Pacific Journal of Mathematics

A RIEMANN SINGULARITIES THEOREM FOR PRYM THETA DIVISORS, WITH APPLICATIONS

Roy Smith and Robert Varley

Volume 201 No. 2 December 2001

PACIFIC JOURNAL OF MATHEMATICS
Vol. 201, No. 2, 2001

A RIEMANN SINGULARITIES THEOREM FOR PRYM THETA DIVISORS, WITH APPLICATIONS

Roy Smith and Robert Varley

Let (P, Ξ) be the naturally polarized model of the Prym variety associated to the étale double cover $\pi : \widetilde{C} \to C$ of smooth connected curves, where $\Xi \subset P \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$, and $g(C) = g$. If L is any "nonexceptional" singularity of Ξ, i.e., a point L on $\Xi \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$ such that $h^0(\widetilde{C}, L) \geq 4$, but which cannot be expressed as $\pi^*(M)(B)$ for any line bundle M on C with $h^0(C, M) \geq 2$ and effective divisor $B \geq 0$ on $\widetilde{C}$, then we prove $\mathrm{mult}_L(\Xi) = (1/2)h^0(\widetilde{C}, L)$. We deduce that if C is nontetragonal of genus $g \geq 11$, then double points are dense in $\mathrm{sing}_{st}\Xi = \{L \text{ in } \Xi \subset \mathrm{Pic}^{2g-2}(\widetilde{C}) \text{ such that } h^0(\widetilde{C}, L) \geq 4\}$. Let $X = \widetilde{\alpha}^{-1}(P) \subset \mathrm{Nm}^{-1}(|\omega_C|)$ where $\mathrm{Nm} : \widetilde{C}^{(2g-2)} \to C^{(2g-2)}$ is the norm map on divisors induced by π, and $\widetilde{\alpha} : \widetilde{C}^{(2g-2)} \to \mathrm{Pic}^{2g-2}(\widetilde{C})$ is the Abel map for $\widetilde{C}$. If $h : X \to |\omega_C|$ is the restriction of Nm to X and $\varphi : X \to \Xi$ is the restriction of $\widetilde{\alpha}$ to X, and if $\dim(\mathrm{sing}\Xi) \leq g-6$, we identify the bundle $h^*(\mathcal{O}(1))$ defined by the norm map h, as the line bundle $\mathcal{T}_\varphi \otimes \varphi^*(K_\Xi)$ intrinsic to X, where $\mathcal{T}_\varphi$ is the bundle of "tangents along the fibers" of φ. Finally we give a proof of the Torelli theorem for cubic threefolds, using the Abel parametrization $\varphi : X \to \Xi$.

Introduction.

If C is a smooth curve of genus g, among the most basic tools for the study of the natural theta divisor $\Theta(C) \subset \mathrm{Pic}^{g-1}(C)$ of the Jacobian of C are Abel's and Riemann's theorems that describe the geometry of the "Abel" map $\alpha : C^{(g-1)} \to \Theta(C)$ parametrizing Θ by the symmetric product of the curve. They say the map α is birational, and that over a point L of multiplicity μ on Θ, the fiber $\alpha^{-1}(L) \cong |L| \cong \mathbb{P}^{\mu-1}$, is smooth and isomorphic to the complete linear system $|L|$, a projective space of dimension $\mu - 1$. The essential point here is that (one plus) the dimension of the fiber $\alpha^{-1}(L)$ computes the multiplicity of the point L on Θ. It follows also (see [**K**]) that the normal bundle to the fiber $\alpha^{-1}(L)$ in $C^{(g-1)}$ maps onto the tangent cone to Θ at L, and that there is a natural determinantal equation for the tangent cone to Θ at L.

In the case of the Prym variety of a connected étale double cover $\pi : \widetilde{C} \to C$ of a smooth curve C of genus g, the natural theta divisor $\Xi(C) = (P \cap \Theta(\widetilde{C}))_{\mathrm{red}} \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$ is parametrized by the restriction $\varphi : X \to \Xi$ of the Abel map $\widetilde{\alpha} : \widetilde{C}^{(2g-2)} \to \Theta(\widetilde{C})$ for $\widetilde{C}$, to the inverse image $\widetilde{\alpha}^{-1}(P) = X$ of the natural translate $P \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$ of the Prym variety of π (see Section 1 below for the precise definitions). Consequently there are two natural ways to study the theta divisor Ξ, either as the intersection $(P \cap \widetilde{\Theta})_{\mathrm{red}}$ or as the image of the Abel map $\varphi : X \to P$. Using the intersection representation $2\Xi = (P \cdot \widetilde{\Theta})$, Mumford in [**M1**, p. 343] gives a Pfaffian equation for the (projectivized) tangent cone $\mathbb{P}C_L\Xi$ of Ξ at a point L by restricting Kempf's equation for $\mathbb{P}C_L\widetilde{\Theta}$. This equation is valid only when the intersection $\mathbb{P}T_LP \cap \mathbb{P}C_L\widetilde{\Theta}$ is proper and hence equal as a set to $\mathbb{P}C_L\Xi$. Mumford gave a necessary and sufficient condition for the intersection $\mathbb{P}T_LP \cap \mathbb{P}C_L\widetilde{\Theta}$ to be proper, but only when $h^0(\widetilde{C}, L) = 2$. I.e., [**M1**, Prop., p. 343], when $h^0(\widetilde{C}, L) = 2$ the intersection $\mathbb{P}T_LP \cap \mathbb{P}C_L\widetilde{\Theta}$ is proper if and only if L is not of form $\pi^*(M)(B)$ for any line bundle M on C with $h^0(C, M) \geq 2$ and divisor $B \geq 0$ on $\widetilde{C}$. Combining the intersection representation with the Abel parametrization of Ξ, in the present paper we deduce (Theorem 2.1) that Mumford's condition for the intersection $\mathbb{P}T_LP \cap \mathbb{P}C_L\widetilde{\Theta}$ to be proper is sufficient without any hypothesis on $h^0(\widetilde{C}, L)$. We also give a counterexample (Example 2.18) with $h^0(\widetilde{C}, L) = 4$, to the necessity of the condition. The Abel parametrization $\varphi : X \to \Xi$ of the theta divisor for Pryms differs from that for Jacobians in that the fiber of the Abel map over a general point on a Prym theta divisor is isomorphic to $\mathbb{P}^1$ rather than $\mathbb{P}^0$, and also that the source space X of the Abel-Prym map is not always smooth. Thus there are two concepts of normal space to a fiber of φ, the Zariski normal space and the normal cone. We show in Corollary 2.8 that the intersection $\mathbb{P}T_LP \cap \mathbb{P}C_L\widetilde{\Theta}$ is the image of the union of the Zariski normal spaces in X at points of the fiber $\varphi^{-1}(L)$, and consequently whenever X is smooth along $\varphi^{-1}(L)$, then $\mathbb{P}T_LP \cap \mathbb{P}C_L\widetilde{\Theta}$ equals $\mathbb{P}C_L\Xi$ as sets. It follows that whenever X is smooth along $\varphi^{-1}(L)$, one can compute the multiplicity of Ξ at L, from the dimension of the fiber $\varphi^{-1}(L)$. I.e., then $\mathrm{mult}_L(\Xi) = (1/2)h^0(\widetilde{C}, L) = (1/2)(1 + \dim \varphi^{-1}(L))$. Finally the smoothness criterion of Beauville and Welters is used to show in Lemma 2.15 that X is singular precisely over "exceptional" singular points of Ξ, those called "case 1" by Mumford in [**M1**, p. 344]. (See Section 1.6 for the definition.) Consequently one can use this analog for Prym varieties of the Riemann singularities theorem (RST), to compute the multiplicity of Ξ at all nonexceptional singular points. In Theorem 3.2 and Corollary 3.3 we prove, by generalizing an argument of Welters, a criterion for the fiber $\varphi^{-1}(L)$ over a generic point L of a component of $\mathrm{sing}\Xi$ to be $\cong \mathbb{P}^3$. Combining this with a result of Debarre, we deduce that if C is nontetragonal of genus $g \geq 11$, and $\dim(P) = p = g-1$, then on every component Z of singΞ of dimension $\geq p-6$, double points of Ξ are dense, and at every double point L on Z, the quadric tangent cone $\mathbb{P}C_L\Xi$ contains the Prym canonical curve $\varphi_\eta(C)$. Since it is known that $\dim(\text{sing}\Xi) \geq p-6$, this adds further evidence at least when $g \geq 11$, for a "modified Donagi's conjecture", (see [**Do**, **D1**, **Ve**, **LS**] and Section 1.7 below). In particular, one can ask whether the Prym canonical model of a doubly covered nontetragonal curve C of genus $g \geq 11$ is the unique spanning curve in the base locus of those quadric tangent cones to Ξ at all double points of components Z of singΞ such that $\dim(Z) \geq p-6$. Since Debarre has shown that a general C with $g \geq 8$ can be recovered in this way, and since every Prym canonical model of a curve C with $g \geq 9$ and Clifford index ≥ 3 is determined by the quadrics containing it ([**LS**]), our density result brings the state of knowledge on this problem near that which was provided for Jacobians by the paper [**AM**] of Andreotti and Mayer. A primary problem remaining open is to prove, say for doubly covered nontetragonal curves C of genus $g \geq 11$, that the quadric tangent cones at stable double points generate the ideal of all quadrics containing $\varphi_\eta(C)$, an analog of Mark Green's theorem [**Gr**]. As a further application of the dimension estimate in Proposition 3.1 we deduce Corollary 3.5(i) a criterion for $\varphi^{-1}(\text{sing}\Xi)$ to have codimension ≥ 2 in X, and use this to prove (Theorem 4.2) an intrinsic formula for the line bundle defined by the norm map h on X. In Section 5 we apply the Riemann singularities theorem to a proof of the Torelli theorem for a cubic threefold W. The proof assumes the usual presentation of the intermediate Jacobian of W as the Prym variety for a conic bundle representation of W. The new feature is that it describes the geometry of Ξ via the Abel parametrization, which exists for all Prym varieties, rather than the parametrization via the Fano surface of W, which is somewhat peculiar to the cubic threefold. At the end of the paper we append an outline of the results.

1. Background on Prym varieties.

1.1. General conventions and notation. In this paper all curves considered are smooth, complete, connected, nonhyperelliptic, and defined over $\mathbb{C}$. (This last restriction seems irrelevant in Sections 2 and 4 where any algebraically closed field of characteristic $\neq 2$ should do, but in Section 3, Corollary 3.4, we use results of Debarre [**D1**] where the field is assumed to be $\mathbb{C}$, in Lemma 3.6 we use Bertini's theorem, and in Section 5, Lemma 5.5, we use a result of [**SV1**] which depends on the characteristic zero Kawamata Viehweg vanishing theorem.) The primary source for the definition and basic properties of Prym varieties is [**M1**]. References in textbook form are [**LB**] and [**ACGH**]. We also use the fundamental results of [**B1**], [**D1**], and [**We2**].

For any variety V and a point p on V, we denote by $\mathbb{P}C_pV$ the projectivized tangent cone of V at p, and by $\mathbb{P}T_pV$ the projectivized Zariski tangent space. If $S \subset V$ is a subvariety then $\mathbb{P}NC(S/V)$ denotes the projectivized normal cone of S in V.

1.2. The Prym variety (P, Ξ) of a double cover $\pi : \widetilde{C} \to C$. The fundamental object of study is a connected étale double cover $\pi : \widetilde{C} \to C$ of smooth curves, where if $g = g(C)$ is the genus of C, then $\widetilde{g} = g(\widetilde{C}) = 2g-1$. The map π induces a norm map $\mathrm{Nm} : \mathrm{Pic}^d(\widetilde{C}) \to \mathrm{Pic}^d(C)$ on line bundles for all d, and if $d = 0$, the Prym variety of $\pi : \widetilde{C} \to C$, denoted $P_0(\widetilde{C}/C)$ or simply P_0, is defined to be that connected component of $\mathrm{Nm}^{-1}(0) \subset \mathrm{Pic}^0(\widetilde{C})$ which contains 0. To obtain a polarization on P_0 consider the translate $P \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$ defined as $P = \{L \text{ in } \mathrm{Pic}^{2g-2}(\widetilde{C}) : \mathrm{Nm}(L) = \omega_C$, and $h^0(\widetilde{C}, L)$ is even$\}$. Then the reduced codimension one subvariety $\Xi = \{L \text{ in } P : h^0(L) > 0\} \subset P$ defines a principal polarization on P such that as divisors, $P \cdot \widetilde{\Theta} = 2\Xi$, where $\widetilde{\Theta} = \{L : \deg(L) = 2g-2 = \widetilde{g} - 1, h^0(\widetilde{C}, L) > 0\} \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$ is the canonical theta divisor on $\mathrm{Pic}^{\widetilde{g}-1}(\widetilde{C})$. The principally polarized Prym variety defined by π, is the pair (P_0, Ξ) where Ξ is given only up to translation, or (more often for us) the pair (P, Ξ) where the inclusion $\Xi \subset P$ is canonically defined. If $g = g(C)$ and P is the Prym variety of $\pi : \widetilde{C} \to C$, we denote the dimension of P by $p = \dim(P) = g - 1$.

1.3. The divisor variety X defined by $\widetilde{C} \to C$. The most important geometric tool for study of a Jacobian variety is the family of Abel maps. In particular for $\widetilde{C}$, the principal such map is the birational surjection $\widetilde{\alpha} : \widetilde{C}^{(2g-2)} \to \widetilde{\Theta} \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$, defined by $\widetilde{\alpha}(D) = \mathcal{O}(D)$. Since $\Xi = P \cap \widetilde{\Theta}$ as sets, it is natural to restrict this map over P; we denote the resulting map $\varphi : X \to \Xi \subset P$, the Abel parametrization of the Prym theta divisor Ξ, where $X = \widetilde{\alpha}^{-1}(P) \subset \widetilde{C}^{(2g-2)}$. The question of irreducibility and smoothness of X has been studied by Welters and Beauville in [**We2**] and [**B1**]. When C is nonhyperelliptic, X is a reduced, irreducible, normal, local complete intersection variety, in particular Cohen Macaulay. Moreover by [**M1**], φ is a $\mathbb{P}^1$ bundle over Ξ_{sm} = the subset of smooth points of Ξ, and over each point of Ξ the fiber of φ is isomorphic to some $\mathbb{P}^n$ with n odd. Indeed, if L is a point of $\Xi \subset \widetilde{\Theta} \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$, then L is a line bundle on $\widetilde{C}$ and $\varphi^{-1}(L) \cong |L|$, so $\dim \varphi^{-1}(L) = h^0(\widetilde{C}, L) - 1$, where by definition of Ξ, $h^0(\widetilde{C}, L)$ is even and positive. It is possible for the fiber dimension of φ to be one also over some "exceptional" singular points of Ξ.

