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In one of their early works, Miranda and Persson have classified all possible configurations of singular fibers for semistable extremal elliptic fibrations on $K3$ surfaces. They also obtained the Mordell-Weil groups in terms of the singular fibers except for 17 cases where the determination and the uniqueness of the groups were not settled. In this paper, we settle these problems completely. We also show that for all cases with 'larger' Mordell-Weil groups, this group, together with the singular fibre type, determines uniquely the fibration structure of the $K3$ surface (up to based fibre-space isomorphisms).

0. Introduction.

Let $f : X \to C$ be an elliptic surface over a smooth projective curve C with a section O, i.e., a Jacobian elliptic fibration over C. Throughout this paper, we always assume that

($*$) f has at least one singular fiber.

Let $MW(f)$ be the Mordell-Weil group of $f : X \to C$, i.e., the group of sections, O being the zero. Under the assumption ($*$), it is known that $MW(f)$ is a finitely generated Abelian group (the Mordell-Weil theorem). More precisely, if we let R be the subgroup of the Néron-Severi group $\mathrm{NS}(X)$ of X generated by O and all the irreducible components in fibers of f, then (i) $\mathrm{NS}(X)$ is torsion-free, and (ii) $MW(f) \cong \mathrm{NS}(X)/R$ (see [**S**], for instance). Note that the Shioda-Tate formula $\operatorname{rank} MW(f) = \rho(X) - \operatorname{rank} R$ easily follows from the second statement.

We call $f : X \to C$ *extremal* if

(i) the Picard number $\rho(X)$ of X is equal to $h^{1,1}$ and
(ii) $\operatorname{rank} MW(f) = 0$.

If $f : X \to C$ is extremal, then the Shioda-Tate formula implies $\operatorname{rank} R = \rho(X)$. Hence, in other words, $f : X \to C$ is extremal if and only if $\rho(X) = \operatorname{rank} R = h^{1,1}(X)$. Also, taking the isomorphism $MW(f) \cong NS(X)/R$ into account, it seems that we can say a lot about $MW(f)$ only from the data of types of singular fibers.

In [**MP1**], Miranda and Persson studied extremal rational elliptic surfaces. They gave a complete classification and proved the uniqueness of such surfaces.

Suppose that $f : X \to C$ is a semi-stable elliptic $K3$ surface, i.e., f has only I_n type singular fibers with Kodaira's notation [**Ko**]. In this case, $C = \mathbf{P}^1$, $\mathrm{NS}(X) = \mathrm{Pic}\, X$, and f is extremal if and only if f has exactly six singular fibers. For a semi-stable elliptic $K3$ surface, the configuration of singular fibers is said to be $[n_1, \dots, n_s]$ ($n_1 \le n_2 \le \cdots \le n_s$) if it has singular fibers $I_{n_1}, \dots, I_{n_s}$. In [**MP2**], Miranda and Persson gave a complete list for realizable s-tuples $[n_1, \dots, n_s]$; and their list shows that there are 112 extremal cases. In [**MP3**], they go on to study $MW(f)$ for those extremal elliptic $K3$ surfaces.

We say that $f : X \to \mathbf{P}^1$ is of *type* m if the corresponding $[n_1, n_2, \dots, n_6]$ appears as the No. m case in the table of [**MP3**]. Suppose that f is of type m. What Miranda and Persson did in [**MP3**] are that

(i) if $m \neq 2, 4, 9, 11, 13, 27, 31, 32, 35, 37, 38, 44, 48, 53, 55, 69$ and 92, $MW(f)$ is determined by the 6-tuples $[n_1, n_2, \dots, n_6]$, and
(ii) if $MW(f) \supseteq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, then the corresponding elliptic $K3$ surface is unique.

The main purpose of this paper is

(i) to determine $MW(f)$ for the remaining cases, and
(ii) to consider the uniqueness problem for other kinds of $MW(f)$; more precisely, this problem may be formulated as follows:

Question 0.1. Let $f_1 : X_1 \to \mathbf{P}^1$ and $f_2 : X_2 \to \mathbf{P}^1$ be semi-stable extremal elliptic $K3$ surfaces such that

(i) both X_1 and X_2 have the same configuration of singular fibers, and
(ii) their Mordell-Weil groups are isomorphic.

Then is it true that there exists an isomorphism $\varphi : X_1 \to X_2$ such that

(a) φ preserves the fibrations, and
(b) the zero section of f_1 maps to that of f_2 with φ?

Now let us state our result concerning the first problem.

Theorem 0.2. *Let $f : X \to \mathbf{P}^1$ be of type m, m being one of the 17 exceptional cases as above. Then we have the following table:*

m	*the* 6*-tuple*	$MW(f)$	m	*the* 6*-tuple*	$MW(f)$
2	$[1,1,1,1,2,18]$	(0), $\mathbf{Z}/3\mathbf{Z}$	4	$[1,1,1,1,4,16]$	$\mathbf{Z}/4\mathbf{Z}$
9	$[1,1,1,1,10,10]$	(0), $\mathbf{Z}/5\mathbf{Z}$	11	$[1,1,1,2,3,16]$	(0), $\mathbf{Z}/2\mathbf{Z}$
13	$[1,1,1,2,5,14]$	(0), $\mathbf{Z}/2\mathbf{Z}$	27	$[1,1,1,5,6,10]$	(0), $\mathbf{Z}/2\mathbf{Z}$
31	$[1,1,2,2,2,16]$	$\mathbf{Z}/4\mathbf{Z}$	32	$[1,1,2,2,3,15]$	(0), $\mathbf{Z}/3\mathbf{Z}$
35	$[1,1,2,2,6,12]$	$\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/6\mathbf{Z}$	37	$[1,1,2,2,9,9]$	(0), $\mathbf{Z}/3\mathbf{Z}$
38	$[1,1,2,3,3,14]$	(0), $\mathbf{Z}/2\mathbf{Z}$	44	$[1,1,2,4,4,12]$	$\mathbf{Z}/4\mathbf{Z}$
48	$[1,1,2,4,8,8]$	$\mathbf{Z}/8\mathbf{Z}$	53	$[1,1,3,3,4,12]$	$\mathbf{Z}/3\mathbf{Z}$, $\mathbf{Z}/6\mathbf{Z}$
55	$[1,1,3,3,8,8]$	(0), $\mathbf{Z}/2\mathbf{Z}$	69	$[1,2,2,3,4,12]$	$\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/4\mathbf{Z}$
92	$[1,3,4,4,4,8]$	$\mathbf{Z}/4\mathbf{Z}$			

Moreover, all the above possibilities for $MW(f)$ in each of these 17 *types are realizable.*

Once we have settled the problem on $MW(f)$, we next consider Question 0.1. Our result is the following:

Theorem 0.3. *Let $f : X \to \mathbf{P}^1$ be an extremal semi-stable elliptic $K3$ surface. If $\sharp(MW(f)) \geq 4$, then Question 0.1 has a positive answer except for $m = 49$.*

Remark 0.4. Let ϕ be the homomorphism from $MW(f)$ to $\mathbf{Z}/n_1\mathbf{Z} \times \cdots \times \mathbf{Z}/n_6\mathbf{Z}$ given in [**MP3**, §2], i.e., $\phi(s) = (a_1, \dots, a_6)$, where a_i is the component number of the irreducible component that s hits at the corresponding singular fiber. Since ϕ is injective, we can identify $MW(f)$ with its image by ϕ. Then we have:

(1) Let $g_m : Y_m \to \mathbf{P}^1$ be any Jacobian elliptic fibration of type m with $MW(g_m) = (0)$ and fitting one of the nine cases in Theorem 0.2. Let $\{I_{n_1}, I_{n_2}, \dots, I_{n_k}, I_{n_{k+1}}, \dots, I_{n_6}\}$ be the set of types of singular fibers of g_m so that $1 = n_1 = n_2 = \cdots = n_{k-1} < n_k \leq n_{k+1} \leq \cdots \leq n_6$. Then the Picard lattice $\operatorname{Pic} Y_m$ is identical to $U \oplus A_{n_k-1} \oplus \cdots \oplus A_{n_6-1}$ with the $\mathbf{Q}/2\mathbf{Z}$-valued discriminant quadratic form $q_{\operatorname{Pic} Y_m}$ equal to (cf. [**Mo**]):

$$(-(n_k - 1)/n_k) \oplus \cdots \oplus (-(n_6 - 1)/n_6).$$

Here $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the dual $(\operatorname{Pic} Y_m)^\vee = \operatorname{Hom}_{\mathbf{Z}}(\operatorname{Pic} Y_m, \mathbf{Z})$ naturally contains $\operatorname{Pic} Y_m$ as a sublattice with $\mathbf{Z}/n_k\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/n_6\mathbf{Z}$ as the factor group (see §1 for definitions).

An easy case-by-case check, using the fact that $q_{(T_{Y_m})} = -q_{(\mathrm{Pic}\, Y_m)}$, shows that the intersection matrix of the transcendental lattice T_{Y_m} is, modulo the action of $SL_2(\mathbf{Z})$, uniquely determined by the data $[n_1, \ldots, n_6]$ (see [**Ni**, Prop. 1.6.1] or [**Mo**, Lemma 2.4]). So the intersection matrix of T_{Y_m} is equal to the corresponding one in the proof of Lemma (3.3). Thus, for each of these 9 of type m, there is exactly one $K3$ surface (modulo isomorphisms of abstract surfaces without the fibered structure being taken into consideration) which has a Jacobian elliptic fibration of type m with trivial Mordell-Weil group.

Also, for both $(m, G_m) = (35, \mathbf{Z}/2\mathbf{Z}), (53, \mathbf{Z}/3\mathbf{Z})$, there is a unique $K3$ surface X_m, which has a Jacobian elliptic fibration f_m of type m and $MW(f_m) = G_m$, because we can prove that the transcendental lattice T_{X_m} is unique in each pair case and identical to the corresponding one in the proof of Lemma (3.3).

The authors suspect that if $(f_m)_i : (X_m)_i \to \mathbf{P}^1$ are two Jacobian elliptic surfaces of the same type m and with $MW((f_m)_1) \cong MW((f_m)_2)$ then $(X_m)_1 \cong (X_m)_2$, though there may not be any fibered surface isomorphism between $((X_m)_i, (f_m)_i)$ $(i = 1, 2)$; see the fourth remark below and our Proposition (4.9). The importance of Lemma (3.3) is that its proof can be used, we guess, to lattice-theoretically show the existence of all cases of m and possibly to give an affirmative answer to this question. See [**SZ**] and [**Y**] for the non-semistable cases.

(2) When $m = 49$, we have $MW(f) = \mathbf{Z}/5\mathbf{Z}$ with $s_1 = (0, 0, 0, 2, 2, 2)$ or $s_2 = (0, 0, 0, 1, 1, 4)$ as its generator (cf. the Table in [**MP3**]). However, we have $2s_2 = (0, 0, 0, 2, 2, 10 - 2)$. So we may assume that $MW(f)$ always has s_1 as its generator after suitable relabeling of fiber components if necessary.

(3) When $m = 110$, we have $MW(f) = \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ with

$$G_1 = \{s_1 = (0, 0, 1, 1, 2, 2), s_2 = (1, 1, 2, 2, 0, 2)\}$$

or

$$G_2 = \{s_1 = (0, 0, 1, 1, 2, 2), s_3 = (1, 1, 1, 1, 0, 4)\}$$

as its set of generators (cf. the Table in [**MP3**]). Note that G_2 can be replaced by the new generating set $G_2' := \{s_1, 2s_3 = (3 - 1, 3 - 1, 2, 2, 0, 2)\}$. So we may assume that $MW(f)$ always has G_1 as its set of generators after suitable relabeling of fiber components if necessary.

(4) When $m = 46$, we have $MW(f) = \mathbf{Z}/2\mathbf{Z}$ with $s_1 = (0, 0, 0, 0, 3, 5)$ or $s_2 = (0, 0, 1, 2, 0, 5)$ as its generator (cf. the Table in [**MP3**]). As in the proof of Lemma (3.8), one can show that there are pairs (X_i, f_i) $(i = 1, 2)$ of the same type $m = 46$ with $MW(f_i) = \{O, s_i\}$. Moreover, the minimal resolution Y_i of $X_i/\langle s_i \rangle$ for $i = 1$ (resp. $i = 2$) has an

elliptic fibration $g_i : Y_i \to \mathbf{P}^1$, *induced from* f_i, of type $m = 101$ (resp. $m = 66$). Hence there is no isomorphism between the pairs (X_i, f_i).

(5) For $m = 69$, we have either $MW(f) = \mathbf{Z}/2\mathbf{Z}$ with $s = (0, 1, 1, 0, 0, 6)$ as its generator, or $MW(f) = \mathbf{Z}/4\mathbf{Z}$ with $s = (0, 1, 1, 0, 1, 3)$ as its generator (cf. Lemma (3.7)).

The contents of this article are as follows: In §1, we explain our technique and we give a brief summary of the facts we need. In §2, we give a method to construct (or show the nonexistence) of elliptic fibrations and give several examples of extremal elliptic $K3$ surfaces with trivial Mordell-Weil groups. §3 and §4 are devoted to proving Theorems 0.2 and 0.3, respectively.

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Conventions. In this article, the ground field is always the complex numbers field $\mathbf{C}$.

To describe the type of simple singularities of plane curves, we use bold capital letters, $\mathbf{A}$, $\mathbf{D}$ and $\mathbf{E}$.

We use capital italic letters A, D and E to describe the type of lattices, but we always multiply the value of intersection form by -1 for such lattices.

1. Preliminaries.

1.1. Cremona transformations and its applications.

We fix notation about Cremona transformations related with two-dimensional families of conics.

Let V be the vector space of homogeneous polynomials of degree 2 in three variables. Let $P, Q, R \in \mathbf{P}^2$ be three different points in general position and let $V_{P,Q,R}$ be the subspace of elements of V vanishing at P, Q and R; it is a 3-dimensional vector space. It is classical to define a rational map $CR_{P,Q,R}$: $\mathbf{P}^2 \dashrightarrow \check{\mathbf{P}}(V_{P,Q,R})$ where if $P_0 \in \mathbf{P}^2$, its image is the hyperplane of elements of $V_{P,Q,R}$ which also vanish at P_0. By a suitable choice of coordinates and the identification of $\check{\mathbf{P}}(V_{P,Q,R})$ with $\mathbf{P}^2$ this map may be written as:

$$\begin{array}{ccc} \mathbf{P}^2 & \dashrightarrow & \mathbf{P}^2 \\ [x : y : z] & \mapsto & [yz : xz : xy] \,. \end{array}$$

The map $CR_{P,Q,R}$ is not defined at P, Q, R, which are called the centers of the Cremona transformation. Outside the lines joining P, Q, R, this map is an isomorphism.

