Pacific Journal of Mathematics

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Volume 202 No. 1 January 2002

PACIFIC JOURNAL OF MATHEMATICS
Vol. 202, No. 1, 2002

SPECTRUM AND ASYMPTOTICS OF THE BLACK–SCHOLES PARTIAL DIFFERENTIAL EQUATION IN (L^1, L^∞)-INTERPOLATION SPACES

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Let E be an (L^1, L^∞)-interpolation space. Then $(T_E(t)f)(x) = f(e^{-t}x)$ defines a group on E. It is strongly continuous if and only if E has order continuous norm. In any case, a generator A_E can be associated with T_E. It is shown that its spectrum is the strip $\{\underline{\alpha}_E \leq Re\,\lambda \leq \overline{\alpha}_E\}$, where $\underline{\alpha}_E$ and $\overline{\alpha}_E$ are the Boyd indices of E. The operator $B_E = (A_E)^2$ generates a holomorphic semigroup which governs the Black–Scholes partial differential equation $u_t = x^2 u_{xx} + x u_x$, whose well-posedness, spectrum and asymptotics in E are studied.

0. Introduction.

Let E be an (L^1, L^∞)-interpolation space on $(0, \infty)$, $\mathbb{R}$ or $\mathbb{T}$. Then the upper and lower Boyd indices $\underline{\alpha}_E$ and $\overline{\alpha}_E$ are of great importance. For example, the Hilbert transform is bounded on E if and only if $0 < \underline{\alpha}_E$ and $\overline{\alpha}_E < 1$. Also norm convergence of the Fourier series can be expressed in terms of the Boyd indices (see [**BS**]). In his paper [**Bo**], Boyd computes the spectrum of the Cesaro operator in terms of the Boyd indices. Here we consider a natural one-parameter group of dilations $(T_E(t))_{t\in\mathbb{R}}$ on E. It turns out that the Boyd indices are just the growth bounds (or exponential bounds) of this group.

To be more precise, we consider an (L^1, L^∞)-interpolation space E on $(0, \infty)$ througout this article. The group T_E on E is defined by

$$(T_E(t)f)(x) = f(e^{-t}x)$$

for all $f \in E,\ t \in \mathbb{R},\ x > 0$. Now the first problem is that T_E is not strongly continuous, in general. In fact, one of our main results says that T_E is a C_0-group if and only if E has order continuous norm.

Still it is possible to associate a generator A_E to T_E without any further condition on the space, and we show that its spectrum is the strip

$$\sigma(A_E) = \{\lambda \in \mathbb{C} : \underline{\alpha}_E \leq Re\,\lambda \leq \overline{\alpha}_E\}.$$

Thus the spectrum of A_E varies very much in function of the space E. It turns out that the Cesaro operator is just $(1 - A_E)^{-1}$. Thus it is bounded if and only if $\overline{\alpha}_E < 1$. In that case we obtain its spectrum just by applying the result above on the spectrum of A_E.

Of particular interest is the operator $B_E = (A_E)^2$. In fact, B_E is a degenerate elliptic operator given by $(B_E f)(x) = x^2 f''(x) + x f'(x)$ with suitable domain. As a consequence of the results on A_E we obtain much information on B_E. It always generates a generalized holomorphic semigroup V_E on E. So this semigroup gives the solution of the Black-Scholes partial differential equation

$$u_t = x^2 u_{xx} + x u_x. \tag{BS}$$

We show that the semigroup V_E is strongly continuous if and only if E has order continuous norm. Nevertheless, one of the main results says that T_E as well as V_E are always $\sigma(E, E'_n)$-continuous, where E'_n is the Köthe dual of E; i.e., the space of all functionals given by a measurable function. This allows us to formulate precisely well-posedness for (BS) in E. Finally, we consider perturbations of the operator B_E. The results imply in particular well-posedness of the more general equation

$$u_t = \alpha x^2 u_{xx} + \beta x u_x + \gamma u$$

where $\alpha > 0$ is a constant and $\beta, \gamma \in L^\infty(0, \infty)$.

Because of its importance in mathematical finance (see [**BlSc**]), the Black-Scholes partial differential equation has been investigated most recently. We refer to Gozzi, Monte, Vespri [**GMV**], Barucci, Gozzi, Vespri [**BGV**] and Colombo, Giuli, Vespri [**CGV**] for further information. We would like to emphasize that the motivation for this work lies in the interesting relations between properties of interpolation spaces and the semigroups considered here. It is not at all a contribution to modelling in mathematical finance.

The paper is organized in the following way: After some preliminaries we show in Section 2 that the semigroup T_E is strongly continuous if and only if E has order continuous norm. In that case we can use a result of Greiner [**G**] to determine the spectrum of E. In the less conventional situation where E does not have order continuous norm we use the theory of resolvent positive operators and integrated semigroups. Now the situation is much more complicated, and Section 3 is devoted to the generalization of Greiner's decomposition theorem to resolvent bipositive operators. In Section 4 we prove the results on the spectrum in the general case. Here it is also shown that the semigroup T_E is $\sigma(E, E'_n)$-continuous. In Section 5 we investigate the Black-Scholes operator $B_E = (A_E)^2$. Its perturbations are studied in Section 6.

1. Preliminaries.

On the interval $(0,\infty)$ we consider Lebesgue measure (dm or dx). For a Borel measurable function $f : (0,\infty) \to \mathbb{C}$ the **distribution function** is defined by $d_{|f|}(\lambda) = m\{t \in (0,\infty) : |f(t)| > \lambda\}$ for $\lambda > 0$. We will consider only functions f for which $d_{|f|}(\lambda) < \infty$ for some $\lambda > 0$. The space of all such functions will be denoted by $\mathcal{S}_0(0,\infty)$. For $f \in \mathcal{S}_0(0,\infty)$ we define

$$f^*(t) = \inf\{\lambda > 0 : d_{|f|}(\lambda) \leq t\} \quad \text{for} \quad t > 0.$$

Then $f^* : (0,\infty) \to (0,\infty)$ is decreasing, right-continuous and equimeasurable with $|f|$ (i.e., $d_{f^*} = d_{|f|}$). The function f^* is called the **decreasing rearrangement** of $|f|$ (see e.g., [**BS**]). In particular we recall that

$$\int_0^t f^*(s)ds = \sup\left\{ \int\limits_A |f|dm : A \subset (0,\infty) \text{ measurable and } m(A) \leq t \right\}$$

(by [**BS**, Prop. 3.3., p. 53]).

Suppose that E is a linear subspace of $\mathcal{S}_0(0,\infty)$, which is a Banach space with respect to the norm $\|\cdot\|_E$. Then E will be called a **rearrangement invariant Banach function space** if

$$f \in E\,,\ g \in \mathcal{S}_0(0,\infty) \ \text{ and } \ g^* \leq f^* \ \text{ imply that } \ g \in E \ \text{ and } \ \|g\|_E \leq \|f\|_E$$

(see e.g., [**KPS**]). If E is such a rearrangement invariant space on $(0,\infty)$, we always have the continuous embeddings

$$L^1 \cap L^\infty(0,\infty) \subseteq E \subseteq (L^1 + L^\infty)(0,\infty).$$

Here the spaces $L^1 \cap L^\infty$ and $L^1 + L^\infty$ are equipped with the norms

$$\begin{aligned} \|f\|_{L^1\cap L^\infty} &= \max\{\|f\|_1\,,\, \|f\|_\infty\}, \\ \|f\|_{L^1+L^\infty} &= \inf\{\|g\|_1 + \|h\|_\infty : f = g + h, \\ &\qquad g \in L^1(0,\infty),\, h \in L^\infty(0,\infty)\}, \end{aligned}$$

respectively.

Given $f,\, g \in \mathcal{S}_0(0,\infty)$, we say that g is **submajorized** by f (in the sense of Hardy-Littlewood-Polya) if

$$\int_0^t g^*(s)ds \leq \int_0^t f^*(s)ds \quad \text{for all} \quad t > 0,$$

which is denoted by $g \lll f$.

Using this submajorization relation the exact (L^1, L^∞)-interpolation spaces can be characterized. In fact, it is a result of A.P. Calderon (e.g., see [**BS**, Theorem 2.12]) that a Banach space $(E, \|\ \|_E)$, with $E \subseteq (L^1 + L^\infty)(0,\infty)$, is an exact (L^1, L^∞)-interpolation space if and only if,

$$f \in E,\, g \in \mathcal{S}_0(0,\infty) \ \text{ and } \ g \lll f \ \text{ imply that } \ g \in E \ \text{ and } \ \|g\|_E \leq \|f\|_E\,.$$

In particular, such interpolation spaces are rearrangement invariant Banach function spaces. Although some of the results in this paper hold for more general rearrangement invariant spaces, we will assume that the spaces we consider are exact (L^1, L^∞)-interpolation spaces. This class includes many of the classical function spaces (e.g., L^p-spaces, Orlicz spaces, Lorenz spaces, Marcinkiewiecz spaces).

If E is a rearrangement invariant Banach function space on $(0,\infty)$ which is **monotone complete** (i.e., $0 \le f_n \in E$, $f_n \le f_{n+1}$ *a.e.*, $\sup_n \|f_n\|_E < \infty$ implies that there exists $0 \le f \in E$ such that $f_n \uparrow f$ *a.e.* and $\|f\|_E = \sup_n \|f_n\|_E$), then E is an exact (L^1, L^∞)-interpolation space (see e.g., [**BS**, Theorem 2.2, p. 106]).

Similarly, any rearrangement invariant Banach function space with order continuous norm is an exact (L^1, L^∞)-interpolation space.

Since every interpolation space can be renormed in such a way that it becomes an exact interpolation space (see [**BS**]), in the following we will assume that the interpolation space is exact, throughout the paper.

For $s > 0$ the dilation operator D_s, acting on measurable functions f on $(0,\infty)$, is defined by

$$D_s f(t) = f(t/s),\ t > 0.$$

Clearly, the operators D_s are bounded on any (L^1, L^∞)-interpolation space E and satisfy $\|D_s\|_E \le \max(1,s)$ for all $s > 0$. Note that $(D_s f)^* = D_s f^*$ for all $s > 0$ and all $f \in E$, so in particular $\|D_s f\|_E$ is an increasing function of s.

For such a space E the **upper and lower Boyd indices** are defined by

$$\overline{\alpha}_E = \lim_{s\to\infty} \frac{\log\|D_s\|}{\log s},\ \ \underline{\alpha}_E = \lim_{s\downarrow 0} \frac{\log\|D_s\|}{\log s}$$

respectively, and satisfy $0 \le \underline{\alpha}_E \le \overline{\alpha}_E \le 1$ (see e.g., [**BS**], [**KPS**]). By way of example, if $E = L^p \cap L^q(0,\infty)$, $1 \le p \le q \le \infty$, (equipped with the norm $\|f\|_E = \max(\|f\|_p, \|f\|_q)$), then $\underline{\alpha}_E = 1/q$, $\overline{\alpha}_E = 1/p$.

In Section 4 we will use the following result.

Lemma 1.1. *Let E be an (L^1, L^∞)-interpolation space on $(0,\infty)$ and μ a (positive) Borel measure on $(0,\infty)$.*

Suppose that $f \in E$ satisfies $\int_0^\infty \|D_s f\|_E \, d\mu(s) < \infty$.

Then $\int_0^\infty D_s f(x) d\mu(s)$ is absolutely convergent for almost all $x > 0$,

$$\int_0^\infty D_s f(\cdot) d\mu(s) \in E \quad \text{and} \quad \left\| \int_0^\infty D_s f(\cdot) d\mu(s) \right\|_E \le \int_0^\infty \|D_s f\|_E \, d\mu(s).$$

In particular, if $\int_0^\infty \|D_s\|_E \, d\mu(s) < \infty$, then

$$T_\mu f(x) = \int_0^\infty D_s f(x) d\mu(s),\ \textbf{a.e.}\ \ x \in (0,\infty),$$

defines a bounded linear operator in E satisfying

$$\|T_\mu\|_E \le \int_0^\infty \|D_s\|_E \, d\mu(s).$$

Proof. The proof is divided in two parts.

1. Suppose that $f \in (L^1 + L^\infty)(0,\infty)$ is such that $\int_0^\infty D_s f^*(\cdot) d\mu(s) \in (L^1 + L^\infty)(0,\infty)$. We claim that $\int_0^\infty D_s f(x) d\mu(s)$ is absolutely convergent for a.e. $x \in (0,\infty)$, and that

$$\int_0^\infty D_s f(\cdot) d\mu(s) \prec\!\!\prec \int_0^\infty D_s f^*(\cdot) d\mu(s).$$

Indeed, for any measurable set $A \subseteq (0,\infty)$ with $m(A) < \infty$ we have

$$\int_A \left(\int_0^\infty |D_s f(x)| d\mu(s)\right) dx = \int_0^\infty \left(\int_A D_s f(x) dx\right) d\mu(s) \le$$

$$\int_0^\infty \left(\int_0^{m(A)} D_s f^*(x) dx\right) d\mu(s) = \int_0^{m(A)} \left(\int_0^\infty |D_s f^*(x)| d\mu(s)\right) dx < \infty.$$

This shows in particular that $\int_0^\infty |D_s f(x)| d\mu(s) < \infty$ for a.e. $x \in (0,\infty)$. Moreover,

$$\begin{aligned}
&\int_0^t \left(\int_0^\infty |D_s f(\cdot)| d\mu(s)\right)^* (x) dx \\
&= \sup\left\{\int_A \left(\int_0^\infty |D_s f(x)| d\mu(s)\right) dx \,:\, m(A) \le t\right\} \\
&\le \int_0^t \left(\int_0^\infty D_s f^*(x) d\mu(s)\right) dx \quad \text{for all} \quad t > 0,
\end{aligned}$$

and since $|\int_0^\infty D_s f(x) d\mu(s)| \le \int_0^\infty |D_s f(x)| d\mu(s)$ the claim follows.