1.4. The restricted norm map $h : X \to |\omega_C|$. In addition to the Abel map $\varphi : X \to \Xi$, the other important map on X is the restriction to X of the norm map $\mathrm{Nm} : \widetilde{C}^{(2g-2)} \to C^{(2g-2)}$ on divisors, denoted $h : X \to |\omega_C|$. Note that by definition of X, Nm maps X onto the canonical linear system

$|\omega_C|$ on C. Indeed X is defined as a scheme by Welters [**We2**] and Beauville [**B1**] as a connected component of the inverse image of $|\omega_C|$ under the norm map. Thus if $\alpha : C^{(2g-2)} \to \Theta(C)$ is the Abel map for C, since P and $|\omega_C|$ inherit their reduced scheme structures as components of the fibers $\mathrm{Nm}^{-1}(\omega_C)$ and $\alpha^{-1}(\omega_C)$ respectively, and since the compositions $\alpha \circ \mathrm{Nm}$ and $\mathrm{Nm} \circ \widetilde{\alpha}$ are equal, the scheme structure of X is induced either from $X \subset \mathrm{Nm}^{-1}(|\omega_C|) = (\alpha \circ \mathrm{Nm})^{-1}(\omega_C)$, as a connected component of the fiber over ω_C of the composition $\widetilde{C}^{(2g-2)} \to C^{(2g-2)} \to \mathrm{Pic}^{2g-2}(C)$, or from $X = \widetilde{\alpha}^{-1}(P) \subset (\mathrm{Nm} \circ \widetilde{\alpha})^{-1}(\omega_C)$, as a connected component of the fiber over ω_C of the composition $\widetilde{C}^{(2g-2)} \to \mathrm{Pic}^{2g-2}(\widetilde{C}) \to \mathrm{Pic}^{2g-2}(C)$. Thus to study X, one extracts from the diagram below:

$$\begin{array}{ccc} \widetilde{\alpha} : \widetilde{C}^{(2g-2)} & \longrightarrow & \mathrm{Pic}^{2g-2}(\widetilde{C}) \\ \big\downarrow \mathrm{Nm} & & \mathrm{Nm} \big\downarrow \\ \alpha : C^{(2g-2)} & \longrightarrow & \mathrm{Pic}^{2g-2}(C) \end{array}$$

the following diagram of subvarieties and restrictions:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \Xi \\ \big\downarrow h & & \big\downarrow \\ |\omega_C| & \longrightarrow & \{\omega_C\} \end{array}$$

The map $h : X \to |\omega_C| \cong \mathbb{P}^{g-1}$ is a finite surjection, hence defines an ample line bundle on X.

Definition 1.4.1. Denote by $\mathcal{O}_X(1)$ the line bundle $h^*(\mathcal{O}(1))$, where $\mathcal{O}(1)$ is the standard ample line bundle on the projective space $|\omega_C|$.

In Theorem 4.2 below we give a formula for the line bundle $\mathcal{O}_X(1)$, in terms of data intrinsically defined by X, at least for curves C with $\dim(\mathrm{sing}\Xi) \leq p-5$, i.e., those C not on Mumford's list in [**M1**, p. 344]. Now the canonical model of the curve C is the dual variety of the branch divisor of the map h and the curve $\widetilde{C}$ parametrizes the irreducible components $\mathcal{D}_p$ (see proof of Theorem 4.2 for the definition of the $\mathcal{D}_p$) of the divisors $h^*(H)$ for hyperplanes H tangent to the branch locus of h in $|\omega_C|$. [Using [**SV3**] for the irreducibility of the divisors $\mathcal{D}_p$, the arguments in [**SV4**, pp. 357, 360], generalize exactly]. Since the linear system defining h recovers $\pi : \widetilde{C} \to C$, it is of interest to know when it is complete, i.e., when $h^0(X, \mathcal{O}_X(1)) = g(C)$. We conjecture this is true when C is nonhyperelliptic, but this remains open for $g \geq 4$ (see [**SV4**, p. 359] when $g = 3$).

1.5. Prym canonical curves. The double cover $\pi : \widetilde{C} \to C$ defines a unique square-trivial line bundle η on C by $\ker(\pi^* : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(\widetilde{C})) = \{0, \eta\}$. The linear series $\omega_C \otimes \eta$ is base point free when C is nonhyperelliptic and the image $\varphi_\eta(C)$ of the associated projective map $\varphi_\eta : C \to |\omega_C \otimes \eta|^*$ is called the Prym canonical model of C. The line bundle $\omega_C \otimes \eta$ is very ample when C is nontetragonal and also when C is a generic tetragonal curve; see [**D1**] for a precise analysis of those tetragonal curves for which $\omega_C \otimes \eta$ is not very ample.

1.6. Stable and exceptional singularities. A point L of $\Xi \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$ will be called (cf. [**D1**], [**T1**]) a "stable singularity" of Ξ (with respect to the double cover $\pi : \widetilde{C} \to C$) if and only if $h^0(\widetilde{C}, L) \geq 4$, and an "exceptional singularity" of Ξ (again with respect to π) if and only if $L = \pi^*(M)(B)$, where M is a line bundle on C with $h^0(C, M) \geq 2$ and $B \geq 0$ is an effective divisor on $\widetilde{C}$. When a double cover $\widetilde{C} \to C$ representing (P, Ξ) is given or understood, the set of stable singularities is denoted $\mathrm{sing}_{\mathrm{st}}\Xi$, and the set of exceptional singularities is denoted $\mathrm{sing}_{\mathrm{ex}}\Xi$. According to [**M1**, p. 343], for every Prym representation of (P, Ξ), we have $\mathrm{sing}\Xi = \mathrm{sing}_{\mathrm{st}}\Xi \cup \mathrm{sing}_{\mathrm{ex}}\Xi$. Thus a Prym representation of (P, Ξ) defines a decomposition of $\mathrm{sing}\Xi$ into two generally overlapping subsets, since in particular any line bundle $L = \pi^*(M)(B)$ on Ξ, where M is a line bundle on C with $h^0(C, M) \geq 3$, is both stable and exceptional. For example, the unique singularity on the theta divisor of the intermediate Jacobian of a cubic threefold W (see Section 5 below), is both stable and exceptional, for any Prym representation associated to a general line on W. Debarre has shown in [**D1**] that Prym representations of the same abelian variety (P, Ξ) by different double covers of tetragonal curves, can lead to different decompositions of $\mathrm{sing}\Xi$ into stable and exceptional subsets. For $g(C) \geq 7$, i.e., for $p = \dim(P) \geq 6$, $\mathrm{sing}_{\mathrm{st}}\Xi$ is always nonempty and every irreducible component of $\mathrm{sing}_{st}\Xi$ has dimension $\geq p - 6$, [**D2**]. By [**M1**], on any Prym theta divisor Ξ, all components of $\mathrm{sing}\Xi$ of dimension $\geq p - 4$ lie entirely in $\mathrm{sing}_{\mathrm{ex}}(\Xi)$, but for a general curve of any genus $\mathrm{sing}_{\mathrm{ex}}(\Xi)$ is empty [see [**LB**, p. 389], and Prop. 2.19 below].

1.7. Donagi's conjecture. In [**Do**] Donagi made his famous "tetragonal conjecture", which implies that two smooth connected étale double covers $\widetilde{C}_1 \to C_1, \widetilde{C}_2 \to C_2$ of nontetragonal curves C_i, are isomorphic as double covers if and only if they define isomorphic polarized Prym varieties (P_i, Ξ_i). Verra found in [**Ve**] a lovely counterexample where C_i are generic smooth plane sextics (hence of genus 10). He noted that plane sextics are the only curves with the same Clifford index as tetragonal curves and suggested that consequently these may be the only counterexamples. The conjecture must then be modified [cf. [**LS**]] to assume at least that $\mathrm{Cliff}(C_i) \geq 3$. One approach to proving the modified Donagi's conjecture, analogous to Green's result in [**Gr**] which refines Andreotti Mayer's approach for Jacobians, is to try to show that a Prym canonical model $\varphi_\eta(C)$ of a nontetragonal curve is determined by the base locus of the quadric tangent cones at appropriately determined double points of Ξ. This approach has several complications.

First only the "stable" double points on Ξ have tangent cones which always contain $\varphi_\eta(C)$. Secondly since the subset $\mathrm{sing}_{\mathrm{st}}\Xi$ depends on the double cover, one does not prove the conjecture simply by showing that $\varphi_\eta(C)$ is determined by the base locus of tangent cones to $\mathrm{sing}_{\mathrm{st}}\Xi$. For example, although there are generally three doubly covered tetragonal curves with the same Prym variety, it is entirely possible that each double cover is determined by the tangent cones to Ξ at those double points which are stable for that double cover. However, there are good reasons to believe this approach will eventually succeed.

Debarre shows in [**D1**] that for Prym varieties of doubly covered nontetragonal curves C of genus $g \geq 11$, the locus $\mathrm{sing}_{\mathrm{st}}\Xi$ is intrinsically defined by Ξ, independently of which double cover is considered to represent (P, Ξ). In particular then $\mathrm{sing}_{\mathrm{st}}\Xi$ is the union of all irreducible components of $\mathrm{sing}\Xi$ having dimension $\geq p-6$, where $p = \dim(P)$. In [**LS**] it is shown using results of Green and Lazarsfeld that for all doubly covered curves C with $\mathrm{Cliff}(C) \geq 3$ and $g(C) \geq 9$, that $\varphi_\eta(C)$ is determined by the base locus of the quadrics containing it. Debarre shows in [**D2**] for $g \geq 7$, and C general, that the quadric tangent cones to Ξ at its double points (all of which are stable when C is general), generate the ideal of quadrics containing $\varphi_\eta(C)$. The prerequisite existence and density result for stable double points on generic Ξ follows from Welters' "generic Riemann singularities theorem" for Prym varieties in [**We1**]. In the present paper we provide another step in this approach to Donagi's conjecture, by proving a precise Riemann singularities theorem for Prym varieties, Theorem 2.1 below, and deducing in Corollary 3.4 that double points are dense in $\mathrm{sing}_{\mathrm{st}}\Xi$, for every doubly covered nontetragonal curve C of genus $g \geq 11$. In Corollary 2.22 below we also deduce Welters' generic RST for Prym varieties from the precise version in Theorem 2.1.

2. A Riemann singularities theorem for Prym varieties.

In this section, assume $\widetilde{C} \to C$ is an étale connected double cover of a smooth nonhyperelliptic curve C, $\Xi \subset \widetilde{\Theta} \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$ is the natural model $\Xi = (\widetilde{\Theta} \cap P)_{\mathrm{red}}$ for the theta divisor of the associated Prym variety $P \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$, and L is a point of Ξ. Thus L is a line bundle on $\widetilde{C}$ with a positive even number of global sections, and $\mathrm{Nm}(L) = \omega_C$. In particular $(1/2)h^0(\widetilde{C}, L)$ is an integer. Since as divisors $2\Xi = (\widetilde{\Theta} \cdot P)$, it follows that $\mathrm{mult}_L\Xi \geq (1/2)\mathrm{mult}_L\widetilde{\Theta} = (1/2)h^0(\widetilde{C}, L)$, and $(1/2)h^0(\widetilde{C}, L)$ is the "expected" multiplicity of Ξ at L. Our goal is a simple criterion for $\mathrm{mult}_L\Xi$ to equal this expected multiplicity.

2.0. Terminology. If $\text{mult}_L\Xi = (1/2)h^0(\widetilde{C}, L)$, we say the "Riemann singularities theorem" (RST) holds at L. (In that case and that case only, the Pfaffian described by Mumford in [**M1**, p. 343], the square root of the restriction of Kempf's equation for $\mathbb{P}C_L\widetilde{\Theta}$, gives an equation for the tangent cone $\mathbb{P}C_L\Xi$ to the Prym theta divisor.) Recall that L is an "exceptional singularity" of Ξ if and only if it falls in case 1 of Mumford's description [**M1**, p. 344] of singularities of Ξ, i.e., if and only if L lies in Ξ and $L = \pi^*(M)(B)$, where M is a line bundle on C with $h^0(C, M) \geq 2$ and $B \geq 0$ is an effective divisor on $\widetilde{C}$.

Theorem 2.1. *Assume $\widetilde{C} \to C$ is an étale connected double cover of a smooth nonhyperelliptic curve C of genus $g \geq 3$, and L a point of $\Xi \subset P \subset \text{Pic}^{2g-2}(\widetilde{C})$. If L is not an exceptional singularity of Ξ, then* $\text{mult}_L\Xi = (1/2)h^0(\widetilde{C}, L)$*; i.e., RST holds at every nonexceptional L on Ξ.*

Corollary 2.2. *If C is not tetragonal, $g = g(C) \geq 11$, and $\widetilde{C} \to C$ is any étale connected double cover, then RST holds at a general point L of every component of the locus* $\text{sing}_{\text{st}}\Xi$ *of stable singularities of Ξ, i.e., at a general point of every component of Ξ of dimension $\geq g - 7$.*

Remarks 2.3. (i) Mumford [**M1**, p. 343] originally proved Theorem 2.1 and its converse when $h^0(\widetilde{C}, L) = 2$. In [**Sh**, Lemma 5.7, p. 121] Shokurov generalizes this argument to give a sufficient criterion for the RST condition to fail as follows: If $H^0(\widetilde{C}, L)$ contains a subspace of dimension greater than $(1/2)h^0(\widetilde{C}, L)$ which is isotropic for the form $\langle s, t\rangle = s \otimes \iota^*(t) - t \otimes \iota^*(s)$ [**M1**, p. 343], where $\iota : \widetilde{C} \to \widetilde{C}$ is the involution associated to the double cover $\pi : \widetilde{C} \to C$, then $\text{mult}_L\Xi > (1/2)h^0(\widetilde{C}, L)$. He applies this to show if C is a general bielliptic curve, then Ξ has too many triple points for (P, Ξ) to be the Jacobian of a curve.

(ii) The converse of Theorem 2.1 can fail when $h^0(\widetilde{C}, L) > 2$, as we will show below by giving an example of an exceptional singularity L at which the RST does hold, i.e., one with $\text{mult}_L\Xi = (1/2)h^0(\widetilde{C}, L)$.

(iii) Theorem 2.1 implies (Corollary 2.22 below) Welters' theorem [**We1**] that RST holds at every point of Ξ when C is a general curve, using only the classical Gieseker Petri theorem [**G**], [**ACGH**, p. 215], which implies that there are no exceptional singularities on Ξ when C is general.

(iv) We will apply Corollary 2.2 to prove (in Corollary 3.4 below) that if C is nontetragonal and $g \geq 11$, then a general point L of any component of the locus $\text{sing}_{\text{st}}\Xi$ of stable singularities of Ξ is a double point. This should be a fundamental initial step in any attempt to generalize the method of Andreotti-Mayer [**AM**] and Green [**Gr**] to prove a suitable form of the conjecture of Donagi [**Do**, **Ve**, **LS**], e.g., that the Prym map is injective on the set of doubly covered smooth nontetragonal curves with $g(C) = g \geq 11$.

(v) Theorem 2.1 is true also when the curve C is hyperelliptic. In fact if $g(C) \leq 5$, or if C is either hyperelliptic or trigonal, the theorem is immediate since then by [**M1**, **R**, **AM**] either $\dim P = p \leq 4$, or (P, Ξ) is a Jacobian, so every component of $\mathrm{sing}\Xi$ has dimension $\geq p-4$. Then by [**M1**, Lemma, p. 345] the only nonexceptional points of Ξ are smooth points, and at smooth points the conclusion of the theorem follows immediately from the equation $2\Xi = \widetilde{\Theta}\cdot P$, [**M1**, Cor., p. 342]. Pryms of generic doubly covered plane quintic curves are either Jacobians of genus five curves or intermediate Jacobians of cubic threefolds, and then all singularities on Ξ are exceptional as well.

Proof of Corollary 2.2. Debarre has shown [**D1**, Th. 3.1(i), p. 548] that with the hypotheses of Corollary 2.2 every component of $\mathrm{sing}_{\mathrm{ex}}\Xi$ has lower dimension than any component of $\mathrm{sing}_{\mathrm{st}}\Xi$, so Corollary 2.2 follows immediately from Theorem 2.1. □

Proof of Theorem 2.1. The first observation is that the problem is purely set theoretic. □

Lemma 2.4. *With the hypotheses of Theorem* 2.1*, the following statements are equivalent.*

(i) $\mathrm{mult}_L\Xi = (1/2)h^0(\widetilde{C}, L)$.
(ii) $\mathbb{P}C_L\Xi = \mathbb{P}C_L\widetilde{\Theta} \cap \mathbb{P}T_LP$ *as sets.*
(iii) $2[\mathbb{P}C_L\Xi] = [\mathbb{P}C_L\widetilde{\Theta}]\cdot[\mathbb{P}T_LP]$ *as cycles.*
(iv) $\mathbb{P}C_L\widetilde{\Theta} \not\supset \mathbb{P}T_LP$.