Let us consider now $P, Q \in \mathbf{P}^2$ and a line L through P such that $Q \notin L$. In the same way we define $V_{P,L,Q}$ as the space of equation of conics passing through P and Q and tangent to L at P. We define in the same way $CR_{P,L,Q}$. We can choose coordinates such that we have:

$$\begin{array}{ccc} \mathbf{P}^2 & \dashrightarrow & \mathbf{P}^2 \\ [x:y:z] & \mapsto & \left[y^2 : xy : xz\right]. \end{array}$$

This map is not defined at P and Q and it is an isomorphism outside L and the line joining P and Q. We say that the centers are Q and the two first infinitely near points of L at P; sometimes we will replace in the notation L by any curve through P whose only tangent at P is L.

There is a third type of Cremona transformation associated to a conic. Let C be a smooth conic passing through a point P; we denote $V_{P,C}$ as the space of equations of conics C' such that $(C \cdot C')_P = 3$. We denote $CR_{P,C}$ the associated Cremona transformation. It is not defined at P and is an isomorphism outside the tangent line to C at P. We say that the centers at P are the three first infinitely near points of C at P; sometimes we will replace in the notation Q by any curve through P such that Q is the only conic with highest contact at P. We can choose equations to write it down as:

$$\begin{array}{ccc} \mathbf{P}^2 & \dashrightarrow & \mathbf{P}^2 \\ [x:y:z] & \mapsto & \left[x^2 : xy : y^2 - xz\right]. \end{array}$$

1.2. Some lattice theory.

We here briefly review Nikulin's lattice theory. Details are found in [**Ni**]. Let L be a lattice, i.e.,

(i) L is a free finite $\mathbf{Z}$-module and
(ii) L is equipped with a nondegenerate bilinear symmetric pairing $\langle\ ,\ \rangle$.

For a given lattice L, $\operatorname{disc} L$ is the determinant of the intersection matrix. Note that it is independent of the choice of a basis. We call L unimodular if $\operatorname{disc} L = \pm 1$. Let J be a sublattice of L. We denote its orthogonal complement with respect to $\langle\ ,\ \rangle$ by $J^{\perp}$.

For a lattice L, we denote its dual lattice by $L^{\vee}$. Note that, by using the pairing, L is embedded in $L^{\vee}$ as a sublattice with same rank. Hence the quotient group $L^{\vee}/L$ is a finite Abelian group, which we denote by G_L.

L is called even if $\langle x, x\rangle$ is even for all $x \in L$. For an even lattice L, we define a quadratic form q_L with values in $\mathbf{Q}/2\mathbf{Z}$ as follows:

$$q_L(x \bmod L) = \langle x, x\rangle \bmod 2\mathbf{Z}.$$

Then we have the following lemma:

Lemma 1.1. *Let L be an even unimodular lattice. Let J_1 and J_2 be sublattices of L such that $J_1^{\perp} = J_2$ and $J_2^{\perp} = J_1$. Then*

(i) $G_{J_1} \cong G_{J_2}$ *and*
(ii) $q_{J_1} = -q_{J_2}$.

For a proof, see [**Ni**].

A sublattice M of L is called primitive if L/M is torsion-free.

Example 1.2. For a $K3$ surface X, $H^2(X, \mathbf{Z})$ is an even unimodular lattice with respect to the intersection pairing. The Picard group, $\operatorname{Pic} X$, is a primitive sublattice of $H^2(X, \mathbf{Z})$, and $T_X := (\operatorname{Pic} X)^{\perp}$ is called the transcendental lattice of X.

We shall end this subsection with the following lemma.

Lemma 1.3. *For $j = 1, 2$, let $\Delta_j = \Delta(1)_j \oplus \cdots \oplus \Delta(r_j)_j$ be a lattice where each $\Delta(i)_j$ is of Dynkin type A_a, D_d or E_e.*

(1) *Suppose that $\Phi : \Delta_1 \to \Delta_2$ is a lattice-isometry. Then $r_1 = r_2$ and $\Phi(\Delta(i)_1) = \Delta(i)_2$ after relabeling.*
(2) *Let $\mathbf{A} = A_{m_1} \oplus \cdots \oplus A_{m_k}$ be a direct sum of lattices of Dynkin type A_{m_i}. Suppose that $\mathbf{A}$ is an index-n $(n > 1)$ sublattice of $\Delta := \Delta_2$ and that $(m_1, \ldots, m_k) = (1, 1, 5, 11), (2, 2, 3, 11)$. Then one of the following three cases occurs* (*the first two are quite unlikely but the authors do not have a proof yet*) :

(2-1) $\mathbf{A} = A_1 \oplus (A_1 \oplus A_5 \oplus A_{11}), \Delta = A_1 \oplus D_{17}$, *and* $(A_1 \oplus A_5 \oplus A_{11}) \subseteq D_{17}$ *is an index-*6 *extension.*
(2-2) $\mathbf{A} = A_2 \oplus (A_2 \oplus A_3 \oplus A_{11}), \Delta = A_2 \oplus D_{16}$, *and* $(A_2 \oplus A_3 \oplus A_{11}) \subseteq D_{16}$ *is an index-*6 *extension.*
(2-3) $\mathbf{A} = A_1 \oplus A_{11} \oplus (A_1 \oplus A_5), \Delta = A_1 \oplus A_{11} \oplus E_6$, *and* $(A_1 \oplus A_5) \subseteq E_6$ *is an index-*2 *extension.*

Proof. We observe that

$$|\det(A_n)| = n + 1, \quad |\det(D_n)| = 4, \quad |\det(E_6)| = 3,$$
$$|\det(E_7)| = 2, \quad |\det(E_8)| = 1.$$

We also note that for an index n lattice extension $L \subseteq M$ one has $|\det(L)| = n^2 |\det(M)|$.

(1) is true when $r_1 = r_2 = 1$. In general, for a generating root e in $\Delta(1)_1$ with $e^2 = -2$, one has $(\Phi(e))^2 = -2$ and hence $\Phi(e) \in \Delta(1)_2$ say, because Δ_2 is even and negative definite. Now the connectedness of $\Delta(1)_1$ implies that $\Phi(\Delta(1)_1) \subseteq \Delta(1)_2$. Thus to prove (1), we may assume that $r_2 = 1, \Delta_2 = \Delta(1)_2$. The same argument applied to Φ^{-1} shows that $r_1 = 1$.

(2) The argument in (1) applied to the inclusion $\mathbf{A} \hookrightarrow \Delta_2$, implies that each $\Delta(i)_1$ contains a finite-index sublattice which is a sum of a few summands of $\mathbf{A}$. Now it follows from the observations at the beginning of the proof of this lemma, that either (2) is true or one of the following two cases occurs:

Case (2-4) $\mathbf{A} = A_{11} \oplus (A_2 \oplus A_2 \oplus A_3), \Delta = A_{11} \oplus D_7$, and $(A_2 \oplus A_2 \oplus A_3) \subseteq D_7$ is an index-3 extension.

Case (2-5) $\mathbf{A} = A_2 \oplus A_3 \oplus (A_2 \oplus A_{11}), \Delta = A_2 \oplus A_3 \oplus D_{13}$, and $(A_2 \oplus A_{11}) \subseteq D_{13}$ is an index-3 extension.

In the following, if e_i's form a canonical $\mathbf{Z}$-basis of A_n we let $h_n = (1/(n+1)) \sum_{i=1}^{n} ie_i \pmod{A_n}$ be the generator of $(A_n)^\vee / A_n \cong \mathbf{Z}/(n+1)\mathbf{Z}$. Note that $(h_n)^2 = -n/(n+1)$.

Suppose the contrary that Case (2-4) occurs. Set $\mathbf{B} = A_2 \oplus A_2 \oplus A_3$. Then $D_7 \subseteq \mathbf{B}^\vee := \mathrm{Hom}_{\mathbf{Z}}(\mathbf{B}, \mathbf{Z})$. and the latter is generated by h_2, h_2', h_3 with $(h_2)^2 = -2/3 = (h_2')^2, (h_3)^2 = -3/4$. Since D_7 is generated by roots and contains an index-3 sublattice $\mathbf{B}$, there is a root $t \in D_7 - \mathbf{B}$, and we can write $t = ah_2 + bh_2' + A$ where $a, b \in \mathbf{Z}, A \in \mathbf{B}$. Then $-2 = t^2 = (-2/3)(a^2 + b^2) + A^2 - 2s_1$ for some $s_1 \in \mathbf{Z}$. Since $\mathbf{B}$ is even and negative definite, $A^2 = -2s_2$ for some $s_2 \in \mathbf{Z}$. Denote by $s = s_1 + s_2$. Then $3 = a^2 + b^2 + 3s$, $3|(a^2 + b^2)$. Hence $a = 3a_1, b = 3b_1$ for some $a_1, b_1 \in \mathbf{Z}$. This leads to that $t = a_1(3h_2) + b_1(3h_2') + A \in \mathbf{B}$, a contradiction.

Suppose the contrary that Case (2-5) occurs. Set $\mathbf{B} = A_2 \oplus A_{11}$. Then $D_{13} \subseteq \mathbf{B}^\vee$ and the latter is generated by h_2, h_{11}. As in Case (2-4), there is a root $t \in D_{13} - \mathbf{B}$, and we can write $t = ah_2 + 4bh_{11} + A$ where $a, b \in \mathbf{Z}, A \in \mathbf{B}$. Then $-2 = t^2 = (-2/3)(a^2 + 22b^2) - 2s$ for some $s \in \mathbf{Z}$. Hence $3 = a^2 + 22b^2 + 3s$, $3|(a^2 + b^2)$ and $a = 3a_1, b = 3b_1$ for some $a_1, b_1 \in \mathbf{Z}$. This leads to that $t \in \mathbf{B}$, a contradiction. □

1.3. Review on elliptic surfaces with large torsion group.

We here give a brief summary on the results in [**CP**] and [**C**]. Let $f : X \to C$ be an elliptic surface over a curve C with a section O. Let $MW(f)$ be its Mordell-Weil group, the group of sections, O being the zero element. We denote its torsion part by $MW(f)_{\mathrm{tor}}$. Suppose that $MW(f)_{\mathrm{tor}} \supset \mathbf{Z}/m\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z}$, $m|n$, $mn \geq 3$, and the j-invariant of X is not constant. Then it is known that one obtains $f : X \to C$ in a certain universal way, which we describe below. For that purpose, we need some notations.

Set

$$\Gamma_m(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod n, \; b \equiv 0 \bmod m \right\}.$$

Let $X_m(n) = \Gamma_m(n) \backslash \mathcal{H}^*$, where $\mathcal{H}$ is the upper halfplane in $\mathbf{C}$, and let $E_m(n)$ be the elliptic modular surface of $\Gamma_m(n)$. By definition, $E_m(n)$ is an elliptic surface over $X_m(n)$; and we denote the morphism from $E_m(n)$ to $X_m(n)$ by $\psi_{m,n}$.

Suppose that $MW(f)_{\rm tor} \supset \mathbf{Z}/m\mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z}$, $m|n$, $mn \geq 3$. Then we have a commutative diagram

$$\begin{array}{ccc} C & \stackrel{g}{\to} & X_1(N) \\ & j \searrow & \downarrow j_{m,n} \\ & & \mathbf{P}^1 \end{array}$$

where j and $j_{m,n}$ are the j-invariants of f and $\psi_{m,n}$, respectively. Moreover, this diagram essentially gives $f : X \to C$, i.e., X is obtained as the pull-back surface by g, in the sense of relatively minimal smooth model.

Thus f is determined by g. Hence the uniqueness of X is reduced to that of g, which we consider in §4.

1.4. Comments on pencil of plane curves and nodal cubics.

Let $C = \{f = 0\}$ and $D = \{g = 0\}$ two projective plane curves of degree d without common components. They define a pencil of curves by considering $\{C_{[t:s]}\}_{[t:s]\in\mathbf{P}^1}$, where $C_{[t:s]}$ is the curve of equation $sf - tg = 0$. Let us denote $\mathcal{B} := C \cap D$; it is the set of base points of the pencils; these base points are the intersection points of any couple of elements of the pencil. A base point P is multiple if $(C \cdot D)_P > 1$ (we may replace C and D by any couple of different elements of the pencil). A pencil defines a rational map $\mathbf{P}^2 \dashrightarrow \mathbf{P}^1$ which is well-defined outside the base points. Let $Z \subset \mathbf{P}^2$ be an irreducible curve of degree e which is not a component of any element in the pencil. Let $C_{[t:s]}$ a generic element of the pencil. Then the pencil defines a map $\phi : Z \to \mathbf{P}^1$ of degree

$$d_Z := de - \sum_{P\in\mathcal{B}} (Z \cdot C_{[t:s]})_P;$$

if a base point P is in Z its image is the unique value $\phi(P)$ such that $(Z \cdot C_{\phi(P)})_P$ is greater than the generic intersection number. The critical points of the map are the points $Q \in Z$ such that:

- If Q is not a base point, then $C_{\phi(Q)}$ is either singular at Q or not transversal to Z at Q, i.e., $(Z \cdot C_{\phi(Q)})_Q > 1$.
- If $Q \in \mathcal{B}$, then $(Z \cdot C_{\phi(Q)})_Q > 1 + (Z \cdot C_{[t:s]})_P$, for $[t : s] \neq \phi(Q)$.

Let us consider a nodal cubic N in $\mathbf{P}^2$. We will apply later the following well-known result.

Proposition 1.4. *There exists a homogeneous coordinate system* $[x : y : z]$ *in* $\mathbf{P}^2$ *such that the equation of* N *is* $xyz + x^3 - y^3 = 0$. *The subgroup* G *of* $PGL(3, \mathbf{C})$ *fixing* N *is isomorphic to the dihedral group of order* 6. *Let* $\varphi : \mathbf{C}^* \to \mathrm{Reg}(N)$ *be the mapping defining by* $\varphi(t) := [t : t^2 : t^3 - 1]$. *Let us consider on* N *the geometrical group structure with zero element* $[1 : 1 : 0] = \varphi(1)$. *Then* φ *is a group isomorphism. Each element of* G *is determined by its action on* $\mathrm{Reg}(N)$*; the induced action on* $\mathbf{C}^*$ *is generated by* $t \mapsto t^{-1}$ *and* $t \mapsto \zeta t$ *where* $\zeta^3 = 1$.