2. Now assume that $f \in E$ is such that $\int_0^\infty ||D_s f||_E d\mu(s) < \infty$. Since $||D_s f||_E = ||D_s f^*||_E$, it follows from [**KPS**, II.4.7] that $\int_0^\infty D_s f^*(\cdot) d\mu(s) \in E$ and

$$\left\|\int_0^\infty D_s f^*(\cdot) d\mu(s)\right\|_E \le \int_0^\infty \|D_s f\|_E \, d\mu(s).$$

From 1. above it follows that $\int_0^\infty D_s f(x) d\mu(s)$ is absolutely convergent for a.e. $x \in (0,\infty)$, and

$$\int_0^\infty D_s f(\cdot) d\mu(s) \prec\!\!\prec \int_0^\infty D_s f^*(\cdot) d\mu(s).$$

Since E is an exact (L^1, L^∞)-interpolation space, this implies that $\int_0^\infty D_s f(\cdot) d\mu(s) \in E$ and

$$\left\| \int_0^\infty D_s f^*(\cdot) d\mu(s) \right\|_E \leq \int_0^\infty \|D_s f\| \, D_s f \, \|_E \, d\mu(s).$$

Finally it should be observed that the function $\int_0^\infty D_s f(\cdot) d\mu(s)$ does not depend (modulo Lebesgue null sets) on the choice of the representative f. □

Next we recall some notions and results concerning resolvent positive operators which will be needed later. Let E be a Banach lattice. An operator A on E is called **resolvent positive** if there exists a number $\lambda_0 \in \mathbb{R}$ such that $(\lambda_0, \infty) \subset \varrho(A)$ and $R(\lambda, A) \geq 0$ for all $\lambda > \lambda_0$. Denote by

$$s(A) = \sup\{Re\, \lambda : \lambda \in \sigma(A)\}$$

the **spectral bound** of A. It is known that

$$s(A) = \inf\{\lambda \in \mathbb{R} \cap \varrho(A) : R(\lambda, A) \geq 0\}$$

and that, $s(A) \in \sigma(A)$ if $s(A) > -\infty$. Moreover, one hat

$$0 \leq R(\mu, A) \leq R(\lambda, A) \text{ if } \mu > \lambda > s(A) \tag{1.1}$$

and

$$|R(\lambda, A)x| \leq R(Re\, \lambda, A)|x| \tag{1.2}$$

for all $x \in E$, $Re\ \lambda > s(A)$. We say that A **generates an integrated semigroup**, if there exists a stronlgy continuous increasing function $S : [0, \infty) \to \mathcal{L}(E)$ satisfying $S(0) = 0$ such that

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} \, dS(t) \qquad (\lambda > \lambda_0) \tag{1.3}$$

(as an improper strongly defined Riemann-Stieltjes integral) for some $\lambda_0 \geq \varrho(A)$. In that case S is called **the integrated semigroup generated by** A, and it is known that (1.3) converges whenever $Re\, \lambda > s(A)$. Moreover,

$$R(\lambda, A) = \lambda \int_0^\infty e^{-\lambda t} S(t) \, dt \qquad (Re\, \lambda > \max\{s(A), 0\}). \tag{1.4}$$

We need the following lemma.

Lemma 1.2. *Assume that S is bounded. Then $s(A) < 0$.*

Proof. It follows from [**A2**] Proposition 6.1 that $s(A) \leq 0$. Now (1.4) implies that $\|R(\lambda, A)\| \leq M = \sup_{t \geq 0} \|S(t)\|$ for $\lambda > 0$. This implies that $0 \in \varrho(A)$ and $R(0, A) = \lim_{\lambda \downarrow 0} R(\lambda, A) \geq 0$. Then for small $\mu < 0$

$$R(\mu, A) = \sum_{n=0}^\infty (-\mu)^n R(0, A)^{n+1} \geq 0.$$

This implies that $s(A) < 0$. $\square$

It is known that a resolvent positive operator generates a once integrated semigroup if $D(A)$ is dense or E has order continuous norm. We refer to [**A2**] for this and further information. Without any further assumption, it is known ([**A3**, Corollary 4.5]) that every resolvent positive operator A generates a twice integrated semigroup S_2; i.e., $S_2 : [0,\infty) \to \mathcal{L}(E)$ is strongly continuous increasing function such that

$$R(\lambda, A) = \int_0^\infty \lambda^2 e^{-\lambda t} S_2(t)\, dt \qquad (Re\,\lambda > \max\{s(A), 0\}).$$

Of course, if A generates a C_0-semigroup, then $S(t) = \int_0^t T(s)\, ds$ is the once-integrated semigroup and $S_2(t) = \int_0^t \int_0^s T(r) dr\, ds$ the twice integrated semigroup generated by A.

2. The Cesaro operator in spaces with order continuous norm.

In this section we will show that the theory of strongly continuous positive semigroups provides on efficient framework to compute the spectrum of the Cesaro operator in certain rearrangement invariant Banach function spaces. Let E be an $(L^1, L^\infty)-$interpolation space on $(0,\infty)$. For $t \in \mathbb{R}$ and $f \in E$ let $T(t)f(x) = f(e^{-t}x)$ for a.e. $x \in (0,\infty)$. This defines a bounded linear operator $T(t)$ on E satisfying $\|T(t)\|_E \le \max(1, e^t)$, and $\mathcal{T}_E = \{T(t)\}_{t\in\mathbb{R}}$ is a group. The growth bounds of this group are

$$\omega_0^+(\mathcal{T}_E) := \lim_{t\to\infty} \frac{\log \|T(t)\|_E}{t} = \lim_{s\to\infty} \frac{\log \|D_s\|}{\log s} = \overline{\alpha}_E,$$

$$\omega_0^-(\mathcal{T}_E) := \lim_{t\to\infty} \frac{\log \|T(-t)\|_E}{t} = \lim_{s\downarrow 0} \frac{\log \|D_s\|}{\log s} = -\underline{\alpha}_E.$$

Now assume in addition that E has order continuous norm. Using that stepfunctions on bounded intervals are dense in E, it follows immediately that $\mathcal{T}_E$ is a strongly continous group. Let A_E be the generator of $\mathcal{T}_E$. The spectral bounds of A_E are defined by

$$s^+(A_E) = \sup\{Re\,\lambda \,:\, \lambda \in \sigma(A_E)\},$$

$$s^-(A_E) = \sup\{Re\,\lambda \,:\, \lambda \in \sigma(-A_E)\} = s^+(-A_E).$$

Then $-\,\omega_0^-(\mathcal{T}_E) \le -s^-(A_E) \le s^+(A_E) \le \omega_0^+(A_E)$, and

$$\sigma(A_E) \subseteq \{\lambda \in \mathbb{C} : -s^-(A_E) \le Re\,\lambda \le s^+(A_E)\}.$$

Theorem 2.1. *Let E be a rearrangement invariant Banach function space on $(0,\infty)$ with order continuous norm. Then*

$$\sigma(A_E) = \{\lambda \in \mathbb{C} \,:\, \underline{\alpha}_E \le Re\,\lambda \le \overline{\alpha}_E\}.$$

Proof. The proof is divided in three steps:

1. First we show that $\sigma(A_E)$ is invariant under purely imaginary translations. To this end, for $\tau \in \mathbb{R}$ we define the isometry $M_\tau : E \to E$ by $M_\tau f(x) = x^{i\tau} f(x)$ for a.e. $x \in (0,\infty)$ and all $f \in E$. Then $M_\tau^{-1} T(t) M_\tau = e^{-it\tau} T(t)$ for all $t, \tau \in \mathbb{R}$, and so $M_\tau^{-1} A_E M_\tau = A_E - i\tau$ for all $\tau \in \mathbb{R}$.
Hence $\sigma(A_E) = \sigma(M_\tau^{-1} A_E M_\tau) = \sigma(A_E) - i\tau$ for all $\tau \in \mathbb{R}$.

2. Next we will show that $\sigma(A_E) \cap \mathbb{R} = [-s^-(A_E), s^+(A_E)]$. It is clear that $\sigma(A_E) \cap \mathbb{R} \subseteq [-s^-(A_E), s^+(A_E)]$. Moreover, since $\mathcal{T}_E$ consists of positive operators, $s^+(A_E), -s^-(A_E) \in \sigma(A_E)$ (see e.g., [**N**], C - III, Theorem 1.1). Take $\mu \in \rho(A_E) \cap \mathbb{R}$. We claim that either $\mu > s^+(A_E)$ or $\mu < -s^-(A_E)$. Indeed, defining

$$\begin{aligned} I_\mu &= \{f \in E : R(\mu, A_E)|f| \geq 0\} \quad \text{and} \\ J_\mu &= \{f \in E : R(\mu, A_E)|f| \leq 0\}, \end{aligned}$$

it follows from the Theorem on p. 43 in [**G**] that I_μ and J_μ are $\mathcal{T}_E-$invariant bands satisfying $E = I_\mu \oplus J_\mu$. Since any band in E is of the form $\{f \in E : f = 0 \text{ a.e. on } B\}$ for some measurable subset $B \subseteq (0,\infty)$, it is easy to see that the only $\mathcal{T}_E-$invariant bands are E and $\{0\}$. Hence $I_\mu = E$ or $J_\mu = E$. Suppose that $I_\mu = E$. From the definition of I_μ it then follows that $R(\mu, A_E) \geq 0$, which implies that $\mu > s^+(A_E)$ (see [**N**, C - III, Theorem 1.1.]). If $J_\mu = E$, a similar argument shows that $\mu < -s^-(A_E)$, by which the claim is proved.

3. Finally we show that $s^+(A_E) = \overline{\alpha}_E$ and $s^-(A_E) = -\underline{\alpha}_E$. Take $\lambda > s^+(A_E)$. Then (see e.g., [**N**, C - III, Theorem 1.2.])

$$R(\lambda, A_E)f = \int_0^\infty e^{-\lambda t} T(t) f \, dt \quad \text{for all} \quad f \in E.$$

Fix $f \in E$. Observe that $(T(t)f)^* = T(t)f^*$ and that the function $t \mapsto T(t)f^*$ is increasing for $t \geq 0$. Hence

$$R(\lambda, A_E)f^* = \int_0^\infty e^{-\lambda s} T(s) f^* \, ds \geq \int_t^\infty e^{-\lambda s} T(s) f^* \, ds \geq \frac{e^{-\lambda t}}{\lambda} T(t) f^*$$

for all $t \geq 0$ (note that $\lambda > s^+(A_E) \geq -\omega_0^-(\mathcal{T}_E) = \underline{\alpha}_E \geq 0$). This implies that

$$\|T(t)f\|_E \leq \lambda e^{\lambda t} \|R(\lambda, A_E) f^*\|_E \leq \lambda e^{\lambda t} \|R(\lambda, A_E)\|_E \, ||f||_E$$

for all $t \geq 0$. This shows that $\omega_0^+(\mathcal{T}_E) \leq \lambda$, and consequently $\omega_0^+(\mathcal{T}_E) \leq s^+(A_E)$. Hence $\omega_0^+(\mathcal{T}_E) = s^+(A_E)$, i.e., $\overline{\alpha}_E = s^+(A_E)$. Via a similar argument it follows that $s^-(A_E) = -\underline{\alpha}_E$. Combining the results of (1), (2) and (3) we see that

$$\sigma(A_E) = \{\lambda \in \mathbb{C} : \underline{\alpha}_E \leq Re\, \lambda \leq \overline{\alpha}_E\}.$$

$\square$

Recall that the **Cesaro operator** C on $(0,\infty)$ is given by

$$Cf(x) \,=\, \frac{1}{x}\int_0^x f(u)du\,, \qquad x\,>\,0,$$

defined for functions f on $(0,\infty)$ which are integrable on $(0,x)$ for all $x\,>\,0$. If E is a rearrangement invariant Banach function space on $(0,\infty)$ such that $Cf\in E$ for all $f\in E$, we denote the induced operator in E by C_E. Then C_E is a positive, and so a bounded operator on E.

Corollary 2.2. *Let E be a rearrangement invariant Banach function space on $(0,\infty)$ with order continuous norm. Then the Cesaro operator is bounded on E if and only if $\overline{\alpha}_E<1$. In that case the spectrum $\sigma(C_E)$ of C_E is given by*

$$\sigma(C_E)\,=\,\left\{\lambda\in\mathbb{C}\,:\,1\,-\,\overline{\alpha}_E\,\leq\,Re\left(\frac{1}{\lambda}\right)\,\leq 1\,-\,\underline{\alpha}_E\right\}\cup\{0\}.$$

Proof. Assume that $\overline{\alpha}_E<1$. Then $s(A_E)<1$ by Theorem 2.1. Moreover, we have

$$\begin{aligned}(R(1,A_E)f)(x) &= \left(\int_0^\infty e^{-t}T(t)f\;dt\right)(x)\\ &= \int_0^\infty e^{-t}f(e^{-t}x)\;dt = \frac{1}{x}\int_0^x f(u)du\end{aligned}$$

for almost all $x\in E$ and all $f\in E$.

Conversely, assume that the Cesaro operator is bounded on E. Consider the operators $S(t)=\int_0^t e^{-s}T(s)\;ds$. Then

$$(S(t)f)(x)=\frac{1}{x}\int_{e^{-t}x}^x f(u)du\leq(C_Ef)(x)\;a.e.$$

Hence $\|S(t)\|\leq\|C_E\| \qquad (t\geq 0)$. This implies that $s(A_E)<1$ by Lemma 1.2.