Sketch of Proof. If $\widetilde{\vartheta}$ is a Taylor series at L for the theta function of $\widetilde{\Theta}$, then $\widetilde{\vartheta}$ restricts on T_LP to ξ^2, the square of the Taylor expansion at L of a theta function for Ξ. Consequently, the lowest order term of $\widetilde{\vartheta}$ which does not vanish identically on T_LP equals the square of the lowest nonvanishing term of ξ, i.e., equals the square of an equation for the tangent cone of Ξ at L. In particular the leading term $\widetilde{\vartheta}_h$ of $\widetilde{\vartheta}$ defines $\mathbb{P}C_L\Xi$ as a set if and only if $\widetilde{\vartheta}_h$ does not vanish identically on T_LP. I.e., if $\widetilde{\vartheta}_h$ is the lowest nonvanishing term of $\widetilde{\vartheta}$ on $T_L\mathrm{Pic}^{2g-2}(\widetilde{C})$, hence an equation for the tangent cone of $\widetilde{\Theta}$ at L, then $\widetilde{\vartheta}|T_LP = (\widetilde{\vartheta}_h + \cdots) = \xi^2 = (\xi_{h/2} + \cdots)^2$. Hence $\mathbb{P}TC_L\widetilde{\Theta} \not\supset \mathbb{P}T_LP$ iff $\widetilde{\vartheta}_h|T_LP = (\xi_{h/2})^2$ is not identically zero, iff $\widetilde{\vartheta}_h|T_LP = (\xi_{h/2})^2$ is the square of an equation for the tangent cone of Ξ, iff $\xi_{h/2}$ is the first nonvanishing term of ξ, iff $\mathrm{mult}_L\Xi = h/2$. Since by the classical Riemann singularities theorem for $\widetilde{\Theta}$ we have $h = h^0(\widetilde{C}, L)$, the lemma follows. □

We can now summarize the proof of Theorem 2.1 as follows: If $\varphi : X \to \Xi$ is the Abel parametrization of the Prym theta divisor by the special variety X of divisors on $\widetilde{C}$, we show first in Corollary 2.9 that $\mathbb{P}C_L\Xi = \mathbb{P}C_L\widetilde{\Theta}\cap\mathbb{P}T_LP$ holds as sets whenever X is smooth at every point of the fiber $\varphi^{-1}(L)$. Then we complete the proof by showing in Lemma 2.15 that X is smooth along $\varphi^{-1}(L)$ if and only if L is not an exceptional singularity of Ξ.

Tangent spaces to the divisor variety $X \subset \widetilde{C}^{(2g-2)}$. Recall that for each point L of P, the inclusion $P \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$ induces an inclusion of projective tangent spaces $|\omega_C \otimes \eta|^* = \mathbb{P}T_L P \subset \mathbb{P}T_L \mathrm{Pic}^{2g-2}(\widetilde{C}) = |\omega_{\widetilde{C}}|^*$, and that we want to determine the intersection in $|\omega_{\widetilde{C}}|^*$ of the subspace $\mathbb{P}T_L P$ with the projectivized tangent cone $\mathbb{P}C_L\widetilde{\Theta}$. From the Riemann Kempf singularities theorem [**K**, Thm. 1, p. 178] the cone $\mathbb{P}C_L\widetilde{\Theta}$ is the union of the projectivized images, under the differential of the Abel map, of the tangent spaces to the symmetric product $\widetilde{C}^{(2g-2)}$ at all points $\widetilde{D}$ of the fiber $|L| = \varphi^{-1}(L) \subset \widetilde{C}^{(2g-2)}$. I.e., $\mathbb{P}C_L\widetilde{\Theta} = \cup \mathbb{P}\widetilde{\alpha}_*(T_{\widetilde{D}}\widetilde{C}^{(2g-2)})$, where the union is taken over all $\widetilde{D}$ in $|L|$. To apply Kempf's argument to X we need to describe the Zariski tangent space to X at a point $\widetilde{D}$ of $|L|$.

Lemma 2.5. *If $\widetilde{C} \to C$ is any smooth connected étale double cover, C nonhyperelliptic, $X \subset \widetilde{C}^{(2g-2)}$ is the special variety of divisors on $\widetilde{C}$, and $\widetilde{\alpha}_*$ is the differential of the Abel map $\widetilde{\alpha} : \widetilde{C}^{(2g-2)} \to \mathrm{Pic}^{2g-2}(\widetilde{C})$ for $\widetilde{C}$, then the Zariski tangent space to X at $\widetilde{D}$ is given by:*

(2.5.1) $$T_{\widetilde{D}}X = (\widetilde{\alpha}_{*,\widetilde{D}})^{-1}(T_L P).$$

Proof. The scheme structure of X may be defined by pulling back that of P, $X = \widetilde{\alpha}^{-1}(P)$. Thus the Zariski tangent space to X is also a pull back from that of P, i.e., $T_{\widetilde{D}}X = (\widetilde{\alpha}_*)^{-1}(T_L P)$. □

Since the cone $\mathbb{P}C_L\widetilde{\Theta}$ is ruled by the image spaces $\mathbb{P}\widetilde{\alpha}_*(T_{\widetilde{D}}\widetilde{C}^{(2g-2)}) = \langle\widetilde{D}\rangle =$ the span of the divisor $\widetilde{D}$ in the canonical space $|\omega_{\widetilde{C}}|^*$ of the curve $\widetilde{C}$, in order to intersect $\mathbb{P}T_L P$ with $\mathbb{P}C_L\widetilde{\Theta}$, a natural first step is to intersect $\mathbb{P}T_L P$ with each ruling $\langle\widetilde{D}\rangle$.

Lemma 2.6. *The intersection $\mathbb{P}T_L P \cap \langle\widetilde{D}\rangle$ equals the projectivized image, in $\mathbb{P}T_L P \cong |\omega\otimes\eta|^*$, of the Zariski tangent space $T_{\widetilde{D}}X$ under the derivative $\varphi_{*,\widetilde{D}}$ of the restricted Abel map $\varphi : X \to \Xi$. I.e., $\mathbb{P}\varphi_*(T_{\widetilde{D}}X) = ((\mathbb{P}T_L P) \cap \langle\widetilde{D}\rangle) \subset (\mathbb{P}T_L P \cap \mathbb{P}C_L\widetilde{\Theta})$.*

Proof. Since the map $\widetilde{\alpha}_{*,\widetilde{D}} : \mathbb{P}T_{\widetilde{D}}\widetilde{C}^{(2g-2)} \to \langle\widetilde{D}\rangle$ is surjective by the Riemann Kempf theorem (see also [**MM**]), its restriction to $(\widetilde{\alpha}_{*,\widetilde{D}})^{-1}(\langle\widetilde{D}\rangle \cap \mathbb{P}T_L P) = (\widetilde{\alpha}_{*,\widetilde{D}})^{-1}(\mathbb{P}T_L P) = \mathbb{P}T_{\widetilde{D}}X$ (by (2.5.1)), surjects onto $\langle\widetilde{D}\rangle \cap \mathbb{P}T_L P$. Since $\varphi_{*,\widetilde{D}}$ is the restriction to $T_{\widetilde{D}}X$ of $\widetilde{\alpha}_{*,\widetilde{D}}$, thus $\mathbb{P}\varphi_*(T_{\widetilde{D}}X) = \mathbb{P}T_L P \cap \langle\widetilde{D}\rangle$ as claimed. □

Corollary 2.7. *For any point $\widetilde{D}$ of X, $\dim(\mathbb{P}T_L P \cap \langle\widetilde{D}\rangle) = \dim(\mathbb{P}\widetilde{\alpha}_*(T_{\widetilde{D}}X))$ $= \dim T_{\widetilde{D}}X - \dim|\widetilde{D}| - 1$.*

Proof. This follows from the rank formula for a linear map. I.e., the linear map $\varphi_{*,\widetilde{D}}$ has domain $= T_{\widetilde{D}}X$, projectivized image $= \mathbb{P}T_LP \cap \langle\widetilde{D}\rangle$, and we claim the kernel equals $T_{\widetilde{D}}|\widetilde{D}|$. Indeed, since all fibers of the abel map $\widetilde{\alpha}$ on $\widetilde{C}^{(2g-2)}$ are nonsingular, and $\widetilde{\alpha}^{-1}(\mathcal{O}(\widetilde{D})) = |\widetilde{D}|$, the kernel of $\widetilde{\alpha}_{*,\widetilde{D}}$ equals $T_{\widetilde{D}}|\widetilde{D}|$ which has the same dimension as $|\widetilde{D}|$, and since $|\widetilde{D}| \subset X$, we also have kernel $(\varphi_{*,\widetilde{D}}) = T_{\widetilde{D}}|\widetilde{D}|$. □

Corollary 2.8. *As sets, the intersection $\mathbb{P}C_L\widetilde{\Theta} \cap \mathbb{P}T_LP$ equals the union of the images $\mathbb{P}\varphi_*(T_{\widetilde{D}}X)$ of all the Zariski tangent spaces to X at points $\widetilde{D}$ of $|L| = \varphi^{-1}(L)$.*

Proof. This is immediate from Lemma 2.6 and the Riemann Kempf theorem. □

Thus to determine when the intersection $\mathbb{P}C_L\widetilde{\Theta}\cap\mathbb{P}T_LP$ equals the tangent cone $\mathbb{P}C_L\Xi$ as sets, we only need to determine when that tangent cone is the set theoretic image of the Zariski tangent spaces along the fiber $\varphi^{-1}(L)$. For a proper map between smooth varieties, if the scheme theoretic fiber over a point L of the target variety is also smooth, then the normal bundle to the fiber surjects onto the tangent cone to the image variety at L, [**K**, Lemma p. 179], [**MM**, p. 230]. Since the scheme theoretic fibers of φ are equal to the corresponding fibers of the Abel map $\widetilde{\alpha}$, they are always smooth, and we get the following abstract version of the RST for Prym varieties.

Corollary 2.9. *With the hypotheses of Theorem 2.1, if X is smooth at every point of the fiber $\varphi^{-1}(L)$, then $\mathbb{P}C_L\Xi = \mathbb{P}C_L\widetilde{\Theta} \cap \mathbb{P}T_LP$ as sets.*

Proof. The projective tangent cone $\mathbb{P}C_L\Xi$ is the exceptional fiber over L of the blowup of Ξ at L, and the projective normal cone in X to $\varphi^{-1}(L) = |L|$ is the exceptional fiber of the blowup of X along $|L|$. Since $\Xi = \varphi(X)$ and $\varphi : X \to \Xi \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$ is proper, the map induced by φ on these blowups is surjective. In particular the exceptional fiber over $|L|$ surjects onto the exceptional fiber over L, i.e., the projective normal cone in X to $|L|$ surjects onto the projective tangent cone $\mathbb{P}C_L\Xi$, whether X is smooth or not. Since the scheme theoretic fibers $\varphi^{-1}(L) = \widetilde{\alpha}^{-1}(L)$ are equal, and the fibers $\widetilde{\alpha}^{-1}(L)$ are always smooth by the Mattuck Mayer version of the Riemann Roch theorem [**MM**], the fibers $\varphi^{-1}(L)$ are also smooth. Thus whenever X is smooth at $\widetilde{D}$, the projective normal space $\mathbb{P}N_{\widetilde{D}}(|L|/X)$ is a fiber of the projective normal cone $\mathbb{P}NC(|L|/X)$ and the induced map is defined there by the derivative $\widetilde{\alpha}_*$. Since the tangent space $T_{\widetilde{D}}X$ and the normal space $N_{\widetilde{D}}(|L|/X) = T_{\widetilde{D}}X/T_{\widetilde{D}}(|L|) = T_{\widetilde{D}}X/\ker\widetilde{\alpha}_*$, have the same image under $\widetilde{\alpha}_*$, Corollary 2.9 follows from Corollary 2.8. □

We deduce the following abstract version of Mumford's result:

Corollary 2.10. *If L is a point of Ξ such that $h^0(\widetilde{C}, L) = 2$, then Ξ is singular at L iff the RST theorem fails for Ξ at L, iff $L = \mathcal{O}(\widetilde{D}) = \varphi(\widetilde{D})$ is the image of some singular point $\widetilde{D}$ of X.*

Proof. If $h^0(\widetilde{C}, L) = 2$, then the RST holds at L iff $\mathrm{mult}_L \Xi = 1$, iff Ξ is smooth at L. If X is smooth at every point $\widetilde{D}$ in $\varphi^{-1}(L)$, then Corollary 2.9 implies the RST holds at L. If X is singular at $\widetilde{D}$, since $\dim X = g - 1$, then $\dim(T_{\widetilde{D}}X) \geq \dim(X) + 1 \geq g$. If $L = \mathcal{O}(\widetilde{D})$ and $h^0(\widetilde{C}, L) = 2$, then $\dim |\widetilde{D}| = 1 = \dim(\ker(\varphi_*))$, hence by Corollary 2.7 $\dim \mathbb{P}\varphi_*(T_{\widetilde{D}}X) \geq g-2 = p-1 = \dim \mathbb{P}T_L P$. Since by Lemma 2.6, $\mathbb{P}\varphi_*(T_{\widetilde{D}}X) = \mathbb{P}T_L P \cap \langle \widetilde{D} \rangle$, it follows that $\dim \mathbb{P}T_L P = \dim(\mathbb{P}T_L P \cap \langle \widetilde{D} \rangle)$. Thus $\mathbb{P}T_L P = (\mathbb{P}T_L P \cap \langle \widetilde{D} \rangle) \subset (\mathbb{P}T_L P \cap \mathbb{P}C_L \widetilde{\Theta}) \subset \mathbb{P}C_L \widetilde{\Theta}$. Hence RST fails at $L = \mathcal{O}(\widetilde{D})$, by Lemma 2.4 (iv). □

Detecting singularities of X. To complete the proof of Theorem 2.1 we will relate the smoothness of X to the existence of exceptional singularities on Ξ, in particular we show that L is an exceptional singularity of Ξ if and only if X is singular at some point $\widetilde{D}$ of $\varphi^{-1}(L)$. We will use formula (2.5.1) for the tangent space to X to deduce a smoothness criterion for X, and then relate it to Beauville's formulation of Welters' criterion. First of all, to measure when X is singular at $\widetilde{D}$ we need to compute the dimension of the tangent space $T_{\widetilde{D}}X$. Denote $H^0(C, \omega_C)$ by Ω_C and $H^0(\widetilde{C}, \omega_{\widetilde{C}})$ by $\Omega_{\widetilde{C}}$.

Lemma 2.11. *For any point $\widetilde{D}$ of X, $\dim T_{\widetilde{D}}X = g - 2 + \dim\{\omega$ in Ω_C such that $(\pi^*(\omega)) \geq \widetilde{D}\}$.*

Proof. By formula (2.5.1) $T_{\widetilde{D}}X = (\widetilde{\alpha}_{*,\widetilde{D}})^{-1}(T_L P)$ is the tangent space to X at $\widetilde{D}$. Hence $T_{\widetilde{D}}X$ is defined as a subspace of $T_{\widetilde{D}}\widetilde{C}^{(2g-2)}$ by the pullback of those linear equations in $T_L^*\mathrm{Pic}^{2g-2}(\widetilde{C})$ which vanish on $T_L P$ where $L = \mathcal{O}(\widetilde{D})$. Now $T_L^*\mathrm{Pic}^{2g-2}(\widetilde{C}) \cong \Omega_{\widetilde{C}}$ and the subspace of equations vanishing on $T_L P$ corresponds to the subspace $\pi^*(\Omega_C) = (\Omega_{\widetilde{C}})^+ \subset \Omega_{\widetilde{C}}$. Hence the codimension of $T_{\widetilde{D}}X$ in $T_{\widetilde{D}}\widetilde{C}^{(2g-2)}$ equals the number of equations in $\pi^*(\Omega_C)$ minus the number which pull back trivially to $T_{\widetilde{D}}(\widetilde{C}^{(2g-2)})$, i.e., it equals $g - \dim\{\omega$ in $\Omega_C : \pi^*(\omega)$ vanishes on $\widetilde{\alpha}_* T_{\widetilde{D}}(\widetilde{C}^{(2g-2)})\}$. Since $\mathbb{P}\widetilde{\alpha}_* T_{\widetilde{D}}(\widetilde{C}^{(2g-2)}) = \langle \widetilde{D} \rangle$ = the span of the divisor $\widetilde{D}$ on the canonical model of $\widetilde{C}$ in $\mathbb{P}^*(\Omega_{\widetilde{C}}) \cong \mathbb{P}T_L \mathrm{Pic}^{2g-2}(\widetilde{C})$, the linear form $\pi^*(\omega)$ vanishes on $\widetilde{\alpha}_* T_{\widetilde{D}}(\widetilde{C}^{(2g-2)})$ if and only if $(\pi^*(\omega)) \geq \widetilde{D}$. The following sequence thus summarizes the calculation.