2. Some extremal elliptic $K3$ surfaces with trivial Mordell-Weil group.

2.1. Elliptic fibrations and sextic curves.

Relationship between extremal elliptic fibrations and maximizing sextic curves was intensively studied in Persson's paper [**P**]. We explain in this section how to apply this method to construct or discard extremal elliptic fibrations. Let (X, f) be a pair such that X is a $K3$ surface and $f : X \to \mathbf{P}^1$ is a relatively minimal elliptic fibration with a fixed section O.

Step 1. Fix O as the zero element of the Mordell-Weil group $MW(f)$. It determines a group law on each regular fiber and it extends to a group law in the regular part of any fiber. For a fiber F of type I_n, there is a short exact sequence

$$0 \to \mathbf{C}^* \to \operatorname{Reg}(F) \to \mathbf{Z}/n\mathbf{Z} \to 0$$

where the kernel corresponds to the part of $\operatorname{Reg}(F)$ in the irreducible component which intersects O.

Step 2. On the regular part of any fiber F we can consider the map $P \mapsto -P$, (where $F \cap O$ is the zero element). These maps are the restriction of a morphism $\sigma : X \to X$, which is clearly an involution. By definition $f \circ \sigma = f$. Then, there is a natural map $\widetilde{\rho} : X/\sigma \to \mathbf{P}^1$; if F is an elliptic fiber of π, F/σ is the quotient of an elliptic curve by an involution with four fixed points (the 2-torsion), i.e., a smooth rational curve.

Then $\widetilde{\rho} : X/\sigma \to \mathbf{P}^1$ is a morphism from a smooth (rational) surface onto $\mathbf{P}^1$ whose generic fiber is $\mathbf{P}^1$. If F is a fiber of type I_{2n+1} (resp. I_{2n}), F/σ is a curve with normal crossings and $n+1$ irreducible components which are smooth and rational.

Step 3. For any singular fiber F, we contract all of the irreducible components of $\widetilde{\rho}(F)$ but the one which intersects $\widetilde{\rho}(O)$. We obtain a holomorphic fiber bundle $\rho : \Sigma \to \mathbf{P}^1$ with fiber isomorphic to $\mathbf{P}^1$ (Σ smooth) and a map $\tau : X \to \Sigma$ such that $\rho \circ \tau = \pi$. This map is generically $2:1$.

The map τ is a 2-fold covering ramified on the image of the fixed points of σ, i.e., on the image of the 2-torsion. We can write this curve as $E \cup R$ where $E := \tau(O)$, $R \cap E = \emptyset$ and R has intersection number three with the fibers of ρ. The number of irreducible components of R depends on the 2-torsion $T_2(MW(f))$ of the Mordell-Weil group of X (one irreducible component if $T_2(MW(f)) = 0$, two if $T_2(MW(f)) = \mathbf{Z}/2\mathbf{Z}$ and three if $T_2(MW(f)) = \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$).

If the configuration of π is $[1, \dots, n_1, \dots, n_r]$, $1 < n_1 \leq \cdots \leq n_r$, then R has exactly r singular points of type $\mathbf{A}_{n_1-1}, \dots, \mathbf{A}_{n_r-1}$.

Remark 2.1. Let us suppose that $n_r > 7$, and let us call F the fiber of ρ containing this point $\mathbf{A}_{n_r-1}$; R intersects also F at another point P. Then we can perform three Nagata elementary transformations on the first three infinitely near points of R at $\mathbf{A}_{n_r-1}$. We call Σ' the result of this operation and we do not change the notation for the strict transforms; it induces a new fibration $\rho' : \Sigma' \to \mathbf{P}^1$ where E is a section of self-intersection -1. The curve R has a singular point $\mathbf{A}_{n_r-7}$ and $(R \cdot E)_P = 3$, and R is smooth at P. We can contract E and we obtain a projective plane where the contraction of R is a curve of degree 6 (also denoted by R) which has $r+1$ singular points of type $\mathbf{A}_{n_1-1}, \mathbf{A}_{n_2-1}, \dots, \mathbf{A}_{n_r-7}$ and $\mathbf{E}_6$; the image of F is the tangent line to R at $\mathbf{E}_6$ and passes through $\mathbf{A}_{n_1-7}$. The pencil which induces the elliptic fibration (the *preferred pencil*) is the pencil of lines through $\mathbf{E}_6$. This fibration is called the standard fibration in [**P**] and in this case $\mathbf{E}_6$ is its center.

We can consider some kind of converse of this construction. Let $R \subset \mathbf{P}^2$ be a reduced curve (maybe reducible) of degree six such that its singular points are simple. Let P be a singular point of R. Then if X is the minimal resolution of the ramified double covering of $\mathbf{P}^2$ ramified on R and $\pi : X \to \mathbf{P}^1$ is the mapping induced by the pencil of lines through P, then π is a relatively minimal elliptic fibration of the $K3$-surface X. We call (X, π) the elliptic fibration associated to (R, P) and we will call the pencil of lines at P the preferred pencil; we will denote $\sigma : X \to \mathbf{P}^2$ the double covering. The following result is easy and useful.

Proposition 2.2. *Let $\pi : X \to \mathbf{P}^1$ be the elliptic fibration associated to (R, P) as above. Let E be a section of X; let $C := \sigma(E)$. Then either C is an irreducible component of R, either the intersection number of C and E at any point in $C \cap R$ is an even number.*

In both cases C is a curve of degree d having at P a singular point of multiplicity $d-1$. In the first case there is exactly one section over C and in the second case there are exactly two such sections.

We study now the existence of elliptic fibrations with trivial Mordell-Weil group in the cases of ambiguity which appear in the list of Miranda and Persson. In fact, we have applied this method to all cases of ambiguity in the list. As it is very long, we present only a few cases, where interesting phenomena occur.

2.2. Type $m = 9$.

Proposition 2.3. *There exist elliptic $K3$ surfaces of type* 9*, i.e., with configuration* $[1, 1, 1, 1, 10, 10]$*, and trivial Mordell-Weil group.*

This proposition gives one ambiguity case as such a fibration with Mordell-Weil group of order 5 appears in [**MP3**].

We look for an irreducible curve R of degree 6 having three singular points of type $\mathbf{E}_6, \mathbf{A}_3, \mathbf{A}_9$ and such that the tangent line to R at $\mathbf{E}_6$ passes through

$\mathbf{A}_3$. As in the case above the line through $\mathbf{A}_3$ and $\mathbf{A}_9$ intersects R at two other points.

Step 1. *First Cremona transformation.*

We consider $CR_{\mathbf{E}_6,\mathbf{A}_3,\mathbf{A}_9}$. We denote R_1 the strict transform of R; R_1 is a quintic curve. We have a smooth point Q such that the tangent line T to R_1 at Q verifies that $(R_1 \cdot Q)_Q = 4$. We denote Q' the other point in $R_1 \cap T$.

The other singular points of R_1 are $\mathbf{A}_7$ (coming from $\mathbf{A}_9$), P_1 (an ordinary double point coming from $\mathbf{A}_3$) and another ordinary double point denote P_2. The preferred pencil of lines has its center at P_1. The line joining P_1 and P_2 intersects R_1 at Q. The line joining P_1 and $\mathbf{A}_7$ passes through Q'. The ramification locus is $R_1 \cup T$.

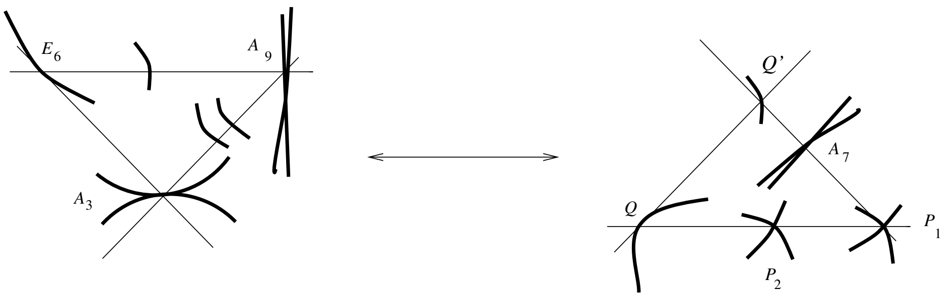

Figure 1.

Step 2. *Second and third Cremona transformations.*

We perform $CR_{P_1,P_2,\mathbf{A}_7}$. We obtain a quartic curve R_2 with one singular point $\mathbf{A}_5$ (coming from $\mathbf{A}_7$). The line T becomes a conic T_2 and $R_2 \cap T_2 = \{Q, Q', Q''\}$ where $(R_2 \cdot T_2)_Q = 5$, $(R_2 \cdot T_2)_{Q'} = 2$, $(R_2 \cdot T_2)_{Q''} = 1$, and $\mathbf{A}_5, Q', Q''$ are aligned. The center of the preferred pencil is Q''.

We perform the third Cremona transformation $CR_{\mathbf{A}_5,L,Q''}$, L being the tangent line at $\mathbf{A}_5$. We obtain two cubics R_3 and T_3. The cubic R_3 has an ordinary double point $\mathbf{A}_1$ and T_3 has also a double point denoted S (which is the center of the preferred pencil). The curves R_3 and T_3 have two intersection points Q and Q', with intersection numbers 5 and 4, and the points Q', S and $\mathbf{A}_1$ are aligned.

Question 2.4. Does there exist an irreducible nodal cubic R_3 (with node $\mathbf{A}_1$), an irreducible cubic T_3 with a double point S in $\mathbf{P}^2$ such that $R_3 \cap T_3 = \{Q, Q'\}$, $Q, Q' \neq S, \mathbf{A}_1$, with $(R_3 \cdot T_3)_Q = 5$, $(R_3 \cdot T_3)_{Q'} = 4$ and $Q', S, \mathbf{A}_1$ aligned?

Proposition 2.5. *The answer to Question 2.4 is yes.*

Proof. We proceed by applying Proposition 1.4 to R_3. We suppose that $Q = p(s^{-4})$ and $Q' = p(s^5)$. In this situation the equation of the line joining Q' and $\mathbf{A}_1$ is $y = s^5 x$. Let $f(x, y, z) = 0$ an equation for T_3 such that the coefficient of z^3 in f is 1. Then $f(t, t^2, t^3 - 1) = (t - s^5)^4 (t - s^{-4})^5$. We impose that T_3 intersects the line $y = s^2 x$ at one point outside Q' (with multiplicity 2). We force this point to be singular and we get the conditions on s (again with Maple-V). We obtain that

$$(s^6 - 1)(s^6 + 3s^3 + 1)(s^{12} + 4s^9 + s^6 + 4s^3 + 1) = 0.$$

We consider the action of the dihedral group; in the first term it is enough to retain the cases $s = \pm 1$; the positive case is too degenerate so it remains only $s = -1$. The equation of T_3 in this case is:

$$13\,y^3 + 9\,y^2 x - 5\,y^2 z - 9\,y x^2 - 6\,yxz - yz^2 - 13\,x^3 - 5\,x^2 z + xz^2 + z^3 = 0.$$

For the second term, one can see that we force $S = \mathbf{A}_1$ which is also too degenerate. The last factor gives two different cases (the twelve roots give two orbits by the action of the dihedral group). The equation is:

$$\begin{aligned}
&\left(-\frac{1265\,s^9}{2} - 60\,s^3 - \frac{4671}{2} - 2170\,s^6\right) x^3 \\
&+ \left(1205\,s^8 + 320\,s^{11} + 1285\,s^2\right) zx^2 \\
&+ \left(10080\,s + 135\,s^4 + 9480\,s^7 + 2466\,s^{10}\right) yx^2 \\
&+ \left(60\,s + 60\,s^7 + 16\,s^{10} + 5\,s^4\right) z^2 x \\
&+ \left(15255\,s^2 + 216\,s^5 + 14325\,s^8 + 3780\,s^{11}\right) y^2 x \\
&+ \left(\frac{495\,s^9}{2} + \frac{2103}{2} + 990\,s^6\right) yzx \\
&+ \left(-\frac{1735\,s^9}{2} - 60\,s^3 - \frac{6609}{2} - 3110\,s^6\right) y^3 \\
&- \left(640\,s + 620\,s^7 + 160\,s^{10} + 5\,s^4\right) zy^2 \\
&+ \left(-75\,s^2 - 75\,s^8 - 20\,s^{11} - 4\,s^5\right) z^2 y + z^3 = 0.
\end{aligned}$$

□

We deduce that there are essentially three different answers to Question 2.4. The main feature of the first answer is that the tangent line L to R_3 at Q' passes through Q. The elliptic surface is obtained from the double covering of $\mathbf{P}^2$ ramified along $R_3 + T_3$, and the elliptic fibration comes from the pencil of lines with center at S. One of the singular fibers is produced by the line joining S, $\mathbf{A}_1$ and Q'.

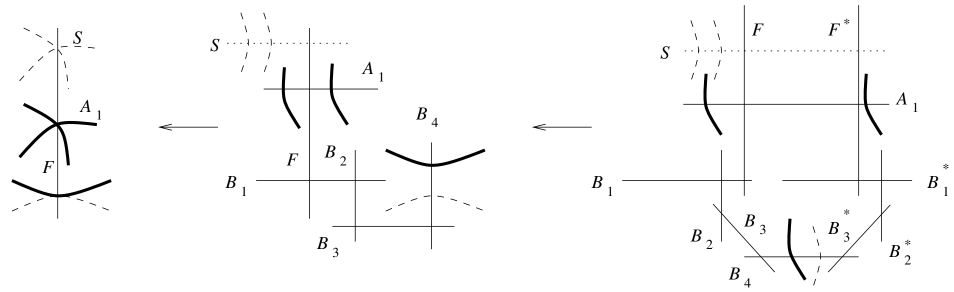

Figure 2.

The other singular fiber is produced by the line joining S and Q.

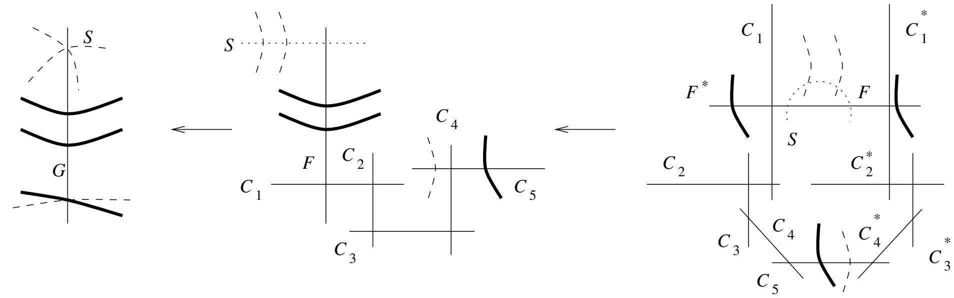

Figure 3.