Now assume that $\overline{\alpha}_E<1$. Then $C_E=R(1,A_E)$. From the spectral mapping theorem for resolvents and Theorem 2.1. it follows that

$$\begin{aligned}\sigma(C_E) \;&=\; \left\{\frac{1}{1-z}\;:\;z\in\sigma(A_E)\right\}\cup\{0\}\\ \;&=\; \left\{\lambda\in\mathbb{C}\,:\,1\,-\,\overline{\alpha}_E\,\leq\,Re\left(\frac{1}{\lambda}\right)\,\leq 1-\,\underline{\alpha}_E\right\}\cup\{0\}.\end{aligned}$$

□

Remark 2.3. 1. For $E\,=\,L^p(0,\infty)\,,\,1\,<\,p\,<\,\infty$, the result of the above corollary was obtained by D.W. Boyd in [**Bo**]. In the same paper the result of the above corollary is announced but, as fas as we know, a proof was never published.

2. For a large class of rearrangement invariant Banach function spaces E on $(0,\infty)$ it is well-known that boundedness of C_E is equivalent with $\overline{\alpha}_E < 1$. For spaces E with the Fatou property, this result is originally due to D.W. Boyd [**Bo**]. Proofs can also be found in e.g., [**BS**]. In a later section of the present paper we will discuss this equivalence for general (L^1, L^∞)-interpolation spaces.

3. In [**A4**] the semigroup T_E has been used on $E = L^p(0,\infty)$ to produce an example of p-dependent spectrum. It is remarkable that on $L^p \cap L^q(1,\infty)$, $p \neq q$, the type of the semigroup is strictly larger than the spectral bound [**A5**].

4. In the proof of Theorem 2.1 it was not necessary to compute the explicit form of the generator A_E of $\mathcal{T}_E$. However, it is not difficult to show that this generator is given by $A_E f(x) = -xf'(x)$, a.e. $x \in (0,\infty)$, with domain

$$D(A_E) = \{f \in E : f \in AC_{\mathrm{loc}}(0,\infty) \quad \text{and} \quad xf'(x) \in E\}.$$

We leave the details to the reader.

Crucial in the above approach is the strong continuity of the group $\mathcal{T}_E$. As we have seen, if E has order continuous norm, then $\mathcal{T}_E$ is strongly continuous. We will show next that strong continuity of $\mathcal{T}_E$ implies that E has order continuous norm. In the theorem which follows we need not assume that E is an (L^1, L^∞)-interpolation space. In fact, if E is any rearrangement invariant Banach function space on $(0,\infty)$, then $\mathcal{T}_E$ is a group of bounded linear operators in E with $\|T(t)\| \leq \max(1, e^t)$ for all $t \in \mathbb{R}$ (this follows from [**KPS**, Section II, 4.3]).

Theorem 2.4. *Let E be a rearrangement invariant Banach function space on $(0,\infty)$. The group $\mathcal{T}_E$ is strongly continuous if and only if E has order continuous norm.*

In the proof of this theorem we will use a criterion for order continuity of the norm which is implicit in [**KPS**, (Section II, 4.5)]. For the sake of convenience we will state this criterion in the next lemma and provide the proof.

Lemma 2.5. *Let E be a rearrangement invariant Banach function space on $(0,\infty)$. Then E has order continuous norm if and only if*

(i) $\left\| f^* \chi_{(0,\frac{1}{n})} \right\|_E \to 0 \ (n \to \infty)$ *for all* $f \in E$;

(ii) $\left\| f^* \chi_{(n,\infty)} \right\|_E \to 0 \ (n \to \infty)$ *for all* $f \in E$.

Proof. It is clear that order continuity of the norm implies (i) and (ii). Now assume that E satisfies (i) and (ii). First observe that (ii) implies that $f^*(t) \to 0$ as $t \to \infty$ for all $f \in E$, i.e., that $m\{x \in (0,\infty) : |f(x)| > \lambda\} <$

∞ for all $\lambda > 0$ and all $f \in E$. Now suppose that $f_n \in E \quad (n = 1, 2, \dots)$ such that $f_n \downarrow 0$ a.e.. Let $\varepsilon > 0$ be given. By (i), (ii) there exists $N \in \mathbb{N}$ such that $\left\| f_1^* \chi_{(0,1/N)} \right\|_E < \varepsilon$ and $\left\| f_1^* \chi_{(N,\infty)} \right\|_E < \varepsilon$. From the above observation it follows that $f_n^*(1/N) \downarrow 0$ as $n \to \infty$. Hence there exists $n_0 \in \mathbb{N}$ such that $f_n^*(1/N) < \varepsilon$ for all $n \geq n_0$. For $n \geq n_0$ we have

$$\begin{aligned} \|f_n\|_E = \|f_n^*\|_E &\leq \left\| f_n^* \chi_{(0,\frac{1}{N})} \right\|_E + \left\| f_n^* \chi_{[\frac{1}{N},N]} \right\|_E + \left\| f_n^* \chi_{(N,\infty)} \right\|_E \\ &\leq \left\| f_1^* \chi_{(0,1/N)} \right\|_E + \varepsilon \left\| \chi_{[1/N,N]} \right\|_E + \left\| f_1^* \chi_{(N,\infty)} \right\|_E \\ &\leq 2\varepsilon + \varepsilon C \left\| \chi_{[1/N,N]} \right\|_{L^1+L^\infty} \leq (2 + C)\varepsilon, \end{aligned}$$

where $C > 0$ is the embedding constant of E into $(L^1 + L^\infty)(0,\infty)$. This shows that $\|f_n\|_E \downarrow 0$ $(n \to \infty)$, and we may conclude that E has order continuous norm. $\square$

Proof of Theorem 2.4. As observed already above, if E has order continuous norm, then $\mathcal{T}_E$ is strongly continuous. Now assume that $\mathcal{T}_E$ is strongly continuous. Fix $f \in E$ and define $g(s) = f^*(s-1)$ for $s > 1$ and $g(s) = 0$ for $0 < s \leq 1$. Then $g^* = f^*$, so $g \in E$. Since $T(t)g(s) = 0$ for $0 < s \leq e^t$, it follows that $|T(t)g - g| \geq g\chi_{(1,e^t]}$ for all $t > 0$, and so $\left\| g\chi_{(1,e^t]} \right\|_E \leq \|T(t)g - g\|_E$ for all $t > 0$. Hence $\left\| g\chi_{(1,e^t]} \right\|_E \to 0$ as $t \downarrow 0$. Now $(g\chi_{(1,e^t]})^* = f^*\chi_{(0,e^t-1]}$ implies that $\left\| f^*\chi_{(0,e^t-1]} \right\|_E \to 0$ as $t \downarrow 0$, which shows that $\left\| f^*\chi_{(0,1/n)} \right\|_E \to 0 \quad (n \to \infty)$. It remains to show that $\left\| f^*\chi_{(n,\infty)} \right\|_E \to 0$ $(n \to \infty)$. Define $n_0 = 0$ and $n_k = 3(n_{k-1} + 1)$ for $k = 1, 2, \dots$, and let

$$h(s) = \begin{cases} f(s + k - n_k - 1) & \text{if } n_k < s \leq n_k + 1\,,\ k = 1, 2 \dots \\ 0 & \text{otherwise.} \end{cases}$$

Then $h^* = f^*$, so $f \in E$. Now let $\varepsilon > 0$ be given. By the strong continuity of $\mathcal{T}_E$, there exists $0 < t_0 \leq 1$ such that $\|T(-t_0)h - h\|_E < \varepsilon$. Take k_0 such that $e^{-t_0} < n_k\ (n_k + 1)^{-1}$ for all $k \geq k_0$. Now observe that h is supported on the set $\bigcup_{k=1}^{\infty} (n_k\,,\, n_k + 1]$ and $T(-t_0)h$ is supported on $\bigcup_{k=1}^{\infty} (e^{-t_0} n_k\,,\, e^{-t_0}(n_k + 1)]$. Since, by the definition of the n_k's and by the choice of k_0 , $n_{k-1} + 1 < e^{-t_0} n_k < e^{-t_0}(n_k + 1) < n_k$ for all $k \geq k_0$, it follows that $h\chi_{(n_{k_0},\infty)}$ and $T(-t_0)\chi_{(n_{k_0},\infty)}$ are disjointly supported. Hence,

$$|T(-t_0)h - h| \geq |T(-t_0)h - h|\chi_{(n_{k_0},\infty)} \geq h\chi_{(n_{k_0},\infty)}$$

which implies that $\left\|h\chi_{(n_{k_0},\infty)}\right\|_E < \varepsilon$. Now

$$(h\chi_{(n_{k_0},\infty)})^* = f^*\chi_{(k_0-1,\infty)}\,, \text{ so } \left\|f^*\chi_{(k_0-1,\infty)}\right\|_E < \varepsilon.$$

This shows that $\left\|f^*\chi_{(n,\infty)}\right\|_E \to 0$ $(n \to \infty)$. Via Lemma 2.5 it now follows that E has order continuous norm. $\square$

The above theorem shows in particular that it is not possible to compute the spectrum of the Cesaro operator using the theory of strongly continuous (semi)groups, as in the proof of Theorem 2.1, if the space E does not have order continuous norm. This is one of the motivations for the investigations in the next section. In particular we will need an appropriate substitute for the spectral decomposition theorem for generators of strongly continuous groups of G. Greiner [**G**].

3. Spectral decomposition.

Througout this section we assume that A is an operator on a complex Banach lattice E such that $\pm A$ is resolvent positive (we say that A is **resolvent bipositive**). Then we know from the proof of [**N**, C-III Corollary 1.6] that $\sigma(A) \neq \emptyset$. Denote by

$$s(A) = \sup\{Re\ \lambda : \lambda \in \sigma(A)\}$$

the **spectral bound** of A. Then we know that

$$\sigma(A) \subset \{\lambda \in \mathbb{C} : -s(-A) \leq Re\ \lambda \leq s(A)\}; \tag{3.1}$$

$$s(A), -s(-A) \in \sigma(A); \tag{3.2}$$

$$R(\lambda, A) \geq 0 \text{ if } \lambda > s(A); \tag{3.3}$$

$$R(\lambda, A) \leq 0 \text{ if } \lambda < -s(-A). \tag{3.4}$$

Definition 3.1. Let $\mu \in (-s(-A),\ s(A))$. We say that A **allows a spectral decomposition with respect to μ** if there exists a band decomposition $E = E_1 \oplus E_2$ such that $R(\lambda, A)E_i = E_i \quad (i = 1, 2)$ for all $\lambda \in \varrho(A)$ and such that the part A_i of A in E_i satisfies

$$\sigma(A_1) = \{\lambda \in \sigma(A) : Re\,\lambda < \mu\},$$
$$\sigma(A_2) = \{\lambda \in \sigma(A) : Re\,\lambda > \mu\}.$$

In particular, in that case $\pm A_i$ is resolvent positive and $s(A_1) < \mu$ and $-s(-A_2) > \mu$.

The main result of this section is the following:

Theorem 3.2. *Let $\mu \in (-s(-A),\ s(A)) \cap \varrho(A)$. Then A allows a spectral decomposition with respect to μ if one of the following two conditions is satisfied.*

(a) *The operator A satisfies*

(K_μ) $x \in D(A)$ *implies* $|x| \in D(A)$ *and* $|(\mu - A)|x|\ | \leq |(\mu - A)x|$;

(b) *The domain $D(A)$ is dense.*

The condition (K_μ) is a weak form of Kato's equality which we will discuss later.

For the proof we can assume that $\mu = 0$ which we will do in the following. It is known that for $\lambda \in \varrho(A) \cap \mathbb{R}$ one has $R(\lambda, A) \geq 0$ if and only if $\lambda > s(A)$ (see [**N**, C-III Theorem 1.1]). In view of this, following Greiner's idea [**N**, C-III Theorem 4.8], we set

$$E_1 = \{x \in E : R(0, A)|x| \geq 0\};$$
$$E_2 = \{x \in E : R(0, A)|x| \leq 0\}.$$

Lemma 3.3. a) *E_1 and E_2 are closed ideals in E.*

b) *The operator A allows a spectral decomposition with respect to* 0 *whenever $E_1 + E_2 = E$.*

Remark 3.4. Lemma 3.3 a) is true without additional hypotheses. Conditions (a), (b) of Theorem 3.2 are used to show that $E = E_1 + E_2$. Our point is to replace in Greiner's argument the semigroup (which does not need to exist here, see Example 3.13) by the twice integrated semigroup. Moreover, we simplify the argument using the following description of the abscissa of the Laplace transform (see [**ANS**, Proposition 1.1] or [**HP**, Sec. 6.2] for a proof).

