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HARMONIC MAPS FROM $\mathbb{R}^n$ TO $\mathbb{H}^m$ WITH SYMMETRY

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It is known that there is no nonconstant bounded harmonic map from the Euclidean space $\mathbb{R}^n$ to the hyperbolic space $\mathbb{H}^m$. This is a particular case of a result of S.-Y. Cheng. However, there are many polynomial growth harmonic maps from $\mathbb{R}^2$ to $\mathbb{H}^2$ by the results of Z. Han, L.-F. Tam, A. Treibergs and T. Wan. One of the purposes of this paper is to construct harmonic maps from $\mathbb{R}^n$ to $\mathbb{H}^m$ by prescribing boundary data at infinity. The boundary data is assumed to satisfy some symmetric properties. On the other hand, it was proved by Han-Tam-Treibergs-Wan that under some reasonable assumptions, the image of a harmonic diffeomorphism from $\mathbb{R}^2$ into $\mathbb{H}^2$ is an ideal polygon with $n + 2$ vertices on the geometric boundary of $\mathbb{H}^2$ if and only if its Hopf differential is of the form ϕdz^2 where ϕ is a polynomial of degree n. It is unclear whether one can find explicit relation between the coefficients of ϕ and the vertices of the image of the harmonic map. The second purpose of this paper is to investigate this problem. We will explicitly demonstrate some families of polynomial holomorphic quadratic differentials, such that the harmonic maps from $\mathbb{R}^2$ into $\mathbb{H}^2$ with Hopf differentials in the same family will have the same image. In proving this, we first study the asymptotic behaviors of harmonic maps from $\mathbb{R}^2$ into $\mathbb{H}^2$ with polynomial Hopf differentials ϕdz^2. The result may have independent interest.

0. Introduction.

Let $\mathbb{R}^n$ be the Euclidean space, and $\mathbb{H}^n$ be the hyperbolic space. In [**HTTW**], it was proved that under some reasonable assumptions, the image of a harmonic diffeomorphism from $\mathbb{R}^2$ into $\mathbb{H}^2$ is an ideal polygon with $n+2$ vertices on the geometric boundary of $\mathbb{H}^2$ if and only if its Hopf differential is of the form ϕdz^2 where ϕ is a polynomial of degree n. Note that ϕ is a polynomial of degree n if and only if the harmonic map is of polynomial growth of order $\frac{n}{2}+1$, see [**TW**] for example. In [**LW**], it is shown that the closure of the image of a harmonic map from $\mathbb{R}^n$ into $\mathbb{H}^m$ with polynomial growth of order l will intersect the geometric boundary of $\mathbb{H}^m$ at no more than Cl^{n-1} points, where C is a constant independent of l. Moreover, the image lies in the convex hull of these points. In higher dimensions, unlike harmonic maps from hyperbolic space to hyperbolic space, there are very few examples of nontrivial harmonic maps from $\mathbb{R}^n$ into $\mathbb{H}^m$. In fact, if the image of a harmonic map from $\mathbb{R}^n$ to $\mathbb{H}^m$ is bounded, then the harmonic map must be constant [**Cg**]. Also, there is no rotationally symmetric harmonic map from $\mathbb{R}^n$ into $\mathbb{H}^n$ [**T**]. On the other hand, in [**WA**], (see also [**TW**]), it was shown that orientation preserving harmonic diffeomorphisms from $\mathbb{R}^2$ into $\mathbb{H}^2$ can be parametrized by their Hopf differentials, provided that the harmonic diffeomorphisms satisfy some natural conditions. In particular, one can construct harmonic diffeomorphisms from $\mathbb{R}^2$ to $\mathbb{H}^2$ with prescribed Hopf differentials. In [**HTTW**], harmonic diffeomorphisms with prescribed images had been constructed via the Gauss maps of constant mean curvature cuts in Minkowski three space. Both methods of constructions cannot be applied to higher dimensions. In this paper, we will use a more direct method to construct harmonic maps from $\mathbb{R}^n$ to $\mathbb{H}^m$ with prescribed boundary data at infinity. The boundary data is assumed to satisfy some symmetric properties. It should be remarked that if u is a harmonic map from $\mathbb{R}^2$ into $\mathbb{H}^{m-1}$ then u can be considered as a harmonic map from $\mathbb{R}^2$ into $\mathbb{H}^m$ by embedding $\mathbb{H}^{m-1}$ into $\mathbb{H}^m$. Also if u is a harmonic map from $\mathbb{R}^{n-1}$ into $\mathbb{H}^m$, then the map v from $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ into $\mathbb{H}^m$ defined by $v(x,t) = u(x)$ for $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ is harmonic. The harmonic maps we are going to construct are not in these categories, and are said to be nontrivial. Each of the constructed harmonic maps has polynomial growth, and the closure of its image in $\mathbb{H}^m \cup \partial\mathbb{H}^m$ intersects $\partial\mathbb{H}^m$ at finitely many points, where $\partial\mathbb{H}^m$ is the geometric boundary of $\mathbb{H}^m$. This can be considered as the first step to understand boundary value problem for harmonic maps from $\mathbb{R}^n$ into $\mathbb{H}^m$. The idea of construction is to find an approximate initial map with symmetry. Using the symmetry of the initial map, one can construct a harmonic map by compact exhaustion. The resulting harmonic map will be of bounded distance from the initial map.