(2.11.1)

$$0 \to \{\omega \text{ in } \Omega_C : (\pi^*(\omega)) \geq \widetilde{D}\} \to (\Omega_{\widetilde{C}})^+ \to T^*_{\widetilde{D}}(\widetilde{C}^{(2g-2)}) \to T^*_{\widetilde{D}}(X) \to 0.$$

Thus $\dim T_{\widetilde{D}}(X) = \dim T^*_{\widetilde{D}}(X) = (2g-2) - g + \dim\{\omega$ in Ω_C such that $(\pi^*(\omega)) \geq \widetilde{D}\} = g - 2 + \dim\{\omega$ in Ω_C such that $(\pi^*(\omega)) \geq \widetilde{D}\}$. □

Corollary 2.12 (smoothness criterion)**.** *X is smooth at $\widetilde{D}$ if and only if the only differentials ω on C such that $\pi^*(\omega)$ vanishes on $\widetilde{D}$ are the multiples of $\omega_0 =$ the differential vanishing on $\mathrm{Nm}(\widetilde{D}) = D_0$.*

Proof. X is smooth iff $\dim T_{\widetilde{D}}X = \dim X = g-1$, and by Lemma 2.11, this is equivalent to $\dim\{\omega \text{ in } \Omega_C \text{ such that } (\pi^*(\omega)) \geq \widetilde{D}\} = 1$. □

Next we relate this to Beauville's formulation [**B1**] of Welters' criterion [**We2**] for smoothness of X at $\widetilde{D}$.

Lemma 2.13. *X is singular at $\widetilde{D}$ iff there exists an effective divisor $A \geq 0$ on C such that $h^0(C, A) \geq 2$ and $\pi^*(A) \leq \widetilde{D}$.*

Proof. Let $\widetilde{D} = p_1 + p_1' + \cdots + p_r + p_r' + q_1 + \cdots + q_s$, where each pair $\{p_i, p_i'\}$ is a conjugate pair, and the set $p_1, \ldots, p_r, q_1, \ldots, q_s$ contains no conjugate pairs. Then for any divisor A on C, $\pi^*(A) \leq \widetilde{D}$ iff $A \leq \overline{p}_1 + \cdots + \overline{p}_r$, where $\overline{p} = \pi(p)$. Moreover for any differential ω on C, $\pi^*(\omega) \geq \widetilde{D}$ iff $(\omega) \geq \overline{p}_1 + \cdots + \overline{p}_r + \overline{q}_1 + \cdots + \overline{q}_s$. Now if X is singular at $\widetilde{D}$, by Corollary 2.12 there are two independent differentials ω_1, ω_2 on C such that $\pi^*(\omega_i) \geq \widetilde{D}$, and if we define $A = \overline{p}_1 + \cdots + \overline{p}_r$, and $B = \overline{q}_1 \cdots + + \overline{q}_s$, then for $i = 1, 2$ we have $(\omega_i) \geq A + B$, hence $h^0(K - A - B) \geq 2$. Since also $\pi_*(\widetilde{D}) = 2A + B$ is a canonical divisor, $h^0(A) = h^0(K - A - B) \geq 2$, and $\pi^*(A) \leq \widetilde{D}$, so that the Beauville - Welters criterion is satisfied. Conversely, if there is an $A \geq 0$ such that $h^0(A) \geq 2$ and $\pi^*(A) \leq \widetilde{D}$, then since $A \leq \overline{p}_1 + \cdots + \overline{p}_r$, the same two properties hold for $\overline{p}_1 + \cdots + \overline{p}_r$, so we may as well assume $A = \overline{p}_1 + \cdots + \overline{p}_r$. Then again, $h^0(K - A - B) = h^0(A) \geq 2$, so there are at least 2 independent differentials ω such that $(\omega) \geq A + B$, and hence such that $\pi^*(\omega) \geq \widetilde{D}$, whence by Corollary 2.12 X is singular at $\widetilde{D}$. □

This yields the following alternate dimension formula for $T_{\widetilde{D}}X$.

Corollary 2.14. *At any point $\widetilde{D}$ of X, if $A \geq 0$ is the largest effective divisor on C such that $\pi^*(A) \leq \widetilde{D}$, then $\dim T_{\widetilde{D}}X = g - 2 + h^0(A)$.*

Proof. In the notation of the previous proof, Serre duality yields $h^0(A) = h^0(K - A - B) = \dim\{\omega \text{ in } \Omega_C \text{ such that } (\pi^*(\omega)) \geq \widetilde{D}\}$. □

The usefulness of the B-W formulation of singularity of X at $\widetilde{D}$, is its close connection with the concept of exceptional singularities.

Lemma 2.15. *A point L on Ξ is an exceptional singularity iff $L = \mathcal{O}(\widetilde{D}) = \varphi(\widetilde{D})$ for some singular point $\widetilde{D}$ on X. I.e., $\mathrm{sing}_{\mathrm{ex}}(\Xi) = \varphi(\mathrm{sing}X)$.*

Proof. If $L = \mathcal{O}(\widetilde{D})$ for some $\widetilde{D}$ at which X is singular, then by Lemma 2.13, $\widetilde{D} = \pi^*(A)(B)$ where $h^0(A) \geq 2$ and $B \geq 0$, and then taking $M = \mathcal{O}(A)$, L is exceptional. Conversely, if $L = \pi^*(M)(B)$ for some line bundle M on C

with $h^0(M) \geq 2$ and $B \geq 0$, and if A is any divisor in $|M|$, then $\pi^*(A)(B) = \widetilde{D}$ belongs to $|L| = \varphi^{-1}(L) \subset X$, and X is singular at $\widetilde{D} \geq \pi^*(A)$, by Lemma 2.13. □

Now Corollary 2.9, Lemma 2.4 and Lemma 2.15 imply Theorem 2.1.

Corollary 2.16. *If L is a point of Ξ which is not an exceptional singularity, then the RST holds at L, i.e., $\mathbb{P}C_L\Xi = \mathbb{P}C_L\widetilde{\Theta} \cap \mathbb{P}T_LP$ as sets, and hence* $\mathrm{mult}_L\Xi = (1/2)h^0(\widetilde{C}, L)$.

Proof. If L is a point of Ξ which is not an exceptional singularity, then X is smooth at every point $\widetilde{D}$ of $\varphi^{-1}(L)$, so RST holds at L by Corollary 2.9. and Lemma 2.4. □

In particular we recover Mumford's result in its original form.

Corollary 2.17. *If L is a point of Ξ such that $h^0(\widetilde{C}, L) = 2$, then Ξ is singular at L iff L is an exceptional singularity.* (*In particular, "exceptional singularities" are really singular points of* Ξ.)

Proof. This follows from Corollary 2.10 and Lemma 2.15. □

The converse of Corollary 2.16 can fail when $h^0(\widetilde{C}, L) \geq 4$.

Example 2.18. Let C be a nonhyperelliptic genus 5 curve with two vanishing even theta nulls M_1, M_2 with $h^0(M_i) = 2$, and let the line bundle η associated to the double cover be defined by their difference $\eta = M_1 - M_2$. Then $L = \pi^*(M_1)$ implies $\mathrm{Nm}(L) = 2M_1 = \omega_C$, and $h^0(L) = h^0(M_1) + h^0(M_1 + \eta) = h^0(M_1) + h^0(M_2) = 4$, so L is a stable and exceptional singularity on Ξ. However by [**V**, p. 948, ll. 1-3], L is then a vanishing even theta null on (P, Ξ) so $\mathrm{mult}_L\Xi =$ either 2 or 4. Since C is nonhyperelliptic, hence by [**M1**, p. 344] singΞ is zero dimensional, P is indecomposable so $\mathrm{mult}_L\Xi \leq 3$ by [**SV1**, p. 319]. Hence $\mathrm{mult}_L\Xi = 2 = h^0(L)/2$, and L is both a stable and exceptional double point on Ξ at which RST holds.

Gieseker's theorem and exceptional singularities. We recall the following proof from [**LB**], modifying it slightly to conform to our definition of exceptional singularity.

Proposition 2.19. *If C is a general curve of genus ≥ 2, then for every double cover $\widetilde{C} \to C$, there are no exceptional singularities on* Ξ.

Proof [**LB**, Remark (6.7) p. 389]. If C has a double cover with an exceptional singularity L on Ξ, then by definition $L = \pi^*(M)(B)$, for some line bundle M on C with $h^0(M) \geq 2$ and some divisor $B \geq 0$ on $\widetilde{C}$. Then $\mathrm{Nm}(L) = (2M)(\mathrm{Nm}(B)) = \omega_C$, where $\mathrm{Nm}(B) \geq 0$. Hence $h^0(\omega_C - 2M) = h^0(\mathrm{Nm}(B)) \geq 1$. We can deduce that C is special in moduli. For then we can choose a 2 dimensional subspace $W \subset H^0(C, M)$ and consider the cup product map $\mu : W \otimes H^0(C, K - M) \to H^0(C, K)$. If E is the base locus of the pencil $|W|$, the base point free pencil trick [**ACGH**, p. 126] implies the kernel of μ is isomorphic to $H^0(K - 2M + E)$. Since $h^0(K-2M+E) \geq h^0(K-2M) \geq 1$, the cup product map μ above is not injective, thus neither is Petri's map $\mu_0 : H^0(M) \otimes H^0(C, K-M) \to H^0(C, K)$, of which μ is a restriction. Then by Gieseker's theorem, [**G**], [**ACGH**, Thm (1.7), p. 215], the curve C is not general. □

Remarks 2.20. The apparent contradiction between the two statements: (i) that for $g(C) \geq 2$ there are in general no exceptional singularities on Ξ, and (ii) the theorem of Mumford that for $g(C) \leq 4$, all singularities on Ξ are exceptional, is of course resolved by the fact that in this range a general Ξ has no singularities at all.

Proposition 2.19 gives a proof of the following result, whose statement was communicated privately to us by Debarre.

Corollary 2.21. *For any double cover of a general curve C of genus $g \geq 2$, the special variety of divisors X is smooth.*

Proof. By Lemma 2.15, $\mathrm{sing} X \subset \varphi^{-1}(\mathrm{sing}_{\mathrm{ex}}\Xi)$, and by Proposition 2.19, for general C, $\mathrm{sing}_{\mathrm{ex}}\Xi = \emptyset$. □

Corollary 2.22 ([**We1**]). *If C is a general curve of genus $g \geq 2$, then for any connected étale double cover $\pi : \widetilde{C} \to C$, RST holds everywhere on Ξ.*

Proof. Since for any double cover of a general curve C, Ξ has no exceptional singularities, RST holds everywhere on Ξ. □

3. On the density of double points in $\mathrm{sing}_{\mathrm{st}}\Xi$.

As always, assume C is a smooth nonhyperelliptic curve and $\pi : \widetilde{C} \to C$ a connected étale double cover. For potential use in the Andreotti-Mayer-Green approach to Donagi's conjecture, we want to give a criterion for the existence of as many stable double points on Ξ as can be hoped for. We will show in Corollary 3.4 below that if C is nontetragonal and of genus $g \geq 11$, then for all double covers of C, double points are dense in every component of $\mathrm{sing}_{\mathrm{st}}\Xi$. By Corollary 2.2 it would suffice to show the existence of points L with $h^0(\widetilde{C}, L) = 4$, or equivalently with $h^0(\widetilde{C}, L) \leq 4$ on every component of $\mathrm{sing}_{\mathrm{st}}\Xi$. Note that if C is nonhyperelliptic, and $3 \leq g(C) \leq 6$, then X is irreducible and $2 \leq \dim(X) \leq 5$. Hence for any L on Ξ, we have $h^0(\widetilde{C}, L) - 1 = \dim \varphi^{-1}(L) \leq \dim(X) - 1 \leq 4$, so that $h^0(\widetilde{C}, L) \leq 5$, and since $h^0(\widetilde{C}, L)$ is even, in fact then $h^0(\widetilde{C}, L) \leq 4$ for all points L on Ξ. Hence giving a criterion for $h^0(\widetilde{C}, L) \leq 4$ to hold at a general point L of a component of $\mathrm{sing}\Xi$ is a challenge only when $g(C) \geq 7$. We are not in fact able to rule out the possible existence of small components of $\mathrm{sing}\Xi$ on which

h^0 is always ≥ 6, but we can obtain the estimate $h^0(\widetilde{C}, L) \leq 4$ at general points of relatively large components of singΞ. To do this we globalize an argument of Welters [**We1**, Lemma 3.2, p. 681] for changing arbitrary points L of $\mathrm{sing}_{\mathrm{st}}\Xi$ into ones with $h^0(\widetilde{C}, L) = 4$, using the "parity trick" of Mumford [**M2**, bottom of p. 186]. I.e., if $\mathrm{Nm}(L) = \omega_C, h^0(L) \geq 1$, and p is not a base point of $|L|$, then $\mathrm{Nm}(L(p' - p)) = \omega_C$ and $h^0(L(p' - p)) = h^0(L) - 1$. Applying this principle twice changes a point L of Ξ with $h^0(L) \geq 4$ into another point L' of Ξ with $h^0(L') = h^0(L) - 2$. As just described, this trick gives no information on whether the new point L' lies on the same component of singΞ as the original point L. To show every component of $\mathrm{sing}_{\mathrm{st}}\Xi$ contains a point with $h^0 = 4$ we use the following global version of the parity trick (whose hypotheses are vacuous for $g \leq 4$).

Proposition 3.1. *Assume C is smooth, nonhyperelliptic, of genus $g \geq 5$, and $\pi : \widetilde{C} \to C$ any étale connected double cover. Let $Z \subset \varphi^{-1}(\mathrm{sing}\Xi)$ be any irreducible component of $\varphi^{-1}(\mathrm{sing}\Xi)$ on which the generic fiber of φ is $\cong \mathbb{P}^r$, with $r \geq 3$. Then there exists a closed irreducible subvariety $Z' \subset X$ such that $\dim(Z') = \dim(Z)$, and $|D'| \cong \mathbb{P}^{r-2}$ for D' general on Z'. In particular $\dim(\varphi(Z')) \geq \dim(\varphi(Z)) + 2$.*

Assuming Proposition 3.1, we deduce the following results.

