Then $Z = L^\infty(X, \mu)$, where (X, μ) is a probability space and $\phi|_Z = \mu$. Thus

$$M \cong Z \otimes \mathrm{Mat}_n(\mathbb{C}) = L^\infty(X, \mathrm{Mat}_n(\mathbb{C})), \quad \phi = \int_X^\oplus \phi_x d\mu(x),$$

where $\phi_x = \mathrm{Tr}(\cdot\, Q_x)$ is a state on $\mathrm{Mat}_n(\mathbb{C})$, Tr the canonical trace on $\mathrm{Mat}_n(\mathbb{C})$. We first assume $Q_x \geq c > 0$ for all x.

If $s \in M^+$, s is a function in $L^\infty(X, \mathrm{Mat}_n(\mathbb{C}))$. Define the ϕ-preserving conditional expectation $E\colon M \to Z$ by $E(s)(x) = \phi_x(s(x))$. Then

$$E(s)(x) = \mathrm{Tr}(s(x)Q_x) \geq c\mathrm{Tr}(s(x)) \geq cs(x),$$

so $E(s) \geq cs$, and it follows from Corollary 2 that $h_\phi(\alpha) = h_\phi(\alpha|_Z)$.

If there is no $c > 0$ such that $Q_x \geq c$ for all x, let $X_c = \{x \in X \mid Q_x \geq c\}$, $(c > 0)$,

$$N_c = L^\infty(X_c, \mathrm{Mat}_n(\mathbb{C})) \ \text{ and } \ M_c = N_c + \mathbb{C}\chi_{X\setminus X_c},$$

where $\chi_{X\setminus X_c}$ is the characteristic function of $X\setminus X_c$. Since ϕ is α-invariant so is M_c, so by the above argument and Lemma 3, letting $\phi_c = \frac{1}{\mu(X_c)}\phi|_{N_c}$ and $\mu_c = \frac{1}{\mu(X_c)}\mu|_{X_c}$, we obtain

$$h_\phi(\alpha|_{M_c}) = \mu(X_c) h_{\phi_c}(\alpha|_{N_c}) = \mu(X_c) h_{\mu_c}(T|_{X_c}) \leq h_\mu(T),$$

where T is the automorphism of (X, μ) induced by α. Letting $c \to 0$ and using [**SV**, Lemma 3.3] we obtain the Theorem when M is finite.

If M is homogeneous of type I_∞, we have $M \cong L^\infty(X, \mu) \otimes B(H)$, where H is a separable Hilbert space. Let Tr denote the canonical trace on $B(H)$. Write again

$$\phi = \int_X^\oplus \phi_x d\mu(x), \quad \phi_x = \mathrm{Tr}(\cdot\, Q_x),$$

and let $E_x(U)$ denote the spectral projection of Q_x corresponding to a Borel set U. Let $P_c \in M = L^\infty(X, B(H))$ be the projection defined by $P_c(x) = E_x([c, +\infty))$, where $c > 0$. Then P_c is an α-invariant finite projection. Let

$$M_c = P_c M P_c + \mathbb{C}(1 - P_c).$$

Then M_c is a finite type I von Neumann algebra. Its center is isomorphic to $L^\infty(X_c, \mu_c) \oplus \mathbb{C}$, and the restriction of ϕ to it is $\phi(P_c)\mu_c \oplus \phi(1 - P_c)$, where $X_c = \{x \in X \mid P_c(x) \neq 0\}$ and

$$\int_{X_c} f(x) d\mu_c(x) = \frac{1}{\phi(P_c)} \int_{X_c} f(x)\phi_x(P_c(x)) d\mu(x).$$

So we can apply the first part of the proof to M_c. Since $d\mu_c/d\mu \leq \frac{1}{\phi(P_c)}$, applying Lemma 4 we get

$$h_\phi(\alpha|_{M_c}) = \phi(P_c) h_{\mu_c}(T|_{X_c}) \leq h_\mu(T).$$

Now letting $c \to 0$ we conclude that $h_\phi(\alpha) = h_\mu(T)$. □

It should be remarked that in a special case the above theorem was proved in [**GS**, Proposition 2.4].

