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QUASIREGULAR MAPPINGS AND $\mathcal{WT}$-CLASSES OF DIFFERENTIAL FORMS ON RIEMANNIAN MANIFOLDS

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The purpose of this paper is to study the relations between quasiregular mappings on Riemannian manifolds and differential forms. Four classes of differential forms are introduced and it is shown that some differential expressions connected in a natural way to quasiregular mappings are members in these classes.

1. Introduction.

Let Ω be a domain in $R^n, n \geq 2$. A mapping $f : \Omega \to R^n$ is called a quasiregular mapping, if $f = (f_1, f_2, \dots, f_n) \in W^1_{n,\mathrm{loc}}(\Omega)$ and if there exists a constant $K \in [1, \infty)$ such that

$$|f'(x)|^n \leq K \det f'(x), \quad \text{for a.e. } x \in \Omega.$$

The following result is well-known in [**Re**] and [**HKM**].

Each of the functions

$$u = f_i(x) \quad (i = 1, 2, \dots, n), \qquad u = \log |f(x)|,$$

is a generalized solution of a quasilinear elliptic equation

$$\operatorname{div} A(x, \nabla u) = 0, \qquad A = (A_1, A_2, \dots, A_n), \tag{1.1}$$

where

$$A_i(x, \xi) = \frac{\partial}{\partial \xi_i} \left(\sum_{i,j=1}^{n} \theta_{i,j}(x) \xi_i \xi_j \right)^{n/2}, \tag{1.2}$$

and $\theta_{i,j}$ are some functions, which can be expressed in terms of the derivative $f'(x)$, and satisfy

$$c_1(K)|\xi|^2 \leq \sum_{i,j}^{n} \theta_{i,j}(x) \xi_i \xi_j \leq c_2(K)|\xi|^2, \tag{1.3}$$

for some constants $c_1(K)$, $c_2(K) > 0$.

This important proposition connects two large sections of analysis namely, quasiregular mapping theory and the theory of partial differential equations. Much progress in quasiregular mapping theory has resulted from the study of Equations (1.1)-(1.3). On the other hand many investigations of solutions of quasilinear equations in the form (1.1)-(1.3) were stimulated by this connection with quasiregular mapping theory.

However, many theorems about quasiregular mappings, obtained in this way for example, in the monograph [**HKM**] do not make use of the special form (1.2) of functions $A_i(x,\xi)$. In fact, what is important is the divergence form of the Equation (1.1) and the existence of constants $c_1(K)$, $c_2(K)$ – the values of these constants are not significant.

We do not know who was the first turning attention to this fact. Possibly, it was first observed in the paper [**Mi**], where the following fact was recorded and used.

Proposition. *The function $u \in W^1_{n,\mathrm{loc}}(\Omega)$ is the solution of some equation of the form (1.1) with Condition (1.3) if and only if there exists a differential $(n-1)$-form*

$$\theta(x) = \sum_{i=1}^{n} \theta_i(x)\, dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \in L^{n/(n-1)}_{\mathrm{loc}}(\Omega),$$

with the properties:

α) *For every function $\phi \in W^1_n(\Omega)$ with compact support we have*

$$\int\limits_{\Omega} d\phi \wedge \theta = 0\ ,$$

β) *almost everywhere on Ω the following inequalities are true*

$$\nu_1 |du(x)|^n \leq *(du(x) \wedge \theta(x))$$

where $*$ denotes the orthogonal complement of a form and

$$|\theta(x)| \leq \nu_2 |du(x)|^{n-1},$$

with constants $\nu_1, \nu_2 > 0$.

The proof for this proposition is obvious. The above statement concerning the coordinate functions of a quasiregular mapping f also follows from this proposition. For the case $u = f_1(x)$ we put

$$\theta = df_2 \wedge df_3 \wedge \ldots \wedge df_n\, .$$

In order to show that $u = \log|f(x)|$ satisfies (1.1) it suffices to choose

$$\theta = \frac{1}{|f(x)|^n} \sum_{i=1}^{n} df_1 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_n.$$

Looking carefully at Conditions α) and β) on the function u we see that these conditions are on the 1-form $w = du$ and the $(n-1)$-form θ. Some simple differential forms w, $1 \leq \deg w \leq n$, satisfying Conditions α) and β)

in domains $\Omega \subset R^n$ were studied in [**Zh1**] and [**Zh2**]. Similar results have been given in [**Iw**], [**FW**], [**MMV1**], [**MMV2**] and [**Sc**].

The purpose of this paper is to study the relations between quasiregular mappings on Riemannian manifolds and differential forms suggested by the aforementioned proposition. We introduce four classes of differential forms and prove membership in these classes of some differential expressions connected in a natural way to quasiregular mappings.

2. Preliminaries.

2.1. Euclidean space. Let X be a topological space. We denote by $\overline{A}$ the closure of a set $A \subset X$, by $\mathrm{int}A$ the interior of A, and by $\partial A = \overline{A} \setminus \mathrm{int}A$ the boundary of A.

By R^n we denote the Euclidean vector space consisting of elements of the form $x = (x^1, \dots, x^n)$, $x^i \in R$, the field of real numbers. In R^n we use the standard inner product $\langle x, y\rangle = \sum\limits_{i=1}^{n} x^i y^i$ and the norm $|x| = \sqrt{\langle x, x\rangle} = \Big(\sum\limits_{i=1}^{n} x_i^2\Big)^{1/2}$.

The boundary of the n-dimensional ball with center at x and radius r

$$B(x, r) = \{y \in R^n : |y - x| < r\}$$

is the sphere

$$S(x, r) = \{y \in R^n : |y - x| = r\}.$$

For $E \subset R^n$ and for an integer $k = 1, 2, \dots, n$ we denote by $H_k(E)$ the k-dimensional Hausdorff measure of E.

2.2. Differential forms on R^n. The mutually dual spaces $\bigwedge_k(R^n)$ and $\bigwedge^k(R^n)$ of k-vectors and k-forms (k-covectors) are associated with the Euclidean space R^n. Here one has $\bigwedge^0(R^n) = R = \bigwedge_0(R^n)$, and $\bigwedge_k(R^n) = \{0\} = \bigwedge^k(R^n)$ in the case $k > n$ or $k < 0$. The direct sums

$$\bigwedge\nolimits_*(R^n) = \oplus_k \bigwedge\nolimits_k(R^n), \quad \bigwedge\nolimits^*(R^n) = \oplus_k \bigwedge\nolimits^k(R^n)$$

generate contravariant and covariant Grassmann algebras on R^n with the exterior multiplication operator $\wedge$.