Theorem 3.2. *If C is smooth, not hyperelliptic, $g(C) = g \geq 3, \widetilde{C} \to C$ is an étale connected double cover, $\varphi : X \to \Xi$ is the Abel map, and Z an irreducible component of $\varphi^{-1}(\mathrm{sing}\Xi)$ such that $\dim(\varphi(Z)) \geq \dim(\mathrm{sing}\Xi) - 1$, then $h^0(\widetilde{C}, L) \leq 4$ at a general point L on $\varphi(Z)$.*

Proof. We have shown in the remarks just above Proposition 3.1 that $h^0(\widetilde{C}, L) \leq 4$ is true everywhere on Ξ if $g(C) \leq 6$. Assuming $g \geq 7$ and that the theorem is false, there is an irreducible component Z of $\varphi^{-1}(\mathrm{sing}\Xi)$ such that $\dim(\varphi(Z)) \geq \dim(\mathrm{sing}\Xi) - 1$, and at a general point L on $\varphi(Z)$ we have $h^0(\widetilde{C}, L) \geq 6$. Then $|L| \cong \mathbb{P}^r$ where $r \geq 5$, whence $|D'| \cong \mathbb{P}^{r-2}$, with $r - 2 \geq 3$, where D' is a generic point of the variety Z' constructed in Proposition 3.1. Then $\varphi(Z') \subset \mathrm{sing}_{\mathrm{st}}\Xi$, but $\dim(Z') \geq \dim \mathrm{sing}\Xi + 1$, a contradiction. ☐

Corollary 3.3. *Assume C is smooth, nonhyperelliptic, $g(C) = g \geq 6$ and $\dim(\mathrm{sing}\Xi) \leq p - 5 = g - 6$, i.e., C not on Mumford's list in* [**M1**, Thm., p. 344]. *Then for $W \subset \mathrm{sing}_{\mathrm{st}}\Xi$ any component of stable singularities, and a general point L on W, we have $h^0(\widetilde{C}, L) = 4$.*

Proof. If $\widetilde{C} \to C$ is an étale connected double cover such that $\dim(\mathrm{sing}\Xi) \leq p - 5$, and $W \subset \mathrm{sing}_{\mathrm{st}}\Xi$ is any component of stable singularities, then $\dim W \geq p - 6 \geq \dim(\mathrm{sing}\Xi) - 1$. Thus by Theorem 3.2, for a general point L on W, we have $h^0(\widetilde{C}, L) \leq 4$. Since also $h^0(\widetilde{C}, L) \geq 4$ by definition of stable singularities, it follows that $h^0(\widetilde{C}, L) = 4$. ☐

Corollary 3.4. *If C is nontetragonal and $g(C) = g \geq 11$, and $W \subset \mathrm{sing}_{\mathrm{st}}\Xi$ is any component of stable singularities, then at a general point L on W, $\mathrm{mult}_L\Xi = 2$.*

Proof. With these hypotheses, $\dim(W) \geq p-6$, and $\dim(\mathrm{sing}_{\mathrm{ex}}\Xi) \leq p-7$ by [**D1**, Th. 3.1(i), pp. 547-8]. Hence a general point L on W is not exceptional, so RST holds at L by Theorem 2.1. Since C is not on Mumford's list in [**M1**, p. 344], also $\dim(W) \leq \dim(\mathrm{sing}\Xi) \leq p-5$. Thus by Corollary 3.3, $\mathrm{mult}_L\Xi = 2$. □

Corollary 3.5. *Assume C is a smooth nonhyperelliptic curve and $\pi : \widetilde{C} \to C$ a connected étale double cover.*

(i) *If $g(C) \geq 5$, and $\dim(\mathrm{sing}\Xi) \leq p-5$, then $\dim \varphi^{-1}(\mathrm{sing}\Xi) \leq p-2$.*
(ii) *If $g \geq 6$, and $\dim(\mathrm{sing}\Xi) = p-6$, then $\dim \varphi^{-1}(\mathrm{sing}\Xi) \leq p-3$.*

Proof. (i) If $g \geq 5$, $\dim(\mathrm{sing}\Xi) \leq p-5$, and if there were a component Z of $\varphi^{-1}(\mathrm{sing}\Xi)$ of dimension $p-1$, then the generic fiber dimension of φ on Z must be ≥ 4, hence ≥ 5, (since all fibers are odd dimensional). But by Proposition 3.1 there would be a subvariety Z' of X of the same dimension as Z, such that for D' general on Z', we have $|D'| \cong \mathbb{P}^{r-2}$. Then $r-2 \geq 3$ implies that $\varphi(Z') \subset \mathrm{sing}\Xi$ also, but $\dim Z' = \dim Z$ implies that Z' is also a component of $\varphi^{-1}(\mathrm{sing}\Xi)$, a contradiction.

(ii) The same proof works again. □

Proposition 3.1 will be proved in Lemmas 3.6 through 3.10.

Lemma 3.6. *If L is a line bundle on a curve $\widetilde{C}$ such that $\dim|L| \geq 2$, then there exists a divisor D in $|L|$ of form $D = E + p + q$, with p and q each occurring simply in D, and such that p is not a base point of $|L|$ and q is not a base point of $|L-p|$.*

Proof. If $\dim|L| = r \geq 2$, and B_1 is the base divisor of $|L|$, choose (by Bertini) a divisor D_1 in $|L-B_1|$ such that (i) D_1 consists of distinct points and (ii) $\mathrm{supp}(D_1) \cap \mathrm{supp}(B_1) = \emptyset$, and let p be any point of D_1. Then p is not a base point of $|L|$ so $\dim|L-p| = r-1 \geq 1$. Since p does not belong to the divisor $B_1 + D_1 - p$ of $|L-p|$, then p does not belong to the base divisor B_2 of $|L-p|$. Then choose a divisor D_2 in $|L-p-B_2|$ consisting of distinct points and such that $\mathrm{supp}(D_2) \cap \mathrm{supp}(B_2+p) = \emptyset$, and let q be any point of D_2. Then $D = p + B_2 + D_2 = p + q + E$ satisfies the requirements of the Lemma. □

Lemma 3.7. *Let $\widetilde{C}$ be a curve, let $d \geq 2$ be an integer, and define $\mathcal{D}$ as follows: $\mathcal{D} = \{(p,q,D) : D \geq p+q\} \subset \widetilde{C} \times \widetilde{C} \times \widetilde{C}^{(d)}$. Then the projection $\mathcal{D} \to \widetilde{C}^{(d)}$ is a finite map of degree $d(d-1)$ étale at (p,q,D) if p and q occur simply in D.*

Proof. Since $\widetilde{C}$ is assumed complete the map is proper and quasi finite, hence finite, of the stated degree. If p, q occur simply in $D = E + p + q$, the addition map $\widetilde{C} \times \widetilde{C} \times \widetilde{C}^{(d-2)} \to \widetilde{C}^{(d)}$ is étale at (p, q, E), so it suffices to show the map $\mathcal{D} \to \widetilde{C} \times \widetilde{C} \times \widetilde{C}^{(d-2)}$ taking (x, y, F) to $(x, y, F - x - y)$ is étale at (p, q, D). But the map $\widetilde{C} \times \widetilde{C} \times \widetilde{C}^{(d-2)} \to \mathcal{D}$ taking (x, y, H) to $(x, y, H + x + y)$ is a local analytic inverse from an analytic neighborhood of (p, q, E) to an analytic neighborhood of (p, q, D). (Since the varieties are smooth any local analytic bijection is a local analytic isomorphism.) $\square$

Lemma 3.8. *Let $\widetilde{C}$ and $\mathcal{D}$ be as in Lemma 3.7 and assume further that $\widetilde{C}$ has a fixed point free involution ι . If $Z \subset \mathcal{D}$ is any subvariety, define $Z' = \{(p', q', D')$ for all (p, q, D) in $Z\}$ = the "flip" of Z, where $p' = \iota(p), q' = \iota(q)$, and $D' = D - p - q + p' + q'$. Then* $\dim Z = \dim Z'$.

Proof. Since the map $\widetilde{C} \times \widetilde{C} \times \widetilde{C}^{(d-2)} \to \mathcal{D}$ taking (x, y, H) to $(x, y, H + x + y)$ is an analytic bijection, it preserves the dimension of subvarieties, so it suffices to check that in $\widetilde{C} \times \widetilde{C} \times \widetilde{C}^{(d-2)}$ the map taking (x, y, H) to (x', y', H) preserves the dimension of subvarieties. Since the map is a regular involution, hence an isomorphism, it does indeed preserve dimension. $\square$

Lemma 3.9. *Let $\pi : \widetilde{C} \to C$ be any connected étale double cover of a smooth curve C, with associated Prym theta divisor Ξ, and Abel map $\varphi : X \to \Xi$. If Z is an irreducible component of $\varphi^{-1}(\mathrm{sing}\Xi)$, and L in $\varphi(Z)$ is a general point of the image of Z, then $|L| = \varphi^{-1}(L) \subset Z_{sm}$, and Z is the only component of $\varphi^{-1}(\mathrm{sing}\Xi)$ which dominates $\varphi(Z)$.*

Proof. Let $U \subset \varphi(Z)_{sm}$ be the (irreducible) open subset of $\varphi(Z)_{sm} \subset \mathrm{sing}\Xi$ on which $\dim |L|$ is minimal. Then over U, the map $\varphi^{-1}(U) \to U$ is a locally trivial projective bundle over a smooth irreducible base, hence $\varphi^{-1}(U)$ is a smooth irreducible subset of $\varphi^{-1}(\mathrm{sing}\Xi)$ containing an open, dense, subset of Z. Thus $Z \subset \mathrm{cl}(\varphi^{-1}(U))$, and since Z is a maximal irreducible subset of $\varphi^{-1}(\mathrm{sing}\Xi)$, we must have $Z = \mathrm{cl}(\varphi^{-1}(U))$. This proves both statements of the lemma. I.e., for L in U, $\varphi^{-1}(L) = |L| \subset \varphi^{-1}(U) \subset Z_{sm}$. Moreover we have shown that any component of $\varphi^{-1}(\mathrm{sing}\Xi)$ dominating $\varphi(Z)$ equals $\mathrm{cl}(\varphi^{-1}(U))$. Indeed this proof shows that for every subvariety $W \subset \mathrm{sing}\Xi$, exactly one component of $\varphi^{-1}(W)$ dominates W. $\square$

Now we can deduce Proposition 3.1.

Lemma 3.10. *Let $Z \subset \varphi^{-1}(\mathrm{sing}\Xi)$ be any component of $\varphi^{-1}(\mathrm{sing}\Xi)$ on which the generic fiber of φ is $\cong \mathbb{P}^r$, $r \geq 3$. Then there exists $Z' \subset X$ such that Z' is irreducible, $\dim Z' = \dim Z$, and such that for D' general on Z', we have $|D'| \cong \mathbb{P}^{r-2}$, and $\dim(\varphi(Z')) \geq \dim(\varphi(Z)) + 2$.*

Proof. Choose L a general point of $\varphi(Z)$ and using Lemma 3.6, choose D_1 in $|L| \subset Z_{sm}$ of form $p_1 + q_1 + E_1 = D_1$, where p_1 and q_1 occur simply in

$|L|$, p_1 is not a base point of $|L|$ and q_1 not a base point of $|L-p_1|$. Define $\widetilde{Z} = \{(p,q,D) : D \geq p+q$, and D belongs to $Z\} \subset \widetilde{X} = \{(p,q,D) : D \geq p+q$, and D belongs to $X\}$. Then $\widetilde{Z} \to Z$ is finite and étale at (p_1, q_1, D_1). Thus since D_1 is in $|L| \subset Z_{sm}$, $\widetilde{Z}$ is also smooth at (p_1, q_1, D_1). Hence there is a unique component $\widetilde{Z}_1$ of $\widetilde{Z}$ containing the point (p_1, q_1, D_1). Now "flip" $\widetilde{Z}_1$ as in Lemma 3.8, to $\widetilde{Z}_1' = \{(p', q', D')$ for all (p,q,D) in $\widetilde{Z}_1\}$ and let Z' = image of $\widetilde{Z}_1'$ under projection to X. Since by Mumford's parity trick, [**M2**, p. 188, step III], replacing two points changes the parity of D twice, hence leaves it even, $\widetilde{Z}_1'$ is contained in $\widetilde{X}$. Hence the flipped set Z' lies in X. Then $D_1' \in Z'$ and $|D_1'| \cong \mathbb{P}^{r-2}$, by the choice of p_1 and q_1 in D_1. So since no divisor in $\widetilde{Z}_1$ can have its dimension lowered by more than 2 through flipping 2 points, thus for D' generic in Z', we have $|D'| \cong \mathbb{P}^{r-2}$. Since the fibers of φ are contained in complete linear series, hence the restricted Abel map $\varphi : Z' \to \Xi$ has generic fiber dimension $\leq r-2$. Thus $\dim(\varphi(Z')) \geq \dim(\varphi(Z)) + 2$. ☐

Remarks 3.11. (i) When C is nonhyperelliptic, the conclusion of Corollary 3.3 that $h^0(\widetilde{C}, L) = 4$ for a general stable singularity L on Ξ, holds vacuously when $3 \leq g(C) \leq 4$ since then $\mathrm{sing}_{\mathrm{st}}\Xi = \emptyset$. The equation $h^0(\widetilde{C}, L) = 4$ holds for all points L on $\mathrm{sing}_{\mathrm{st}}\Xi$ when $g(C) = 5$ by the remarks above Proposition 3.1. Welters' argument [**We1**, Lemma 3.2, p. 681], for the existence of at least one point L on $\mathrm{sing}_{\mathrm{st}}\Xi$ with $h^0(\widetilde{C}, L) = 4$ whenever $\mathrm{sing}_{\mathrm{st}}\Xi \neq \emptyset$, already implies the conclusion of Corollary 3.3 for any double cover such that $\mathrm{sing}_{\mathrm{st}}\Xi$ is irreducible.

(ii) Since it is known (see Proposition 5.1 below) that the tangent cone to Ξ at a stable double point contains the Prym canonical model $\varphi_\eta(C)$ of C, Corollary 3.4 provides as many such quadrics as possible for nontetragonal curves with $g(C) \geq 11$. I.e., then for all L in a dense open subset of $\mathrm{sing}_{\mathrm{st}}\Xi, \mathbb{P}C_L(\Xi)$ is a quadric such that $\varphi_\eta(C) \subset \mathbb{P}C_L(\Xi)$. Considering the results of [**LS**], a primary open question concerning Donagi's conjecture then is whether in this case these quadrics generate the space of all quadrics containing $\varphi_\eta(C)$.

For tetragonal curves, Debarre [**D1**] has shown that the Prym variety of a generic tetragonal curve of genus $g \geq 13$ arises as Prym variety of exactly three doubly covered curves C_i, all tetragonal. Further, $\mathrm{sing}\Xi$ has dimension $p-6$ and has 3 components of that dimension, and the generic point of each component is nonexceptional for the representation of P as a tetragonal Prym associated to exactly two of the C_i. Hence Corollary 3.3 implies that the generic singular point on any one of the three components is a double point, and it follows that the base locus of the quadric tangent cones to any one of these components must contain the Prym canonical models of the two curves C_i for which this is a stable, nonexceptional component. It is conceivable that the quadric tangent cones at double points of the union of two of these components determines the unique curve C_i for which both these components are stable.

(iii) The conclusion of Corollary 3.5(i) that $\dim \varphi^{-1}(\mathrm{sing}\Xi) \le p-2$, holds also for all nonhyperelliptic curves C with $g = 3, 4$; for $g = 3$, it holds since $\mathrm{sing}\Xi = \emptyset$ and $\dim X = p = 2$, and for $g(C) = 4$ it holds by irreducibility of X, since $\dim X = 3$ and all fibers of φ are odd dimensional. For $g = 5$, Corollary 3.5(i) holds since then $\dim(\mathrm{sing}\Xi) \le p-5$ implies $\mathrm{sing}\Xi = \emptyset$, while $\dim X = 4$. The hypothesis is necessary here however since by Example 2.18, there is a doubly covered nonhyperelliptic curve C with $g(C) = 5, \dim(\mathrm{sing}\Xi) = 0 = p-4$, and $h^0(L) = 4$. Thus $\dim \varphi^{-1}(L) = 3 = p-1$.

In Corollary 3.5(ii), the hypotheses cannot hold for $g \le 5$ and the conclusion can fail as we have seen. When $g = 6$, the hypothesis is necessary since a plane quintic curve C with an odd double cover [**M1**, p. 348, line 1] such that $(P, \Xi) \cong (J(W), \Theta(W))$ where W is a smooth cubic threefold, gives an example of Ξ with $\dim(\mathrm{sing}\Xi) = 0 = p-5$, and $\varphi^{-1}(\mathrm{sing}\Xi) \cong \mathbb{P}^3$ has dimension $3 = p-2$.

(iv) Corollary 3.5 is useful for comparing line bundles on X with pull backs of line bundles from Ξ. This will be applied in Section 4 to describe the fundamental line bundle $\mathcal{O}_X(1)$ associated to the divisor variety X. An open question concerning the relation of X to the Prym Torelli problem is to compute $h^0(X, \mathcal{O}_X(1))$.