If A is a C*-algebra and ϕ a state on A, the central measure μ_ϕ of ϕ is the measure on the spectrum $\hat{A}$ of A defined by $\mu_\phi(F) = \phi(\chi_F)$, where ϕ is regarded as a normal state on A'', see [**P**, 4.7.5]. Thus by Theorem 5 and [**P**, 4.7.6] we have the following:

Corollary 6. *Let (A, ϕ, α) be a C*-dynamical system with A a separable unital type* I *C*-algebra. Then $h_\phi(\alpha) = h_{\mu_\phi}(\hat{\alpha})$, where $\hat{\alpha}$ is the automorphism of the measure space $(\hat{A}, \mu_\phi)$ induced by α.*

Since inner automorphisms act trivially on the center we have:

Corollary 7. *If (M, ϕ, α) is a W*-dynamical system with M of type* I *and α an inner automorphism then $h_\phi(\alpha) = 0$.*

Note that in the finite case the above corollary also follows from a result of N. Brown [**Br**, Lemma 2.2].

The next result was shown in [**S**] when ϕ is a trace.

Corollary 8. *Let R denote the hyperfinite* II_1*-factor. Let A be a Cartan subalgebra of R and u a unitary operator in A. If ϕ is a normal state such that u belongs to the centralizer of ϕ then $h_\phi(\mathrm{Ad}\, u) = 0$.*

Proof. As in [**S**], it follows from [**CFW**] that there exists an increasing sequence of full matrix algebras $N_1 \subset N_2 \subset \dots$ with union weakly dense in R such that $A \cong A_n \otimes B_n$, where $A_n = N_n \cap A$ and $B_n = (N_n' \cap R) \cap A$ for all $n \in \mathbb{N}$. Let $M_n = N_n \otimes B_n$. Then M_n is of type I and contains u. Hence $h_\phi(\mathrm{Ad}\, u|_{M_n}) = 0$. Since $(\cup_n M_n)^- = R$, $h_\phi(\mathrm{Ad}\, u) = 0$ by [**SV**, Lemma 3.3]. □

If (A, ϕ, α) and (B, ψ, β) are C*-dynamical systems we always have

$$h_{\phi\otimes\psi}(\alpha \otimes \beta) \geq h_\phi(\alpha) + h_\psi(\beta),$$

see [**SV**, Lemma 3.4]. Equality does not always hold, see [**NST**] or [**Sa**]. However, we have:

Theorem 9. *Let (A, ϕ, α) and (B, ψ, β) be W*-dynamical systems. Suppose that A is of type* I*, and B is injective. Then*

$$h_{\phi\otimes\psi}(\alpha \otimes \beta) = h_\phi(\alpha) + h_\psi(\beta).$$

Proof. We shall rather prove that $h_{\phi\otimes\psi}(\alpha \otimes \beta) = h_\phi(\alpha|_{Z(A)}) + h_\psi(\beta)$. For this it suffices to consider the case when A is abelian; the general case will follow by the same arguments as in the proof of Theorem 5. (Note that the mapping $x \mapsto \mathrm{Tr}(x) - x$ on $\mathrm{Mat}_n(\mathbb{C})$ is not completely positive, but the mapping $x \mapsto \mathrm{Tr}(x) - \frac{1}{n}x$ is by the Pimsner-Popa inequality. Thus replacing M with $M \otimes B$ and Z with $Z \otimes B$ in the proof of Theorem 5 we have to replace the inequality $E(s) \geq cs$ in the proof with $E(s) \geq \frac{c}{n}s$.)

So suppose that A is abelian. It is clear that it suffices to prove that if $A_1, \dots, A_n$ are finite-dimensional subalgebras of A, and $B_1, \dots, B_n$ are finite-dimensional subalgebras of B, then

$$H_{\phi\otimes\psi}(A_1 \otimes B_1, \dots, A_n \otimes B_n) = H_\phi(A_1, \dots, A_n) + H_\psi(B_1, \dots, B_n).$$

We always have the inequality "$\geq$", [**SV**, Lemma 3.4]. To prove the opposite inequality consider a decomposition

$$\phi \otimes \psi = \sum_{i_1,\dots,i_n} \omega_{i_1\dots i_n}.$$

Let $H_{\{\phi\otimes\psi=\sum\omega_{i_1\dots i_n}\}}(A_1 \otimes B_1, \dots, A_n \otimes B_n)$ be the entropy of the corresponding abelian model, so

$$H_{\{\phi\otimes\psi=\sum\omega_{i_1\dots i_n}\}}(A_1 \otimes B_1, \dots, A_n \otimes B_n)$$
$$= \sum_{i_1,\dots,i_n} \eta\omega_{i_1\dots i_n}(1) + \sum_{k=1}^{n} \sum_i S\left(\phi \otimes \psi|_{A_k\otimes B_k}, \sum_{i_k=i} \omega_{i_1\dots i_n}|_{A_k\otimes B_k}\right).$$