Let $\omega \in \bigwedge^k(R^n)$ be a covector. We denote by $\Lambda(k, n)$ the set of ordered multi–indices $I = (i_1, i_2, \dots, i_k)$, of integers $1 \le i_1 < \dots < i_k \le n$. The form ω can be written in a unique way as the linear combination

$$\omega = \sum_{I \in \Lambda(k,n)} \omega_I \, dx_I.$$

Here ω_I are the coefficients of ω with respect to the standard basis of $\bigwedge^k(R^n)$

$$dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}, \quad I = (i_1, i_2, \dots, i_k) \in \Lambda(k, n).$$

Let $I = (i_1, \dots, i_k)$ be a multi–index from $\Lambda(k, n)$. The complement I^* of the multi–index I is the multi-index $I^* = (j_1, \dots, j_{n-k})$ in $\Lambda(n-k, n)$ where the components j_p are in $\{1, \dots, n\} \setminus \{i_1, \dots, i_k\}$. We have

$$dx_I \wedge dx_{I^*} = \sigma dx_1 \wedge \dots \wedge dx_n \tag{2.3}$$

where $\sigma = \sigma(I)$ is the signature of the permutation $(i_1, \dots, i_k, j_1, \dots, j_{n-k})$ in the set $\{1, 2, \dots, n\}$. Note that $\sigma(I^*) = (-1)^{k(n-k)}\sigma(I)$.

Let $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$ be a differential form of the standard basis of $\bigwedge^k(R^n)$. We set

$$\star\, dx_I = \sigma(I) dx_{I^*}. \tag{2.4}$$

For $\omega \in \bigwedge^k(R^n)$ with $\omega = \sum_{I \in \Lambda(k,n)} \omega_I \, dx_I$, we set

$$\star\, \omega = \sum_{I \in \Lambda(k,n)} \omega_I \star dx_I. \tag{2.5}$$

Then $\star\omega$ belongs to $\bigwedge^{n-k}(R^n)$. The differential form $\star\omega$ is called the orthogonal complement of the differential form ω.

The operator $\star : \bigwedge^*(R^n) \to \bigwedge^*(R^n)$, also called Hodge star operator, has the following properties:

If $\alpha, \beta \in \bigwedge^k(R^n)$ and $a, b \in R$, then

$$\star\,(a\alpha + b\beta) = a \star \alpha + b \star \beta. \tag{2.6}$$

For every ω with $\deg \omega = k$ we have

$$\star\,(\star\omega) = (-1)^{k(n-k)}\omega. \tag{2.7}$$

We introduce the following notation. Let ω be a differential form of degree k. We set

$$\star^{-1} \omega = (-1)^{k(n-k)} \star \omega. \tag{2.8}$$

The operator $\star^{-1}$ is an inverse to $\star$ in the sense that $\star^{-1}(\star\omega) = \star(\star^{-1}\omega) = \omega$.

The inner or scalar product of the differential forms α and β of the same degree is defined as

$$\langle \alpha, \beta \rangle = \star^{-1}(\alpha \wedge \star\beta) = \star(\alpha \wedge \star\beta). \tag{2.9}$$

The scalar product of differential forms has the usual properties of the scalar product. We set

$$|\omega| = \sqrt{\langle \omega, \omega \rangle}.$$

A differential form ω of degree k is called simple if there are differential forms $\alpha_1, \dots, \alpha_k$ of degree 1 such that

$$\omega = \alpha_1 \wedge \dots \wedge \alpha_k.$$

We note the following useful property of the Euclidean norm: If $\alpha, \beta \in \bigwedge^*(R^n)$, then

$$|\alpha \wedge \beta| \le |\alpha||\beta|,$$

if at least one of the differential forms α, β is simple. If α and β are simple and nonzero, then equality holds if and only if the subspaces associated with α and β are orthogonal. More generally, if $\deg \alpha = p$, $\deg \beta = q$, then

$$|\alpha \wedge \beta| \leq (C^p_{p+q})^{1/2} |\alpha||\beta|, \tag{2.10}$$

see [**Fe**] §1.7.

The linear isomorphism $\mathrm{Hom}\big(\bigwedge_k(R^n), R\big) \simeq \bigwedge^k(R^n)$, that defines the duality of the spaces $\bigwedge_k(R^n)$ and $\bigwedge^k(R^n)$, associates a k-vector with a differential form. A vector $a = (a_1, \dots, a_n) \in R^n$ defines a differential form of degree 1

$$\omega = a_1 dx^1 + a_2 dx^2 + \ldots + a_n dx^n. \tag{2.11}$$

We denote it by Ω_a. Let $u = (u_1, \dots, u_k)$, $u_i \in \bigwedge_1(R^n)$, be a nondegenerated frame. The set of all k-dimensional frames is identified with the set of simple k-vectors. One can prove that the differential form

$$\Omega_u = \Omega_{u_1} \wedge \ldots \wedge \Omega_{u_k}$$

does not depend on the choice of the particular frame from the class of frames equivalent with u. This fact produces a one-to-one correspondence $u \mapsto \Omega_u$ of the set of simple polyvectors onto the set of simple differential forms.

3. Differential forms on Riemannian manifolds.

3.1. Riemannian manifolds. Let $\mathcal{M}$ be an n-dimensional Riemannian manifold with boundary or without boundary. Throughout the sequel we will assume that the manifold $\mathcal{M}$ is orientable and of class C^p where p is at least 3. By $T(\mathcal{M})$ we denote the tangent bundle and by $T_m(\mathcal{M})$ the tangent space at the point $m \in \mathcal{M}$. For each pair of vectors $x, y \in T_m(\mathcal{M})$ the symbol $\langle\, ,\rangle$ denotes their scalar product. The Riemannian connection on $T_m(\mathcal{M})$ gives the natural connection for tensors of every type. This connection preserves the scalar product mentioned above.

Below we shall use standard notation for function classes on manifolds. Thus, for example, the symbol $L^p_{\mathrm{loc}}(D)$ stands for the set of all Lebesgue measurable functions on an open set $D \subset \mathcal{M}$, locally integrable to the power p, $1 \leq p \leq \infty$, on D. The symbol $W^1_{p,\mathrm{loc}}(D)$ stands for the set of functions that have generalized partial derivatives in the sense of Sobolev of class $L^p_{\mathrm{loc}}(D)$ and $\mathrm{Lip}\,(D)$ denotes the class of all Lipschitz functions on D.

Let $\mathcal{M}$ and $\mathcal{N}$ be Riemannian manifolds of class C^k, $k \geq 3$, and $F : D \to \mathcal{N}$, $D \subset \mathcal{M}$, a mapping. We shall say that $F \in L^p_{\mathrm{loc}}(D)$ if for an arbitrary function $\phi \in C^0(\mathcal{N})$ we have $\phi \circ F \in L^p_{\mathrm{loc}}(D)$. The mapping F is in the class $W^1_{p,\mathrm{loc}}(D)$, if $\phi \circ F \in W^1_{p,\mathrm{loc}}(D)$ for every $\phi \in C^1(\mathcal{N})$.