Proposition 2.6. *The solution for $s=-1$ produces the elliptic fibration such that MW is cyclic of order 5. The solutions $s^{12}+4s^9+s^6+4s^3+1=0$ produce elliptic fibrations with trivial Mordell-Weil group; this case was not previously known.*

Proof. We note that the exceptional curve of the blowing-up of S never produces a section. In both cases the strict preimage of T_3 produces a section.

In the case $s=-1$, the intersection numbers of the line T with the curve R_3+T_3 are always even; then the preimage of L is reducible and produces two sections. We note also that Q is in this case an inflection point for both R_3 and T_3; the common tangent line has also even intersection numbers with R_3+T_3 and then it produces two sections. We have found five different sections, then all of them.

Let us consider now the second case. We know already a section. By Proposition 2.2, any other section should come from a section to the pencil of lines through S having always even intersection numbers with the ramification curve R_3+T_3. Then the problem is as follows:

Is there a curve D of degree d having a point of multiplicity $d-1$ at S and such that $(S\cdot R_3)_P \equiv (S\cdot T_3)_P \mod 2$ for any $P\in \mathbf{P}^2$ and any branch of D at S has even intersection number with T_3?

Let us suppose that such a curve exists. It gives two different sections D_0 and D_1 in the elliptic surface. From [**MP3**], D_0 and D_1 are torsion sections, and then they must be disjoint. In particular, D cannot intersect $R_3\cup T_3$ outside $S, \mathbf{A}_1, Q, Q'$ and no branch of D at S is tangent to any branch of T_3 at S.

D_0 and D_1 belong to the 5-torsion, so by the structure of the singular fibers, we have:

- $\mathbf{A}_1 \notin D$;
- $(T_3\cdot D)_{Q'} = (R_3\cdot D)_{Q'} = a = 0,2,4$;
- $(T_3\cdot D)_{Q'} = (R_3\cdot D)_{Q} = b = 1,3,5$.

Then, putting all these conditions together, we obtain that $S\notin D$ and so D is a line; then $3=a+b$. The two possibilities appear in the previous case, but not in this one. □

2.3. Case $m=11$.

The method to find or discard the fibrations in the other cases is the same one. As the answers are positive, we will give the results that may be verified by the reader. Let us consider the polynomial

$$\begin{aligned}
&p_1(x,y,z)\\
&:= \left(\frac{11593}{95004009}-\frac{4027\,v}{190008018}\right)y^4x^2+\left(\frac{4705}{10556001}-\frac{2183\,v}{10556001}\right)zxy^4\\
&+\left(-\frac{1493\,v}{4691556}+\frac{803}{2345778}\right)z^2y^4+\left(-\frac{48226}{5000211}+\frac{1475\,v}{5000211}\right)zy^3x^2\\
&+\left(\frac{1174\,v}{185193}-\frac{4736}{185193}\right)z^2xy^3+\left(\frac{635\,v}{123462}-\frac{755}{61731}\right)z^3y^3\\
&+\left(\frac{20153}{87723}+\frac{1081\,v}{175446}\right)z^2y^2x^2+\left(\frac{854}{3249}-\frac{187\,v}{3249}\right)z^3y^2x\\
&+\left(-\frac{427}{6498}+\frac{187\,v}{12996}\right)z^4y^2+\left(-\frac{22612}{13851}+\frac{386\,v}{13851}\right)z^3yx^2\\
&+\left(\frac{1412}{1539}+\frac{20\,v}{1539}\right)z^4xy+x^3z^3+\left(-\frac{11\,v}{729}-\frac{485}{729}\right)z^4x^2
\end{aligned}$$

where $v^2+2=0$.

Proposition 2.7. *The curve $p_1(x,y,z)=0$ is an irreducible curve with singularities $\mathbf{E}_6$ (at $[1:0:0]$ and tangent line $z=0$), $\mathbf{A}_1$ (at $[0:0:1]$), $\mathbf{A}_9$ (at $[0:1:0]$) and $\mathbf{A}_2$ (at $[1:1:1]$). The pencil of lines through the* *triple point determine after a double covering an elliptic $K3$ fibration of type* $[1, 1, 1, 2, 3, 16]$ *with trivial Mordell-Weil group.*

Proof. The computations have been performed with Maple-V. We note that the curve is irreducible as the line $x = 0$ joining $\mathbf{A}_9$ and $\mathbf{A}_1$ is not a component. The Miranda-Persson classification finishes the result. □

2.4. Case $m = 13$.

Proposition 2.8. *The curve* $p_2(x, y, z) = 0$ (*see below*) *is an irreducible curve with singularities* $\mathbf{E}_6$ (*at* $[1 : 0 : 0]$ *and tangent line* $y = 0$), $\mathbf{A}_7$ (*at* $[0 : 0 : 1]$), $\mathbf{A}_4$ (*at* $[0 : 1 : 0]$) *and* $\mathbf{A}_1$ (*at* $[1 : 1 : 1]$). *The pencil of lines through the triple point determine after a double covering an elliptic $K3$ fibration of type* $[1, 1, 1, 2, 5, 14]$ *with trivial Mordell-Weil group.*

Proof. As before, computations have been performed with Maple-V. We note that the curve is irreducible as the line $x = y$ joining $\mathbf{A}_7$ and $\mathbf{A}_1$ is not a component. The Miranda-Persson classification finishes the result. □

We have:

$$
\begin{aligned}
&p_2(x, y, z)\\
&:= y^3x^3 + \left(-\frac{24284}{130321} + \frac{10287\,v}{260642} + \frac{144295\,v^2}{1824494}\right) y^4x^2\\
&+ \left(-\frac{6071515\,v^2}{130321} - \frac{2851308\,v}{130321} + \frac{13668817}{130321}\right) zx^2y^3\\
&+ \left(\frac{38660279\,v}{260642} + \frac{161684215\,v^2}{521284} - \frac{179634441}{260642}\right) z^2x^2y^2\\
&+ \left(-\frac{252208635\,v^2}{521284} - \frac{60782001\,v}{260642} + \frac{277127879}{260642}\right) z^3x^2y\\
&+ \left(\frac{55758423\,v}{521284} + \frac{460287135\,v^2}{2085136} - \frac{125694751}{260642}\right) z^4x^2\\
&+ \left(-\frac{10473}{6859} + \frac{2326\,v}{6859} + \frac{32860\,v^2}{48013}\right) zxy^4\\
&+ \left(-\frac{361050\,v^2}{6859} - \frac{176895\,v}{6859} + \frac{1579285}{13718}\right) z^2xy^3\\
&+ \left(\frac{725753\,v}{13718} + \frac{1458065\,v^2}{13718} - \frac{1564472}{6859}\right) z^3xy^2\\
&+ \left(\frac{1625477}{13718} - \frac{191737\,v}{6859} - \frac{3045105\,v^2}{54872}\right) z^4xy\\
&+ \left(-\frac{268}{361} + \frac{141\,v}{722} + \frac{3495\,v^2}{10108}\right) z^2y^4 + \left(\frac{825}{722} - \frac{255\,v}{361} - \frac{1175\,v^2}{1444}\right) z^3y^3
\end{aligned}
$$

$$+\left(-\frac{686}{361}+\frac{1099\,v}{1444}+\frac{6055\,v^2}{5776}\right)z^4y^2,$$

where $5\,v^3-4\,v^2-14\,v+14=0$.

Let us remark that this condition has exactly one real solution.

2.5. Case $m=27$.

In this cases we only state the result concerning the existence and unicity of curves and we give the equation of the polynomial. The proofs and methods of computations are very similar to the previous ones.

Proposition 2.9. *The curve* $p_3(x,y,z)=0$ (*see below*) *is an irreducible curve with singularities* $\mathbf{E}_6$ (*at* $[0:0:1]$ *and tangent line* $y=0$), $\mathbf{A}_3$ (*at* $[1:0:0]$), $\mathbf{A}_5$ (*at* $[0:1:0]$) *and* $\mathbf{A}_4$ (*at* $[1:1:1]$). *The pencil of lines through the triple point determine after a double covering an elliptic* $K3$ *fibration of type* $[1,1,1,5,6,10]$ *with trivial Mordell-Weil group.*

We have

$$\begin{aligned}
&p_3(x,y,z)\\
&:=\left(-\frac{200\,v^2}{297}-\frac{425}{297}-\frac{110\,v}{27}\right)y^4x^2+\left(\frac{125}{396}+\frac{5\,v}{9}-\frac{13\,v^2}{396}\right)zy^4x\\
&+\left(\frac{5\,z^2}{528}-\frac{5}{264}+\frac{5\,v}{48}\right)z^2y^4+\left(\frac{115\,v^2}{81}+\frac{220}{81}+\frac{875\,v}{81}\right)y^3x^3\\
&+\left(\frac{655}{108}+\frac{493\,v}{54}+\frac{133\,v^2}{108}\right)zy^3x^2+\left(\frac{5\,v^2}{36}-\frac{115}{36}-\frac{5\,v}{9}\right)z^2y^3x+z^3y^3\\
&+\left(-\frac{2225}{972}-\frac{3275\,v}{486}-\frac{725\,v^2}{972}\right)y^2x^4+\left(-\frac{2831}{324}-\frac{2032\,v}{81}-\frac{797\,v^2}{324}\right)zy^2x^3\\
&+\left(-\frac{37\,v^2}{72}-\frac{35}{36}-\frac{215\,v}{72}\right)z^2y^2x^2+\left(\frac{1225\,z^2}{972}+\frac{5215}{972}+\frac{7495\,v}{486}\right)zyx^4\\
&+\left(\frac{1105}{324}+\frac{788\,v}{81}+\frac{193\,v^2}{324}\right)z^2yx^3+\left(-\frac{893\,v^2}{3888}-\frac{4333}{1944}-\frac{24499\,v}{3888}\right)z^2x^4
\end{aligned}$$

where $25+75\,v+15\,v^2+v^3=0$.

2.6. Case $m=32$.

Let us consider the polynomial

$$\begin{aligned}
&p_4(x,y,z)\\
&:=y^3z^3+\left(\frac{5625\,v}{668168}-\frac{33625}{334084}\right)z^2x^4+\left(\frac{3475\,v}{58956}+\frac{39275}{29478}\right)yz^2x^3\\
&+\left(-\frac{1465\,v}{1734}-\frac{1775}{867}\right)y^2x^2z^2+\left(\frac{173\,v}{204}-\frac{299}{102}\right)y^3xz^2
\end{aligned}$$

$$+\left(-\frac{v}{40}+\frac{17}{20}\right)y^4z^2+\left(\frac{19675\,v}{501126}-\frac{188825}{501126}\right)yzx^4$$
$$+\left(\frac{350\,v}{4913}+\frac{23110}{4913}\right)y^2x^3z+\left(-\frac{1580\,v}{867}-\frac{5900}{867}\right)y^3x^2z$$
$$+\left(\frac{11\,v}{15}-5/3\right)y^4xz+\left(\frac{29555\,v}{668168}-\frac{232705}{668168}\right)y^2x^4$$
$$+\left(-\frac{1885\,v}{29478}+\frac{116975}{29478}\right)y^3x^3+\left(-\frac{1205\,v}{1734}-\frac{33517}{8670}\right)y^4x^2$$

where $v^2-v+34=0$.

Proposition 2.10. *The curve $p_4(x,y,z)=0$ is an irreducible curve with singularities $\mathbf{E}_6$ (at $[0:0:1]$ and tangent line $y=0$), $\mathbf{A}_8$ (at $[1:0:0]$), $\mathbf{A}_2$ (at $[0:1:0]$) and two points of type $\mathbf{A}_1$ in the line $x+y+z=0$. The pencil of lines through the triple point determine after a double covering an elliptic $K3$ fibration of type $[1,1,2,2,3,15]$ with trivial Mordell-Weil group.*

2.7. Case $m=37$.

Proposition 2.11. *The curve $p_5(x,y,z)=0$ (see below) is an irreducible curve with singularities $\mathbf{E}_6$ (at $[0:0:1]$ and tangent line $x=0$), $\mathbf{A}_2$ (at $[0:1:0]$), $\mathbf{A}_8$ (at $[1:0:0]$) and two points of type $\mathbf{A}_1$ in the line $x+y+z=0$. The pencil of lines through the triple point determine after a double covering an elliptic $K3$ fibration of type $[1,1,2,2,9,9]$ with trivial Mordell-Weil group.*

We have:

$$p_5(x,y,z)$$
$$:=\left(\frac{3970803\,v}{130438}-\frac{345557847\,v^2}{65219}+\frac{8058927}{130438}\right)y^4x^2$$
$$+\left(-\frac{82574784\,v^2}{5929}+\frac{37159110\,v}{5929}-\frac{3105297}{5929}\right)zy^4x$$
$$+\left(-\frac{653967}{2156}+\frac{3545235\,v}{1078}-\frac{5380479\,v^2}{1078}\right)z^2y^4$$
$$+\left(\frac{5894214\,v}{9317}-\frac{295704\,v^2}{9317}-\frac{650011}{9317}\right)y^3x^3$$
$$+\left(-\frac{278076\,v^2}{847}+\frac{808926\,v}{847}-\frac{86286}{847}\right)zy^3x^2$$
$$+\left(-\frac{105723\,v^2}{77}+\frac{80505\,v}{77}-\frac{15255}{154}\right)z^2xy^3$$

$$+\left(\frac{14286}{1331}-\frac{136113\,v}{1331}+\frac{65742\,v^2}{1331}\right)y^2x^4$$
$$+\left(-\frac{24048\,v}{121}+\frac{30018\,v^2}{121}+\frac{4599}{242}\right)zy^2x^3$$
$$+\left(-\frac{2199\,v}{11}+\frac{3966\,v^2}{11}+\frac{195}{11}\right)z^2y^2x^2$$
$$+\left(-\frac{309}{121}+\frac{3711\,v}{121}-\frac{8358\,v^2}{121}\right)zyx^4$$
$$+\left(\frac{471\,v}{11}-\frac{903\,v^2}{11}-\frac{87}{22}\right)z^2yx^3+\left(-\frac{42\,v^2}{11}+\frac{159\,v}{44}-\frac{15}{44}\right)z^2x^4+z^3x^3$$

where $28\,v^3-30\,v^2+12\,v-1=0$.