(v) We do not know, even when $\dim(\mathrm{sing}\Xi) \le p-5$, whether any components Z of $\varphi^{-1}(\mathrm{sing}\Xi)$ exist that do not dominate components of $\mathrm{sing}\Xi$. In particular we do not know whether there exist any components Z of $\varphi^{-1}(\mathrm{sing}\Xi)$ on which the generic fiber dimension of φ is ≥ 5.

4. A formula for the line bundle $\mathcal{O}_X(1)$ defined by the norm map $h : X \to |\omega_C|$.

Recall that if P is the Prym variety associated to a smooth connected étale double cover $\widetilde{C} \to C$, and $g = g(C)$, then $p = \dim(P) = \dim(X) = g-1$, where $\varphi : X \to \Xi$ is the restriction of the Abel map $\widetilde{\alpha} : \widetilde{C}^{(2g-2)} \to \mathrm{Pic}^{2g-2}(\widetilde{C})$ over $\Xi \subset P \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$, and $\dim(\Xi) = p-1 = g-2$. The restriction to X of the norm map $\mathrm{Nm} : \widetilde{C}^{(2g-2)} \to C^{(2g-2)}$ is denoted $h : X \to |\omega_C|$, and the associated line bundle $h^*(\mathcal{O}_{|\omega_C|}(1))$ is denoted $\mathcal{O}_X(1)$. We will show with mild genericity hypotheses that this line bundle is obtained from the pullback $\varphi^*(K_\Xi)$ of the canonical bundle on Ξ by twisting with the "tangent bundle along the fibers of φ". We must first give a definition of this relative tangent sheaf for our present situation in which $\varphi : X \to \Xi$ is not necessarily a $\mathbb{P}^1$ -bundle over $\mathrm{sing}(\Xi)$.

Definition 4.1. Given $\varphi : X \to \Xi$ as above, define $\mathcal{T}_\varphi$ on all of X to be the coherent sheaf $\mathcal{T}_\varphi = \mathcal{H}om(\Omega^1_\varphi, \mathcal{O}_X)$ = the dual $\mathcal{O}_X$-module of the relative Kähler differentials Ω^1_φ.

Note that on the open set $U = \varphi^{-1}(\Xi_{sm})$, the restriction $U \to \Xi_{sm}$ is a Zariski locally trivial $\mathbb{P}^1$ bundle, hence $\mathcal{T}_\varphi$ is the intuitive relative tangent bundle at least on U; in particular its restriction to U is a subbundle of $\mathcal{T}_X$ whose restriction to each fiber of $\varphi : U \to \Xi_{sm}$ is the tangent bundle to the fiber.

Theorem 4.2. *Assume C is nonhyperelliptic, $\pi : \widetilde{C} \to C$ any étale connected double cover and (P, Ξ) the Prym variety. If $g(C) = 3$ or 4, or if $g(C) \geq 5$ and* $\dim(\text{sing}\Xi) \leq g - 6$, *then $\mathcal{O}_X(1) \cong \mathcal{T}_\varphi \otimes \varphi^*(K_\Xi)$, where $\mathcal{O}_X(1)$ is the line bundle associated to the norm map $h : X \to |\omega_C|$.*

Corollary 4.3. (i) *Under the hypotheses of Theorem 4.2, the line bundle $\mathcal{O}_X(1)$ is intrinsically defined on X, i.e., $\mathcal{O}_X(1)$ is determined by X as an abstract variety.*

(ii) *Under the hypotheses of Theorem 4.2, the sheaf $\mathcal{T}_\varphi = \mathcal{H}om(\Omega^1_\varphi, \mathcal{O}_X)$ is the unique line bundle on X which on $X - \varphi^{-1}(\text{sing}\Xi)$ equals the bundle of tangents along the fibers of φ; in particular the hypotheses of Theorem 4.2 imply that $\mathcal{T}_\varphi$ is a line bundle on all of X.*

Proof of Corollary 4.3(i). First we will show the map φ is determined intrinsically by X.

Claim. Two points of X lie in the same fiber of φ if and only if they can be joined by a smooth rational curve λ on X.

Since the fibers are projective spaces any two points in the same fiber are joined by a curve isomorphic to $\mathbb{P}^1$. Moreover since P contains no rational curves, every smooth rational curve on X is collapsed to a point by φ, hence lies in some fiber of φ. Thus two points which are joined by a smooth rational curve do lie in the same fiber of φ. QED for the Claim. □

(Note that φ can be regarded as the extremal contraction cont_R defined by any smooth rational curve on X. I.e., if λ is a general fiber of φ on X, then φ induces an exact sequence in homology $0 \to \mathbb{R}[\lambda] \to H_2(X, \mathbb{R}) \to H_2(\Xi, \mathbb{R}) \to 0$, by Leray s.s.)

Thus the fibers of φ are characterized by X. Since $\Xi = \varphi(X)$ is normal, we claim Ξ is characterized as a scheme by the fibers of φ in X. First Ξ has the quotient topology induced by φ, since φ is proper, so Ξ is determined as a topological space. Then since Ξ is normal, $\mathcal{O}_\Xi = \varphi_*(\mathcal{O}_X)$, so the regular functions on open subsets U of Ξ are functions on $\varphi^{-1}(U)$ which are constant on the fibers of φ. Hence $\mathcal{O}_\Xi$ and the fibers of φ are determined by X. Since X determines $\varphi : X \to \Xi$, by Theorem 4.2 X determines $\mathcal{O}_X(1)$. □

Proof of Corollary 4.3(ii). Since $\mathcal{O}_X(1) \cong \mathcal{T}_\varphi \otimes \varphi^*(K_\Xi)$, $\mathcal{T}_\varphi = \mathcal{O}_X(1) \otimes (\varphi^*(K_\Xi))^*$ is the tensor product of two line bundles. QED Corollary 4.3. □

Proof of Theorem 4.2. If $g = 3$, the Prym is a 2 dimensional Jacobian and $\varphi : X \to \Xi$ is a $\mathbb{P}^1$ bundle over a smooth genus 2 curve, and in this case the formula has been proved in [**SV4**, p. 358]. If $g \geq 4$, we claim it suffices, by a "Hartogs" argument, to show that $\mathcal{O}_X(1)$ and $\mathcal{T}_\varphi \otimes \varphi^*(K_\Xi)$ are isomorphic on the open subset $U = \varphi^{-1}(\Xi_{sm})$ of X.

Definition 4.4. We say a sheaf $\mathcal{F}$ on an irreducible scheme X has the "Hartogs property" if its sections extend uniquely across closed sets of codimension ≥ 2. I.e., if for every closed subset $Z \subset X$ all of whose components are of codimension ≥ 2 in X, and every open set $V \subset X$, the restriction $H^0(V, \mathcal{F}) \to H^0(V - (Z \cap V), \mathcal{F})$ is an isomorphism.

Lemma 4.5. *If X is an irreducible Cohen Macaulay scheme and $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$ modules, then the sheaf $\mathcal{F}^* = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ has the Hartogs property.* (*In particular $\mathcal{O}_X$ itself, all locally free $\mathcal{O}_X$ modules, and all "reflexive" $\mathcal{O}_X$ modules, have the Hartogs property on a Cohen Macaulay variety.*)

Proof. This follows from some properties of depth, which we recall.

(i) If $\mathcal{F}$ is a coherent sheaf on a scheme X and $Z \subset X$ is a closed subset, then local sections of $\mathcal{F}$ extend uniquely across Z if and only if $\mathcal{F}$ has depth ≥ 2 along Z, ([**Gro**, Prop. 1.11, pp. 11-12, Thm. 3.8, p. 44] or see [**SV5**, Prop. 18, p. 391], for a summary statement).

(ii) If X is an algebraic scheme, and $Z \subset X$ a closed subset, such that $\mathcal{O}_X$ has depth ≥ 2 along Z, then for any coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$ modules, the coherent sheaf $\mathcal{F}^* = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ also has depth ≥ 2 along Z, ([**SV5**], Lemma 22, p. 392; similar to lemma, p. 21, of [**S**]).

(iii) If X is an irreducible noetherian Cohen Macaulay scheme, and $Z \subset X$ a closed subset, then $\mathcal{O}_X$ has depth $\geq k$ along Z if and only if every irreducible component of Z has codimension $\geq k$ in X [**H**, p. 184]. QED for Lemma 4.5. □

Since when C is nonhyperelliptic X is an irreducible normal local complete intersection, in particular Cohen Macaulay, and since $\mathcal{O}_X(1)$ and $\varphi^*(K_\Xi)$ are line bundles on X, it follows from Lemma 4.5 that both $\mathcal{O}_X(1)$ and $\mathcal{T}_\varphi \otimes \varphi^*(K_\Xi)$ have the Hartogs property on X. If $g = 4$, then (P, Ξ) is a 3 dimensional Jacobian hence Ξ is singular only when (P, Ξ) is a hyperelliptic Jacobian and then Ξ has one singular point. Then since X is irreducible of dimension 3 and every fiber of φ is an odd dimensional projective space, $\varphi^{-1}(\text{sing}\Xi) \cong \mathbb{P}^1$ hence has codimension two in X. Now using Corollary 3.5(i), under the hypotheses of Theorem 4.2, also if $g \geq 4$ then

$Z = \varphi^{-1}(\mathrm{sing}(\Xi))$ has codimension at least 2 in X. Hence if we have an isomorphism between $\mathcal{O}_X(1) \otimes (\varphi^*(K_\Xi))^*$ and $\mathcal{T}_\varphi$ over $X - Z = U = \varphi^{-1}(\Xi_{sm})$, the isomorphism extends uniquely to an isomorphism over all of X.

Thus, from now on we will primarily consider U and the map $\varphi : U \to \Xi_{sm}$. We claim that on U, $\mathcal{O}_X(1) \otimes (\mathcal{T}_\varphi)^*$ is the pullback of a line bundle on Ξ_{sm}. To see this, take any point z in Ξ_{sm} and consider the preimage $\varphi^{-1}(z) \cong \mathbb{P}^1$; we will check that the restrictions of $\mathcal{O}_X(1)$ and $\mathcal{T}_\varphi$ to $\varphi^{-1}(z)$ are both line bundles of degree two. For $\mathcal{O}_X(1)|\varphi^{-1}(z)$, consider the map $h : X \to |\omega_C|$ followed by the injective linear map of projective spaces $\pi^* : |\omega_C| \to |\omega_{\widetilde{C}}|$; then the degree of this composition will equal the degree of the restriction of h to $\varphi^{-1}(z)$, i.e., the degree of $\mathcal{O}_X(1)|\varphi^{-1}(z)$. The composition $X \to |\omega_{\widetilde{C}}|$ is given by $D \mapsto D + \iota^*(D)$, where $\iota : \widetilde{C} \to \widetilde{C}$ is the involution. On a line $\varphi^{-1}(z) = |D_z| = \mathbb{P}H^0(\widetilde{C}, L_z)$, this map is induced by the map $H^0(\widetilde{C}, L_z) \to H^0(\widetilde{C}, \omega_{\widetilde{C}})$ on sections: $s \mapsto s \otimes \iota^*(s)$, where $s \in H^0(\widetilde{C}, L_z)$, $\iota^*(s) \in H^0(\widetilde{C}, \iota^*(L_z))$, and $L_z \otimes \iota^*(L_z) \cong \pi^*(Nm(L_z)) \cong \pi^*(\omega_C) \cong \omega_{\widetilde{C}}$; thus the map $\varphi^{-1}(z) \cong \mathbb{P}H^0(\widetilde{C}, L_z) \to \mathbb{P}H^0(\widetilde{C}, \omega_{\widetilde{C}}) = |\omega_{\widetilde{C}}|$ is homogeneous of degree 2, hence is given on $\varphi^{-1}(z) \cong \mathbb{P}^1$ by sections of $\mathcal{O}_{\mathbb{P}^1}(2)$. On the other hand, the restriction $(\mathcal{T}_\varphi)|\varphi^{-1}(z)$ is the tangent bundle of $\varphi^{-1}(z) \cong \mathbb{P}^1$ which has degree 2. Thus both $\mathcal{O}_X(1)$ and $\mathcal{T}_\varphi$ restrict to the line bundle $\mathcal{O}(2)$ on each fiber $\varphi^{-1}(z) \cong \mathbb{P}^1$ (since $z \in \Xi_{sm}$) and hence the line bundle $\mathcal{O}_X(1) \otimes (\mathcal{T}_\varphi)^*$ on U is trivial on each fiber of φ.

It follows that there exists a line bundle, say $\mathcal{M}$, on Ξ_{sm} such that $\mathcal{O}_X(1) \otimes (\mathcal{T}_\varphi)^* \cong \varphi^*(\mathcal{M})$. Indeed, if we set $\mathcal{M} = \varphi_*(\mathcal{O}_X(1) \otimes (\mathcal{T}_\varphi)^*)$, then $\mathcal{M}$ is a line bundle on Ξ_{sm} by Grauert's theorem [**H**, Cor. 12.9, p. 288], since $\mathcal{F} = \mathcal{O}_X(1) \otimes (\mathcal{T}_\varphi)^*$ is flat over Ξ_{sm} and for each $z \in \Xi_{sm}$, $h^0(\varphi^{-1}(z), (\mathcal{O}_X(1) \otimes (\mathcal{T}_\varphi)^*)|\varphi^{-1}(z)) = h^0(\varphi^{-1}(z), \mathcal{O}_{\varphi^{-1}(z)}) = 1$. Then the natural homomorphism of line bundles on U, $\varphi^*(\mathcal{M}) = \varphi^*(\varphi_*(\mathcal{O}_X(1) \otimes (\mathcal{T}_\varphi)^*)) \to \mathcal{O}_X(1) \otimes (\mathcal{T}_\varphi)^*$ is an isomorphism since it is evidently an isomorphism on each fiber.

It remains to show that the line bundle $\mathcal{M}$ on Ξ_{sm} is isomorphic to $K_\Xi|\Xi_{sm}$. For this, we will show how to express divisors in both series $|\mathcal{O}_X(1)|$ and $|K_\Xi|$ in terms of the "standard divisors" $\{\mathcal{D}_p\}$ on X. Recall that for any point p on $\widetilde{C}$, the divisor $\mathcal{D}_p = \{$those D in X such that $D \geq p\}$. Then for all points $\overline{p}$ in C, if $H_{\overline{p}} \subset |\omega_C|$ is the hyperplane of $|\omega_C| \cong (\mathbb{P}^{g-1})^*$ corresponding to the point $\varphi_\omega(\overline{p})$ of $|\omega_C|^*$ on the canonical curve $\varphi_\omega(C) \subset |\omega_C|^* = \mathbb{P}^{g-1}$, and if $\pi^{-1}(\overline{p}) = \{p, p'\}$, then $h^{-1}(H_{\overline{p}}) = \mathcal{D}_p + \mathcal{D}_{p'}$. Thus $\mathcal{O}_X(1) \cong \mathcal{O}_X(\mathcal{D}_p + \mathcal{D}_{p'})$. Now consider a general point $p \in \widetilde{C}$; then one knows ([**SV2**]) the following: $\mathcal{D}_p \subset X$ is irreducible, maps birationally onto Ξ, and the formula $\varphi(\mathcal{D}_p \cap \mathcal{D}_{p'}) = \Gamma_{\overline{p}}$ holds, where $\Gamma_{\overline{p}}$ is the Gauss divisor on Ξ defined by the Prym canonical image $\varphi_\eta(\overline{p})$ of the point $\overline{p} = \pi(p) \in C$. Since $\varphi(\mathcal{D}_p \cap \mathcal{D}_{p'}) = \Gamma_{\overline{p}} \in |\mathcal{O}_\Xi(\Xi)|$, and $\mathcal{O}_\Xi(\Xi) \cong K_\Xi$ by adjunction, we see that the divisor $\varphi(\mathcal{D}_p \cap \mathcal{D}_{p'})$ on Ξ is a canonical divisor, i.e., $\mathcal{O}_\Xi(\varphi(\mathcal{D}_p \cap \mathcal{D}_{p'})) \cong K_\Xi$. Let $F \subset \Xi$ be the locus of points over which $\varphi : \mathcal{D}_p \to \Xi$ is not an isomorphism; since $\mathcal{D}_p \to \Xi$ is birational and Ξ is normal, $\operatorname{codim}_\Xi(F) \geq 2$. Then $V = (\Xi_{sm} - F)$ is an open subset of Ξ with complement of codimension ≥ 2, so it will suffice to show that $\mathcal{M}|_V \cong K_\Xi|_V$.