Lemma 3.5. *Let X be a Banach space and $f : [0, \infty) \to X$ be continuous. Then abs $(f) \leq 0$ (i.e., $\hat{f}(\lambda) := \lim_{t\to\infty} \int_0^t e^{-\lambda s} f(s)\, ds$ exists whenever $Re\, \lambda > 0$) if and only if*

$$\sup_{t \geq 0} e^{-wt} \left\| \int_0^t f(s)\ ds \right\| < \infty, \quad \textit{for all}\ \ w > 0.$$

Proof of Lemma 3.3. Let S be the twice integrated semigroup generated by A; that is, $S : [0, \infty) \to \mathcal{L}(E)$ is strongly continuous and there exists $\omega \geq 0$ such that $(\omega, \infty) \subset \varrho(A)$, $\sup_{t \geq 0} \|e^{-\omega t} S(t)\| < \infty$ and,

$$R(\lambda, A)x = \lambda^2 \int_0^\infty e^{-\lambda t} S(t)x\ dt \quad (Re\, \lambda > \omega,\ x \in E).$$

Then $S(t) \geq 0$ and $S(t)R(\lambda, A) = R(\lambda, A)S(t)$ for all $\lambda \in \varrho(A),\ t \geq 0$ and for all $x \in E$,

$$\int_0^t S(s)x\ ds \in D(A) \text{ and } A \int_0^t S(s)x\ ds = S(t)x - \frac{t^2}{2} x. \tag{3.5}$$

See Section 1 and [**A3**] for these notions and results. We claim that for $x \in E$,

$$x \in E_1 \text{ if and only if } \mathit{abs}\ (S(\cdot)|x|) \le 0. \tag{3.6}$$

In fact, if $x \in E_1$, then by (3.5) (since $0 \in \varrho(A)$),

$$\begin{aligned}\int_0^t S(s)|x|\, ds &= \frac{t^2}{2}R(0,A)|x| - S(t)R(0,A)|x| \\ &\le \frac{t^2}{2}R(0,A)|x| \qquad (t \ge 0).\end{aligned}$$

Hence $\mathit{abs}\ (S(\cdot)|x|) \le 0$ by Lemma 3.5. Conversely, assume $\mathit{abs}\ (S(\cdot)|x|) \le 0$. Then $r(\lambda) = \lambda^2 \int_0^\infty e^{-\lambda t} S(t)|x|\, dt \quad (Re\ \lambda > 0)$ is holomorphic and for $\lambda > \omega$ one has $r(\lambda) = R(\lambda, A)|x|$, or equivalently, $\lambda A^{-1} r(\lambda) - r(\lambda) = A^{-1}|x|$. This remains true for $Re\ \lambda > 0$ by the uniqueness of holomorphic extensions. Hence $R(\lambda, A)|x| = r(\lambda) \ge 0$ for $\lambda \in (0, \varepsilon)$ where $\varepsilon > 0$ such that $(0,\varepsilon) \subset \varrho(A)$. This implies $R(0,A)|x| \ge 0$.

a) It follows from (3.6) and Lemma 3.5 that E_1 is an ideal. Closedness follows from the definition. Replacing A by $-A$ we see that also $E_2 = \{x : R(0,-A)|x| \ge 0\}$ is a closed ideal.

b) It is clear that $E_1 \cap E_2 = \{0\}$. Now assume that $E_1 + E_2 = E$. Then E_1 and E_2 are projection bands.

Let $\lambda_0 > s(A)$. Since $R(\lambda_0, A) \ge 0$ and $R(\lambda_0, A)S(t) = S(t)R(\lambda_0, A)$ it follows from (3.6) and Lemma 3.5 that $R(\lambda_0, A)E_i \subset E_i \quad (i = 1, 2)$. Hence $R(\lambda_0, A)P_1 = P_1 R(\lambda_0, A)$ where P_1 denotes the band projection onto E_1. It follows easily that $x \in D(A)$ implies $P_1 x \in D(A)$ and $AP_1 x = P_1 Ax$; and this in turn implies $R(\lambda, A)P_1 = P_1 R(\lambda, A)$ for all $\lambda \in \varrho(A)$. Thus $R(\lambda, A)E_i \subset E_i \quad (i = 1, 2)$ for all $\lambda \in \varrho(A)$. Hence $\varrho(A) = \varrho(A_1) \cap \varrho(A_2)$. Finally, by the first part of the proof, $Q(\lambda)x = \lambda^2 \int_0^\infty e^{-\lambda t} S(t)x\, dt$ exists for all $x \in E_1$ and $Re\,\lambda > 0$. Thus $Q(\lambda) \in \mathcal{L}(E_1)$, $A^{-1}Q(\lambda) = Q(\lambda)A^{-1}$ and $\lambda A^{-1}Q(\lambda)x - Q(\lambda)x = A^{-1}x$ if $Re\,\lambda > \omega$ and so for $Re\,\lambda > 0$ by holomorphy. This implies that $\lambda \in \varrho(A_1)$ and $Q(\lambda) = (\lambda - A_1)^{-1}$ if $Re\ \lambda > 0$. Similarly, $\{\lambda : Re\,\lambda < 0\} \subset \varrho(A_2)$. ☐

Lemma 3.6. *If* (K_0) *holds, then* $E = E_1 + E_2$.

Proof. Let $0 \le x \in E$ and $y = R(0,A)x$. Then $|y| \in D(A)$ and $|A|y|\,| \le |Ay| = x$. Thus $x_1 := \frac{1}{2}(x - A|y|) \ge 0$ and $x_2 := \frac{1}{2}(x + A|y|) \ge 0$. Moreover, $R(0,A)x_1 = \frac{1}{2}(R(0,A)x + |y|) = \frac{1}{2}(y + |y|) = y^+ \ge 0$. Thus $y_1 \in E_1$. Similarly, $R(0,A)x_2 = -y^- \le 0$ so that $x_2 \in E_2$. Clearly, $x = x_1 + x_2$. We have shown that $E_+ \subset E_{1+} + E_{2+}$. This implies the claim. ☐

Now we prove Theorem 3.2. Under the hypothesis (a), the proof is complete. Case (b) follows from the following lemma.

Lemma 3.7. *Assume that $D(A)$ is dense. Then $E = E_1 + E_2$.*

Proof. We can assume that E is a real Banach lattice.

a) Let $x \in D(A^3)$, $\sigma = \operatorname{sign} q(x) \in \mathcal{L}(E'')$ where $q : E \to E''$ is the canonical embedding. We show that

$$\langle \sigma Ax,\ \varphi\rangle = \langle |x|,\ A'\varphi\rangle \qquad (\varphi \in D(A'^3)). \tag{3.7}$$

In fact, it follows from the resolvent equation that $R(\lambda, A)$ is decreasing on $(s(A), \infty)$. Consequently, $\lambda R(\lambda, A)y = R(\lambda, A)Ay + y$ is bounded on $[s(A)+1, \infty)$ and so $R(\lambda, A)y \to 0 \quad (\lambda \to \infty)$ for $y \in D(A)$. This implies that $\lambda R(\lambda, A)y \to y \quad (\lambda \to \infty)$ if $y \in D(A^2)$. Finally,

$$\lambda^2 R(\lambda, A)y - \lambda y = \lambda R(\lambda, A)Ay \to Ay \qquad (\lambda \to \infty)$$

if $y \in D(A^3)$. For the same reason,

$$\lambda^2 R(\lambda, A)'\varphi - \lambda\varphi \to A'\varphi \qquad (\lambda \to \infty)$$

if $\varphi \in D(A'^3)$. Consequently, if $0 \le \varphi \in D(A'^3)$, then

$$\begin{aligned}
\langle \sigma Ax, \varphi\rangle &= \lim_{\lambda\to\infty} \langle \sigma(\lambda^2 R(\lambda, A)x - \lambda x)\ ,\ \varphi\rangle \\
&= \lim_{\lambda\to\infty} \langle \sigma(\lambda^2 R(\lambda, A)x) - \lambda |x|,\ \varphi\rangle \\
&\le \limsup_{\lambda\to\infty}\ \langle \lambda^2 R(\lambda, A)|x| - \lambda|x|,\ \varphi\rangle \\
&= \limsup_{\lambda\to\infty}\ \langle |x|,\ \lambda^2 R(\lambda, A)'\varphi - \lambda\varphi\rangle \\
&= \langle |x|,\ A'\varphi\rangle.
\end{aligned}$$

Replacing A by $-A$ gives (3.7) for $0 \le \varphi \in D(A'^3)$. Let $\mu_0 > s(A)$. Since $D(A'^3) = R(\mu_0, A)'^3 E' = D(A'^3) \cap E'_+ - D(A'^3) \cap E'_+$ we obtain (3.7) for all $\varphi \in D(A'^3)$.

b) Next we assume that $\mu = 0 \in \varrho(A)$ as before. Given $y \in D(A^2)$, we show that there exists $z'' \in E''$ such that $|z''| \le |y|$ and

$$|R(0, A)y| = R(0, A)''z''. \tag{3.8}$$

In fact, let $x = R(0, A)y$, $\sigma = \operatorname{sign} q(x)$, $z'' = \sigma y$. Let $\psi \in D(A'^2)$, $\varphi = R(0, A)'\psi$. Then by (3.7),

$$\begin{aligned}
\langle R(0, A)''z'',\ \psi\rangle &= \langle z'',\ R(0, A)'\psi\rangle \\
&= \langle z'', \varphi\rangle = -\langle \sigma Ax,\ \varphi\rangle = -\langle |x|,\ A'\varphi\rangle \\
&= \langle |x|,\ \psi\rangle.
\end{aligned}$$

Since $D(A'^2) = (R(0, A)')^2 E'$ separates points, (3.8) follows.

c) Let $y \in D(A^2)_+$. Then $(R(0,A)y)^+ \in E_1$. In fact,

$$\begin{aligned}(R(0,A)y)^+ &= 1/2(|R(0,A)y| + R(0,A)y) \\ &= 1/2(R(0,A)''z'' + R(0,A)y) \\ &= R(0,A)''y_1''\end{aligned}$$

where $y_1'' = 1/2(y + z'') \geq 0$. It follows from (3.5) that

$$\left(\int_0^t S(s)\, ds\right)'' = t^2/2\ R(0,A)'' - S(t)''R(0,A)''.$$

Hence

$$\begin{aligned}\left(\int_0^t S(s)\, ds\right)'' y_1'' &= t^2/2\ R(0,A)''y_1'' - S(t)''R(0,A)''y_1'' \\ &= t^2/2\ (R(0,A)y)^+ - S(t)(R(0,A)y)^+ \\ &\leq t^2/2\ (R(0,A)y)^+.\end{aligned}$$

Hence

$$\begin{aligned}\left\|\int_0^t S(s)(R(0,A)y)^+\, ds\right\| &= \left\|\int_0^t S(s)R(0,A)''\ y_1''\ \ ds\right\| \\ &\leq t^2/2\ \|R(0,A)\|\ \|(R(0,A)y)^+\|.\end{aligned}$$

Thus abs $(S(\cdot)(R(0,A)y)^+) \leq 0$ by Lemma 3.5.

It follows from (3.6) that $(R(0,A)y)^+ \in E_1$.

d) Let $y \in D(A^2)_+$. Then, applying c) to $(-A)$ we have

$$(R(0,A)y)^- = (R(0,-A)y)^+ \in E_2.$$

Thus $R(0,A)y = (R(0,A)y)^+ - (R(0,A)y)^- \in E_1 + E_2$.

Since for $\mu > s(A)$, $D(A^2) = R(\mu,A)^2E = R(\mu,A)^2E_+ - R(\mu,A)^2E_+$ one has $D(A^2) = D(A^2)_+ - D(A^2)_+$. Thus $D(A^3) = R(0,A)D(A^2) \subset E_1 + E_2$. Consequently, $E = \overline{D(A^3)} \subset \overline{E_1 + E_2} = E_1 + E_2$, the sum of closed ideals being closed [**S**, III.1.2]. □

Let $T \in \mathcal{L}(E)$. A band B of E is called **reducing for** T if $TB \subset B$ and $TB^d \subset B^d$ (equivalently, T commutes with the band projection onto B).

Corollary 3.8. *Let A be an operator on E such that*

(a) *$\pm A$ is resolvent positive;*
(b) *$R(\lambda, A)$ has no nontrivial reducing band for some (equivalently all) $\lambda \in \varrho(A)$.*
(c) *$D(A)$ is dense or A satisfies (K_μ) for all $\mu \in \mathbb{R}$.*

Then $\sigma(A) \cap \mathbb{R} = [-s(-A),\ s(A)]$.

Next we give several comments concerning the inequality (K_μ). Greiner [**G**] (see also [**N**, C-III Section 4]) uses Kato's equality

$$A|x| = Re\,((\text{sign }\bar{x})Ax) \tag{K}$$

in his proof of the decomposition theorem. It holds for all $x \in D(A)$ if A is the generator of a positive C_0-group on a σ-order complete Banach lattice. In particular, $D(A)$ is a sublattice of E. Here, for $x \in E$, $\bar{x} = Re\,x - iIm\,x$ denotes the complex conjugate of x. Moreover, for $x \in E$, the operator sign $\bar{x} \in \mathcal{L}(E)$ is uniquely determined by the properties

$$\begin{aligned}
(\text{sign }\bar{x})x &= |x| \\
|(\text{sign }\bar{x})y| &\le |y| \qquad (y \in E) \\
(\text{sign }\bar{x})y &= 0 \quad \text{if } \ y \perp x.
\end{aligned}$$

It is clear that $A - \mu$ satisfies (K) for all $\mu \in \mathbb{R}$ if A satisfies (K). Thus condition (K) implies condition (K_μ) for all $\mu \in \mathbb{R}$.

However, the converse is not true. In fact, in the following proposition we show that the adjoint A' of the generator A of a positive C_0-group always satisfies (K_μ) for all $\mu \in \mathbb{R}$. However, we show by an example that (K) may be violated.