2.8. Case $m=38$**.**

Let us consider the polynomial

$$\begin{aligned}&p_6(x,y,z)\\ &:=\frac{1404\,x^2y^4}{1445}-\frac{9\,xy^4z}{85}+\frac{17\,z^2y^4}{60}+\frac{10800\,x^3y^3}{4913}+\frac{1980\,x^2y^3z}{289}\\ &-\frac{37\,z^2y^3x}{102}+y^3z^3+\frac{105840\,x^4y^2}{83521}+\frac{4410\,x^3y^2z}{289}+\frac{13965\,z^2y^2x^2}{1156}\\ &+\frac{720300\,x^4yz}{83521}+\frac{780325\,z^2yx^3}{29478}+\frac{14706125\,z^2x^4}{1002252}.\end{aligned}$$

Proposition 2.12. *The curve $p_6(x,y,z)=0$ is an irreducible curve with singularities $\mathbf{E}_6$ (at $[0:0:1]$ and tangent line $y=0$), $\mathbf{A}_7$ (at $[1:0:0]$), $\mathbf{A}_1$ (at $[0:1:0]$) and two points of type $\mathbf{A}_2$ in the line $x+y+z=0$. The pencil of lines through the triple point determine after a double covering an elliptic $K3$ fibration of type $[1,1,2,3,3,14]$ with trivial Mordell-Weil group.*

2.9. Case $m=55$**.**

Let us consider the polynomial

$$\begin{aligned}&p_7(x,y,z)\\ &:=\left(\frac{139}{176}+\frac{175\,v}{176}\right)y^4z^2+.\left(-\frac{837\,v}{242}+\frac{7101}{968}\right)y^4zx\\ &+\left(\frac{30537}{10648}-\frac{29565\,v}{10648}\right)y^4x^2+\left(-\frac{151\,v}{44}+\frac{155}{44}\right)y^3z^2x\\ &+\left(\frac{675}{242}+\frac{837\,v}{242}\right)y^3zx^2+\left(-\frac{669\,v}{2662}+\frac{2765}{1331}\right)y^3x^3\\ &+\left(-\frac{81\,v}{22}+\frac{243}{44}\right)y^2z^2x^2+\left(\frac{441\,v}{242}-\frac{183}{242}\right)y^2zx^3\end{aligned}$$

$$+\left(-\frac{1107}{1331}+\frac{2025\,v}{1331}\right)y^2x^4+\left(-\frac{17}{11}+\frac{107\,v}{22}\right)yz^2x^3$$
$$+\left(\frac{153\,v}{121}+\frac{18}{121}\right)yzx^4+z^3x^3+\left(\frac{13}{22}-\frac{5\,v}{22}\right)z^2x^4$$

where $3v^2-4v+2=0$.

Proposition 2.13. *The curve $p_7(x,y,z)=0$ is an irreducible curve with singularities $\mathbf{E}_6$ (at $[0:0:1]$ and tangent line $x=0$), $\mathbf{A}_1$ (at $[0:1:0]$), $\mathbf{A}_7$ (at $[1:0:0]$) and two points of type $\mathbf{A}_2$ in the line $x+y+z=0$. The pencil of lines through the triple point determine after a double covering an elliptic $K3$ fibration of type $[1,1,3,3,8,8]$ with trivial Mordell-Weil group.*

3. The complete determination of the Mordell-Weil group for each type of semi-stable extremal fibrations.

In this section, we shall show Theorem 0.2 which will follow from the Table in [**MP3**], and the Lemmas below. We recall Lemma 1.3 and Shioda-Inose's result that the isomorphism class of a $K3$ surface X of Picard number 20 is uniquely determined by the transcendental lattice T_X, modulo the action of $SL_2(\mathbf{Z})$ [**SI**].

Lemma 3.1. *Let S be an even symmetric lattice of rank 20 and signature $(1,19)$ and T a positive definite even symmetric lattice of rank 2. Assume that $\varphi: T^\vee/T \to S^\vee/S$ is an isomorphism which induces the following equality involving $\mathbf{Q}/2\mathbf{Z}$-valued discriminant (quadratic) forms: $q_S=-q_T$.*

Let X be the unique $K3$ surface (up to isomorphisms) with the transcendental lattice $T_X=T$. Then the Picard lattice $\operatorname{Pic} X$ is isometric to S.

Proof. Consider the overlattice L of $S\oplus T$ obtained by adding all elements $\varphi(x)+x$, $x\in T^\vee$, where $\varphi(x)\in S^\vee$ denotes one representative of $\varphi(x+T)\in S^\vee/S$. The (even) intersection form on $S\oplus T$ is naturally extended to a $\mathbf{Q}$-valued one on $S^\vee\oplus T^\vee$. For each $x\in T^\vee$, we have, modulo $2\mathbf{Z}$, $(\varphi(x)+x,\varphi(x)+x)=-q_T(x)+q_T(x)=0$, i.e., $(\varphi(x)+x,\varphi(x)+x)\in 2\mathbf{Z}$. Also for $x_i\in T^\vee$, combining $(\varphi(x_1+x_2),\varphi(x_1+x_2))=-(x_1+x_2,x_1+x_2)$ (mod $2\mathbf{Z}$) and $(\varphi(x_i),\varphi(x_i))=-(x_i,x_i)$ (mod $2\mathbf{Z}$), we see that $(\varphi(x_1),\varphi(x_2))=-(x_1,x_2)$ (mod $\mathbf{Z}$), whence mod $\mathbf{Z}$ we have $(\varphi(x_1)+x_1,\varphi(x_2)+x_2)=0$. Thus L is an even (integral) symmetric lattice of rank 22 and signature $(1+2,19+0)$. Clearly, $L/(S\oplus T)\cong T^\vee/T$ and hence $|\det(L)|=|\det(S\oplus T)|/|T^\vee/T|^2=1$. Now by the classification of indefinite unimodular even symmetric lattices, L is isometric to the $K3$ lattice (cf. [**Se**]).

On the other hand, by [**SI**], there is a unique $K3$ surface X (modulo isomorphisms) with the intersection form of the transcendental lattice T_X equal to T (modulo $SL_2(\mathbf{Z})$). We identify L with $H^2(X,\mathbf{Z})$ and T with T_X. Note that there are two embeddings $\iota_k: T_X\to H^2(X,\mathbf{Z})$: $\iota_1: T_X\hookrightarrow$

$H^2(X, \mathbf{Z})$ as the transcendental sublattice, and $\iota_2 : T_X = T \hookrightarrow S \oplus T \hookrightarrow L = H^2(X, \mathbf{Z})$.

The embedding ι_1 (resp. ι_2) is primitive by the definition of T_X (resp. of L). Now Nikulin's uniqueness theorem of primitive embedding implies that there is an isometry Ψ of $H^2(X, \mathbf{Z})$ such that $\iota_1 = \Psi \circ \iota_2$ [**Mo**, Cor. 2.10]. Note that the Picard lattice $\text{Pic}X = (\iota_1(T_X))^\perp = (\Psi(\iota_2(T_X)))^\perp = \Psi(T^\perp) = \Psi(S) \cong S$. □

Lemma 3.2. *Let $f : X \to \mathbf{P}^1$ be of type $m = 4$ as in Theorem 0.2. Then $MW(f) \neq (0)$.*

Proof. Suppose the contrary that $f : X \to \mathbf{P}^1$ is of type $m = 4$ with $MW(f) = (0)$. Then Pic X is a direct sum $U \oplus A_3 \oplus A_{15}$ of lattices, where $U = (a_{ij})$ satisfies $a_{ii} = 0, a_{12} = a_{21} = 1$. Let (b_{ij}) be the intersection matrix of the transcendental lattice $T = T_X$. Then $b_{ii} > 0$ and $\det(b_{ij}) = |\det(\text{Pic}\, X)| = 64$ (cf. [**BPV**]). After conjugation by $SL(2, \mathbf{Z})$, we may assume that $-b_{11} < 2|b_{12}| \le b_{11} \le b_{22}$, and that $b_{12} \ge 0$ when $b_{11} = b_{22}$. An easy calculation shows that one of the following cases occurs:

(1) $(b_{ij}) = \text{diag}\,[2, 32]$,
(2) $(b_{ij}) = \text{diag}\,[4, 16]$,
(3) $(b_{ij}) = \text{diag}\,[8, 8]$, and
(4) $b_{11} = 8, b_{22} = 10, b_{12} = 4$.

Embed T, as a sublattice, naturally into $T^\vee = \text{Hom}_{\mathbf{Z}}(T, \mathbf{Z})$. Then $T^\vee/T \cong (\text{Pic}X)^\vee/\ (\text{Pic}X) \cong \mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/16\mathbf{Z}$. Note that $(\text{Pic}X)^\vee/(\text{Pic}X)$ is generated by $\varepsilon_1 = (1/4)\sum_{i=1}^{3} iv_i$ and $\varepsilon_2 = (1/16)\sum_{i=4}^{18}(i-3)v_i$, modulo Pic X, where v_i's form a canonical basis of $A_3 \oplus A_{15} \subseteq \text{Pic}\ X$. So the discriminantal quadratic form $q_T : T^\vee/T \to \mathbf{Q}/2\mathbf{Z}$ is equal to $-q_{\text{Pic}\,X} = (-\varepsilon_1^2) \oplus (-\varepsilon_2^2) = (3/4) \oplus (15/16)$.

On the other hand, in Case (4), $T^\vee$ has a $\mathbf{Z}$-basis $(e_1\ e_2)(b_{ij})^{-1} = (g_1\ g_2)$, where e_1, e_2 form a canonical basis of T, where $g_1 = (1/32)(5e_1 - 2e_2), g_2 = (1/16)(-e_1 + 2e_2)$. This leads to that $\text{ord}(g_1)$ is equal to 32 in $T^\vee/T$, a contradiction.

In Cases (1)-(3) where $T = \text{diag}\,[s, t]$, with $(s, t) = (2, 32), (4, 16)$ or $(8, 8)$, the discriminantal quadratic form q_T is equal to $(1/s) \oplus (1/t)$. This leads to that $(1/s) \oplus (1/t) \cong (3/4) \oplus (15/16)$, which is impossible by an easy check. □

Lemma 3.3. *Consider the pairs below:*

$$(m, G_m) = (2, \langle 0\rangle), (9, \langle 0\rangle), (11, \langle 0\rangle), (13, \langle 0\rangle), (27, \langle 0\rangle), (32, \langle 0\rangle),$$
$$(37, \langle 0\rangle), (38, \langle 0\rangle), (55, \langle 0\rangle), (35, \mathbf{Z}/2\mathbf{Z}), (53, \langle \mathbf{Z}/3\mathbf{Z}\rangle).$$

For each of these eleven pairs (m, G_m), there is a Jacobian elliptic $K3$ surface $f_m : X_m \to \mathbf{P}^1$ of type m as in Theorem 0.2 such that $(m, MW(f_m)) = (m, G_m)$.

Proof. The existence of the pairs where $m = 2, 35$ is proved constructively in [**AT**]. The rest is also constructively proved in §2. In the paragraphs below, we will give an independent lattice-theoretical proof.

Let T_m, $m = 2, 9, 11, 13, 27, 32, 37, 38, 55, 35, 53$, be the positive definite even symmetric lattice of rank 2 with the following intersection form, respectively:

$$\begin{pmatrix}4 & 2\\ 2 & 10\end{pmatrix}, \begin{pmatrix}10 & 0\\ 0 & 10\end{pmatrix}, \begin{pmatrix}10 & 2\\ 2 & 10\end{pmatrix}, \begin{pmatrix}2 & 0\\ 0 & 70\end{pmatrix}, \begin{pmatrix}10 & 0\\ 0 & 30\end{pmatrix}, \begin{pmatrix}12 & 6\\ 6 & 18\end{pmatrix},$$

$$\begin{pmatrix}18 & 0\\ 0 & 18\end{pmatrix}, \begin{pmatrix}6 & 0\\ 0 & 42\end{pmatrix}, \begin{pmatrix}24 & 0\\ 0 & 24\end{pmatrix}, \begin{pmatrix}6 & 0\\ 0 & 12\end{pmatrix}, \begin{pmatrix}4 & 0\\ 0 & 12\end{pmatrix}.$$

For the first nine m above, let S_m be the even lattice of rank 20 and signature (1,19) with the following intersection form, respectively

$$U \oplus A_1 \oplus A_{17}, U \oplus A_9 \oplus A_9, U \oplus A_1 \oplus A_2 \oplus A_{15},$$

$$U \oplus A_1 \oplus A_4 \oplus A_{13}, U \oplus A_4 \oplus A_5 \oplus A_9, U \oplus A_1 \oplus A_1 \oplus A_2 \oplus A_{14},$$

$$U \oplus A_1 \oplus A_1 \oplus A_8 \oplus A_8, U \oplus A_1 \oplus A_2 \oplus A_2 \oplus A_{13}, U \oplus A_2 \oplus A_2 \oplus A_7 \oplus A_7.$$

We now define S_m for $m = 35, 53$. Let Γ_{35} be the lattice $U \oplus A_1 \oplus A_1 \oplus A_5 \oplus A_{11}$, with $G, H, J_i (1 \le i \le 5), \theta_i (1 \le i \le 11)$ as the canonical basis of $A_1 \oplus A_1 \oplus A_5 \oplus A_{11}$, and $\mathcal{O}, F$ as a basis of U such that $\mathcal{O}^2 = -2, F^2 = 0, \mathcal{O} \cdot F = 1$.

We extend Γ_{35} to an index-2 integral over lattice $S_{35} = \Gamma_{35} + \mathbf{Z} s_{35}$, where

$$s_{35} = \mathcal{O} + 2F - G/2 - H/2 - (1/2)\left(\sum_{i=1}^{6} i\theta_i + \sum_{i=7}^{11}(12-i)\theta_i\right).$$

It is easy to see that the intersection form on Γ_{35} can be extended to an integral even symmetric lattice of signature $(1, 19)$. Indeed, setting $s = s_{35}$, we have

$$s^2 = -2, s \cdot F = s \cdot G = s \cdot H = s \cdot \theta_6 = 1, s \cdot \mathcal{O} = s \cdot J_i = s \cdot \theta_j = 0 \ (\forall i; j \ne 6).$$

Moreover, $|\det(S_{35})| = |\det(\Gamma_{35})|/2^2 = 72$.