Now $\mathcal{D}_p \subset X$ provides a section of the $\mathbb{P}^1$-bundle $\varphi : U \to \Xi_{sm}$ over $V \subset \Xi_{sm}$, and therefore $N(\mathcal{D}_p/U)|(\mathcal{D}_p \cap \varphi^{-1}(V)) \cong \mathcal{T}_\varphi|(\mathcal{D}_p \cap \varphi^{-1}(V))$. I.e., at a point x of $\mathcal{D}_p$ in $\varphi^{-1}(V)$, $\mathcal{D}_p$ is transverse to the fiber $\varphi^{-1}(z)$, where $z = \varphi(x)$, so the tangent space $T_x(\varphi^{-1}(z)) \subset T_x(U)$ maps isomorphically onto the normal space $N_x(\mathcal{D}_p) = T_x(U)/T_x(\mathcal{D}_p)$. (Note that although the bundles $\mathcal{T}_\varphi$ and $\mathcal{O}_X(\mathcal{D}_p)$ are different both on X and on the open set $U \subset X$, indeed they have different restrictions to fibers of φ, they have the same restrictions to the section $(\mathcal{D}_p \cap \varphi^{-1}(V))$ over V.) The normal bundle $N(\mathcal{D}_p/U)$ is the restriction (to $\mathcal{D}_p \cap U$) of $\mathcal{O}_{\mathcal{D}_p}(\mathcal{D}_p) = \mathcal{O}_X(\mathcal{D}_p)|\mathcal{D}_p$, so in $\varphi^{-1}(V)$, $(T_\varphi)|\mathcal{D}_p \cong \mathcal{O}_{\mathcal{D}_p}(\mathcal{D}_p)$. On the other hand, restricting $\mathcal{O}_X(1)$ to $\mathcal{D}_p$ gives $\mathcal{O}_X(1)|\mathcal{D}_p \cong \mathcal{O}_X(\mathcal{D}_p + \mathcal{D}_{p'})|\mathcal{D}_p \cong \mathcal{O}_{\mathcal{D}_P}(\mathcal{D}_p + \mathcal{D}_{p'})$. Thus, in the open set $\varphi^{-1}(V) \subset X$, we have the following restrictions to $\mathcal{D}_p$: $\mathcal{O}_X(1)|\mathcal{D}_p \cong \mathcal{O}_{\mathcal{D}_p}(\mathcal{D}_p + \mathcal{D}_{p'})$ and $(\mathcal{T}_\varphi)|\mathcal{D}_p \cong \mathcal{O}_{\mathcal{D}_p}(\mathcal{D}_p)$. Hence, taking the "difference", we obtain the formula $(\mathcal{O}_X(1) \otimes (\mathcal{T}_\varphi)^{-1})|\mathcal{D}_p \cong \mathcal{O}_{\mathcal{D}_p}(\mathcal{D}_{p'})$, and it remains to determine the line bundle $\mathcal{O}_{\mathcal{D}_p}(\mathcal{D}_{p'}) \cong \mathcal{O}_{\mathcal{D}_p}(\mathcal{D}_p \cap \mathcal{D}_{p'})$ on $\mathcal{D}_p$. The map $(\varphi^{-1}(V) \cap \mathcal{D}_p) \to V$ is an isomorphism, so it suffices to determine the line bundle $\mathcal{M}_1 = \mathcal{O}_V(\varphi(\varphi^{-1}(V) \cap (\mathcal{D}_p \cap \mathcal{D}_{p'})))$ on V. As noted above, this line bundle $\mathcal{M}_1$ on V is the restriction to V of the line bundle $\mathcal{O}_\Xi(\varphi(\mathcal{D}_p \cap \mathcal{D}_{p'})) = K_\Xi$. Thus, $\mathcal{M}_1 \cong K_\Xi|_V$, and hence $(\mathcal{O}_X(1) \otimes (\mathcal{T}_\varphi)^{-1})|\varphi^{-1}(V) \cap \mathcal{D}_p \cong \varphi^*(K_\Xi|_V)$. QED Theorem 4.2. □

Remarks 4.6. (i) Since Mumford has classified all cases in which $\dim \Xi \geq g - 5$ [**M1**, p. 344], explicit criteria on $\pi : \widetilde{C} \to C$ can be given for the hypothesis $\dim(\operatorname{sing}\Xi) \leq g - 6$ of Theorem 4.2 to hold, in particular it holds for all C of genus $g \geq 7$ which are neither hyperelliptic, trigonal nor bielliptic. We expect if C is also assumed to be nontetragonal, that then $\dim(\operatorname{sing}\Xi) = g - 7$, at least for $g(C) \geq 11$. Since Debarre [**D1**] gave a list of those tetragonal curves with $\dim(\operatorname{sing}\Xi) = g - 6$, this would give a good account of the dimension of $\operatorname{sing}\Xi$.

(ii) The formula, $\mathcal{O}_X(1) \cong \mathcal{T}_\varphi \otimes \varphi^*(K_\Xi)$, gives a simple way to think of $\mathcal{O}_X(1)$ in terms of the canonical bundle K_X. Namely, consider in general a $\mathbb{P}^1$-bundle $\varphi : X \to \Xi$ over a variety Ξ of general type; then K_X would have (in additive notation) the form $\Omega^1_\varphi + \varphi^*(K_\Xi)$ and the relative canonical bundle Ω^1_φ is negative on the fibres of φ , so if one "changes the sign of K_X along the fibres" (i.e., replaces $\Omega^1_\varphi + \varphi^*(K_\Xi)$ by $\mathcal{T}_\varphi + \varphi^*(K_\Xi)$), then one obtains an ample line bundle on X intrinsic to the $\mathbb{P}^1$-bundle structure $\varphi : X \to \Xi$. The proof of Theorem 4.2 given here generalizes the one in [**SV4**], and is our original proof of the $\mathcal{O}_X(1)$ formula. The formula relating reducible divisors in $|\mathcal{O}_X(1)|$ to Gauss divisors on Ξ, needed for this generalization, is in [**SV2**].

(iii) It is possible to prove the result of Corollary 4.3(i), that the line bundle $\mathcal{O}_X(1)$, is intrinsically defined by X, without any hypotheses on $\operatorname{codim}(\varphi^{-1}(\operatorname{sing}\Xi))$. In this generality, the line bundle K_X is more convenient to work with than the coherent sheaf $\mathcal{H}om(\Omega^1_\varphi, \mathcal{O}_X)$, and then (see [**SV3**]) we can compute $K_X = 2\varphi^*(K_\Xi) - \mathcal{O}_X(1)$, which yields the formula $\mathcal{O}_X(1) = 2\varphi^*(K_\Xi) - K_X$. Since φ is determined by X in general (another proof of this is also in [**SV3**]) this implies that X always determines the line bundle $\mathcal{O}_X(1)$. Note that if $\varphi : X \to \Xi$ were a $\mathbb{P}^1$ bundle, so that we had $K_X = \Omega^1_\varphi + \varphi^*(K_\Xi)$ and $\mathcal{T}_\varphi = \mathcal{H}om(\Omega^1_\varphi, \mathcal{O}_X)$, these sheaves would be line bundles, and this new version of the formula would be equivalent to the one proved above in Theorem 4.2. In particular, the formulas are always equivalent over $X - \varphi^{-1}(\operatorname{sing}\Xi)$ so that when $\varphi^{-1}(\operatorname{sing}(\Xi))$ has codimension ≥ 2 in X, by using the depth argument above to extend such an isomorphism, the more general formula in [**SV3**] gives another proof of Theorem 4.2.

(iv) Recently Izadi and Pauly (see [**IP**]) have given a proof of a formula for $\mathcal{O}_{X^-}(1)$, where X^- is the "odd" half of the divisor variety for the Prym, analogous to the one in Theorem 4.2 above. The proof in [**SV3**] of the formula for $\mathcal{O}_X(1)$ applies to X^- as well, hence gives a version of their formula. I.e., in that case we get again $\mathcal{O}_{X^-}(1) = (\varphi^*(K_\Xi))^2 \otimes (K_{X^-})^*$, but since $\varphi : X^- \to P^-$ is birational onto a (smooth) abelian variety, a canonical divisor of X^- is the pullback of a canonical divisor of P^- plus a divisor E whose support is the exceptional divisor of $\varphi : X^- \to P^-$, and thus one gets $\mathcal{O}_X(1) = (\varphi^*(K_\Xi))^2 \otimes (\mathcal{O}_{X^-}(E))^*$. The proof in [**IP**] (Lemma 2.2, p. 6, [**IP**]) identifies the divisor E more precisely.

5. A proof of the Torelli theorem for cubic threefolds.

If W is a smooth cubic threefold, then associated to a general line λ on W there is a conic bundle representation of W and consequently a Prym representation of the intermediate Jacobian $(J(W), \Theta(W))$ as a Prym variety (P, Ξ) associated to an "odd" double cover of a smooth plane quintic C [**M1**, pp. 347-8], [**CG**, App.], [**B2**], [**T2**]. Moreover Ξ has a unique singular point, a triple point at which the projective tangent cone is W. This unpublished result of Mumford is treated particularly clearly in [**B2**]. Prym theory is used there to establish that there is only one singular point L and then the theory of the Abel Jacobi map on the Fano surface F of W, in particular the "tangent bundle theorem" and the parametrization of Ξ by $F \times F$, is used to deduce the multiplicity and the structure of the tangent cone of Ξ at L. The following argument computes the structure of Ξ at its unique singular point, including the multiplicity and tangent cone, using the Abel parametrization $\varphi : X \to \Xi$ (which exists for all Prym varieties), and the explicit form of the Prym canonical map defined by a conic bundle structure on a cubic threefold.

First we give a criterion for the tangent cone at a point of Ξ to contain the Prym canonical model $\varphi_\eta(C) \subset |\omega_C \otimes \eta|^*$ of C.

Proposition 5.1. *Let $\pi : \widetilde{C} \to C$ be any connected étale double cover of a smooth nonhyperelliptic curve C of genus $g \geq 4$, L any stable singular point of Ξ (possibly also exceptional), and* $\mathrm{mult}_L \Xi = r$. *If $(1/2)h^0(\widetilde{C}, L) = r$, i.e., if RST holds at L, or if L is base point free and $(1/2)h^0(\widetilde{C}, L) \leq r \leq h^0(\widetilde{C}, L) - 1$, then as sets $\varphi_\eta(C) \subset \mathbb{P}C_L(\Xi)$.*

Lemma 5.2. *If p and p' are conjugate points on the canonical model of $\widetilde{C}$ in $|\omega_{\widetilde{C}}|^*$, i.e., if the double cover $\pi : \widetilde{C} \to C$ maps $\pi(p) = \pi(p') = \overline{p}$, and if $L_{p,p'}$ is the line in $|\omega_{\widetilde{C}}|^*$ joining p to p', and $\varphi_\eta : C \to \varphi_\eta(C) \subset |\omega_C \otimes \eta|^*$ is the Prym canonical map, then $\varphi_\eta(\overline{p}) = L_{p,p'} \cap |\omega_C \otimes \eta|^*$.*

Proof of Lemma 5.2. See [**T1**], p. 957, line 11, or [**SV2**], proof of part 1 of main theorem, claim 1. □

Proof of Proposition 5.1. Lemma 5.2 implies $\varphi_\eta(C) \subset \mathrm{Sec}(\varphi_{\widetilde{K}}(\widetilde{C})) \cap \mathbb{P}T_L P$. Moreover $2\Xi = \widetilde{\Theta} \cdot P$ implies $\mathbb{P}C_L(\Xi) = \{\widetilde{\vartheta}_{2r} = 0\} \cap \mathbb{P}T_L P$ when $r = \mathrm{mult}_L \Xi$. Hence we have only to show that, under the hypotheses of Proposition 5.1, the inclusion $\mathrm{Sec}(\varphi_{\widetilde{K}}(\widetilde{C})) \cap \mathbb{P}T_L P \subset \{\widetilde{\vartheta}_{2r} = 0\} \cap \mathbb{P}T_L P$ holds. I.e., we are assuming either $4 \leq \mathrm{mult}_L \widetilde{\Theta} = 2r$, or L is base point free and $4 \leq \mathrm{mult}_L \widetilde{\Theta} \leq 2r \leq 2\mathrm{mult}_L \widetilde{\Theta} - 2$. Hence, in the first case by [**K**, p. 183] and in the second case by [**ACGH**, Thm. 1.6(ii), p. 232], we have $\mathrm{Sec}(\varphi_{\widetilde{K}}(\widetilde{C})) \subset \{\widetilde{\vartheta}_{2r} = 0\}$, thus as desired we get $\varphi_\eta(C) \subset \mathrm{Sec}(\varphi_{\widetilde{K}}(\widetilde{C})) \cap \mathbb{P}T_L P \subset \{\widetilde{\vartheta}_{2r} = 0\} \cap \mathbb{P}T_L P = \mathbb{P}C_L(\Xi)$. □

Remark 5.3. Since RST holds at every stable double point of Ξ, if C is nontetragonal and $g(C) = g \geq 11$, then in light of Proposition 5.1 and Corollary 3.4, for all L in a dense open subset of $\mathrm{sing}_{\mathrm{st}} \Xi$, $\mathbb{P}C_L(\Xi)$ is a quadric such that $\varphi_\eta(C) \subset \mathbb{P}C_L(\Xi)$.

The Torelli theorem. We assume the following facts about the double cover representing $J(W)$ as a Prym variety, [**B2**], [**B3**]. For a general line λ on W, the family C of triangles on W having λ as one side, is a smooth curve C doubly covered by the smooth connected curve $\widetilde{C}$ of lines on W distinct from λ but incident to λ. Associating each triangle to the plane it spans embeds C as a quintic curve in the $\mathbb{P}^2$ of planes containing λ in $\mathbb{P}^4$, the double cover $\widetilde{C} \to C$ is "odd" in the sense that $h^0(C, H \otimes \eta)$ is odd, where $H = \mathcal{O}(1)$ defines the plane embedding of C and η defines the double cover, and the Prym variety (P, Ξ) associated to the double cover $\widetilde{C} \to C$ is isomorphic to $(J(W), \Theta(W))$. The Prym canonical map $\varphi_{\omega \otimes \eta} : C \to \mathbb{P}^4$ takes a point of

C to that vertex of the corresponding triangle on W which is "opposite" λ. To prove Torelli for W it suffices to recover W from (P, Ξ). Since every singularity on Ξ is either exceptional or stable, we want to classify these. The proof of the next lemma is in [**B2**] but we include it for completeness.

Lemma 5.4. *There is exactly one exceptional singularity on* Ξ, $L = \pi^*(\mathcal{O}_C(1))$, *which is also stable and at which* $h^0(\widetilde{C}, L) = 4$.