Set $C = \vee_{k=1}^n A_k$. Let $p_1, \dots, p_r$ be those atoms p of C for which $\phi(p) > 0$. Define positive linear functionals $\psi_{m,i_1\dots i_n}$ on B,

$$\psi_{m,i_1\dots i_n}(b) = \frac{\omega_{i_1\dots i_n}(p_m \otimes b)}{\phi(p_m)}.$$

Let also ϕ_m be the linear functional on C defined by the equality $\phi_m(a) = \phi(ap_m)$. Then

$$\omega_{i_1\dots i_n} = \sum_{m=1}^r \phi_m \otimes \psi_{m,i_1\dots i_n} \quad \text{on} \quad C \otimes B,$$

and

$$\psi = \sum_{i_1,\dots,i_n} \psi_{m,i_1\dots i_n} \quad \text{for} \quad m = 1, \dots, r.$$

Since the supports of the positive functionals ϕ_m are mutually orthogonal minimal projections in C, we have

$$\begin{aligned}
&\sum_{k=1}^n \sum_i S\left(\phi \otimes \psi|_{A_k \otimes B_k}, \sum_{i_k = i} \omega_{i_1\dots i_n}|_{A_k\otimes B_k}\right) \\
&\le \sum_{k=1}^n \sum_i S\left(\phi \otimes \psi|_{C \otimes B_k}, \sum_{i_k = i} \omega_{i_1\dots i_n}|_{C\otimes B_k}\right) \\
&= \sum_{k=1}^n \sum_i S\left(\phi \otimes \psi|_{C \otimes B_k}, \sum_{m=1}^r \phi_m \otimes \left(\sum_{i_k = i} \psi_{m,i_1\dots i_n}\right)|_{C\otimes B_k}\right) \\
&= \sum_{k=1}^n \sum_i \sum_{m=1}^r \phi(p_m) S\left(\psi|_{B_k}, \sum_{i_k = i} \psi_{m,i_1\dots i_n}|_{B_k}\right).
\end{aligned}$$

If $a_i \ge 0$ then $\eta\left(\sum_i a_i\right) \le \sum_i \eta(a_i)$. Hence we have

$$\begin{aligned}
&\sum_{i_1,\dots,i_n} \eta\omega_{i_1\dots i_n}(1) \\
&\le \sum_{m=1}^r \sum_{i_1,\dots,i_n} \eta(\phi_m \otimes \psi_{m,i_1\dots i_n})(1) \\
&= \sum_{m=1}^r \eta\phi(p_m) \sum_{i_1,\dots,i_n} \psi_{m,i_1\dots i_n}(1) + \sum_{m=1}^r \phi(p_m) \sum_{i_1,\dots,i_n} \eta\psi_{m,i_1\dots i_n}(1) \\
&= \sum_{m=1}^r \eta\phi(p_m) + \sum_{m=1}^r \phi(p_m) \sum_{i_1,\dots,i_n} \eta\psi_{m,i_1\dots i_n}(1).
\end{aligned}$$

Thus

$$H_{\{\phi\otimes\psi=\sum\omega_{i_1\dots i_n}\}}(A_1\otimes B_1,\dots,A_n\otimes B_n)$$
$$\leq \sum_{m=1}^{r}\eta\phi(p_m)+\sum_{m=1}^{r}\phi(p_m)H_{\{\psi=\sum\psi_{m,i_1\dots i_n}\}}(B_1,\dots,B_n).$$

Since $\sum_m \eta\phi(p_m)=H_\phi(C)=H_\phi(A_1,\dots,A_n)$, we conclude that

$$H_{\phi\otimes\psi}(A_1\otimes B_1,\dots,A_n\otimes B_n)\leq H_\phi(A_1,\dots,A_n)+H_\psi(B_1,\dots,B_n),$$

completing the proof of the Theorem. □

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