Proposition 3.9. *Let B be the generator of a positive C_0-group T on a Banach lattice F and let $A = B'$ on $E = F'$. Then A satisfies (K_μ) for all $\mu \in \mathbb{R}$.*

Proof. We can assume $\mu = 0$. Recall that $D(B') = \text{Fav }(B') = \{\varphi \in F' : \limsup_{t\downarrow 0} 1/t\|T(t)'\varphi - \varphi\| < \infty\}$, see [**EN**, Chapter II.5.19] or [**CH**]. Let $\varphi \in D(B')$. Let $0 \le x \in E$, $1 \ge t \ge 0$. Then

$$\begin{aligned}
\langle |T(t)'\varphi - \varphi|,\ x\rangle &= \sup_{|y|\le x} |\langle T(t)'\varphi - \varphi,\ y\rangle| \\
&= \sup_{|y|\le x} \left| \int_0^t \langle B'\varphi, T(s)y\rangle\, ds \right| \\
&\le \int_0^t \langle |B'\varphi|, T(s)x\rangle\, ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left\langle \frac{1}{t}(T(t)'|\varphi| - |\varphi|),\ x \right\rangle &= \left\langle \frac{1}{t}(|T(t)'\varphi| - |\varphi|),\ x \right\rangle \\
&\le \left\langle \frac{1}{t}|T(t)'\varphi - \varphi|,\ x \right\rangle \\
&\le \frac{1}{t}\int_0^t \langle |B'\varphi|, T(s)x\rangle\, ds \\
&\le \|B'\varphi\| \sup_{0<t\le 1} \|T(s)x\|.
\end{aligned}$$

Thus, $|\varphi| \in \text{Fav }(B') = D(B')$. Moreover,

$$\begin{aligned}\langle B'|\varphi|\ ,\ x\rangle &= \lim_{t\downarrow 0t}\frac{1}{t}\langle T(t)'|\varphi| - |\varphi|\ ,\ x\rangle \\ &\leq \overline{\lim_{t\downarrow 0t}}\ \frac{1}{t}\int_0^t \langle |B'\varphi|\ ,\ T(s)x\rangle\ ds \\ &\leq \langle |B'\varphi|,\ x\rangle.\end{aligned}$$

Hence $B'|\varphi| \leq |B'\varphi|$. □

Remark 3.10. It follows from Proposition 3.9 that Theorem 3.2 also holds if A is the adjoint of a generator B of a positive C_0-group. But of course, this can be directly seen by applying Theorem 3.2 to B.

Next we show that in the situation of Proposition 3.7 it can happen that $|B'|\varphi|\ | \neq |B'\varphi|$ for some $\varphi \in D(B')$; in particular, B' does not satisfy (K) in general.

Example 3.11. Consider in the space $E = C_0(\mathbb{R})$, equipped with the sup-norm, the C_0-group $(T(t))_{t\in\mathbb{R}}$ given by $T(t)f(x) = f(x+t)$ for all $x, t \in \mathbb{R}$. The generator B of this group is given by $Bf = f'$ with $D(B) = \{f \in C_0^1(\mathbb{R}) : f' \in C_0(\mathbb{R})\}$. Identifying the dual space $C_0(\mathbb{R})'$ with the space $M_b(\mathbb{R})$ of all bounded Borel measures on $\mathbb{R}$, it is easy to see that $D(B') = \{\mu \in M_b(\mathbb{R}) : D\mu \in M_b(\mathbb{R})\}$ and $B'\mu = -D\mu$ for all $\mu \in D(B')$, where $D\mu$ denotes the distributional derivate of the measure μ. As is well-known, every $\mu \in D(B')$ is absolutely continuous with respect to Lebesgue measure and is of the form $\mu = fdx$ with $f \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Moreover, for such measures μ we have $D\mu = df$ (where df denotes the Borel measure induced by f). Now take $f = -1_{(-1,0]} + 1_{(0,1]}$ and $\mu = fdx$. Then $\mu \in D(B')$ and $B'\mu = -D\mu = \delta_{-1} - 2\delta_0 + \delta_1$, hence $|B'\mu| = \delta_{-1} + 2\delta_0 + \delta_1$ (here δ_p denotes the Dirac measure at the point p). Since $|\mu| = |f|dx$, it follows that $B'|\mu| = -D|\mu| = -\delta_{-1} + \delta_1$, hence $|B'|\mu|| = \delta_{-1} + \delta_1$. This shows that $|B'|\mu|| \neq |B'\mu|$, so B' does not satisfy the Kato equality.

Remark 3.12. a) In Example 3.11 one has $\overline{D(B')} = L^1(\mathbb{R})$, and the part A of B' in $L^1(\mathbb{R})$ generates a positive C_0-group (given by the right shift). Thus the part of B' in $\overline{D(B')}$ does satisfy (K).

b) More generally, $\overline{D(A)}$ is a band if A is a resolvent positive operator on a KB-space ([**AB**, Appendice]).

We conclude giving an example where $\pm A$ is resolvent positive, E is a reflexive Banach lattice, but neither A nor $-A$ generate C_0-semigroups.

Example 3.13. a) Let $(F, \|\ \|_F)$ be a Banach function space on $(0,\infty)$ corresponding to the function norm $\|\ \|_F$ given by

$$\|f\|_F = \|f\|_{L^p(0,\infty)} + \|f\|_{L^q(1,\infty)}$$

where $1 < p < q < \infty$. Then $(T(t)f)(x) = f(e^t x)$ defines a lattice C_0-semigroup on F. Let B be its generator. Then $\sigma(B) = \{\lambda \in \mathbb{C} : Re\ \lambda = -\frac{1}{p}\}$ and $R(\lambda, B) \geq 0$ for $\lambda > -\frac{1}{p}$, $R(\lambda, B) \leq 0$ for $\lambda < -\frac{1}{p}$. But $-B$ is not generator of a C_0-semigroup.

b) Taking $E = F \oplus F$ and $A = B \oplus (-B)$ one obtains the desired example.

Proof of a). Let $G = L^p(0,\infty)$. Then $(U(t)f)(x) = f(e^t x)$ defines a positive C_0-group on G. Let A be its generator. Then $\sigma(A) = \{\lambda \in \mathbb{C} : Re\ \lambda = -\frac{1}{p}\}$ and $(R(\lambda, A)f)(x) = \int_0^\infty e^{-\lambda t} f(e^t x)\ dt = x^\lambda \int_x^\infty f(s)s^{-\lambda-1}\ ds$ for $\lambda > -\frac{1}{p}$.

One has $U(t)F \subset F$ and $T(t) = U(t)_{|F} \quad (t \geq 0)$. Thus B is the part of A in F.

Observe that $R(\lambda, A)G \subset F \quad (\lambda > -\frac{1}{p})$. In fact, let $0 \leq f \in G$, $g(x) = (R(\lambda, A)f)(x) = x^\lambda \int_x^\infty f(s)s^{-\lambda-1}\ ds$. Then

$$g(x) \leq x^\lambda \|f\|_p \left(\int_x^\infty s^{(-\lambda-1)p'}\ ds \right)^{\frac{1}{p'}} \leq \text{const}\ \cdot \|f\|_p \cdot x^{-\frac{1}{p}}$$

for $x \geq 1$ (where $\frac{1}{p} + \frac{1}{p'} = 1$). Thus $g \cdot 1_{(1,\infty)} \in L^\infty(1,\infty) \cap L^p(1,\infty) \subset L^q(1,\infty)$.

It follows that $(-\frac{1}{p}, \infty) \subset \varrho(B)$ and $R(\lambda, B) = R(\lambda, A)_{|F} \geq 0 \quad (\lambda > -\frac{1}{p})$.

Since, for $\lambda > -\frac{1}{p}$, $D(A) = R(\lambda, A)G \subset F$, we have $R(\lambda, A)G \subset F$ for all $\lambda \in \varrho(A)$. Thus, for $\lambda < -\frac{1}{p}$, $\lambda \in \varrho(B)$ and $R(\lambda, B) = R(\lambda, A)_{|F} \leq 0$.

Assume that $-B$ generates a C_0-semigroup $(T(-t))_{t\geq 0}$. Then $T(t)f = \lim\limits_{n\to\infty} (I + \frac{t}{n}B)^{-n} f$ in F for all $f \in F$. Since F is continuously embedded into G, it follows that $T(t) = U(-t)$.

However, $U(-t)F \not\subset F$, $t > 0$, which is a contradiction. In fact, let $-\frac{1}{p} < \alpha < -\frac{1}{q}$ and $f(x) = (1-x)^\alpha 1_{(0,1)}(x)$. Then

$$\int_0^\infty |f(x)|^p\ dx = \int_0^1 (1-x)^{\alpha p}\ dx = \int_0^1 y^{\alpha p}\ dy = \frac{1}{\alpha p + 1} < \infty.$$

Thus $f \in F$. However, for $t > 0$, $U(-t)f \notin L^q(1,\infty)$. In fact,

$$\begin{aligned} \|U(-t)f\|^q_{L^q(1,\infty)} &= \int_1^\infty f(e^{-t}x)^q\ dx = \int_{e^{-t}}^1 (1-y)^{\alpha q}\ dy\ e^t \\ &= \int_0^{1-e^{-t}} y^{\alpha q}\ dy\ e^t = \infty \end{aligned}$$

since $\alpha q + 1 < 0$. $\square$

In Section 4 a whole class of operators A is given for which $\pm A$ is resolvent positive but $D(A)$ is not dense. The preceding example has the additional remarkable property that the semigroup operators $T(t)$ which always exist as operators from $D(A^2)$ into E are not bounded on E. Recall, that A

generates a twice integrated semigroup S and $T(t)x = \frac{d^2}{dt^2}S(t)x$ exists for all $x \in D(A^2)$.

4. The Cesaro operator on arbitrary interpolation spaces.

In this section we shall illustrate how the theory developed in the previous section can be used to obtain results analogous to the ones in Section 2, but now for a much larger class of function spaces.

Let E be an exact (L^1, L^∞)-interpolation space on $(0,\infty)$. As in Section 2 we denote by $\mathcal{T}_E = \{T(t)\}_{t\in\mathbb{R}}$ the group defined by $T(t)f(x) = f(e^{-t}x)$. Since we do not assume that E has order continuous norm, the group $\mathcal{T}_E$ need not be strongly continuous (see Theorem 2.4). For $t \geq 0$ define

$$S_+(t)f(x) = \int_0^t T(s)f(x)\ ds = \int_0^t f(e^{-s}x)\ ds\ ,\ x > 0\ ,\ f \in E.$$

Using that E is an (L^1, L^∞)-interpolation space, it follows that $S_+(t)$ is a bounded linear operator in E and $\|S_+(t)\|_E \leq e^t - 1$ for all $t \geq 0$. Moreover, $\|S_+(t+h) - S_+(t)\|_E \leq he^{t+h}$ for all $t, h \geq 0$. Similarly, if we define

$$S_-(t)f(x) = \int_0^t T(-s)f(x)\ ds = \int_0^t f(e^s x)\ ds,\ x > 0,\ f \in E,$$

then $\|S_-(t)\|_E \leq \max(1 - e^{-t}, t)$ and $\|S_-(t+h) - S_-(t)\|_E \leq he^{-t}$ for all $t, h \geq 0$. We show next that $\{S_+(t)\}_{t\geq 0}$ and $\{S_-(t)\}_{t\geq 0}$ are actually integrated semigroups in E. To this end, for $\lambda \in \mathbb{C}$ with $Re\ \lambda > 1$ define

$$R(\lambda)f(x) = x^{-\lambda}\int_0^x u^{\lambda-1}f(u)du,\ x > 0,\ f \in E.$$

Via interpolation, $R(\lambda)$ is a bounded linear operator in E and

$$\|R(\lambda)\|_E \leq (Re\,\lambda - 1)^{-1}. \tag{4.1}$$

Similarly, for $Re\,\lambda < 0$ we define

$$R(\lambda)f(x) = -x^{-\lambda}\int_x^\infty u^{\lambda-1}f(u)du,\ x > 0,\ f \in E;$$

then

$$\|R(\lambda)\|_E \leq (-Re\,\lambda)^{-1}. \tag{4.2}$$

Now it is not difficult to verify that $R(\lambda) = R(\lambda, A_E)$ on $\{Re\,\lambda > 1\} \cup \{Re\,\lambda < 0\}$, where $A_E : D(A_E) \to E$ is given by

$$D(A_E) = \{f \in E : f \in AC_{\text{loc}}(0,\infty),\ xf'(x) \in E\},\ A_E f(x) = -xf'(x).$$

Moreover, integration by parts shows that

$$R(\lambda, A_E)f = \lambda \int_0^\infty e^{-\lambda t} S_+(t) f dt,\ Re\,\lambda > 1,\ f \in E$$

and

$$R(\lambda, -A_E)f = \lambda \int_0^\infty e^{-\lambda t} S_-(t) f dt,\ Re\,\lambda > 0,\ f \in E.$$

Hence, $\{S_+(t)\}_{t\geq 0}$ and $\{S_-(t)\}_{t\geq 0}$ are the integrated semigroups generated by A_E and $-A_E$ respectively. In particular, $\pm A_E$ are resolvent positive.

Remark 4.1. From the estimates (4.1) and (4.2) on $R(\lambda, A_E)$ above, it follows that the part of A_E in $\overline{D(A_E)}$ generates a strongly continuous group (cf. [**A1**, Corollary 4.2]). It is easy to see that this group is the restriction of $\mathcal{T}_E$ to $\overline{D(A_E)}$. This implies that $\overline{D(A_E)} = \{f \in E : \lim_{t\to 0} \|T(t)f - f\|_E = 0\}$. In combination with Theorem 2.4, this shows that $D(A_E)$ is dense if and only if E has order continuous norm.

Theorem 4.2. *Let E and A_E be as above. Then*

$$\sigma(A_E) = \{\lambda \in \mathbb{C} : \underline{\alpha}_E \leq Re\,\lambda \leq \overline{\alpha}_E\},$$

where $\underline{\alpha}_E$ and $\overline{\alpha}_E$ denote the lower- and upper-Boyd indices of E.

Proof. We divide the proof in five steps.

(1) As before, we denote $s^+(A_E) = \sup\{Re\,\lambda : \lambda \in \sigma(A_E)\}$ and $s^-(A_E) = s^+(-A_E)$. Then $\sigma(A_E) \subseteq \{\lambda \in \mathbb{C} : -s^-(A_E) \leq Re\,\lambda \leq s^+(A_E)\}$. Moreover, $s^+(A_E),\ -s^-(A_E) \in \sigma(A_E)$ as $\pm A_E$ are resolvent positive.