Note that $\Gamma_{35}^{\vee} = \mathrm{Hom}_{\mathbf{Z}}(\Gamma_{35}, \mathbf{Z})$ contains naturally Γ_{35} as a sublattice with $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/6\mathbf{Z} \oplus \mathbf{Z}/12\mathbf{Z}$ as the factor group, and is generated by the following, modulo Γ_{35}:

$$h_1 = G/2,\ h_2 = H/2,\ h_3 = (1/6)\sum_{i=1}^{5} iJ_i,\ h_4 = (1/12)\sum_{i=1}^{11} i\theta_i.$$

Since $(S_{35})^{\vee}$ is an (index-2) sublattice of $(\Gamma_{35})^{\vee}$, an element x is in $(S_{35})^{\vee}$ if and only if $x = \sum_{i=1}^{4} a_i h_i \pmod{\Gamma_{35}}$ such that x is integral on S_{35}, i.e.,

$x \cdot s = (a_1 + a_2 + a_4)/2$ is an integer. Hence $(S_{35})^\vee$ is generated by the following, modulo Γ_{35}:

$$h_3,\ 2h_i,\ h_1 + h_2,\ h_1 + h_4,\ h_2 + h_4.$$

Noting that $2h_1, 2h_2 \in S_{35}$ and $(h_1 + h_2) + 6h_4$ is equal to s (mod Γ_{35}) and hence contained in S_{35}, we can see easily that $(S_{35})^\vee$ is generated by the following, modulo S_{35}:

$$\varepsilon_1 := h_3, \quad \varepsilon_2 := h_1 - h_4.$$

Now the fact that $|(S_{35})^\vee/S_{35}| = 72$ and that $6\varepsilon_1, 12\varepsilon_2 \in S_{35}$ imply that $(S_{35})^\vee/S_{35}$ is a direct sum of its cyclic subgroups which are of order 6, 12, and generated by $\varepsilon_1, \varepsilon_2$, modulo S_{35}.

We note that the negative of the discriminant form

$$-q_{(S_{35})} = (-(\varepsilon_1)^2) \oplus (-(\varepsilon_2)^2) = (5/6) \oplus ((1/2) + (11/12)) = (5/6) \oplus (-7/12).$$

Next we define S_{53}. Let Γ_{53} be the lattice $U \oplus A_2 \oplus A_2 \oplus A_3 \oplus A_{11}$, with $G_i (i = 1, 2), H_i (i = 1, 2), J_i (i = 1, 2, 3), \theta_i (1 \le i \le 11)$ as the canonical basis of $A_2 \oplus A_2 \oplus A_3 \oplus A_{11}$, and $\mathcal{O}, F$ as a basis of U as in the case of S_{35}.

Extend Γ_{53} to an index-3 integral over lattice $S_{53} = \Gamma_{53} + \mathbf{Z}s_{53}$, where

$$s_{53} = \mathcal{O} + 2F - (1/3)(2G_1 + G_2 + 2H_1 + H_2) - (2/3)\sum_{i=1}^{11} i\theta_i + \sum_{i=5}^{11} (i-4)\theta_i,$$

$$(\text{set } s = s_{53})\ s^2 = -2, s \cdot F = s \cdot G_1 = s \cdot H_1 = s \cdot \theta_4 = 1,$$
$$s \cdot \mathcal{O} = s \cdot G_2 = s \cdot H_2 = s \cdot J_i = s \cdot \theta_j = 0\ (\forall i; j \ne 4).$$

Moreover, $|\det(S_{53})| = |\det(\Gamma_{53})|/3^2 = 48$.

Note that $\Gamma_{53}^\vee$ is generated by the following, modulo Γ_{53}:

$$h_1 = (1/3)\sum_{i=1}^{2} iG_i, \quad h_2 = (1/3)\sum_{i=1}^{2} iH_i,$$
$$h_3 = (1/4)\sum_{i=1}^{3} iJ_i, \quad h_4 = (1/12)\sum_{i=1}^{11} i\theta_i.$$

$(S_{53})^\vee$ is generated by the following, modulo Γ_{53}:

$$h_3,\ 3h_i,\ h_1 + h_2 + h_4,\ h_1 - h_2,\ h_1 - h_4,\ h_2 - h_4.$$

Noting that $3h_1, 3h_2 \in S_{53}$ and $3h_4 + (h_1 + h_2 + h_4)$ is equal to s (mod Γ_{53}) and hence contained in S_{53}, we see that $(S_{53})^\vee$ is generated by $\varepsilon_1 := h_3, \varepsilon_2 := h_1 - h_4$, modulo S_{53}. As in the case of S_{35}, $(S_{53})^\vee/S_{53}$ is a direct sum of its cyclic subgroups, which are of order 4, 12, and generated by $\varepsilon_1, \varepsilon_2$, modulo S_{53}.

The negative of the discriminant form

$$\begin{aligned}-q_{(S_{53})} &= (-(\varepsilon_1)^2) \oplus (-(\varepsilon_2)^2) \\ &= (3/4) \oplus ((2/3) + (11/12)) = (3/4) \oplus (-5/12).\end{aligned}$$

Claim 3.4. The pair (S_m, T_m) satisfies the conditions of Lemma (3.1) and hence if we let X_m be the unique $K3$ surface with $T_{X_m} = T_m$ then $\operatorname{Pic} X_m = S_m$ (both two equalities here are modulo isometries).

Proof of the claim. We need to show that $q_{T_m} = -q_{S_m}$. Note that $A_n^\vee/A_n = \mathbf{Z}/(n+1)\mathbf{Z}$ and $q_{(A_n)} = (-n/(n+1))$. For the first nine m, if we write $S_m = U \oplus A_{n_1-1} \oplus \cdots A_{n_k-1}$, then

$$q_{S_m} = (-(n_1-1)/n_1) \oplus \cdots \oplus (-(n_k-1)/n_k);$$

moreover, $S_m^\vee/S_m$ is generated by two elements ε_i $(i=1,2)$ (ε_i is a simple sum of the natural generators of $S_m^\vee/S_m$) such that for every $a, b \in \mathbf{Z}$ one has $-q_{(S_m)}(a\varepsilon_1 + a\varepsilon_2) = -a^2(\varepsilon_1)^2 - b^2(\varepsilon_2^2)$. For all eleven m, ε_i can be chosen such that $(-\varepsilon_1^2, -\varepsilon_2^2)$ is respectively given as follows:

$$\begin{gathered}(1/2, 17/18), (9/10, 9/10), (1/2, -19/48), (1/2, 121/70), \\ (9/10, 49/30), (-5/6, -17/30), (25/18, 25/18), (-5/6, -17/42), \\ (-11/24, -11/24), (5/6, -7/12), (3/4, -5/12).\end{gathered}$$

On the other hand, $T_m^\vee$ is generated by $(g_1\ g_2) = (e_1\ e_2)T_m^{-1}$, where e_1, e_2 form a canonical basis of T_m which gives rise to the intersection matrix of T_m shown before this claim. Now, the claim follows from the existence of the following isomorphism, which induces $q_{T_m} = -q_{S_m}$:

$$\varphi : T_m^\vee/T_m \to S_m^\vee/S_m; \quad (g_1\ g_2) \mapsto (\varepsilon_1\ \varepsilon_2)B_m.$$

Here B_m is respectively given as:

$$\begin{gathered}\begin{pmatrix}1 & 1\\ 2 & 5\end{pmatrix}, \begin{pmatrix}7 & 0\\ 0 & 7\end{pmatrix}, \begin{pmatrix}0 & 1\\ 11 & 17\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 51\end{pmatrix}, \begin{pmatrix}7 & 0\\ 0 & 17\end{pmatrix}, \begin{pmatrix}-2 & 1\\ 1 & 3\end{pmatrix}, \\ \begin{pmatrix}7 & 0\\ 0 & 7\end{pmatrix}, \begin{pmatrix}2 & 3\\ 21 & 10\end{pmatrix}, \begin{pmatrix}2 & 3\\ 3 & -2\end{pmatrix}, \begin{pmatrix}3 & 2\\ 4 & 3\end{pmatrix}, \begin{pmatrix}0 & 1\\ 3 & 4\end{pmatrix}.\end{gathered}$$

Write S_m (resp. Γ_m) as $U \oplus \mathbf{A}(m)$ with $\mathbf{A}(m) = A_{n_1-1} \oplus \cdots \oplus A_{n_k-1}$, for the first nine m (resp. $m = 35, 53$) as in the definitions of them. Let $\mathcal{O}, F$ be a $\mathbf{Z}$-basis of U for all m, as in the definition of S_{35}. By [**PSS**, p. 573, Th. 1], after an (isometric) action of reflections on $S_m = \operatorname{Pic} X_m$, we may assume at the beginning that F is a fiber of an elliptic fibration $f_m : X_m \to \mathbf{P}^1$. Since $\mathcal{O}^2 = -2$, Riemann-Roch Theorem implies that $\mathcal{O}$ is an effective divisor because $\mathcal{O} \cdot F > 0$. Moreover, $\mathcal{O} \cdot F = 1$ implies that $\mathcal{O} = \mathcal{O}_1 + F'$ where $\mathcal{O}_1$ is a cross-section of f_m and F' is an effective divisor contained in fibers. So f_m is a Jacobian elliptic fibration and we can choose $\mathcal{O}_1$ as the zero element of $MW(f_m)$.

Let Λ_m be the lattice generated by all fiber components of f_m. Clearly, $\Lambda_m = \mathbf{Z}F \oplus \Delta, \Delta = \Delta(1) \oplus \cdots \oplus \Delta(r)$ (depending on m), where each $\Delta(i)$ is a negative definite even lattice of Dynkin type A_p, D_q, or E_r, contained in a single reducible singular fiber F_i of f_m and spanned by smooth components of F_i disjoint from $\mathcal{O}_1$.

Claim 3.5. We have:

(1) $\mathrm{Span}_{\mathbf{Z}}\{x \in S_m | x \cdot F = 0, x^2 = -2\} = \Lambda_m = \mathbf{Z}F \oplus \mathbf{A}(m)$; in particular, $r = k$, and there are lattice-isometries: $\Delta \cong \mathbf{A}(m)$ and $\Delta(i) \cong A_{n_i}$ $(i = 1, 2, \dots, k)$, after relabeling.

(2) There are k singular fibers F_i of type $\widetilde{A}_{n_i-1}$ $(1 \le i \le k)$ of f_m, and any fiber other than F_i is irreducible.

(3) $MW(f_m) = (0)$ (resp. $\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/3\mathbf{Z}$) for the first nine m (resp. $m = 35, 53$).

Proof. The assertion (2) follows from (1) (see also [**K**, Lemma 2.2]).

The first equality in (1) is clear from Kodaira's classification of elliptic fibers and the Riemann Roch Theorem as used prior to this claim to deduce $\mathcal{O} \ge 0$. The second equality is clear for the cases of the first nine m because then $\mathrm{Pic}\, X_m = S_m = (\mathbf{Z}\mathcal{O} + \mathbf{Z}F) \oplus \mathbf{A}(m)$.

Let $m = 35, 53$. We now show the second equality using Lemma 1.3. Clearly, $\mathbf{Z}F \oplus \mathbf{A}(m)$ is contained in the first term of (1) and hence in Λ_m. One notes that $19 = \text{rank } S_m - 1 \ge \text{rank } \Lambda_m = 1 + \text{rank } \Delta \ge 1 + \text{rank } \mathbf{A}(m) = 1 + \sum_{i=1}^{k}(n_i - 1) = 19$. Hence $\Delta = \Delta(1) \oplus \cdots \oplus \Delta(r) \cong \Lambda_m/\mathbf{Z}F$ contains a finite-index sublattice $(\mathbf{Z}F \oplus \mathbf{A}(m))/\mathbf{Z}F \cong \mathbf{A}(m) = A_{n_1-1} \oplus \cdots \oplus A_{n_k-1}$.

Suppose the contrary that the second equality in (1) is not true. Then $\mathbf{A}(m)$ is an index-n $(n > 1)$ sublattice of Δ. By Lemma 1.3, one of Cases (2-1) - (2-3) there occurs.

Case (2-1). Then $m = 35$, f_m has reducible singular fibers of types $\widetilde{A}_1, I_{13}^*$ and no other reducible fibers. This leads to that $72 = |\,\mathrm{Pic}\, X_m| = (2 \times 4)/|MW(f_m)|^2$, a contradiction (cf. [**S**]).

Case (2-2). Then $m = 53$, f_m has reducible singular fibers of types $\widetilde{A}_2, I_{12}^*$ and no other reducible fibers. This leads to that $48 = |\,\mathrm{Pic}\, X_m| = (3 \times 4)/|MW(f_m)|^2$, a contradiction.

Case (2-3). Then $m = 35$, f_m has reducible singular fibers of types $\widetilde{A}_1, I_{12}, IV^*$ and no other reducible fibers. Since $72 = |\,\mathrm{Pic}\, X_m| = (2 \times 12 \times 3)/|MW(f_m)|^2$, we have $MW(f_m) = (0)$ and $S_m = \mathrm{Pic}\, X_m = \mathbf{Z}\mathcal{O}_1 + \Lambda_m = \mathbf{Z}\mathcal{O}_1 + (\ \mathbf{Z}F \oplus \Delta) = \mathbf{Z}\mathcal{O}_1 + (\ \mathbf{Z}F \oplus A_1 \oplus A_{11} \oplus E_6)$.

By the Riemann-Roch theorem and the fact that $(s_m)^2 = -2$, $s_m.F = 1$ and $MW(f_m) = (0)$, we see that $s_m = \mathcal{O}_1 \pmod{\Lambda_m}$. This, together with the fact that $\mathcal{O} = \mathcal{O}_1 \pmod{\Lambda_m}$ and the definition of s_m, implies that $(1/2)(G + H + D) \in \Lambda_m$, where $D = \sum_{i=1}^{6} i\theta_i + \sum_{i=7}^{11}(12 - i)\theta_i$.

Consider the index-2 extension

$$A_1 \oplus A_{11} \oplus (A_1 \oplus A_5) = \mathbf{A}(m) \cong (\mathbf{Z}F \oplus \mathbf{A}(m))/\mathbf{Z}F \subseteq (\mathbf{Z}F \oplus \Delta)/\mathbf{Z}F \\ \cong \Delta = A_1 \oplus A_{11} \oplus E_6.$$

The proof of Lemma 1.3 shows that (the first summand A_1 in this rearranged $\mathbf{A}(m)$) $\oplus\mathbf{Z}F$ = (the summand A_1 in Δ) $\oplus\mathbf{Z}F$, (the summand A_{11} in $\mathbf{A}(m)$) $\oplus\mathbf{Z}F$ = (the summand A_{11} in Δ) $\oplus\mathbf{Z}F$, and (the summand $(A_1 \oplus A_5)$ in $\mathbf{A}(m)$) $\oplus\mathbf{Z}F \subseteq$ (the summand E_6 in Δ) $\oplus\mathbf{Z}F$. So we may assume that, mod $\mathbf{Z}F$, G is the $\mathbf{Z}$-generator of the first summand A_1 in Δ, θ_i $(1 \le i \le 11)$ form a $\mathbf{Z}$-basis of the summand A_{11} in Δ, and H is contained in the summand E_6 in Δ.