Let $V(\mathcal{M})$ be a vector bundle on $\mathcal{M}$. Let in the elements of this bundle be given a Euclidean scalar product and let the linear connection on $V(\mathcal{M})$ preserve this scalar product. In this case we may say that $V(\mathcal{M})$ is a Riemannian vector bundle over $\mathcal{M}$.

By $\bigwedge^k(\mathcal{M})$ and $\bigwedge_k(\mathcal{M})$ we denote Riemannian vector bundles $\bigwedge^k(T_m(\mathcal{M}))$ and $\bigwedge_k(T_m(\mathcal{M}))$. The sections of these bundles are the fields of k-covectors (k-forms) and k-vectors, which we shall discuss now in some detail.

3.2. Basic properties of differential forms. Let $x^1, \dots, x^n$ be local coordinates in the neighborhood of a point $m \in \mathcal{M}$. The square of a line element on $\mathcal{M}$ has the following expression in terms of the local coordinates $x^1, \dots, x^n$

$$ds^2 = \sum_{i,j=1}^{n} g_{ij} dx^i \, dx^j.$$

By the symbol g^{ij} we shall denote the contravariant tensor defined by the equality

$$(g^{ik})(g_{kj}) = (\delta^i_j), \quad i,j = 1, \dots, n,$$

where δ^j_i is the Kronecker symbol.

Each section α of the bundle $\bigwedge^k(\mathcal{M})$ (that is a differential form) can be written in terms of the local coordinates $x^1, \dots, x^n$ as the linear combination

$$\alpha = \sum_{I \in \Lambda(k,n)} \alpha_I \, dx_I = \sum_{1 \le i_1 < \ldots < i_k \le n} \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}. \tag{3.3}$$

Let α be a differential form defined on an open set $D \subset \mathcal{M}$. If $\mathcal{F}(D)$ is a class of functions defined on D then we say that the differential form α is in this class provided that $\alpha_I \in \mathcal{F}(D)$. For instance, the differential form α is in the class $L^p(D)$ if all its coefficients are in this class.

The orthogonal complement of a differential form α on a Riemannian manifold $\mathcal{M}$ will be denoted by $\star\alpha$. If $\deg \alpha = 1$ then in the local orthonormal system of coordinates $x^1, \dots, x^n$ at m we can write

$$\star\alpha(m) = \star \sum_{i=1}^{n} \alpha_i(m) \, dx^i = \sum_{i=1}^{n} (-1)^{i-1} \alpha_i(m) \, dx^1 \wedge \ldots \widehat{dx^i} \ldots \wedge dx^n ,$$

where the sign $\widehat{\ }$ means that the expression under $\widehat{\ }$ is omitted. We remark that the differential form dv is the volume element on $\mathcal{M}$.

If α, $\deg \alpha = k$, $0 \le k \le n$, is a differential form whose coefficients are in $C^1(\mathcal{M})$ then $d\alpha$, $\deg(d\alpha) = k+1$, denotes its differential defined by

$$d\alpha = \sum_{I \in \Lambda(k,n)} d\alpha_I \wedge dx_I.$$

The differentiation is a linear operation for which the following properties hold:

If α and β are arbitrary differential form that are differentiable in a domain $U \subset \mathcal{M}$ then

(i) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$,

(ii) $d(d\alpha) = 0$,

where k is the degree of the differential form α.

The operator $\star$ and the exterior differentiation d define the codifferential operator δ by the formula

$$\delta\alpha = (-1)^k \star^{-1} d \star \alpha \tag{3.4}$$

for a differential form α of degree k. Clearly, $\delta\alpha$ is a differential form of degree $k-1$.

Let $\mathcal{M}$ be a compact n-dimensional orientable Riemannian manifold with nonempty piecewise smooth boundary $\partial\mathcal{M}$. The following Stokes formula holds

$$\int\limits_{\partial\mathcal{M}} \alpha = \int\limits_{\mathcal{M}} d\alpha,$$

for an arbitrary form $\alpha \in C^1(\mathcal{M})$, $\deg\alpha = n-1$.

3.5. . A differential form α of degree k on the manifold $\mathcal{M}$ with coefficients $\alpha_{i_1\dots i_k} \in L^p_{\mathrm{loc}}(\mathcal{M})$ is called weakly closed if for each differential form β, $\deg\beta = k+1$, with

$$\operatorname{supp}\beta \cap \partial\mathcal{M} = \emptyset, \quad \operatorname{supp}\beta = \overline{\{m \in \mathcal{M} : \beta \neq 0\}} \subset \mathcal{M},$$

and with coefficients in the class $W^1_{q,\mathrm{loc}}(\mathcal{M})$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p, q \leq \infty$, we have

$$\int\limits_{\mathcal{M}} \langle \alpha, \delta\beta \rangle dv = 0. \tag{3.6}$$

For smooth differential forms α Condition (3.6) agrees with the traditional condition of closedness $d\alpha = 0$. In fact, if $\alpha, \beta \in C^1(\mathcal{M}), \operatorname{supp}\beta \cap \partial\mathcal{M} = \emptyset$, then we have

$$\int\limits_{\mathcal{M}} d\alpha \wedge \star\beta = \int\limits_{\mathcal{M}} d(\alpha \wedge \star\beta) + (-1)^{k+1} \int\limits_{\mathcal{M}} \alpha \wedge d \star \beta.$$

Because the differential form β has compact support on the orientable manifold $\mathcal{M}$ the first integral on the right hand side is zero by the Stokes formula. Thus we get

$$\int\limits_{\mathcal{M}} d\alpha \wedge \star\beta = (-1)^{k+1} \int\limits_{\mathcal{M}} \alpha \wedge \star \star^{-1} d \star \beta = \int\limits_{\mathcal{M}} \alpha \wedge \star\delta\beta = \int\limits_{\mathcal{M}} \langle \alpha, \delta\beta \rangle dv.$$

We fix an arbitrary point $m \in \mathcal{M}$ and pass to the local coordinates on $\mathcal{M}$ in a neighborhood of this point. Using Condition (3.6) and the fundamental lemma of the variational calculus, the *du Bois-Reymond Lemma*, we conclude that everywhere in this neighborhood of m the coefficients of the differential form $d\alpha$ are zero. Thus the validity of (3.6) under the given conditions on β is equivalent to the requirement $d\alpha = 0$ understood in the classical sense.

We next introduce the following very useful theorem.