Proof (cf. [**B2**]). An exceptional singularity on Ξ is a line bundle L on $\widetilde{C}$ of form $L = \pi^*(M)(B)$ where M is a line bundle on C with $h^0(M) \geq 2$ and $B \geq 0$ is an effective divisor on $\widetilde{C}$, and where $h^0(\widetilde{C}, L)$ is even and $\mathrm{Nm}(L) = \omega_C$. In particular M is an effective line bundle on C with $\omega_C - 2M = \mathcal{O}(\pi(B))$ also effective, hence $\deg(M) \leq 5$. Since C is a smooth plane quintic, C is neither hyperelliptic nor trigonal so $\deg(M) = 4$ or 5, and the canonical series $|\omega_C| = |\mathcal{O}_C(2)|$ is cut out on C by conics. If $\deg(M) = 4$ then $h^0(C, M) \leq 2$ by Clifford so $h^0(M) = 2$. Since C is not trigonal M has no base point and thus there is a divisor D of 4 distinct points in $|M|$. By RRT, $h^0(C, K - M) = 3$ and hence there are 3 independent conics passing through D. Then the 4 points of D fail by one to impose independent conditions on conics so all 4 points of D lie on a line and $|\mathcal{O}(1) - D|$ is effective, i.e., $M = \mathcal{O}(1)(-p)$ for some point p on C. If $\deg(M) = 5$, then M has at most one base point and again there is a divisor D in $|M|$ with 5 distinct points and $2 \leq h^0(M) \leq 3$ by Clifford, implies $h^0(K - M) \geq 2$. Again at least 4 points of M lie on a line and we have either $M = \mathcal{O}(1)$ or $M = \mathcal{O}(1)(\overline{p} - \overline{q})$ where $\overline{p}$, $\overline{q}$ are on C. Thus if $H = \mathcal{O}_C(1)$ the only possibilities for M are H, $H - \overline{p}$ or $H + \overline{p} - \overline{q}$, corresponding to series of form g^2_5, g^1_4, or g^1_5 on C. The case $M = H$ gives the one actual exceptional singularity on $\Xi, L = \pi^*(H)$. This has $h^0(\widetilde{C}, L) = 4$, since $h^0(\widetilde{C}, L) = h^0(C, H) + h^0(C, H \otimes \eta)$, $h^0(C, H) = 3$, and $h^0(C, H \otimes \eta)$ is odd and less than 3, hence $= 1$. If $M = H + \overline{p} - \overline{q}$, with $\overline{p} \neq \overline{q}$, then $\deg(M) = 5$ so $L = \pi^*(M)$ and $\mathrm{Nm}(L) = 2M = K$, so M is a theta characteristic. But for $M = H + \overline{p} - \overline{q}$ to be a theta characteristic, we must have $K = 2H - 2\overline{p} + 2\overline{q} = K - 2\overline{p} + 2\overline{q}$, hence $-2\overline{p} + 2\overline{q} = 0$, and our plane quintic would be hyperelliptic, a contradiction.

If $M = H - \overline{p}$ then we would have $L = \pi^*(H - \overline{p})(B)$ and $\pi_*(\pi^*(H - \overline{p})(B)) = 2(H - \overline{p})(\pi_* B) = K - 2\overline{p} + \pi_*(B) = K$. Thus we would need $B = 2p$ or $2p'$ or $p + p'$. If $B = p + p'$ then $L = \pi^*(H)$, the singular point we already have. If $B = 2p$, then the parity is opposite to that of $\pi^*(H)$ by Mumford's parity trick [**M2**, p. 186] so $h^0(\widetilde{C}, L)$ is not even. Hence there are no exceptional singularities on Ξ other than the one coming from $L = \pi^*(H)$. □

Lemma 5.5. *The point* $L = \pi^*(H)$ *is a triple point on* Ξ *such that* $\varphi_\eta(C) \subset \mathbb{P}C_L(\Xi)$.

Proof. Since both H and $H \otimes \eta$ are odd theta characteristics on C and $h^0(C, M) = 3$, it follows by [**V**, p. 948] that L is a singular odd theta characteristic on P, in particular Ξ has odd multiplicity ≥ 3 at L. By Lemma 5.4 above and the Lemma on p. 345 of [**M1**], $\dim(\mathrm{sing}\Xi) = 0$, so the Prym variety (P, Ξ) is not a polarized product of elliptic curves, hence by [**SV1**, p. 319], $\mathrm{mult}_L\Xi \leq 4$, hence $\mathrm{mult}_L\Xi = 3$. That $\varphi_\eta(C) \subset \mathbb{P}C_L(\Xi)$ then follows from Proposition 5.1 since $h^0(\widetilde{C}, L) = 4$, and $L = \pi^*(H)$ is base point free since H is. □

Lemma 5.6. *There are no nonexceptional singularities on* Ξ.

Proof. By Theorem 2.1 the RST holds at every nonexceptional singularity of Ξ. Since the source space X of the Abel map $\varphi : X \to \Xi$ is 5 dimensional and irreducible, the largest possible fiber of φ is $\mathbb{P}^3$, so by Theorem 2.1 all nonexceptional singularities of Ξ are stable double points. Thus the tangent cone at any such point is a quadric containing the Prym canonical curve $\varphi_\eta(C)$. The same argument proves this for every Prym representation of $J(W)$, i.e., for every choice of general line λ on W, hence the tangent quadric at every nonexceptional singular point contains the Prym canonical model of every plane quintic C_λ associated to every general line λ on W. Since the Prym canonical model $\varphi_\eta(C_\lambda)$ is the locus of vertices of residual pairs of lines in all triangles lying on W and having λ as one side [**B3**, Remarque 6.27], the union of these Prym canonical curves is dense in W. Since the tangent cone at a double point cannot contain the smooth cubic hypersurface W, there are no double points on Ξ, and $L = \pi^*(H)$ is in fact the only singular point on Ξ. □

Lemma 5.7. *The theta divisor* Ξ *has a unique singular point* L*, at which* $\mathbb{P}C_L\Xi = W$.

Proof. We know the triple point $L = \pi^*(\mathcal{O}_C(1))$ is the unique singular point on Ξ, and by the argument of Lemma 5.6 that $\mathbb{P}C_L\Xi$ contains the union of the Prym canonical curves $\varphi_\eta(C_\lambda)$ for every general line λ on W. Hence $\mathbb{P}C_L\Xi \supset W$, and since these are both cubic hypersurfaces and W is smooth, we conclude $\mathbb{P}C_L\Xi = W$.

This proves the Torelli theorem for W. □

6. Outline of the RST and its corollaries.

Theorem.

(1) *For all* L *in* Ξ, *we have* $\cup_{|L|}\mathbb{P}\varphi_*(T_D X) = \mathbb{P}C_L\widetilde{\Theta} \cap \mathbb{P}T_L P$.
(2) *For all* L *such that* $|L| \subset X_{sm}$, *we have* $\cup_{|L|}\mathbb{P}\varphi_*(T_D X) = \mathbb{P}C_L\Xi$.
(3) $\mathrm{Sing}_{ex}\Xi = \varphi(\mathrm{sing}X)$, *hence for all* L *in* $\Xi - \mathrm{sing}_{\mathrm{ex}}\Xi, |L| \subset X_{sm}$.
(4) *For all* L *in* $\Xi - \mathrm{sing}_{\mathrm{ex}}\Xi$, *we have* $\mathbb{P}C_L\Xi = \mathbb{P}C_L\widetilde{\Theta} \cap \mathbb{P}T_L P$, *and thus* $\mathrm{mult}_L\Xi = (1/2)h^0(\widetilde{C}, L)$, *i.e., "RST holds" at every* L *in* $\Xi - \mathrm{sing}_{\mathrm{ex}}\Xi$.

(5) *If* $g(C) \geq 11$ *and* C *not tetragonal, then for any double cover of* C*,* $\mathrm{sing}_{2,\mathrm{st}}(\Xi)$ *is dense in* $\mathrm{sing}_{\mathrm{st}}\Xi$*.*

(6) *If* $2 \leq (1/2)h^0(\widetilde{C}, L) = \mathrm{mult}_L\Xi$*, i.e., if RST holds at* L *in sing* Ξ*, or if* L *is base point free and if* $\mathrm{mult}_L\Xi = r$*, where* $2 \leq (1/2)h^0(\widetilde{C}, L) \leq r \leq h^0(\widetilde{C}, L) - 1$*, then* $\varphi_\eta(C) \subset \mathrm{Sec}(\varphi_{\widetilde{K}}(\widetilde{C})) \cap \mathbb{P}T_LP \subset \{\widetilde{\vartheta}_{2r} = 0\} \cap \mathbb{P}T_LP = \mathbb{P}C_L(\Xi)$*, as sets.*

(7) **Cor:** *If* $g(C) \geq 11$ *and* C *not tetragonal, then for any double cover of* C*,* $\varphi_\eta(C) \subset \mathbb{P}C_L(\Xi)$ *for all* L *in a dense open subset of* $\mathrm{sing}_{\mathrm{st}}\Xi$*.*

(8) **Cor:** *If* W *is a smooth cubic threefold and* (P, Ξ) *the Prym variety associated to the odd cover of the discriminant plane quintic* C *for the conic bundle structure on* W *defined by any general line on* W*, then* $\varphi_\eta(C) \subset \mathbb{P}C_L(\Xi)$*, where* L *is the unique singular point on* Ξ*. Since the union of these Prym canonical models is dense in* W*,* $W \subset \mathbb{P}C_L(\Xi)$*, and since by* [**SV1**, **V**] L *is a triple point,* $W = \mathbb{P}C_L(\Xi)$*.*

(9) **Cor:** *The restricted norm map* $h : X \to |\omega_C|$ *is defined by a linear subsystem of* $|\mathcal{O}_X(1)|$*, where* $\mathcal{O}_X(1) \cong \mathcal{T}_\varphi \otimes \varphi^*(K_\Xi)$*, and* $\mathcal{T}_\varphi$ *is the bundle of tangents "along the fibers" of* φ*, in case* $\dim(\mathrm{sing}\Xi) \leq p - 5$*.*

References

[AM] A. Andreotti and A. Mayer, *On period relations for abelian integrals on algebraic curves*, Ann. Scuola Norm. Sup. Pisa, **21** (1967), 189-238, MR 36 #3792, Zbl 222.14024.

[ACGH] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, *Geometry of Algebraic Curves*, Springer-Verlag, N.Y., 1984, MR 86h:14019, Zbl 559.14017.

[B1] A. Beauville, *Sous-variétés spéciales des variétés de Prym*, Comp. Math., **45** (1982), 357-383, MR 83f:14025, Zbl 504.14022.

[B2] ———, *Les singularités du diviseur* Θ *de la Jacobienne intermédiare de l'hypersurface cubique dans* $\mathbb{P}^4$, Algebraic threefolds (Proc. Varenna 1981), Lecture Notes in Math., **947**, Springer-Verlag, New York, (1982), 190-208, MR 84c:14030, Zbl 492.14033.

[B3] ———, *Variétés de Prym et Jacobiennes intermédiares*, Ann. Scient. Éc. Norm. Sup., **10** (1977), 309-391, MR 57 #12532, Zbl 368.14018.

[CG] C.H. Clemens and P.A. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math., **95** (1972), 281-356, MR 46 #1796, Zbl 231.14004.

[D1] O. Debarre, *Sur les variétés de Prym des courbes tetragonales*, Ann. Scient. Éc. Norm. Sup., **21** (1988), 545-559, MR 90b:14034, Zbl 674.14034.

[D2] ———, *Sur le problème de Torelli pour les variétés de Prym*, Am. J. Math., **111** (1989), 111-134, MR 90b:14035, Zbl 699.14052.

[Do] R. Donagi, *The tetragonal construction*, B.A.M.S., **4**(**2**) (1981), 181-185, MR 82a:14009, Zbl 491.14016.

[G] D. Gieseker, *Stable curves and special divisors*, Invent. Math., **66** (1982), 251-275, MR 83i:14024, Zbl 522.14015.

[Gr] M. Green, *Quadrics of rank four in the ideal of the canonical curve*, Invent. Math., **75** (1984), 85-104, MR 85f:14028, Zbl 542.14018.

[Gro] A. Grothendieck, *Local Cohomology*, Lecture Notes in Math., **41**, Springer-Verlag, New York, 1967, MR 37 #219, Zbl 185.49202.

[H] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977, MR 57 #3116, Zbl 367.14001.

[IP] E. Izadi and C. Pauly, *Some properties of second order theta functions on Prym varieties*, eprint math.AG/9904001.

[K] G. Kempf, *On the geometry of a theorem of Riemann*, Ann. of Math., **98** (1973), 178-185, MR 50 #2180, Zbl 275.14023.

[LB] H. Lange and C. Birkenhake, *Complex Abelian Varieties*, Springer-Verlag, Berlin Heidelberg, 1992, MR 94j:14001, Zbl 779.14012.

[LS] H. Lange and E. Sernesi, *Quadrics containing a Prym canonical curve*, J. Algebraic Geometry, **5** (1996), 387-399, MR 96k:14021, Zbl 859.14009.

[MM] A. Mattuck and A. Mayer, *The Riemann Roch theorem for algebraic curves*, Ann. Scuola Norm. Sup. Pisa, serie III (Mathematics), **XVII** (1963), 223-237, MR 29 #102, Zbl 118.15902.

[M1] D. Mumford, *Prym Varieties* I, in 'Contributions to Analysis', L. Ahlfors ed., Academic Press, N.Y., (1974), 325-350, MR 52 #415, Zbl 299.14018.

[M2] ———, *Theta characteristics of an algebraic curve*, Ann. Scient. Éc. Norm. Sup., **4** (1971), 181-192, MR 45 #1918, Zbl 216.05904.

[R] S. Recillas, *Jacobians of Curves with a g^1_4 are Prym varieties of trigonal curves*, Bol. Soc. Mat. Mexicana, **19** (1974), 9-13, MR 58 #666, Zbl 343.14012.

[S] M. Schlessinger, *Rigidity of quotient singularities*, Invent. Math., **14** (1971), 17-26, MR 45 #1912, Zbl 232.14005.

[Sh] V.V. Shokurov, *Prym varieties: Theory and Applications*, Math. USSR Izvestija, **23**(**1**) (1984), 83-147, MR 85e:14040, Zbl 572.14025.

[SV1] R. Smith and R. Varley, *Multiplicity g points on theta divisors*, Duke Math Journal, **82**(**2**) (1996), 319-326, MR 97c:14043, Zbl 903.14017.

[SV2] ———, *The curve of "Prym canonical" Gauss divisors on a Prym theta divisor*, to appear in T.A.M.S.

[SV3] ———, *The special subvariety of divisors associated to an étale double cover of curves: A factorization of the Prym map*, preprint, 1999.

[SV4] ———, *On the geometry of two dimensional Prym varieties*, Pacific J. of Math., **188**(**2**) (1999), 353-368, MR 2000f:14045

[SV5] ———, *Deformations of theta divisors and the rank four quadrics problem*, Compositio Mathematica, **76** (1990), 367-398, MR 92a:14025, Zbl 941.14009.

[T1] (i) A. Tjurin, *The geometry of the Poincaré theta divisor of a Prym variety*, Math. USSR Izvestija, **9**(**5**), (1975), 951-986;
(ii) A. Tjurin, *Correction to the Paper "The geometry of the Poincaré theta divisor of a Prym variety"*, Math. USSR Izvestija, **12**(**2**) (1978), 438, MR 57 #16314.

[T2] ———, *The geometry of the Fano surface of a nonsingular cubic $F \subset \mathbb{P}^4$ and Torelli theorems for Fano surfaces and cubics*, Math. USSR. Izvestija, **5** (1971), 517-546, MR 44 #2757, Zbl 252.14004.

[V] R. Varley, *Weddle's surfaces, Humbert's Curves, and a certain four dimensional abelian variety*, Am. J. Math., **108** (1986), 931-952, MR 87g:14050, Zbl 604.14031.

[Ve] A. Verra, *The Prym map has degree two on plane sextics*, preprint, Università di Genova, 1992.

[We1] G. Welters, *A theorem of Gieseker-Petri type for Prym varieties*, Ann. Scient. Éc. Norm. Sup., **18** (1985), 671-683, MR 88a:14034, Zbl 628.14036.

[We2] ______, *Abel-Jacobi Isogenies for certain types of Fano threefolds*, thesis for degree of Doctor of Science, Mathematisch Centrum, Amsterdam, 1981, MR 84k:14035, Zbl 474.14028.

Received December 30, 1999 and revised June 13, 2000.

Department of Mathematics
University of Georgia
Athens, GA 30602
E-mail address: roy@math.uga.edu

Department of Mathematics
University of Georgia
Athens, GA 30602
E-mail address: rvarley@math.uga.edu