In particular, for $(G+H+D)/2 \in \Lambda_m = \mathbf{Z}F \oplus \Delta = \mathbf{Z}F \oplus (A_1 \oplus A_{11} \oplus E_6)$, we have, mod $\mathbf{Z}F$, $G/2 \in A_1$, $H/2 \in E_6$, and $D/2 \in A_{11}$. We reach a contradiction to the above observation that the A_1 in Δ is generated by G over $\mathbf{Z}$.

Therefore, the second equality of (1) is true. So there is an isometry $\Phi : \Delta \cong \Lambda_m/\mathbf{Z}F \cong \mathbf{A}(m)$. Now the rest of (1) follows from Lemma 1.3.

The assertion (3) follows from the fact in [**S**, Th. 1.3], that $MW(f_m)$ is isomorphic to the factor group of $\operatorname{Pic} X_m$ modulo $(\mathbf{Z}\mathcal{O}_1 + \mathbf{Z}F) \oplus \Delta$, where the latter is equal to $(\mathbf{Z}\mathcal{O} + \mathbf{Z}F) + \Delta = (\mathbf{Z}\mathcal{O} + \mathbf{Z}F) \oplus \mathbf{A}(m) = U \oplus \mathbf{A}(m)$. This proves the claim.

The existence of singular fibers F_i $(i = 1, 2, \ldots, k)$ of type I_{n_i}, the fact that the sum of Euler numbers of singular fibers of f_m is 24, the fact that each fiber other than F_i is irreducible, and [**MP3**, Lemma 3.1 and Proposition 3.4] imply that f_m is semi-stable. Hence F_i $(i = 1, 2, \ldots, k)$ is of type I_{n_i}, there are $\chi(X_m) - \sum_i(n_i - 1) - k = 6 - k$ fibers of type I_1, and f_m is of type $[1, 1, \ldots, 1, n_1, \ldots, n_k]$, i.e., of type m after an easy case-by-case check. Moreover, $(m, MW(f_m)) = (m, G_m)$ for all eleven m by the last claim. □

Remark 3.6. We note that $S_{35} = U \oplus A_1 \oplus A_{11} \oplus E_6$. This is because the lattices T_{35} and the one on the right hand side satisfy all conditions of Lemma 3.1 by an easy check. In particular, using [**MP3**, Lemma 3.1 and Proposition 3.4] as in the proof of Lemma 3.3, we can show that there is a Jacobian elliptic fibration $\tau_m : X_m \to \mathbf{P}^1$ $(m = 35)$ with singular fibers $I_1, I_1, I_2, I_{12}, IV^*$ and with $MW(\tau_m) = (0)$.

Lemma 3.7. *Let $f : X \to \mathbf{P}^1$ be of type m as in Theorem 0.2. Then we have:*

(1) *If $m = 48$, then $MW(f) \neq \mathbf{Z}/2\mathbf{Z}$, or $\mathbf{Z}/4\mathbf{Z}$.*
(2) *If $m = 4$, then $MW(f) \neq \mathbf{Z}/2\mathbf{Z}$.*
(3) *If $m = 31$, then $MW(f) \neq \mathbf{Z}/2\mathbf{Z}$.*
(4) *If $m = 44$, then $MW(f) \neq \mathbf{Z}/2\mathbf{Z}$.*
(5) *If $m = 69$, then it is impossible that $MW(f)$ is $\mathbf{Z}/2\mathbf{Z}$ with $s = (0, 0, 0, 0, 2, 6)$ as its generator (see Remark 0.4).*

(6) *If* $m = 92$, *then* $MW(f) \neq \mathbf{Z}/2\mathbf{Z}$.

Proof. Let $f : X \to \mathbf{P}^1$ be of type m as in Theorem 0.2.

(1) Assume that f is of type $m = 48$ and $MW(f) \supseteq \mathbf{Z}/2\mathbf{Z}$. We will show that $MW(f) \supseteq \mathbf{Z}/8\mathbf{Z}$ which will imply (1).

$m = 48$ means that the singular fiber type of f is $I_1, I_1, I_2, I_4, I_8, I_8$. Using the height pairing in [**S**] or the Table in [**MP3**], we may assume that $MW(f)$ contains $s = (0, 0, 0, 0, 4, 4)$ as a 2-torsion section after suitable labeling of fiber components.

Let Y, a $K3$ surface again, be the minimal resolution of the quotient surface $X/\langle s\rangle$. f on X induces a Jacobian semi-stable elliptic fibration $g : Y \to \mathbf{P}^1$ of singular fiber type $I_2, I_2, I_4, I_8, I_4, I_4$ where these 6 ordered singular fibers are respectively "images" of ordered singular fibers on X.

To be precise, let $\sigma : \widetilde{X} \to X$ be the blowing-up of all 8 intersections in the first 4 singular fibers of f of types I_1, I_1, I_2, I_4. Then $Y = \widetilde{X}/\langle s\rangle$ and the $\mathbf{Z}/2\mathbf{Z}$-covering $\pi : \widetilde{X} \to Y$ is branched along 4 disjoint curves $\theta_j^{(i)}$, where $(i, j) = (1, 1), (2, 1), (3, 1), (3, 3), (4, 1), (4, 3), (4, 5), (4, 7)$. Here we choose the common image of the zero section and the 2-torsion section s of f, as the zero section O_1 of g, and label clock or anti-clockwise the i-th singular fiber of g of type I_{n_i} as $\sum_{j=0}^{n_i-1} \theta_j^{(i)}$ so that O_1 passes through $\theta_0^{(i)}$, where $[n_1, \dots, n_6] = [2, 2, 4, 8, 4, 4]$.

Note that (Y, g) is of type $m = 103$ in the Table of [**MP3**] and hence there is a 4-torsion section t of g equal to $(0, 0, 2, 2, 1, 1)$ or $(0, 0, 1, 2, 1, 2)$ or $(0, 0, 1, 2, 2, 1)$, after choosing either clockwise or counterclockwise labeling of fiber components, where for orders of six fibers of g we use the current indexing inherited from that of f.

If $t = (0, 0, 1, 2, 1, 2)$ or $(0, 0, 1, 2, 2, 1)$, then t meets the branch locus of π transversally at one point only so that $\pi^{-1}(t)$ is a smooth irreducible curve and $\pi : \pi^{-1}(t) \to t$ is a double cover with exactly one ramification point, a contradiction to Hurwitz's genus formula applied to the covering map π.

Thus $t = (0, 0, 2, 2, 1, 1)$. A check using height pairing shows that $\pi^{-1}(t)$ is a disjoint union of two 8-torsion sections of f. Hence $MW(f) \supseteq \mathbf{Z}/8\mathbf{Z}$. Indeed, $MW(f) = \mathbf{Z}/8\mathbf{Z}$ by [**MP3**]. This proves (1).

Now assume that f is of type $m = 4$ (resp. $m = 31$, $m = 44$, $m = 69$ with $MW(f) = \langle s = (0, 0, 0, 0, 2, 6)\rangle$, or $m = 92$) and $MW(f) \supseteq \mathbf{Z}/2\mathbf{Z}$. Then $MW(f)$ contains a unique 2-torsion section $s = (0, 0, 0, 0, 0, 8)$ (resp. $s = (0, 0, 0, 0, 0, 8)$, $s = (0, 0, 0, 0, 2, 6)$, $s = (0, 0, 0, 0, 2, 6)$, $s = (0, 0, 0, 2, 2, 4)$) (cf. [**MP3**]). As in (1) we can show that f induces a Jacobian semi-stable elliptic fibration g on the minimal resolution Y of $X/\langle s\rangle$. The singular fiber type of g is $I_{n_1} + \dots + I_{n_6}$ where $[n_1, \dots, n_6]$ is equal to $[2, 2, 2, 2, 8, 8]$ (resp. $[2, 2, 4, 4, 4, 8]$, $[2, 2, 4, 8, 2, 6]$, $[2, 4, 4, 6, 2, 6]$, $[2, 6, 8, 2, 2, 4]$) and hence is of type $m = 94$ (resp. $m = 103$, $m = 97$, $m = 104$, or $m = 97$) in the Table of [**MP3**]. Now the inverse on X of the 2-torsion section $t = (0,0,0,0,4,4)$ (resp. $t = (0,0,0,2,2,4)$, $t = (0,0,0,4,1,3)$, t is one of $(0,2,2,0,1,3)$ and $(1,2,2,3,0,0)$, or $t = (0,0,0,4,1,2)$) on Y is a disjoint union of two 4-torsion sections of f. Hence $MW(f) \supseteq \mathbf{Z}/4\mathbf{Z}$. Indeed, $MW(f) = \mathbf{Z}/4\mathbf{Z}$ by [**MP3**]. This proves (2)-(6). The proof of the lemma is completed. □

Lemma 3.8. *Let $f : X \to \mathbf{P}^1$ be of type m as in Theorem 0.2. Then each of the following pairs $(m, MW(f))$ occurs:*

$$(69, \mathbf{Z}/2\mathbf{Z} = \langle (0,1,1,0,0,6) \rangle), (69, \mathbf{Z}/4\mathbf{Z}), (92, \mathbf{Z}/4\mathbf{Z}),$$
$$(32, \mathbf{Z}/3\mathbf{Z}), (37, \mathbf{Z}/3\mathbf{Z}), (44, \mathbf{Z}/4\mathbf{Z}), (55, \mathbf{Z}/2\mathbf{Z}).$$

Proof. The idea of the proof for the existence of the pair $(m, MW(f)) = (69, \mathbf{Z}/4\mathbf{Z})$ is as follows. By [**MP3**], $s = (0,1,1,0,1,3)$ is the generator of $MW(f) = \mathbf{Z}/4\mathbf{Z}$. As in the proof of Lemma 3.7, the minimal resolution Y of $X/\langle 2s \rangle$ is of type $m = 104$. The detailed proof of the existence is given below.

Let $g : Y \to \mathbf{P}^1$ be of type $m = 104$. By the Table in [**MP3**], $MW(g) = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and we may assume that g has singular fibres $\sum_{j=0}^{n_i-1} \theta(i)_j$ $(i = 1, \ldots, 6)$ of type I_{n_i}, and two 2-torsion sections $t_1 = (0,2,2,0,1,3), t_2 = (1,2,2,3,0,0)$, after suitably indexing singular fibers so that $[n_1, \ldots, n_6] = [2,4,4,6,2,6]$. It is easy to check the following relation (cf. [**S**, Lemma 8.1] or [**M**, Formula (2.5)]), where O_1, F are the zero section and a general fiber of g,

$$2t_2 \sim 2(O_1 + 2F) - (\theta(1)_1 + \theta(2)_1 + 2\theta(2)_2 + \theta(2)_3 + \theta(3)_1 + 2\theta(3)_2 + \theta(3)_3 + \theta(4)_1 + 2\theta(4)_2 + 3\theta(4)_3 + 2\theta(4)_4 + \theta(4)_5).$$

Hence we get a relation

$$D = \theta(1)_1 + \theta(2)_1 + \theta(2)_3 + \theta(3)_1 + \theta(3)_3 + \theta(4)_1 + \theta(4)_3 + +\theta(4)_5 \sim 2L$$

for some integral divisor L. Let $\pi : \widetilde{X} \to Y$ be the $\mathbf{Z}/2\mathbf{Z}$-cover, branched along D and induced from the above relation. Then g induces an elliptic fibration $f : \widetilde{X} \to \mathbf{P}^1$ so that the relatively minimal model (X, f) of $(\widetilde{X}, f)$ is of type $m = 69$. The inverse on X of O_1 is a disjoint union of two sections, one of which will be fixed as O of f. Now the inverse on X of the 2-torsion section t_1 on Y is a disjoint union of two 4-torsion sections of f. Hence $MW(f) = \mathbf{Z}/4\mathbf{Z}$ by the Table in [**MP3**]. This proves the existence of the pair $(m, MW(f)) = (69, \mathbf{Z}/4\mathbf{Z})$.

The existence of other pairs is similar. Here we just show which Y, t_1, t_2 we should choose. To be precise, we let $g : Y \to \mathbf{P}^1$ be of type $m = 52$ (resp. $m = 97$; $m = 91$; $m = 110$; $m = 97$; $m = 104$) with singular fibers of type $I_{n_1} + \cdots + I_{n_6}$ with $[n_1, \ldots, n_6] = [2,1,1,6,8,6]$ (resp. $[2,6,8,2,2,4]$; $[3,3,6,6,1,5]$; $[3,3,6,6,3,3]$; $[2,2,4,8,2,6]$; $[2,2,6,6,4,4]$) and we let

$t_1 = O_1$ be the zero section and $t_2 = (1,0,0,3,4,0)$ the 2-torsion section (resp. $t_1 = (0,0,4,1,1,2)$ and $t_2 = (1,3,4,0,0,0)$ two 2-torsion sections; $t_1 = O_1$ and $t_2 = (1,1,2,2,0,0)$ a 3-torsion section; $t_1 = O_1$ and $t_2 = (1,1,2,2,0,0)$ a 3-torsion section; $t_1 = (0,0,0,4,1,3)$ and $t_2 = (1,1,2,4,0,0)$ two 2-torsion sections; $t_1 = O_1$ and $t_2 = (1,1,3,3,0,0)$ a 2-torsion section). Then as in the above paragraph, the minimal model X of a $\mathbf{Z}/n\mathbf{Z}$-cover with $n = 2$ (resp. $n = 2$; $n = 3$; $n = 3$; $n = 2$; $n = 2$) of Y has an elliptic fibration $f : X \to \mathbf{P}^1$, induced from g, of type $m = 69$ (resp. $m = 92$; $m = 32$; $m = 37$; $m = 44$; $m = 55$) such that the inverse on X of t_1 is a disjoint union of O and $s = (0,1,1,0,0,6)$ (resp. a disjoint union of two 4-torsion sections; a disjoint union of O and two 3-torsion sections; a disjoint union of O and two 3-torsion sections; a disjoint union of two 4-torsion sections; a disjoint union of O and a 2-torsion section), whence $MW(f)$ is equal to $\mathbf{Z}/2\mathbf{Z} = \{O, s\}$ (resp. $\mathbf{Z}/4\mathbf{Z}$; $\mathbf{Z}/3\mathbf{Z}$; $\mathbf{Z}/3\mathbf{Z}$; $\mathbf{Z}/4\mathbf{Z}$; $\mathbf{Z}/2\mathbf{Z}$) by the Table in [**MP3**].

This completes the proof of the lemma and also that of Theorem 0.2. □

4. Uniqueness for extremal elliptic $K3$ surfaces with large torsion groups.

The goal of this section is to prove Theorem 0.3.