(2) Next we may use Corollary 3.8 to conclude that

$$\sigma(A_E) \cap \mathbb{R} = [-s^-(A_E),\ s^+(A_E)].$$

Indeed, from the explicit form of A_E given above it follows immediately that A_E satisfies the Kato equality and hence (K_0). Furthermore, using the representation of $R(\lambda, A_E)$ as an integral operator for $Re\,\lambda > 1$ it is easily seen that $R(\lambda, A_E)$ has no nontrivial reducing bands.

(3) For $\tau \in \mathbb{R}$ we define the isometry M_τ in E by $M_\tau f(x) = x^{i\tau} f(x),\ x > 0$. Then $M_\tau^{-1} A_E M_\tau = A_E - i\tau$, and hence $\sigma(A_E) = \sigma(A_E) - i\tau$ for all $\tau \in \mathbb{R}$.

A combination of (1), (2) and (3) already shows that

$$\sigma(A_E) = \{\lambda \in \mathbb{C} : -s^-(A_E) \leq Re\,\lambda \leq s^+(A_E)\}.$$

(4) We will show now that $s^+(A_E) = \overline{\alpha}_E$. Take $\omega > \overline{\alpha}_E$. From the definition of $\overline{\alpha}_E$ it follows that there exists $M_\omega > 0$ such that $\|D_s\|_E \leq M_\omega s^\omega$ for all $s \geq 1$. Since

$$S_+(t)f(x) = \int_0^t f(e^{-s}x)\ ds = \int_1^{e^t} D_u f(x)\frac{du}{u}$$

for all $f \in E,\ x > 0$ and $t \geq 0$, it follows from Lemma 1.1 that

$$\|S_+(t)\|_E \leq \int_1^{e^t} \|D_u\|_E \frac{du}{u} \leq \frac{M_\omega}{\omega} e^{\omega t}$$

for all $t \geq 0$. Hence, if $Re\,\lambda > w$ then the integral $\int_0^\infty e^{-\lambda t} S_+(t)\ dt$ is convergent and $R(\lambda) = \lambda \int_0^\infty e^{-\lambda t} S_+(t)\ dt$ is analytic on $\{Re\,\lambda > \omega\}$. Therefore $s^+(A_E) \leq \omega$, and this shows that $s^+(A_E) \leq \overline{\alpha}_E$. Now take $\lambda > s^+(A_E)$. For $t \geq 0$ we have (since $\lambda > 0$) :

$$R(\lambda, A_E) = \int_0^\infty e^{-\lambda s} dS_+(s) \geq \int_0^t e^{-\lambda s} dS_+(s) \geq e^{-\lambda t} S_+(t),$$

so $\|S_+(t)\|_E \leq e^{\lambda t} \|R(\lambda, A_E)\|_E$. For $f \in E$ the function $s \mapsto T(s)f^*$ is increasing on $[0, \infty)$, hence

$$0 \leq T(t)f^*(x) \leq \int_t^{t+1} T(s)f^*(x)\ ds \leq S_+(t+1)f^*(x),\ x > 0,$$

and so

$$\|T(t)f\|_E = \|T(t)f^*\|_E \leq \|S_+(t+1)f^*\|_E \leq (e^\lambda \|R(\lambda, A_E)\|_E) e^{\lambda t} \|f\|_E.$$

This shows that $\|T(t)\|_E \leq C_\lambda e^{\lambda t}$ for all $t \geq 0$, which implies (see the beginning of Section 2) that $\overline{\alpha}_E \leq \lambda$. Hence $\overline{\alpha}_E \leq s^+(A_E)$.

(5) Finally we show that $s^-(A_E) = -\underline{\alpha}_E$. To prove that $s^-(A_E) \leq -\underline{\alpha}_E$ we may assume that $\underline{\alpha}_E > 0$, as $s^-(A_E) \leq 0$. Take $-\underline{\alpha}_E < \omega < 0$. From the definition of $\underline{\alpha}_E$ it follows that there exists $M_\omega > 0$ such that $\|D_s\|_E \leq M_\omega s^{-\omega}$ for all $0 < s \leq 1$. Via Lemma 1.1 we see that

$$S_-(\infty)f(x) := \int_0^\infty f(e^s x)\ ds = \int_0^1 D_u f(x)\frac{du}{u},\ f \in E,\ x > 0,$$

defines a bounded linear operator in E with $\|S_-(\infty)\|_E \leq (-\omega)^{-1} M_\omega$. Moreover, using Lemma 1.1 again, $\|S_-(t) - S_-(\infty)\|_E \leq (-\omega)^{-1} M_\omega e^{\omega t}$ for all $t \geq 0$. Hence (cf. [**A2**, Proposition 5.5]; [**HP**, Theorem 6.2.1]) $\lambda \mapsto \int_0^\infty e^{-\lambda t} dS_-(t)$ is analytic on $\{Re\ \lambda > \omega\}$ and so $s^-(A_E) \leq \omega$. This shows that $s^-(A_E) \leq -\underline{\alpha}_E$.

Now we show that $-\underline{\alpha}_E \le s^-(A_E)$. We may assume that $s^-(A_E) < 0$, as $\underline{\alpha}_E \ge 0$. Take $s^-(A_E) < \lambda < 0$. Then

$$R(\lambda, -A_E) = \int_0^\infty e^{-\lambda s} dS_-(s) \ge \int_{t-1}^t e^{-\lambda s} dS_-(s) \ge e^{-\lambda(t-1)}\{S_-(t) - S_-(t-1)\}$$

for all $t \ge 1$. For $f \in E$ the function $s \mapsto T(-s)f^*$ is decreasing on $[0, \infty)$, so

$$0 \le T(-t)f^*(x) \le \int_{t-1}^t T(-s)f^*(x)\, ds = S_-(t)f^*(x) - S_+(t)f^*(x),\ x > 0,$$

and hence

$$\|T(-t)f\|_E = \|T(-t)f^*\|_E \le e^{\lambda(t-1)} \|R(\lambda, -A_E)\|_E \|f\|_E$$

for all $t \ge 1$. From this estimate it follows immediately that $-\underline{\alpha}_E \le \lambda$, and we may conclude that $-\underline{\alpha}_E \le s^-(A_E)$. This completes the proof of the theorem. □

Corollary 4.3. *Let E be an exact (L^1, L^∞)-interpolation space on $(0, \infty)$. Then the Cesaro operator C_E is bounded on E if and only if $\overline{\alpha}_E < 1$. In that case*

$$\sigma(C_E) = \left\{\lambda \in \mathbb{C} : 1 - \overline{\alpha}_E \le Re\left(\frac{1}{\lambda}\right) \le 1 - \underline{\alpha}_E\right\} \cup \{0\}. \tag{4.3}$$

Proof. If $\overline{\alpha}_E < 1$ then, by the above theorem, $1 \in \varrho(A_E)$ and integration by parts gives

$$\begin{aligned} R(1, A_E)f(x) &= \int_0^\infty e^{-t} S_+(t)f(x)\, dt = \int_0^\infty e^{-t} f(e^{-t}x)\, dt \\ &= \frac{1}{x}\int_0^x f(u)du\ , a.e.\ x \in (0, \infty) \end{aligned}$$

for all $f \in E$, i.e., $R(1, A_E) = C_E$. The indentity (4.3) now follows from a combination of Theorem 4.2 with the spectral mapping theorem for resolvents. Conversely, assume that the Cesaro operator is bounded on E. It is easy to see that the integrated semigroup generated by $A - I$ is given by

$$(W(t)f)(x) = \int_0^t e^{-s} f(e^{-s}x)\, ds.$$

Since

$$(W(t)f)(x) \leq \frac{1}{x}\int\limits_0^x f(u)du \qquad (x - a.e.)$$

it follows that $\|W(t)\| \leq \|C_E\|$ for all $t \geq 0$. By Lemma 1.2 this implies that $s(A - I) < 0$. □

Remark 4.4. If we assume that $\underline{\alpha}_E > 0$ then it follows by an argument similar to the above that the operator $\widetilde{C}_E$, defined by

$$\widetilde{C}_E f(x) = \int\limits_x^\infty f(u)\frac{du}{u}, \ a.e.\ x \in (0,\infty), \ f \in E,$$

is bounded on E and

$$\sigma(\widetilde{C}_E) = \left\{\lambda \in \mathbb{C} : \underline{\alpha}_E \leq Re\ \left(\frac{1}{\lambda}\right) \leq \overline{\alpha}_E\right\} \cup \{0\}.$$

Indeed, if $\underline{\alpha}_E > 0$ then $0 \in \sigma(A_E)$ and $\widetilde{C}_E = -R(0, A_E)$. It should be observed that in this general situation (i.e., without any additional assumption on the norm of E) it seems that this last result cannot be obtained via a duality argument from Corollary 4.3.

As before, let E be an exact (L^1, L^∞)-interpolation space on $(0,\infty)$, and we denote by $\{T_E(t)\}_{t\in\mathbb{R}}$ the group of bounded operators in E given by $T_E(t)f(x) = f(e^{-t}x)$ for all $f \in E$. As we have seen, if E does not have order continuous norm, this group is not strongly continuous. However, there is always a natural (locally convex) topology in E with respect to which the group is continuous. For this purpose, let E_n' denote the Köthe dual (or associate space) of E, i.e.,

$$E_n' = \left\{g \in L^0(0,\infty) : \int\limits_0^\infty |fg|dx < \infty \quad \forall\ f \in E\right\}.$$

Every $g \in E_n'$ defines a bounded (order continuous) linear functional φ_g on E, given by $\langle f, \varphi_g\rangle = \int\limits_0^\infty fg dx$ for all $f \in E$. In this way we can identify E_n' with subspace of the norm dual E' (and under the present assumptions on E, this subspace is norming for E). As is known, equipped with the norm $\|g\|_{E_n'} = \|\varphi_g\|_{E_n'}$, the space E_n' is an exact (L^1, L^∞)-interpolation space on $(0,\infty)$.

Proposition 4.5. *The group $\{T_E(t)\}_{t\in\mathbb{R}}$ is continuous with respect to $\sigma(E, E_n')$, i.e., for every $f \in E$ and $g \in E_n'$ the function $t \mapsto \int_0^\infty T(t)f(x)g(x)dx$ is continuous.*

Proof. First we assume that E is satisfies the additional condition

$$(*) \qquad f^*(x) \to 0 \ \text{ as } \ x \to \infty \ \text{ for all } \ f \in E.$$

For every $g \in E_n'$ we define the seminorm p_g on E by $p_g(f) = \int_0^\infty f^*(x)g^*(x)dx$ for all $f \in E$. Note that subadditivity of p_g follows from [**BS**, Proposition 3.6 and (3.10) on p. 54]. Actually we will show that $\{T_E(t)\}_{t\in\mathbb{R}}$ is continuous with respect to the topology σ^* generated by the seminorms $\{p_g : g \in E_n'\}$. Since, by the Hardy-Littlewood quality,

$$\left| \int_0^\infty f(x)g(x)dx \right| \le \int_0^\infty f^*(x)g^*(x)dx \quad \forall\, f \in E,\ g \in E_n',$$

the result of the proposition then follows immediately.

We denote by $\mathcal{S}$ the linear span of all characteristic functions $1_{(a,b]}$ with $0 \le a < b < \infty$. We claim that $\mathcal{S}$ is dense in E with respect to σ^*. Let A be a measurable subset of $(0,\infty)$ such that $A \subseteq (0,R]$ for some $0 < R < \infty$. Then there exists a sequence $\{B_n\}_{n=1}^\infty$ of subsets of $(0,R]$, each B_n being a finite union of intervals, such that $m(A \triangle B_n) \to 0 \qquad (n \to \infty)$. This implies that $(1_A - 1_{B_n})^* \to 0$ on $(0,\infty)$ as $n \to \infty$, and so, by dominated convergence, $p_g(1_A - 1_{B_n}) \to 0 \qquad (n \to \infty)$ for all $g \in E_n'$. Hence $1_A \in \overline{\mathcal{S}}^{\sigma^*}$. Now take $0 \le f \in E$. Then there exists a sequence $\{f_n\}_{n=1}^\infty$ of simple functions on bounded measurable sets such that $0 \le f_n \uparrow f$ a.e. on $(0,\infty)$. Since $f^*(x) \to 0$ as $x \to \infty$, it follows that $(f - f_n)^* \downarrow 0$ on $(0,\infty)$. Hence $p_g(f - f_n) \to 0 \qquad (n \to \infty)$ for all $g \in E_n'$ by dominated convergence. From this we may conclude that $f \in \overline{\mathcal{S}}^{\sigma^*}$, by which the claim is proved.

Now we show that $p_g(T_E(t)f - f) \to 0 \qquad (t \to 0)$ for all $f \in E,\ g \in E_n'$. This is easily verified for $f \in \mathcal{S}$. Take $f \in E$ arbitrary, $h \in \mathcal{S}$ and $g \in E_n'$. Then

$$p_g(T_E(t)f - f) \le p_g(T_E(t)(f-h)) + p_g(T_E(t)h - h) + p_g(f - h).$$

For $-1 \le t \le 1$ we have

$$\begin{aligned} p_g(T_E(t)(f-h)) &= \int_0^\infty [T_E(t)(f-h)]^* g^* dx \\ &= \int_0^\infty T_E(t)(f-h)^* g^* dx \le \int_0^\infty (f-h)^*(e^{-1}x)g^*(x)dx \\ &= \int_0^\infty (f-h)^*(x)g^*(ex)e\,dx = p_{g_1}(f-h), \end{aligned}$$

where $g_1 \in E_n'$ is given by $g_1(x) = eg^*(ex)$. This shows that

$$\limsup_{t\to 0} p_g(T_E(t)f - f) \leq p_{g_1}(f-h) + p_g(f-h)$$

for all $h \in \mathcal{S}$. Since $\overline{\mathcal{S}}^{\sigma *} = E$, we may conclude that $\lim\limits_{t\to 0} p_g(T_E(t)f - f) = 0$. Observe that for $f \in E,\ g \in E_n'$ and $s \in \mathbb{R}$ we have

$$\begin{aligned} p_g(T_E(s)f) &= \int_0^\infty T_E(s)f^* \cdot g^* dx = e^s \int_0^\infty f^*(x)g^*(e^s x)dx \\ &= e^s \int_0^\infty f^*[T_{E_n'}(-s)g]^* dx = p_{g_s}(f), \end{aligned}$$

where $g_s = e^s T_{E_n'}(-s)g$. From this it follows that

$$\lim_{t\to s} p_g(T(t)f - T(s)f) = 0$$

for all $f \in E,\ g \in E_n'$ and $s \in \mathbb{R}$. This concludes the proof of the proposition in the case that E satisfies ($*$).