Theorem 3.7. *Let α and β be differential forms, $\beta \in W_q^1(\mathcal{M})$ with a compact support, and $\alpha \in W_{p,\mathrm{loc}}^1(\mathcal{M})$, $1 \le p, q \le \infty$, $\deg\alpha + \deg\beta = n-1$, $1/p + 1/q = 1$. Then*

$$\int_{\mathcal{M}} d\alpha \wedge \beta = (-1)^{\deg\alpha+1} \int_{\mathcal{M}} \alpha \wedge d\beta. \tag{3.8}$$

In particular, the differential form α is weakly closed if and only if $d\alpha = 0$ a.e. on $\mathcal{M}$.

Proof. Fix α and β with the stated properties. Because the coefficients of the differential form α are in the class $W_{p,\mathrm{loc}}^1(\mathcal{M})$ there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of differential forms with coefficients of class $C^1(\mathcal{M})$ converging in the W_p^1-norm to the coefficients of the differential form α on every compact set $K \subset \mathrm{int}\mathcal{M}$.

Let $\{\beta_n\}_{n=1}^{\infty}$ be a sequence of differential forms of degree $\deg\beta_n = \deg\beta$ in the class $C^1(\mathcal{M})$ having compact supports and converging in the norm of W_q^1 to the form β. We may assume that there exists a smooth submanifold $U \subset\subset \mathcal{M}$ such that $\operatorname{supp}\beta_n \subset U$ for all integers n.

The differential forms $\alpha_n \wedge \beta_n$ have compact supports contained in U. The Stokes formula yields

$$\int_{\mathcal{M}} d(\alpha_n \wedge \beta_n) = \int_U d(\alpha_n \wedge \beta_n) = 0,$$

and hence

$$\int_U d\alpha_n \wedge \beta_n + (-1)^{\deg\alpha} \int_U \alpha_n \wedge d\beta_n = 0.$$

We have

$$\int_U d\alpha \wedge \beta - \int_U d\alpha_n \wedge \beta_n = \int_U (d\alpha - d\alpha_n) \wedge \beta + \int_U d\alpha_n \wedge (\beta - \beta_n).$$

Therefore, using inequality (2.10) we obtain

$$\begin{aligned}
&\left|\int_U d\alpha\wedge\beta-\int_U d\alpha_n\wedge\beta_n\right|\\
&\le\int_U |d(\alpha-\alpha_n)\wedge\beta|dv+\int_U |d\alpha_n\wedge(\beta-\beta_n)|dv\\
&\le C\int_U |d(\alpha-\alpha_n)|\,|\beta|dv+C\int_U |d\alpha_n|\,|\beta-\beta_n|dv\\
&\le C||d(\alpha-\alpha_n)||_{L^p(U)}\,||\beta||_{L^q(U)}+C||d\alpha_n||_{L^p(U)}\,||\beta-\beta_n||_{L^q(U)},
\end{aligned}$$

where $C=\max(C_n^{k+1})^{1/2}$ and $k=\deg\alpha$.

Similarly we obtain

$$\begin{aligned}
&\left|\int_U \alpha\wedge d\beta-\int_U \alpha_n\wedge d\beta_n\right|\\
&\le C_1||\alpha||_{L^p(U)}\,||d(\beta-\beta_n)||_{L^q(U)}+C_1||\alpha-\alpha_n||_{L^p(U)}\,||d\beta||_{L^q(U)},
\end{aligned}$$

where $C_1=(C_n^k)^{1/2}$.

These inequalities easily yield (3.8).

If $d\alpha=0$ a.e. on $\mathcal{M}$ then by (3.8)

$$\int_{\mathcal{M}}\alpha\wedge d\beta=0 \tag{3.9}$$

for an arbitrary differential form $\beta\in W_q^1$ with compact support. This, obviously, implies (3.6).

On the other hand, if we take a weakly closed differential form $\alpha\in W^1_{p,\mathrm{loc}}(\mathcal{M})$ then by (3.8) one has

$$\int_{\mathcal{M}} d\alpha\wedge\beta=0\quad\text{for all }\ \beta\in W_q^1(\mathcal{M})\quad\text{with}\quad\operatorname{supp}\beta\subset\mathcal{M}.$$

We fix an arbitrary point $m\in\mathcal{M}$, pass again to the local coordinates on $\mathcal{M}$ in a neighborhood of m and use again the *du Bois-Reymond Lemma* to conclude that almost everywhere in this neighborhood the form $d\alpha$ is zero. □

4. The $\mathcal{WT}$-classes of differential forms.

In this section we introduce several classes of differential forms with generalized derivatives which first were presented in [**MMV1**] and [**MMV2**].

These classes are used to study the associated classes of quasilinear elliptic partial differential equations.

Let $\mathcal{M}$ be a Riemannian manifold of class C^3, $\dim \mathcal{M} = n$, with a boundary or without boundary and let

$$w \in L^p_{\mathrm{loc}}(\mathcal{M}),\ \deg w = k,\ 0 \le k \le n,\ p > 1, \tag{4.1}$$

be a weakly closed differential form on $\mathcal{M}$.

Definition 4.2. A differential form w (4.1) is said to be of the class $\mathcal{WT}_1$ on $\mathcal{M}$ if there exists a weakly closed differential form

$$\theta \in L^q_{\mathrm{loc}}(\mathcal{M}),\ \deg \theta = n-k,\ \frac{1}{p} + \frac{1}{q} = 1, \tag{4.3}$$

such that almost everywhere on $\mathcal{M}$ we have

$$\nu_0\ |\theta|^q \le\ \langle w, \star\theta\rangle, \tag{4.4}$$

where ν_0 is a constant.

Definition 4.5. The differential form (4.1) is said to be of the class $\mathcal{WT}_2$ on $\mathcal{M}$ if there exists a differential form (4.3) such that almost everywhere on $\mathcal{M}$ the conditions

$$\nu_1\,|w|^p \le\ \langle w, \star\theta\rangle \tag{4.6}$$

and

$$|\theta| \le \nu_2\,|w|^{p-1} \tag{4.7}$$

are satisfied, with constants $\nu_1, \nu_2 > 0$.