In [**HTTW**], it was proved that if u is a harmonic diffeomorphism from $\mathbb{R}^2$ onto an ideal polygon with m vertices on $\partial\mathbb{H}^2$, then its Hopf differential is ϕdz^2 with ϕ to be a polynomial of degree $m-2$. However, it is unclear whether it is possible to find explicit relation between the coefficients of ϕ and these m points. The second purpose of this paper is to investigate this problem. We will explicitly demonstrate some families of polynomial holomorphic quadratic differentials, such that the harmonic maps from $\mathbb{R}^2$ into $\mathbb{H}^2$ with Hopf differentials in the same family will have the same image. In proving this, one needs to study asymptotic behaviors of harmonic maps from $\mathbb{R}^2$ into $\mathbb{H}^2$ with polynomial Hopf differentials. Some results in this direction had been obtained in [**HTTW**], using the techniques introduced in [**Wf**] and [**My**]. We will prove that if ϕ is of degree n, then there are $n+2$ rays, with equal angle between them, so that if u is a an orientation preserving harmonic map from $\mathbb{R}^2$ into $\mathbb{H}^2$ with Hopf differential ϕdz^2, then $u(z)$ will tend to infinity as $z \to \infty$ at the same rate along these rays. The result has its own interest and may be useful in the construction of harmonic maps $\mathbb{R}^2$ into $\mathbb{H}^2$ with prescribed data at infinity.

The structure of the paper is as follows. In §1, we will construct harmonic maps with symmetry from $\mathbb{R}^2$ to $\mathbb{H}^2$. In §2, we will use induction to construct nontrivial harmonic maps from $\mathbb{R}^2$ to $\mathbb{H}^m$, $m \geq 3$, and in §3, we will construct nontrivial harmonic maps from $\mathbb{R}^m$ to $\mathbb{H}^m$. In §4, we will study asymptotic behaviors of harmonic maps. In §5, we obtain some partial results on the explicit relation between the Hopf differential and the image of a harmonic map.

1. Harmonic maps from $\mathbb{R}^2$ to $\mathbb{H}^2$.

It was proved in [**WA**] (see also [**TW**]) that given a holomorphic quadratic differential $\phi(z)dz^2$ on $\mathbb{C}$, one can find a harmonic diffeomorphism from $\mathbb{C}$ into $\mathbb{H}^2$ such that the Hopf differential of the harmonic map is $\phi(z)dz^2$. Under certain conditions, the harmonic map is essentially unique. In particular, if $\phi(z) = z^m$, $m \geq 1$, using the result in [**HTTW**], one should be able to prove that up to an isometry of $\mathbb{H}^2$, the image is a regular ideal polygon of $m+2$ sides, see §5 for details. However, the method cannot be applied to higher dimensions. In this section, we will use another method to construct such harmonic maps. Using similar methods we will construct nontrivial harmonic maps with symmetry from $\mathbb{R}^2$ into $\mathbb{H}^m$, and $\mathbb{R}^m$ into $\mathbb{H}^m$, with $m \geq 2$ in the next two sections.

Let $n \geq 3$ be an integer. In $\mathbb{R}^2$, using polar coordinates the harmonic function

$$f(z) = f(re^{\sqrt{-1}\theta}) = r^{\frac{n}{2}} \sin\left(\frac{n}{2}\theta\right)$$

is zero on the rays $\theta = \theta_k$, where $0 \leq k \leq n-1$, where $\theta_k = \frac{2k\pi}{n}$, and $|f|$ is positive on $\theta_k < \theta < \theta_{k+1}$. Note that the ray $\theta = \theta_0$ is the same as the ray $\theta = \theta_n$. For each k, let W_k be the wedge defined by $\theta_k \leq \theta \leq \theta_{k+1}$.