Now assume that E does not satisfy ($*$). Then $1 \in E$ and so $E_n' \subseteq L^1$, which implies that E_n' satisfies ($*$). Since E is a subspace of $(E_n')_n'$, it follows from the first part of the proof that

$$\lim_{t\to 0} \int_0^\infty f \cdot T_{E_n'}(t)g dx = \int_0^\infty fg dx$$

for all $f \in E$ and $g \in E_n'$. Since

$$\int_0^\infty T_E(t)f \cdot g dx = e^t \int_0^\infty f T_{E_n'}(-t)g dx,$$

this implies that $\lim\limits_{t\to s} \int_0^\infty T_E(t)f \cdot g dx = \int_0^\infty fg dx$ for all $f \in E$ and $g \in E_n'$. This suffices to prove the proposition in this case.

5. The Black-Scholes partial differential equation in (L^1, L^∞)-interpolation spaces.

The Black-Scholes partial differential equation is a degenerate parabolic equation of the form

$$u_t = x^2 u_{xx} + x u_x \qquad (t > 0,\ x > 0). \tag{5.1}$$

The aim of this section is to discuss its well-posedness, spectral properties and asymptotic behaviour in (L^1, L^∞)-interpolation spaces. It is convenient to consider the corresponding operator

$$\begin{aligned} B : \mathcal{D}(0,\infty)' &\to \mathcal{D}(0,\infty)' \\ Bf &= x^2 f'' + xf'; \\ \text{i.e., } \langle Bf, \varphi\rangle &= \langle f, ((m^2\varphi)' - m\varphi)'\rangle \end{aligned}$$

for all $\varphi \in \mathcal{D}(0,\infty)$, $f \in \mathcal{D}(0,\infty)'$ where $m(x) = x \qquad (x > 0)$.

Given an (L^1, L^∞)-interpolation space E we consider the part B_E of B in E; i.e., B_E is the operator on E with domain

$$\begin{aligned} D(B_E) &= \{f \in E : Bf \in E\} \\ B_E f &= Bf\ . \end{aligned}$$

Here we use that $E \subset L^1_{\text{loc}}(0,\infty) \subset \mathcal{D}(0,\infty)'$ with the usual identification of functions with distributions. The following proposition allows us to use the results of the preceding sections.

Proposition 5.1. *Let E be an (L^1, L^∞)-interpolation space. Then $B_E = (A_E)^2$.*

Proof. Recall that $D(A_E) = \{f \in E : mf' \in E\}$, $A_E f = -mf'$.

a) We show that $\lambda^2 - B_E$ is injective for $\lambda > 1$. Let $k \in D(B_E)$ such that $(\lambda^2 - B_E)k = 0$. Let $h = \lambda k + mk' \in \mathcal{D}(0,\infty)'$. Then $\lambda h - mh' = 0$ in $\mathcal{D}(0,\infty)'$. This implies that $h \in C(0,\infty)$ and

$$(x^{-\lambda} h)' = x^{-\lambda-1}(-\lambda h + xh') = 0.$$

Hence $h(x) = cx^\lambda$ for some constant c. Thus $\lambda k(x) + xk'(x) = cx^\lambda \in \mathcal{D}(0,\infty)'$. Hence $k \in C^\infty(0,\infty)$ and

$$(x^\lambda k)' = x^{\lambda-1}(\lambda k + xk') = cx^{2\lambda-1}.$$

This implies that $x^\lambda k = ax^{2\lambda} + b$ for some constants a and b. We have shown that $k(x) = ax^\lambda + bx^{-\lambda}$ which is in $L^1 + L^\infty$ only if $a = b = 0$.

b) Now let $f \in D(B_E)$. Let $\lambda > 1$. Then $\lambda \in \varrho(\pm A_E)$. Hence $\lambda^2 \in \varrho(A_E^2)$ and $R(\lambda^2, A_E^2) = (\lambda - A_E)^{-1}(\lambda + A_E)^{-1}$. Let $k = R(\lambda^2, A_E^2)(\lambda^2 - B_E)f$. Then $k \in D(A_E^2)$. Since A_E^2 is a restriction of B_E we have $(\lambda^2 - B_E)k = (\lambda^2 - B_E)f$. Since $(\lambda^2 - B_E)$ is injective, it follows that $f = k \in D(A_E^2)$. □

As a first consequence we determine the spectrum of B_E.

Theorem 5.2. *Let E be an (L^1, L^∞)-interpolation space with Boyd indices $\underline{\alpha}_E$ and $\overline{\alpha}_E$. Then*

$$\sigma(B_E) = \left\{ r + is : \underline{\alpha}_E^2 - \frac{s^2}{4\underline{\alpha}_E^2} \le r \le \overline{\alpha}_E^2 - \frac{s^2}{4\overline{\alpha}_E^2} \right\};$$

i.e., $\sigma(B_E)$ is the region between two parabolas (with appropriate modification if $\underline{\alpha}_E = 0$ or $\underline{\alpha}_E = \overline{\alpha}_E = 0$).

Proof. By Theorem 4.2 we have

$$\sigma(A_E) = \{\lambda \in \mathbb{C} : \underline{\alpha}_E \leq Re\ \lambda \leq \overline{\alpha}_E\}.$$

Since $\sigma(B_E) = \sigma(A_E)^2$ it follows that

$$\begin{aligned} \sigma(B_E) &= \{\alpha^2 + 2\alpha\beta i - \beta^2 : \beta \in \mathbb{R},\ \underline{\alpha}_E \leq \alpha \leq \overline{\alpha}_E\} \\ &= \left\{\alpha^2 - \frac{s^2}{4\alpha^2} + is : s \in \mathbb{R},\ \underline{\alpha}_E \leq \alpha \leq \overline{\alpha}_E\right\} \end{aligned}$$

which implies the claim. □

Thus the spectrum of B_E varies very much as a function of the (L^1, L^∞)-interpolation space.

Next we consider the semigroup generated by B_E.

Theorem 5.3. *Let E be an (L^1, L^∞)-interpolation space with order continuous norm. Then B_E generates a holomorphic C_0-semigroup V_E on E of angle $\pi/2$. Moreover, the exponential type $\omega(V_E)$ of V_E is given by*

$$\omega(V_E) = (\overline{\alpha}_E)^2.$$

Proof. This follows directly from the fact that $B_E = (A_E)^2$ and that A_E generates a C_0-group (cf. [**N**, Theorem 1.15]). It follows from Theorem 5.2 that $s(B_E) = (\overline{\alpha}_E)^2$. Since V_E is holomorphic, $s(B_E) = \omega(V_E)$. □

If E does not have order continuous norm, then $D(B_E)$ is not dense. Still the holomorphic estimate for the resolvent is valid. This situation is very well studied by E. Sinestrari [**Si**] from which we quote the following result.

Theorem 5.4. *Let A be an operator on a Banach space X. Assume that there exist $w \in \mathbb{R}$, $\theta \in [0, \pi/2]$ such that*

$$\text{(5.2)} \qquad \left\{ \begin{array}{l} w + \Sigma(\theta + \pi/2) \subset \varrho(A) \text{ and} \\ \|\lambda R(\lambda, A)\| \leq M \text{ if } \lambda \in w + \Sigma(\theta + \pi/2). \end{array} \right.$$

Then there exists a holomorphic mapping

$$T : \Sigma(\theta) \to \mathcal{L}(X)$$

such that $T(z + z') = T(z)T(z') \qquad (z, z' \in \Sigma(\theta)),$

$$\text{(5.3)} \qquad \sup_{|Arg z| < \theta'} \|e^{-wz}T(z)\| < \infty \text{ for all } 0 < \theta' < \theta,$$

and

$$\text{(5.4)} \qquad R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t)\ dt \qquad (\lambda > \omega).$$

In that case, we call T the **generalized holomorphic semigroup** *generated by A.*

Here we used the usual notation

$$\Sigma(\theta) = \{re^{i\alpha} : r > 0,\ \alpha \in (-\theta, \theta)\}.$$

The semigroup T has the following regularity property. Considering $D(A^k)$ as a Banach space for the norm $\|x\|_{D(A^k)} = \|x\| + \|Ax\| + \ldots + \|A^k x\|$, one has

$$T(\cdot)x \in C^\infty((0,\infty),\ D(A^k)) \text{ and} \tag{5.5}$$

$$\frac{d}{dt}T(t)x = AT(t)x \qquad (t > 0) \tag{5.6}$$

for all $x \in X,\ k \in \mathbb{N}$, see [**Si**] for this. Denoting by

$$s(A) = \sup\{Re\,\lambda : \lambda \in \sigma(A)\}$$

the spectral bound of A, as before, and by

$$\omega(T) = \inf\left\{w \in \mathbb{R} : \sup_{t\geq 0} \|e^{-\omega t}T(t)\| < \infty\right\}$$

the type of T, one has as in the strongly continuous case

$$s(A) = \omega(T). \tag{5.7}$$

Proof of (5.7). Let $Y = \overline{D(A)} \subset X$ and denote by A_0 the part of A in Y. Then A_0 generates a holomorphic C_0-semigroup $(T_0(t))_{t\geq 0}$ on Y and one has $T_0(t) = T(t)_{|Y}$. Since $D(A) \subset Y$ one has $\sigma(A) = \sigma(A_0)$ (by [**A4**, Proposition 1.1]), and in particular $s(A) = s(A_0)$. Let $w' > s(A)$. Then

$$\|T_0(t)\|_{\mathcal{L}(Y)} \leq M' e^{w' t} \qquad (t \geq 0).$$

Since $T(1)X \subset Y$, it follows that $\omega(T) \leq \omega'$. □

Now we can formulate the following result for the operator B_E.

Proposition 5.5. *Let E be an (L^1, L^∞)-interpolation space. Then B_E generates a generalized holomorphic semigroup V_E on E. The semigroup V_E is strongly continuous if and only if E has order continuous norm. Finally the exponential type of V_E is given by*

$$\omega(V_E) = (\overline{\alpha}_E)^2. \tag{5.8}$$

Proof. It follows from (4.1) and (4.2) that

$$\|R(\lambda, A_E)\| \leq (|Re\,\lambda| - 1)^{-1} \qquad (|Re\,\lambda| \geq 1). \tag{5.9}$$

Now the argument given in [**N**] A-II Theorem 1.14 and 1.15 shows that $B_E = A_E^2$ satisfies (5.2). Hence B_E generates a generalized holomorphic semigroup.

If $D(B_E)$ is dense, then also $D(A_E)$ is dense, since $D(B_E) \subset D(A_E)$. Conversely, assume that $D(A_E)$ is dense. Then $\lambda R(\lambda, A_E) \to I$ strongly as $\lambda \to \infty$. Hence $(\lambda R(\lambda, A_E))^2 \to I$ strongly as $\lambda \to \infty$. Thus $D(B_E) =$

$D(A_E^2)$ is dense. Now the second claim follows from Theorem 2.4. Finally, Theorem 5.2 and (5.7) imply that $\omega(V_E) = s(B_E) = (\overline{\alpha}_E)^2$. $\square$

Next we establish the usual formula for V_E.

Proposition 5.6. *Let E be an (L^1, L^∞)-interpolation space. Then*

$$\langle V_E(t)f,\ \varphi\rangle = (4\pi t)^{-1/2} \int_{\mathbb{R}} e^{-r^2/4t} \langle T_E(r)f, \varphi\rangle dr \tag{5.10}$$

for all $f \in E,\ \varphi \in E'$.

Proof. We use the following formula

$$\frac{e^{-\lambda|r|}}{2\lambda} = \int_0^\infty e^{-\lambda^2 t}(4\pi t)^{-1/2} e^{-r^2/4t}\, dt, \tag{5.11}$$

valid for all $\lambda > 0,\ r \in \mathbb{R}$ (see [**D**, p. 138]).

For $\lambda > 1$ we have

$$\begin{aligned}
\int_0^\infty e^{-\lambda^2 t} V_E(t)\, dt &= R(\lambda^2, A_E^2) \\
&= (\lambda - A_E)^{-1}(\lambda + A_E)^{-1} = -R(\lambda, A_E)R(-\lambda, A_E) \\
&= \frac{R(\lambda, A_E) + R(-\lambda, A_E)}{2\lambda} \\
&= \frac{1}{2\lambda}\left(\int_0^\infty e^{-\lambda t} T_E(t)\, dt + \int_0^\infty e^{-\lambda t} T_E(t)\, dt \right) \\
&= \int_{-\infty}^{+\infty} \frac{e^{-\lambda|r|}}{\lambda} T_E(r)\, dr \\
&= \int_{-\infty}^{+\infty} T_E(r) \int_0^\infty e^{-\lambda^2 t}(4\pi t)^{-1/2} e^{-r^2/4t}\, dt\, dr \\
&= \int_0^\infty e^{-\lambda^2 t} \int_{\mathbb{R}} (4\pi t)^{-1/2} e^{-r^2/4t} T_E(r)\, dr\, dt.
\end{aligned}$$

Here the integrals involving $T_E(t)$ are understood in the $\sigma(E, E')$-duality. Observe that it suffices to evaluate by $f \in E_+$ and $\varphi \in E'_+$ only, so that Fubini's theorem can be applied. Now the claim follows from the uniqueness theorem for Laplace transforms. $\square$

It is easy to deduce a pointwise expression from (5.10):

$$(V_E(t)f)(x) = (4\pi t)^{-1/2} \int_{\mathbb{R}} e^{-r^2/4t} f(e^{-r}x)\, dr = (4\pi t)^{-1/2} \int_0^\infty e^{-(\log x - \log y)^2/4t} f(y) \frac{dy}{y}. \tag{5.12}$$

Thus V_E is an integral operator.