For an arbitrary simple differential form of degree k

$$w = w_1 \wedge \ldots \wedge w_k$$

we set

$$\|w\| = \left(\sum_{i=1}^{k} |w_i|^2\right)^{1/2}.$$

For a simple differential form w we have Hadamard's inequality

$$|w| \le \prod_{i=1}^{k} |w_i|.$$

Taking these and using the inequality between geometric and arithmetic means

$$\left(\prod_{i=1}^{k} |w_i|\right)^{1/k} \le \frac{1}{k}\sum_{i=1}^{k} |w_i| \le \left(\frac{1}{k}\sum_{i=1}^{k} |w_i|^2\right)^{1/2}$$

we obtain

$$|w| \le k^{-\frac{k}{2}}\,\|w\|^k. \tag{4.8}$$

Definition 4.9. A simple differential form of degree k

$$w = w_1 \wedge \ldots \wedge w_k,\ w_i \in L^p_{\text{loc}}(\mathcal{M}),\ 1 \le i \le k,$$

is said to be of the class $\mathcal{WT}_3$ on $\mathcal{M}$ if there is a differential form (4.3) such that almost everywhere on $\mathcal{M}$ the inequality (4.7) holds and

$$\nu_3 \, \|w\|^{kp} \le k^{\frac{kp}{2}} \, \langle w, \star\theta \rangle. \tag{4.10}$$

Definition 4.11. A simple differential form of degree k

$$w = w_1 \wedge \ldots \wedge w_k,\ w_i \in L^p_{\text{loc}}(\mathcal{M}),\ 1 \le i \le k,$$

is said to be of the class $\mathcal{WT}_4$ on $\mathcal{M}$, if there exists a simple differential form (4.3) such that the inequality (4.10) holds almost everywhere on $\mathcal{M}$ and

$$(n-k)^{\frac{-(n-k)}{2}} \, \|\theta\|^{n-k} \le \nu_4 \, |w|^{p-1}. \tag{4.12}$$

Remark 4.13. Because every differential form of degree 1 is simple, for $k = 1$ the class $\mathcal{WT}_2$ coincides with the class $\mathcal{WT}_3$ while for $k = n-1$ the class $\mathcal{WT}_3$ coincides with $\mathcal{WT}_4$.

Theorem 4.14. *The following inclusions hold between these $\mathcal{WT}$-classes*

$$\mathcal{WT}_4 \subset \mathcal{WT}_3 \subset \mathcal{WT}_2 \subset \mathcal{WT}_1.$$

Proof. The first two relations follow in an obvious way from (4.8). For the proof of the last one it is enough to observe that

$$|\theta|^q = |\theta|^{\frac{p}{p-1}} \le \left(\nu_2^{\frac{1}{p-1}} |w| \right)^p \le \nu_2^{\frac{p}{p-1}} \nu_1^{-1} \, \langle w, \star\theta \rangle.$$

□

Example 4.15. Let v be a differential form of the class $L^2_{\text{loc}}(\mathcal{M})$ with $\deg v = k$, $1 \le k \le n$. Following Hodge [**Ho**] we shall say that the differential form v is harmonic if it is simultaneously weakly closed and weakly coclosed, that is

$$dv = \delta v = 0. \tag{4.16}$$

In particular, if $f \in C^2(\mathcal{M})$ then the differential form df of degree 1 is harmonic if and only if $\Delta f = 0$.

Theorem 4.17. *Let v be a differential form of $L^2_{\text{loc}}(\mathcal{M})$, $\deg v = k$. If v is a harmonic differential form then v is of the class $\mathcal{WT}_2$ on $\mathcal{M}$ with structure constants $p = 2$, $\nu_1 = \nu_2 = 1$.*

Proof. Setting $\theta = \star^{-1} v \in L^2_{\text{loc}}(\mathcal{M})$ we have

$$\langle v, \star\theta \rangle = \langle v, v \rangle = |v|^2$$

and $|\theta| = |v|$. The differential form $\star^{-1} v$ is weakly closed because $\star^{-1} v = (-1)^{k(n-k)} \star v$. Therefore Conditions (4.6) and (4.7) indeed hold with the constants $p = 2$, $\nu_2 = \nu_3 = 1$. □

5. Quasilinear elliptic equations.

Let $\mathcal{M}$ be a Riemannian manifold and let

$$A : \bigwedge^k(T(\mathcal{M})) \to \bigwedge^k(T(\mathcal{M}))$$

be a mapping defined almost everywhere on the k-vector tangent bundle $\bigwedge^k(T(\mathcal{M}))$. We assume that for almost every $m \in \mathcal{M}$ the mapping A is defined on the k-vector tangent space $\bigwedge^k(T_m(\mathcal{M}))$, that is for almost every $m \in \mathcal{M}$ the mapping

$$A(m,\ .\ ) : \xi \in \bigwedge^k(T_m(\mathcal{M})) \to \bigwedge^k(T_m(\mathcal{M}))$$

is defined and continuous. We assume that the mapping $m \mapsto A_m(X)$ is measurable for all measurable k-vector fields X. Suppose that for almost every $m \in \mathcal{M}$ and for all $\xi \in \bigwedge^k(T_m(\mathcal{M}))$ we have

$$\nu_0\,|A(m,\xi)|^p \le \langle \xi, A(m,\xi)\rangle \tag{5.1}$$

with the constants $p > 1$ and $\nu_0 > 0$.

Definition 5.2. A differential form $w \in W^{1,p}_{\mathrm{loc}}(\mathcal{M})$ is said to be A-harmonic if it is a solution of the A-harmonic equation

$$\delta A(m, dw) = 0, \tag{5.3}$$

understood in the weak sense, that is

$$\int\limits_{\mathcal{M}} \langle d\Phi, A(m,dw)\rangle\, dv = 0 \tag{5.4}$$

for all differential forms $\Phi \in W^{1,q}_{\mathrm{loc}}(\mathcal{M})$, $1/p+1/q = 1$, with $\operatorname{supp}\Phi \cap \partial\mathcal{M} = \emptyset$.

Theorem 5.5. *If the differential form $w \in W^1_{p,\mathrm{loc}}(\mathcal{M})$ is A-harmonic with the property (5.1) then the differential form dw is in the class $\mathcal{WT}_1$ on $\mathcal{M}$.*

Proof. Let w, $\deg w = k$ be a solution of (5.3) understood in the weak sense. Let the differential form $\alpha(m)$ be associated with the vector field $A(m,dw)$ at the point m and set $\theta = \star\alpha$. The differential form w is weakly closed because of (5.4) and the weak closedness of θ follows from

$$\begin{aligned}(-1)^{nk+1}\int\limits_{\mathcal{M}}\langle\theta,\delta\psi\rangle\, dv &= \int\limits_{\mathcal{M}}\langle\star\alpha, \star d\star\psi\rangle\, dv \\ &= \int\limits_{\mathcal{M}}\langle\alpha, d\star\psi\rangle\, dv = \int\limits_{\mathcal{M}}\langle A(m,dw), d\phi\rangle\, dv \;=\; 0\end{aligned}$$

for all $\psi = \star^{-1}\phi \in W^{1,q}(\mathcal{M})$ with $\operatorname{supp}\psi \cap \partial\mathcal{M} = \emptyset$. Further, by (5.1) we get

$$\nu_0|\theta|^q = \nu_0|A(m,dw)|^q \le \langle dw, A(m,dw)\rangle = \langle dw, \star\theta\rangle,$$

which guarantees (4.4). □

From now on we assume that the vector field $A(m,\xi)$ satisfies the conditions

$$\nu_1\,|\xi|^p \le \langle \xi, A(m,\xi)\rangle, \tag{5.6}$$

and

$$|A(m,\xi)| \le \nu_2\,|\xi|^{p-1} \tag{5.7}$$

with $p > 1$ and for some constants $\nu_1, \nu_2 > 0$. It is clear that we have $\nu_1 \le \nu_2$.