Let us use the Poincaré disk model for $\mathbb{H}^2$. Let $a_k = e^{\frac{(2k+1)\pi\sqrt{-1}}{n}}$, $k = 0, ..., n-1$, which are identified as points on the geometric boundary of $\mathbb{H}^2$. Let o be the origin of the unit disk $\mathbb{D}$, and let γ_k be the geodesic from o to a_k in $\mathbb{H}^2$, parametrized by arc length. Define a map $g : \mathbb{R}^2 \to \mathbb{H}^2$ as follows. In the wedge $\theta_k \leq \theta \leq \theta_{k+1}$, let

$$g(z) = \gamma_k(|f(z)|).$$

Since $f = 0$ on each ray $\{\theta = \theta_k\}$, g is well-defined. g satisfies the following properties:

(i) g is a Lipschitz map, which is smooth and harmonic in the interior of each wedge W_k.

(ii) For any $z \in \mathbb{C}$, $g(e^{2\sqrt{-1}\theta_k}\bar{z}) = e^{2\sqrt{-1}\theta_k}\overline{g(z)}$.
(iii) $g(e^{\sqrt{-1}\theta_1}z) = e^{\sqrt{-1}\theta_1}g(z)$.

Lemma 1.1. *For any $R > 0$, let u_R be the harmonic map from $B(R)$ into $\mathbb{H}^2$, where $B(R)$ is the disk of radius R with center at the origin in $\mathbb{R}^2$, such that $u_R = g$ on $\partial B(R)$. Then there is a constant C_1 which is independent of R, such that*

$$d(u_R(z), g(z)) \leq C_1$$

for all R and for all $z \in B(R)$.

Proof. By (iii) and the uniqueness of harmonic maps, we have

$$u_R(e^{\sqrt{-1}\theta_1}z) = e^{\sqrt{-1}\theta_1}u_R(z).$$

Hence it is sufficient to prove that

$$d(u_R(z), g(z)) \leq C_1$$

for all $z \in B(R) \cap W_0$, where W_0 is the wedge defined above. By the definition of u_R,

$$u_R(z) = g(z) \tag{1.1}$$

for $z \in \partial B(R) \cap W_0$. We want to show that $d(u_R(z), g(z))$ is bounded on $\partial W_0 \cap B(R)$ by a constant independent of R. Since W_0 is bounded by two rays $\theta = \theta_0$, $\theta = \theta_1$, by symmetry it is sufficient to prove that $d(u_R(z), g(z))$ is uniformly bounded on $\{\theta = \theta_0\} \cap B(R)$. By (ii), $g(z) = \overline{g(\bar{z})}$. Hence by the uniqueness theorem on harmonic maps, we have $u_R(z) = \overline{u_R(\bar{z})}$. This implies that $u_R(z)$ lies on the real axis, for all $z \in \{\theta = \theta_0\} \cap B(R)$. Observe that the image of u_R lies inside the convex hull A of the ideal boundary points a_k, $0 \leq k \leq n-1$, and the closure of A in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ intersects $\partial\mathbb{H}^2$ at the points a_k. Suppose n is even, then no a_k is on the real axis. Hence there is a constant C_2 independent of R, such that

$$\begin{aligned} d(u_R(z), g(z)) &= d(u_R(z), o) \\ &\leq C_2 \end{aligned} \tag{1.2}$$

for all $z \in \{\theta = \theta_0\} \cap B(R)$, see Figure 1. Suppose n is odd, we want to show that $u_R(z)$ lies on the positive real axis, for all $z \in \{\theta = \theta_0\} \cap B(R)$. This will imply that (1.2) is still true in this case, because no a_k is on the positive real axis. By the definition of g, we see that g maps the upper half space into the that part of $\mathbb{D}^2$ which lies on the upper half space. Since $u_R(z)$ lies on the real axis if z is real, u_R also maps the upper half space into the that part of $\mathbb{D}^2$ which lies on the upper half space. One can prove similarly that u_R maps the half space bounded by the rays $\theta = \frac{2\pi}{n}$ and $\theta = \pi + \frac{2\pi}{n}$ which containing the positive real axis into the same half space, see Figure 2. In particular, $u_R(z)$ lies on the positive real axis, for all $z \in \{\theta = \theta_0\} \cap B(R)$. So

(1.2) is true for all $z \in \partial(B(R) \cap W_0)$. Since $d(u_R(z), g(z))$ is subharmonic, the lemma follows from the maximum principle.