From Proposition 5.6 we now deduce the following continuity result.

Proposition 5.7. *Let E be an exact interpolation space and E'_n its Köthe dual. Then V_E is $\sigma(E, E'_n)$-continuous, i.e.,*

$$\lim_{t\downarrow 0} \langle V_E(t)f, \varphi\rangle = \langle f, \varphi\rangle$$

for all $f \in E$, $\varphi \in E'_n$.

Proof. Let $f \in E$, $\varphi \in E'_n$. Let $\varepsilon > 0$. By Proposition 4.5 we can choose $\delta > 0$ such that $|\langle T_E(r)f, \varphi\rangle - \langle f, \varphi\rangle| \le \varepsilon$ if $|r| \le \delta$. Then

$$\begin{aligned}
&\limsup_{t\downarrow 0} |\langle V_E(t)f, \varphi\rangle - \langle f, \varphi\rangle| \\
&= \limsup_{t\downarrow 0} (4\pi t)^{-1/2} \left| \int_{\mathbb{R}} e^{-r^2/4t} (\langle T_E(r)f, \varphi\rangle - \langle f, \varphi\rangle) dr \right| \\
&\le \varepsilon + \limsup_{t\downarrow 0} (4\pi t)^{-1/2} \int_{|r|\ge\delta} e^{-r^2/4t} |\langle T_E(r)f, \varphi\rangle - \langle f, \varphi\rangle| dr \\
&= \varepsilon.
\end{aligned}$$

This implies the claim. □

Now we obtain the following final result on existence and uniqueness for the Black & Scholes partial differential equation.

Theorem 5.8. *Let E be an exact (L^1, L^∞)-interpolation space with Köthe dual E'_n. Let $f \in E$, $u(t) = V_E(t)f$. Then u is the unique solution of the Cauchy problem*

$$\text{(CP)} \qquad \begin{cases} u \in C^1((0,\infty); E), & u(t) \in D(B_E) \quad (t > 0); \\ \dot{u}(t) = B_E u(t) & (t > 0) \\ \lim_{t\downarrow 0} u(t) = f & \textit{for } \ \sigma(E, E'_n). \end{cases}$$

Moreover, if we put $u(t,x) = (V_E(t)f)(x) = u(t)(x)$, then $u \in C^\infty(0,\infty) \times (0,\infty)$ and

$$\text{(BS)} \qquad u_t = x^2 u_{xx} + x u_x \qquad (t > 0,\ x > 0).$$

Proof. We know that u is a solution of (CP). In order to prove uniqueness let u be a solution of (CP) with $f = 0$. Let $t > 0$, $v(s) = V_E(t-s)u(s)$, $s \in (0,t)$. Since V_E is holomorphic and $\frac{d}{dt}V_E(t) = B_E V_E(t) \quad (t > 0)$ we have

$$\dot{v}(s) = -B_E V_E(t-s)u(s) + V_E(t-s)\dot{u}(s) = 0.$$

Thus v is constant on $(0,t)$. Moreover, $V_E(t-s) \to V_E(t)$ as $s \downarrow 0$ in $\mathcal{L}(E)$. Let $\varphi \in E_n'$. Then

$$\begin{aligned} \langle v(s), \varphi \rangle &= \langle (V_E(t-s) - V_E(t))u(s), \varphi \rangle \\ &\quad + \langle u(s), V_E(t)'\varphi \rangle \\ &\to 0 \quad (s \downarrow 0). \end{aligned}$$

Here we use that $V_E(t)'\varphi \in E_n'$ which follows from (5.10). Thus $v(s) \equiv 0$ on $(0,t)$. Since $u(s) \to u(t)$ in norm as $s \uparrow t$ and $V_E(t-s)u(t) \to u(t)$ for $\sigma(E, E_n')$ as $s \uparrow t$, it follows that $v(s) = V_E(t-s)(u(s) - u(t)) + V_E(t-s)u(t) \to u(t)$ as $s \uparrow t$ for $\sigma(E, E_n')$. Thus $u(t) = 0$.

It remains to show the regularity result. For $f \in D(A_E)$ we have $f \in L^1_{\mathrm{loc}}(0,\infty)$ and $xf' \in E \subset L^1_{\mathrm{loc}}(0,\infty)$. Hence $f \in C(0,\infty)$. From this one obtains by induction that $D(A_E^{k+1}) \subset C^k(0,\infty)$ for all $k \in \mathbb{N}$. Now we know that $V_E(\cdot)f \in C^\infty((0,\infty), D(B_E^k)) = C^\infty((0,\infty); D(A_E^{2k}))$ for all $f \in E$. It is not difficult to see that this implies that $u \in C^\infty(0,\infty) \times (0,\infty)$. □

6. Perturbation.

Let B be an operator on a Banach space X. An operator $Q : D(B) \to X$ is called a **small perturbation** of B if for all $\varepsilon > 0$ there exists $b \geq 0$ such that

$$\|Qx\| \leq \varepsilon \|Bx\| + b\|x\| \qquad (x \in D(B)). \tag{6.1}$$

The following is well-known.

Proposition 6.1. *Let B be the generator of a (generalized) holomorphic semigroup and let Q be a small perturbation of B. Then $B + Q$ generates a (generalized) holomorphic semigroup.*

Example 6.2. Let E be an (L^1, L^∞)-interpolation space. Then A_E is a small perturbation of B_E.

Proof. We have for $\lambda > 1$,

$$R(\lambda^2, B_E) = \frac{1}{2\lambda}(R(\lambda, A_E) + R(\lambda, -A_E))$$

(see the proof of Proposition 5.6). Hence

$$\begin{aligned}\|A_E R(\lambda^2, B_E)\| &= \frac{1}{2\lambda}\|A_E R(\lambda, A_E) + A_E R(\lambda, -A_E)\| \\ &= \frac{1}{2\lambda}\|\lambda R(\lambda, A_E) - \lambda R(\lambda, -A_E)\| \\ &\leq \frac{1}{2}(\|R(\lambda, A_E)\| + \|R(\lambda, -A_E)\|) \\ &\to 0 \quad (\lambda \to \infty).\end{aligned}$$

Let $\varepsilon > 0$. Choose $\lambda > 1$ such that $\|A_E R(\lambda, B_E^2)\| \leq \varepsilon$. Let $f \in D(B_E)$. Then $\|A_E f\| = \|A_E R(\lambda^2, B_E)(\lambda^2 - B_E)f\| \leq \varepsilon\|(\lambda^2 - B_E)f\| \leq \varepsilon\|B_E f\| + \lambda^2\|f\|$. □

It remains to show that $\sigma(E, E_n')$-continuity is preserved by small perturbations. For this we establish a Tauberian theorem (Proposition 6.4) which is valid for Laplace transforms of functions having a holomorphic extension to a sector. They can be characterized as follows (see Prüß [**P**, Theorem 0.1]).

Proposition 6.3. *Let X be a Banach space and let $0 < \theta_0 \leq \pi/2$.*

a) *Let $r : \Sigma(\theta_0 + \pi/2) \to X$ be a holomorphic function such that*

$$\sup_{\lambda \in \Sigma(\theta + \pi/2)} \|\lambda r(\lambda)\| < \infty \tag{6.2}$$

for all $0 < \theta < \theta_0$. Then there exists a holomorphic function $f : \Sigma(\theta_0) \to X$ satisfying

$$\sup_{z \in \Sigma(\theta)} \|f(z)\| < \infty \tag{6.3}$$

for all $0 < \theta < \theta_0$ such that $r(\lambda) = \hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)dt$ for $Re\,\lambda > 0$.

b) *Conversely, assume that $f : \mathbb{R}_+ \to X$ has a holomorphic extension to $\Sigma(\theta_0)$ satisfying (6.3); then the Laplace transform $\hat{f}$ of f has a holomorphic extension r to $\Sigma(\theta_0 + \pi/2)$ satisfying (6.2).*

Now we describe the asymptotic behaviour of $f(t)$ for $t \downarrow 0$ in terms of the behaviour of $r(\lambda)$ as $\lambda \to \infty$.

Proposition 6.4. *Assume that f and r are as in Proposition 6.3. Let $c \in X$. Then $\lim_{t\downarrow 0} f(t) = c$ if and only if $\lim_{\lambda\to\infty} \lambda r(\lambda) = c$.*

Proof. 1. Assume that $\lim_{\lambda\to\infty} \lambda r(\lambda) = c$. Choose $0 < \theta < \theta_0$. It follows from [**HP**, Theorem 3.14.3] that

$$\lim_{\substack{|\lambda|\to\infty \\ \lambda \in \Sigma(\theta+\pi/2)}} \lambda r(\lambda) = c.$$

Let $\varepsilon > 0$. Choose $\varrho_0 > 0$ such that $\|\lambda r(\lambda) - c\| \le \varepsilon$ for all $\lambda \in \overline{\Sigma}(\theta + \pi/2)$ with $|\lambda| \ge \varrho_0$. Let $t \ge 1/\varrho_0$. Choose a contour Γ consisting of the lines $\{\varrho e^{\pm i(\theta+\pi/2)} : \varrho \ge 1/t\}$ and the arc $\{1/t \cdot e^{i\alpha} : -\theta \le \alpha \le \theta\}$. Then by the proof of [**P**, Theorem 0.1],

$$f(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} r(\lambda) d\lambda.$$

Since $\frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \frac{d\lambda}{\lambda} = 1$,

$$\begin{aligned}
\|f(t) - c\| &= \left\| \frac{1}{2\pi i} \int_\gamma e^{\lambda t} (\lambda r(\lambda) - c) \frac{d\lambda}{\lambda} \right\| \\
&\le \frac{\varepsilon}{2\pi} \left\{ 2 \int\limits_{1/t}^{\infty} e^{-tr\cos\theta} \frac{dr}{r} + \int\limits_{-\theta}^{\theta} e^{\cos\alpha} d\alpha \right\} \\
&\le \frac{\varepsilon}{2\pi} \left\{ 2 \cdot \frac{1}{|\cos\theta|} + \int_{-\theta}^{\theta} e^{\cos\alpha} d\alpha \right\}.
\end{aligned}$$

This proves the claim.

2. The converse is a classical Abelian theorem. □

Proposition 6.5. *Let A be the generator of a generalized holomorphic semigroup T on a Banach space X and let B be a small perturbation of A. Denote by S the generalized holomorphic semigroup generated by $A + B$. Let $x \in X$, $\varphi \in X'$, such that $\lim\limits_{t\downarrow 0} \langle T(t)x, \varphi\rangle = \langle x, \varphi\rangle$. Then $\lim_{t\downarrow 0}\langle S(t)x, \varphi\rangle = \langle x, \varphi\rangle$.*

Proof. Replacing A by $A - w$ if necessary, we can assume that A and $A + B$ satisfy (5.2) with $w = 0$. So we are in the situation of Proposition 6.4. Thus we know that $\lim_{\lambda\to\infty}\langle \lambda R(\lambda, A)x, \varphi\rangle = \langle x, \varphi\rangle$, and it suffices to show that $\lim_{\lambda\to\infty}\langle \lambda R(\lambda, A + B)x, \varphi\rangle = \langle x, \varphi\rangle$. For this it suffices to show that

$$\|\lambda R(\lambda, A + B) - \lambda R(\lambda, A)\| \to 0 \qquad (\lambda \to \infty).$$

Let $M \ge 0$ such that $\|\lambda R(\lambda, A)\| \le M$ $(\lambda > 0)$. Let $\varepsilon > 0$. There exists $b \ge 0$ such that

$$\begin{aligned}
\|BR(\lambda, A)\| &\le \varepsilon\|AR(\lambda, A)\| + b\|R(\lambda, A)\| \\
&\le \varepsilon\|\lambda R(\lambda, A) - I\| + b\|R(\lambda, A)\| \\
&\le \varepsilon(M + 1) + bM/\lambda.
\end{aligned}$$

Thus $\lim_{\lambda\to\infty} \|BR(\lambda, A)\| \le \varepsilon(M + 1)$. □

As a result we now know the following. Let E be an (L^1, L^∞)-interpolation space. Let Q be a small perturbation of B_E. Then $B_E + Q$ generates a generalized holomorphic semigroup on E which is $\sigma(E, E_n')$ continuous. In particular, we obtain the following result.

Theorem 6.6. *Let E be an (L^1, L^∞)-interpolation space. Let $\alpha > 0$ be a constant, and let $\beta, \gamma \in L^\infty(0,\infty)$. Consider the operator G on E given by*

$$\begin{aligned} Gf &= \alpha x^2 f'' + \beta x f' + \gamma f \\ D(G) &= D(B_E). \end{aligned}$$

Then G generates a generalized holomorphic semigroup which is $\sigma(E, E'_n)$-continuous.

Proof. By Example 6.2, the operator A_E is a small perturbation of B_E. Thus $B_E - A_E$ generates a generalized holomorphic semigroup. Since β defines a bounded multiplication operator on E, $\beta A_E + \gamma$ is a small perturbation of $\alpha(B_E - A_E)$. Note that $G = \alpha(B_E - A_E) + \beta A_E + \gamma$. □

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Received January 27, 2000.

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