Theorem 5.8. *A differential form $\omega \in W^{1,p}_{\mathrm{loc}}(\mathcal{M})$ is A-harmonic with properties (5.6) and (5.7) if and only if $d\omega \in \mathcal{WT}_2$.*

Proof. As is the proof of Theorem 5.5 we define θ. The weak closedness of w and θ follows as above. From (5.6) it follows that

$$\nu_1 |dw|^p \le \langle dw, A(m,dw)\rangle = \langle dw, \star\,\theta\rangle$$

and from (5.7)

$$|\theta| = |\star\,\alpha| = |A(m,dw)| \le \nu_2 |dw|^{p-1}.$$

Conversely, if $dw \in \mathcal{WT}_2$, then there exists a weakly closed differential form θ (see (4.3)) such that (4.6) and (4.7) are satisfied. With the vector field $a : \mathcal{M} \to \Lambda_k(\mathbb{R})$ associated to the differential form $\alpha = \star\,\theta$ we define

$$A(m,\xi) = \begin{cases} a(m), & \text{for } \xi = dw(m), \\ \xi|\xi|^{p-2}, & \text{for } \xi \ne dw(m). \end{cases} \tag{5.9}$$

The weak closedness of θ ensures that w is a solution of (5.3) understood in the weak sense. Conditions (5.6) and (5.7) for A are satisfied with (4.6) and (4.7). $\square$

6. Quasiregular mappings.

Let $\mathcal{M}$ and $\mathcal{N}$ be Riemannian manifolds of dimension n. A mapping $F : \mathcal{M} \to \mathcal{N}$ of the class $W^1_{n,\mathrm{loc}}(\mathcal{M})$ is called a quasiregular mapping if F satisfies

$$|F'(m)|^n \le K J_F(m) \tag{6.1}$$

almost everywhere on $\mathcal{M}$. Here $F'(m) : T_m(\mathcal{M}) \to T_{F(m)}(\mathcal{N})$ is the formal derivative of $F(m)$, further, $|F'(m)| = \max_{|h|=1} |F'(m)h|$. We denote by $J_F(m)$ the Jacobian of F at the point $m \in \mathcal{M}$, i.e., the determinant of $F'(m)$.

The best constant $K \ge 1$ in the inequality (6.1) is called the outer dilatation of F and denoted by $K_O(F)$. If F is quasiregular then the least constant $K \ge 1$ for which we have

$$J_F(m) \le K l(F'(m))^n$$

almost everywhere on $\mathcal{M}$ is called the inner dilatation of the mapping $F : \mathcal{M} \to \mathcal{N}$ and denoted by $K_I(F)$. Here

$$l(F'(m)) = \min_{|h|=1} |F'(m)h|.$$

The quantity

$$K(F) = \max\{K_O(F), K_I(F)\}$$

is called the maximal dilatation of F and if $K(F) \le K$ then the mapping F is called K-quasiregular.

If $F : \mathcal{M} \to \mathcal{N}$ is a quasiregular homeomorphism then the mapping F is called quasiconformal. In this case the inverse mapping F^{-1} is also quasiconformal in the domain $F(\mathcal{M}) \subset \mathcal{N}$ and $K(F^{-1}) = K(F)$.

Example 6.2. Some basic examples of quasiregular mappings are provided by mappings $F : \mathcal{M} \to \mathcal{N}$ that distort lengths of curves by a bounded factor. Indeed, following [**HKM**], we shall say that a mapping $F : \mathcal{M} \to \mathcal{N}$, $F \in W^1_{1,\mathrm{loc}}(\mathcal{M})$, is an L–BLD mapping if $J_F(m) \ge 0$ almost everywhere on $\mathcal{M}$ and for some constant $L \ge 1$ and for all $h \in T_m(\mathcal{M})$ and almost every $m \in \mathcal{M}$ we have

$$|h|/L \le |F'(m)h| \le L|h|. \tag{6.3}$$

It is readily shown that every L–BLD map is K-quasiregular with $K = L^{2(n-1)}$ ([**HKM**], Lemma 14.80).

Let $\mathcal{A}$ and $\mathcal{B}$ be Riemannian manifolds of dimensions $\dim \mathcal{A} = k$, $\dim \mathcal{B} = n - k$, $1 \le k < n$, and with scalar products $\langle\, ,\rangle_A$, $\langle\, ,\rangle_B$, respectively. On the Cartesian product $\mathcal{N} = \mathcal{A} \times \mathcal{B}$ we introduce the natural structure of a Riemannian manifold with the scalar product

$$\langle\, ,\rangle = \langle\, ,\rangle_{\mathcal{A}} + \langle\, ,\rangle_{\mathcal{B}}.$$

We denote by $\pi : \mathcal{A} \times \mathcal{B} \to \mathcal{A}$ and $\eta : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ the natural projections of the manifold $\mathcal{N}$ onto submanifolds.

If $w_{\mathcal{A}}$ and $w_{\mathcal{B}}$ are volume forms on A and $\mathcal{B}$, respectively, then the differential form $w_{\mathcal{N}} = \pi^* w_{\mathcal{A}} \wedge \eta^* w_{\mathcal{B}}$ is a volume form on $\mathcal{N}$.

Theorem 6.4. *Let $F : \mathcal{M} \to \mathcal{N}$ be a quasiregular mapping and let $f = \pi \circ F : \mathcal{M} \to \mathcal{A}$. Then the differential form $f^* w_{\mathcal{A}}$ is of the class $\mathcal{WT}_2$ on $\mathcal{M}$ with the structure constants $p = n/k$, $\nu_1 = \nu_1(n, k, K_O)$ and $\nu_2 = \nu_2(n, k, K_O)$.*

Remark 6.5. From the proof of the theorem it will be clear that the structure constants can be chosen to be

$$\nu_1^{-1} = \left(k + \frac{n-k}{\overline{c}^2}\right)^{-n/2} n^{n/2}\, K_O, \quad \nu_2^{-1} = \underline{c}^{\,n-k},$$

where $\overline{c} = \overline{c}(k, n, K_O)$ and $\underline{c} = \underline{c}(k, n, K_O)$ are, respectively, the greatest and least positive roots of the equation

$$(k\xi^2 + (n-k))^{n/2} - n^{n/2} K_O \xi^k = 0. \tag{6.6}$$

Proof. Setting $g = \eta \circ F : \mathcal{M} \to \mathcal{B}$ we choose $\theta = g^* w_{\mathcal{B}}$. The volume form $w_{\mathcal{B}}$ is weakly closed.