In the case where $MW(f) \supseteq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, namely, $m = 94, 97, 98, 103, 104, 112$, the uniqueness problem has already been considered in §7 [**MP3**] by using double sextics, and they are all unique. Hence we need to prove the cases when $MW(f) \cong \mathbf{Z}/4\mathbf{Z}$, $\mathbf{Z}/5\mathbf{Z}$, $\mathbf{Z}/6\mathbf{Z}, \mathbf{Z}/7\mathbf{Z}$, $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$.

As we have seen in §1, if $MW(f)$ has an element of order $N \geq 3$, then $f : X \to \mathbf{P}^1$ is obtained as the pull-back surface of the elliptic surface, $\psi_{1,N} : E_1(N) \to X_1(N)$, by some morphism $g : \mathbf{P}^1 \to X_1(N)$. Since $X_1(N)$ should be isomorphic to $\mathbf{P}^1$ and X is a $K3$ surface in our case, $N \leq 8$ by [**C**]. Thus our proof of Theorem 0.3 is reduced to showing the uniqueness of g up to $\mathrm{Aut}(\mathbf{P}^1)$ for each case. Hence it is enough to prove the following:

Proposition 4.1. *Let $g : \mathbf{P}^1 \to X_1(N)$ be the morphism as above. Then g is unique except $m = 49$.*

By comparing the degree of the j-functions, we can easily check the following table:

$MW(f)$	$\deg g$	m
$\mathbf{Z}/4\mathbf{Z}$	4	$4, 31, 44, 69, 92$
$\mathbf{Z}/5\mathbf{Z}$	2	$9, 49, 105$
$\mathbf{Z}/6\mathbf{Z}$	2	$35, 53, 63, 95, 108$
$\mathbf{Z}/7\mathbf{Z}$	1	30
$\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$	2	110

Table 4.2.

One can see that the uniqueness for the case $MW(f) \cong \mathbf{Z}/7\mathbf{Z}$ $(m = 30)$ immediately from the table.

Let us consider the cases of $\deg g = 2$. Our goal is to show that g is unique up to $\mathrm{Aut}(X_1(N))(\cong \mathrm{Aut}(\mathbf{P}^1))$ except $m = 49$.

Case $m = 9$**.** $f : X \to \mathbf{P}^1$ has two I_{10} fibers. This means that the branch points of g are 2 points over which $\psi_{1,5}$ has I_5 fibers. The choice of such two points is unique and g is determined by the branch points. Hence g is unique.

For cases $m = 35, 53, 63, 95, 105, 108$, we can prove the uniqueness in a similar way to that for $m = 9$. Hence we omit it.

Case $m = 110$**.** In this case, $f : X \to \mathbf{P}^1$ is obtained as the pull-back surface of $\psi_{3,3} : E_3(3) \to X_3(3)$ by a degree 2 map $g : \mathbf{P}^1 \to X_3(3)$. $\psi_{3,3}$ has 4 singular fibers, all of which are of type I_3. By [**MP1**, Table 5.3], $E_3(3)$ is given by the Weierstrass equation as follows:

$$y^2 = x^3 + (-3s^2 + 24s)x + (2s^6 + 40s^3 - 16),$$

where s is an inhomogeneous coordinate of $X_3(3) \cong \mathbf{P}^1$. The four I_3 fibers are over $-1, -\omega, \omega^2$ and ∞, where $\omega = \exp(2\pi\sqrt{-1}/3)$.

Consider two fiber preserving automorphisms of $E_3(3)$:

$$\tau_1 : (x, y, s) \mapsto \left(\frac{-3}{(s+1)^2} x, \frac{3\sqrt{-3}}{(s+1)^3} y, \frac{-s+2}{s+1} \right),$$

and

$$\tau_2 : (x, y, s) \mapsto (\omega x, y, \omega s).$$

These automorphisms induce permutations of the I_3 fibers. Since X is a double covering of $E_3(3)$, it is uniquely determined by the branch locus which is two I_3 fibers. Therefore, using τ_1 and τ_2, we can show that $f : X \to \mathbf{P}^1$ is unique.

Putting the case $m = 49$ the aside, we consider the cases of $\deg g = 4$. There are 5 cases: $m = 4, 31, 44, 69, 92$.

The degree of the j-invariant of $E_1(4)$ is 6, as it has three singular fibers I_1^*, I_4 and I_1. With a suitable affine coordinate of $X_1(4)$, we may assume that these singular fibers are over 0, 1 and ∞, respectively. Since the degree of the j-invariant of $f : X \to \mathbf{P}^1$ is 24, the degree of g is 4, and is branched only at 0, 1 and ∞. By [**MP1**, Table 7.1] and the Riemann-Hurwitz formula for $g : \mathbf{P}^1 \to X_1(4)$, we have the following table on the ramification types over each branch point.

m	The ramification types over 0, 1 and ∞
4	(4), (4), $(1,1,1,1)$
31	$(2,2)$, (4), $(2,1,1)$
44	(4), $(3,1)$, $(2,1,1)$
69	$(2,2)$, $(3,1)$, $(3,1)$
92	(4), $(2,1,1)$, $(3,1)$

Table 4.3.

Here the notation $(e_1, \dots, e_k)$ means that $g^{-1}(p)$ $(p \in \{0, 1, \infty\})$ consists of k points, $q_1, \dots, q_k$, and the ramification index at q_j is e_j.

To show the uniqueness, it is enough to show that g assigned with the ramification types as above is unique up to covering isomorphisms over $X_1(4)$. Let us start with the following lemma.

Lemma 4.4. *Let $g : \mathbf{P}^1 \to X_1(4)$ be one of the degree 4 maps in Table 4.3. Let $\alpha : C \to \mathbf{P}^1$ be the Galois closure, and put $\hat{g} = g \circ \alpha$. Then we have the following:*

$m = 4$: $g = \hat{g}$ and g is a 4-fold cyclic covering.
$m = 31$: $\deg \hat{g} = 8$, $C \cong \mathbf{P}^1$ and $\mathrm{Gal}(\hat{g}) \cong \mathcal{D}_8$.
$m = 44, 92$: $\deg \hat{g} = 24$, $C \cong \mathbf{P}^1$ and $\mathrm{Gal}(\hat{g}) \cong \mathcal{S}_4$.
$m = 69$: $\deg \hat{g} = 12$, $C \cong \mathbf{P}^1$ and $\mathrm{Gal}(\hat{g}) \cong \mathcal{A}_4$.

Proof. The monodromy around the branch points gives a permutation representation of $\pi_1(\mathbf{P}^1 \setminus \{0, 1, \infty\})$ to $\mathcal{S}_4$; the basic loops γ_0, γ_1 and γ_∞ about 0, 1 and ∞, respectively map to permutations σ_0, σ_1 and σ_∞. The cycle structure of each permutation is the same as the ramification type over the corresponding point. These permutations satisfy the identity $\sigma_0\sigma_1\sigma_\infty = 1$ in $\mathcal{S}_4$ and generate a transitive subgroup, G, in $\mathcal{S}_4$. Note that this G is nothing but the Galois group of $\hat{g} : C \to X_1(4)$. We apply this argument to each case, and obtain the following table:

m	The cycle structure of σ_0, σ_1 and σ_∞	G
4	(4), (4), $(1,1,1,1)$	$\mathbf{Z}/4\mathbf{Z}$
31	$(2,2)$, (4), $(2,1,1)$	$\mathcal{D}_8$
44	(4), $(3,1)$, $(2,1,1)$	$\mathcal{S}_4$
69	$(2,2)$, $(3,1)$, $(3,1)$	$\mathcal{A}_4$
92	(4), $(2,1,1)$, $(3,1)$	$\mathcal{S}_4$

Table 4.5.

Now all we need to show Is: $C \cong \mathbf{P}^1$. Our argument is based on the following elementary fact:

Fact 4.6. Let x be a point on C, and put $G_x = \{\tau \in G | \tau(x) = x\}$. Then

G	The order of G_x
$\mathbf{Z}/4\mathbf{Z}$	$1, 2, 3$
$\mathcal{S}_4$	$1, 2, 3, 4$
$\mathcal{A}_4$	$1, 2, 3$
$\mathcal{D}_8$	$1, 2, 4$

We prove $C \cong \mathbf{P}^1$ case by case.

Case $m = 4$**.** As $G = \mathbf{Z}/4\mathbf{Z}$, $\deg \hat{g} = \deg g$, and α is the identity.

Case $m = 31$**.** Since $G = \mathcal{D}_8$, $\deg \alpha = 2$. Let ι be an element of order 2 such that $C/\langle\iota\rangle \cong \mathbf{P}^1$. As g is not Galois, ι is not contained in the center of $\mathcal{D}_8$. If α is branched over $g^{-1}(0)$, then $\hat{g}^{-1}(0)$ consists of two points, each of which has the ramification index 4. This means that ι belongs to the center of $\mathcal{D}_8$, which leads us to a contradiction. Hence the branch points of α are two points in $g^{-1}(\infty)$ which are unramified points of g. Hence $C \cong \mathbf{P}^1$.

Cases $m = 44$, 92**.** By Fact 4.6 and $\mathrm{Gal}(C/\mathbf{P}^1) \cong \mathcal{S}_4$, points over 0, 1 and ∞ have the ramification indices 4, 3 and 2, respectively. By the Riemann-Hurwitz formula, we have $C \cong \mathbf{P}^1$.

Case $m = 69$**.** By Fact 4.6, points over 0, 1 and ∞ have the ramification indices 2, 3 and 3, respectively. By the Riemann-Hurwitz formula, $C \cong \mathbf{P}^1$.

This completes our proof for Lemma 4.4. □

The following classical fact is a key to prove Theorem 0.3 in the case where $MW(f) \cong \mathbf{Z}/4\mathbf{Z}$.

Fact 4.7 ([**Na**, pp. 31-32])**.** For a suitable choice of an affine coordinate, w and z, of $X_1(4)$ and $\mathbf{P}^1$, respectively, the map in Table 4.5 can be given by the rational functions as follows:

$$w = z^4 \qquad m = 4$$

$$w = -\frac{(z^4-1)^2}{4z^2} \qquad m = 31$$

$$w = \left(\frac{z^4+2\sqrt{3}z^2-1}{z^4-2\sqrt{3}z^2-1}\right)^3 \qquad m = 69$$

$$w = \frac{(z^8+14z^4+1)^3}{108z^4(z^4-1)^4} \qquad m = 44, 92.$$

Fact 4.7 implies that the Galois coverings described in Lemma 4.4 are essentially unique up to isomorphisms over $\mathbf{P}^1$. The morphisms g in Lemma 4.4 are corresponding to a subgroup of index 4 of G, and for each case, such subgroups are conjugate to each other. This shows that the pull-back morphisms, g, are unique up to covering isomorphisms over $X_1(4)$. Therefore we have Proposition 4.1 in the case where $MW(f) \cong \mathbf{Z}/4\mathbf{Z}$.

Remark 4.8. We can prove the uniqueness for $m = 94, 98, 103, 112$ in a similar way to the case $MW(f) \cong \mathbf{Z}/4\mathbf{Z}$.

We now go on to show that the uniqueness does not hold for $m = 49$.

For the case $m = 49$, as we have seen before, $f :\to \mathbf{P}^1$ is obtained as the pull-back surface of $\psi_{1,5} : E_1(5) \to X_1(5)$ by a degree 2 map $g : \mathbf{P}^1 \to X_1(5)$. $\psi_{1,5}$ has 4 singular fibers. By [**MP1**, Table 5.3], $E_1(5)$ is given by the following Weierstrass equation:

$$y^2 = x^3 - 3(s^4 - 12s^3 + 14s^2 + 12s + 1)x + 2(s^6 - 18s^5 + 75s^4 + 75s^2 + 18s + 1),$$

where s is an inhomogeneous coordinate of $X_1(5) \cong \mathbf{P}^1$. The two I_5 fibers are over $s = 1$ and $s = \infty$, and the two I_1 fibers are over $s = (11 \pm 5\sqrt{5})/2$.

For $m = 49$, There are 4 possible cases for the pull-back morphism depending on the branch points as follows:

	The branch points of g
(1)	0 and $(11+5\sqrt{5})/2$
(2)	0 and $(11-5\sqrt{5})/2$
(3)	∞ and $(11+5\sqrt{5})/2$
(4)	∞ and $(11-5\sqrt{5})/2$

We denote the pull-back morphisms by g_i $(i = 1, 2, 3, 4)$ corresponding to the cases as above, and let $f_i :\to \mathbf{P}^1$ denote the pull-back surface by g_i. Then we have:

Proposition 4.9. *There exists φ in Question 0.1 between either X_1 and X_4 or X_2 and X_3, while there is no such φ between the two pull-back surfaces for other combinations.*

Proof. Consider an automorphism, τ, of $E_1(5) \to X_1(5)$ given by

$$\tau : (x, y, s) \mapsto \left(\frac{1}{s^2}x, \frac{1}{s^3}y, -\frac{1}{s}\right).$$

With τ, the points 0 and $(11+5\sqrt{5})/2$ map to ∞ and $(11-5\sqrt{5})/2$, respectively. Our first assertion follows from this fact. For the second, by using τ, it is enough to show that there is no φ in Question 0.1 between the pull-back surfaces X_1 and X_2.

Suppose that there exists $\varphi : X_1 \to X_2$ as Question 0.1. Then we have:

Claim 4.10. φ induces an automorphism $\hat{\varphi} : X_1(5) \to X_1(5)$ such that $0 \mapsto \infty$, $\infty \mapsto 0$, $(11+5\sqrt{5})/2 \mapsto (11-5\sqrt{5})/2$, and $(11-5\sqrt{5})/2 \mapsto (11+5\sqrt{5})/2$.

Since there is no fractional linear transformation as above, the second assertion follows.

Proof of the Claim. Let ι_i $(i = 1, 2)$ be fiber preserving involutions on X_i $(i = 1, 2)$ determined by the pull-back morphisms g_i. Let $\overline{\varphi}$ and $\bar{\iota}_i$ $(i = 1, 2)$ be the restrictions of each morphism to the zero sections of X_1 and X_2. $\varphi^{-1} \circ \iota_2 \circ \varphi$ gives rise to another fiber preserving involution on X_1. Under $\varphi^{-1} \circ \iota_2 \circ \varphi$, I_{10}, I_5, I_2 fibers map to I_{10}, I_5, I_2 fibers, respectively. Hence $\overline{\varphi}^{-1} \circ \bar{\iota}_2 \circ \overline{\varphi} = \bar{\iota}_1$ or id, but the latter case does not occur since $\bar{\iota}_2 \neq id$. Thus we have an isomorphism $\hat{\varphi} : X_1(5) \to X_1(5)$, and it is easy to see that $\widetilde{\varphi}$ has the desired property.

This finishes the proof of Proposition 4.1.

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