In fact, if the mapping g is sufficiently regular then

$$d\theta = dg^* w_{\mathcal{B}} = g^* dw_{\mathcal{B}} = 0.$$

In the general case for the verification of Condition (3.6) we approximate the mapping $g : \mathcal{M} \to \mathcal{B}$ in the norm of W^1_n by smooth maps g_l, $l = 1, 2, \ldots$. Because Condition (3.6) holds for each of the differential forms $g_l^* w_{\mathcal{B}}$, it must hold also for the differential form $g^* w_{\mathcal{B}}$.

The weak closedness of the differential form $f^* w_{\mathcal{A}}$ follows similarly.

Fix a point $m \in \mathcal{M}$, at which the relation (6.1) holds. Set $a = f(m)$, $b = g(m)$. Then

$$T_{F(m)}(\mathcal{N}) = T_a(\mathcal{A}) \times T_b(\mathcal{B}).$$

The computations can be conveniently carried out as follows. We first rewrite Condition (6.1) in the form

$$|F'(m)|^n \leq K_O |F^* w_{\mathcal{N}}|, \tag{6.7}$$

where $w_{\mathcal{N}}$ is a volume form on $\mathcal{N}$.

For the points $a \in \mathcal{A}$, $b \in \mathcal{B}$ we choose neighborhoods and local systems of coordinates $y^1, \ldots, y^k$, and $y^{k+1}, \ldots, y^n$, orthonormal at a and b, respectively. We have

$$\begin{aligned} f^* w_{\mathcal{A}} &= f^*(dy^1 \wedge \ldots \wedge dy^k) = f^* dy^1 \wedge \ldots \wedge f^* dy^k \\ &= df^1 \wedge \ldots \wedge df^k, \quad f^i = y^i \circ f, \quad i = 1, \ldots, k. \end{aligned}$$

Because the differential form $w_{\mathcal{A}}$ is simple we obtain by the inequality between the geometric and arithmetic means

$$\begin{aligned} |df^1 \wedge \ldots \wedge df^k|^{1/k} &\leq \left(\prod_{i=1}^k |df^i| \right)^{1/k} \\ &\leq \frac{1}{k} \sum_{i=1}^k |df^i| \leq \left(\frac{1}{k} \sum_{i=1}^k |df^i|^2 \right)^{1/2}. \end{aligned} \tag{6.8}$$

Similarly

$$|dg^{k+1} \wedge \ldots \wedge dg^n|^{1/(n-k)} \leq \left(\frac{1}{n-k} \sum_{i=k+1}^n |dg^i|^2 \right)^{1/2}. \tag{6.9}$$

It is not difficult to see that

$$F^* w_{\mathcal{N}} = F^*(\pi^* w_{\mathcal{A}} \wedge \eta^* w_{\mathcal{B}}) = f^* w_{\mathcal{A}} \wedge g^* w_{\mathcal{B}} = f^* w_{\mathcal{A}} \wedge \theta$$

and further that

$$|F^* w_{\mathcal{N}}| = |f^* w_{\mathcal{A}} \wedge g^* w_{\mathcal{B}}| \le |df^1 \wedge \ldots \wedge df^k||dg^{k+1} \wedge \ldots \wedge dg^n|.$$

We have

$$|dF|^2 = \sum_{i=1}^{k} |df^i|^2 + \sum_{i=k+1}^{n} |dg^i|^2 \le n\,|F'|^2.$$

Therefore we get from (6.7), (6.8) and (6.9)

$$\Big(k|f^* w_{\mathcal{A}}|^{2/k} + (n-k)|g^* w_{\mathcal{B}}|^{2/(n-k)}\Big)^{n/2}$$
$$\le n^{n/2}\, K_O \langle f^* w_{\mathcal{A}}, \star\theta\rangle \le n^{n/2}\, K_O |f^* w_{\mathcal{A}}||g^* w_{\mathcal{B}}|.$$

Set

$$\xi = \frac{|f^* w_{\mathcal{A}}|^{1/k}}{|g^* w_{\mathcal{B}}|^{1/(n-k)}}.$$

The preceding relation takes the form

$$\big(k\xi^2 + (n-k)\big)^{n/2} \le n^{n/2}\, K_O \xi^k.$$

Using the notations $\underline{c}$ and $\overline{c}$ for the least and greatest positive roots of Equation (6.6) we have $\underline{c} \le \xi \le \overline{c}$ and

$$\underline{c}|g^* w_{\mathcal{B}}|^{1/(n-k)} \le |f^* w_{\mathcal{A}}|^{1/k} \le \overline{c}|g^* w_{\mathcal{B}}|^{1/(n-k)}. \tag{6.10}$$

As above, from (6.10) it follows that

$$|f^* w_{\mathcal{A}}|^{n/k} \le \Big(k + \frac{n-k}{\overline{c}^2}\Big)^{-n/2} n^{n/2}\, K_O \langle f^* w_{\mathcal{A}}, \star\theta\rangle.$$

Thus Condition (4.6) for the membership of the differential form $f^* w_{\mathcal{A}}$ of degree k in the class $\mathcal{WT}_2$ is indeed satisfied.

To verify Condition (4.7) it is enough to observe that from (6.10) it follows that

$$\underline{c}^{n-k}|\theta| \le |f^* w_{\mathcal{A}}|^{\frac{n-k}{k}}.$$

□

Let $y^1, y^2, \ldots, y^k$ be an orthonormal system of coordinates in R^k, $1 \le k \le n$. Let $\mathcal{A}$ be a domain in R^k and let $\mathcal{B}$ be an $(n-k)$-dimensional Riemannian manifold. We consider the manifold $\mathcal{N} = \mathcal{A} \times \mathcal{B}$.

Let $F = (f^1, f^2, \ldots, f^k, g) : \mathcal{M} \to \mathcal{N}$ be a mapping of the class $W^1_{n,\mathrm{loc}}(\mathcal{M})$ and $g = \eta \circ F$ as defined above. We have $f^* w_{\mathcal{A}} = df^1 \wedge \ldots \wedge df^k$.

Theorem 6.11. *If the mapping F is quasiregular then the differential form $f^* w_{\mathcal{A}}$ is in the class $\mathcal{WT}_3$ on $\mathcal{M}$ with the structure constants $p = n/k$, $\nu_3 = \nu_3(k, n, K_O)$, $\nu_2 = \nu_2(k, n, K_O)$.*

Remark 6.12. We can choose the constants ν_2, ν_3 to be

$$\nu_2 = \underline{c}_1^{k-n}, \quad \nu_3 = \left(1 + \frac{1}{\overline{c}_1^2}\right)^{n/2} n^{-n/2} k^{n/2} K_O^{-1}$$

where $\underline{c}_1$ is the least and $\overline{c}_1$ the greatest positive root of the equation

$$(\xi^2 + 1)^{n/2} - n^{n/2} k^{-k/2} (n-k)^{-(n-k)/2} K_O\, \xi^k = 0. \tag{6.13}$$

Proof. In contrast to the previous case the k-form $f^* w_{\mathcal{A}}$ has now a global coordinate representation. Because the earlier arguments had local character they are applicable to the present case, too. As in the previous case we can choose $\theta = g^* w_{\mathcal{B}}$. Condition (4.7) holds with the same constant. We now proceed to verify Condition (4.10).

Combining (6.7), (6.8) and (6.9) we get

$$\left(\sum_{i=1}^{k} |df^i|^2 + \sum_{i=k+1}^{n} |dg^i|^2\right)^{n/2}$$
$$\leq k^{-k/2}(n-k)^{-(n-k)/2} n^{n/2} K_O \left(\sum_{i=1}^{k} |df^i|^2\right)^{k/2} \left(\sum_{i=k+1}^{n} |dg^i|^2\right)^{(n-k)/2}.$$

Here we set

$$\xi = \left(\frac{\sum_{i=1}^{k} |df^i|^2}{\sum_{i=k+1}^{n} |dg^i|^2}\right)^{1/2}.$$

We then get

$$(\xi^2 + 1)^{n/2} \leq k^{-k/2}(n-k)^{-(n-k)/2} n^{n/2} K_O \xi^k.$$

If $\underline{c}_1, \overline{c}_1$ are, respectively, the least and greatest of the positive roots of (6.13) then

$$\underline{c}_1 \left(\sum_{i=k+1}^{n} |dg^i|^2\right)^{1/2} \leq \left(\sum_{i=1}^{k} |df^i|^2\right)^{1/2} \leq \overline{c}_1 \left(\sum_{i=k+1}^{n} |dg^i|^2\right)^{1/2}. \tag{6.14}$$

From the relations (6.7) and (6.14) it follows that

$$\left(\frac{1}{\overline{c}_1^2} + 1\right)^{n/2} \left(\sum_{i=1}^{k} |df^i|^2\right)^{n/2} \leq n^{n/2} K_O \langle f^* w_{\mathcal{A}}, \star\theta \rangle,$$

which guarantees the truth of (4.10). □

Theorem 6.15. *If the mapping $F:\mathcal{M}\to R^n$ is quasiregular then the differential form $f^*w_{\mathcal{A}} = df^1\wedge\ldots\wedge df^k$ is of the class $\mathcal{WT}_4$ on $\mathcal{M}$ with the structure constants $p=n/k$, $\nu_3=\nu_3(k,n,K_O)$, $\nu_4=\nu_4(k,n,K_O)$.*

Proof. As above we set $\theta = dg^{k+1}\wedge\ldots\wedge dg^n$. Condition (4.10) has already been proved. By (6.7), (6.9) and (6.14) we have

$$(1+\underline{c}_1^2)^{n/2}\left(\sum_{i=k+1}^{n}|dg^i|^2\right)^{n/2}$$
$$\le (n-k)^{-(n-k)/2}n^{n/2}K_O|f^*w_{\mathcal{A}}|\left(\sum_{i=k+1}^{n}|dg^i|^2\right)^{(n-k)/2}.$$

Therefore

$$\left(\sum_{i=k+1}^{n}|dg^i|^2\right)^{k/2} \le (n-k)^{-(n-k)/2}(1+\underline{c}_1^2)^{-n/2}n^{n/2}K_O|f^*w_{\mathcal{A}}|,$$

which easily yields the desired conclusion. □

Remark 6.16. For the constant ν_3 we can choose the constant of Theorem 6.11 and

$$\nu_4 = \left((n-k)^{-n/2}(1+\underline{c}_1^2)^{-n/2}n^{n/2}K_O\right)^{(n-k)/k}.$$

Theorem 6.17. *Let $f=(f^1,f^2,\ldots,f^{n-1}):\mathcal{M}\to R^{n-1}$ be a mapping of the class $W^1_{n,\mathrm{loc}}(\mathcal{M})$ and let the fundamental group π_1 of the manifold $\mathcal{M}$ be trivial. The mapping f can be extended to a quasiregular mapping*

$$F=(f,f^n)=(f^1,\ldots,f^{n-1},f^n):\mathcal{M}\to R^n$$

if and only if the differential form $w=df^1\wedge\ldots\wedge df^{n-1}$ of degree $n-1$ is in the class $\mathcal{WT}_4$ on $\mathcal{M}$ with $p=n/(n-1)$.

Proof. We assume that $F=(f,f^n)$ is quasiregular. By Theorem 6.15 the differential form w is in the class $\mathcal{WT}_4$ on $\mathcal{M}$.

Conversely, let w be a differential form of the class $\mathcal{WT}_4$. Then there exists a weakly closed differential form θ, $\deg\theta=1$, satisfying Conditions (4.10) and (4.12). Because $\pi_1=\{e\}$ there exists an injective function $f^n:\mathcal{M}\to R^1$ such that $df^n=\theta$. From (4.10) we get

$$\nu_3\left(\sum_{i=1}^{n-1}|df^i|^2\right)^{n/2} \le (n-1)^{n/2}|df^1\wedge\ldots\wedge df^n|.$$

Condition (4.12) implies

$$\nu_4^{-1}|df^n| \leq |df^1 \wedge \ldots df^{n-1}|^{1/(n-1)} \leq \left(\frac{1}{n-1}\sum_{i=1}^{n-1}|df^i|^2\right)^{1/2}.$$

Thus we get

$$\begin{aligned}\left(\sum_{i=1}^{n}|df^i|^2\right)^{n/2} &\leq \left(\sum_{i=1}^{n-1}|df^i|^2 + \frac{\nu_4^2}{n-1}\sum_{i=1}^{n-1}|df^i|^2\right)^{n/2}\\ &\leq \left(1+\frac{\nu_4^2}{n-1}\right)^{n/2}\frac{1}{\nu_3}(n-1)^{n/2}|df^1\wedge\ldots\wedge df^n|,\end{aligned}$$

which implies (6.1) with the constant

$$K_O = (n-1+\nu_4^2)^{n/2}n^{-n/2}\nu_3^{-1}.$$

□

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