By Lemma 1.1, passing to a subsequence if necessary, u_R will converge to a harmonic map u such that $d(u(z), g(z))$ is uniformly bounded. In fact, u is a diffeomorphism. We can prove this fact as follows. For each $R > 0$, let us construct a harmonic map v_R from $B(R)$ into $\mathbb{H}^2$ in the following way. Let $b_k = \gamma_k(R^{\frac{n}{2}})$ and let β_k be the minimal geodesic joining b_k to b_{k-1}. Let α_k be the minimal geodesic joining a_k to a_{k-1}. It is easy to see that the distance from a point on γ_k or γ_{k-1} to α_k is bounded by a constant C_3 which is independent of R. Define a map Π_k from $\gamma_k|[0, b_k]$ and $\gamma_{k-1}|[0, b_{k-1}]$ into the line containing β_k, by nearest point projection. Then

$$d(\gamma_k(s), \Pi_k(\gamma_k(s))) \leq C_1. \tag{1.3}$$

Π_k is surjective and is continuous. Let v_R be the harmonic map from $B(R)$ into $\mathbb{H}^2$, such that on the $\partial B(R) \cap W_k$ $v_R(z) = \Pi_k(g(z))$. Note that the boundary map is a homeomorphism from $\partial B(R)$ onto the boundary of the geodesic polygon with boundary $\cup_k \beta_k$. Here are some properties of v_R. By [**SY**], we have:

Lemma 1.2. *v_R is a diffeomorphism onto its image.*

By Lemma 1.1, and (1.3), there is a constant C_4 which is independent of R such that

$$\sup_{x \in B(R)} d(v_R(z), g(z)) \leq C_4.$$

Hence, passing to a subsequence, v_R converge to a harmonic map v, such that

$$d(v(z), g(z)) \leq C_4. \tag{1.4}$$

Lemma 1.3. *Let ϕdz^2 be the Hopf differential of v. Then ϕ is a polynomial of degree $n-2$.*

Proof. By the construction,

$$d(o, g(z)) \leq |z|^{\frac{n}{2}}.$$

By (1.4), we see that

$$d(o, v(z)) \leq C_4 + |z|^{\frac{n}{2}}.$$

By the energy density estimate [**Cg**], there is a constant C_5 independent of z such that

$$e(u)(z) \leq C_5(|z|^{n-2} + 1).$$

Since $|\phi|(z) \leq e(v)$, we conclude that ϕ is a polynomial of degree at most $n-2$. Suppose the degree of ϕ is less than or equal to $n-3$. Let $\phi_R dz^2$ be the Hopf differential of v_R. Then given any $R_0 > 0$ there is R_1 such that if $R > R_1$, then

$$|\phi_R(z)| \leq C_6(|z|^{n-3} + 1),$$

in $B(R_0)$ for some constant C_6 which is independent of R_0, where ϕ_R is the Hopf differential of v_R. Using an argument of [**TW**], we conclude that in $B(\frac{R_0}{2})$,

$$e(v_R)(z) \le C_7(|z|^{n-3} + 1)$$

for some constant C_7 independent of R_0, if R is large enough. Let $R \to \infty$, and then let $R_0 \to \infty$, we have

$$e(v)(z) \le C_7(|z|^{n-3} + 1).$$

This would imply

$$d(o, v(z)) \le C_8(|z|^{(n-1)/2} + 1)$$

for some constant C_8. By (1.4), and the definition of g, this is impossible. Hence the degree of ϕ must be $n - 2$.

Lemma 1.4. *v is a diffeomorphism onto its image.*

Proof. Since the Jacobian J_R of v_R is positive in $B(R)$, the Jacobian J of v satisfies $J \ge 0$. First we want to show that $J > 0$ somewhere. Suppose not, then $J \equiv 0$. Since $J = ||\partial v||^2 - ||\bar\partial v||^2$, where $||\partial v|| = \sigma|\frac{\partial v}{\partial z}|$, and $||\bar\partial v|| = \sigma|\frac{\partial v}{\partial \bar z}|$, $\sigma^2|dv|^2$ is the metric on $\mathbb{H}^2$. we have

$$||\partial v||^2 \equiv ||\bar\partial v||^2.$$

On the other hand,

$$|\phi|^2 = ||\partial v||^2 \cdot ||\bar\partial v||^2.$$

We have

$$|\phi| = ||\partial v||^2.$$

Since ϕ is a polynomial of degree $n-2$, there is $R_0 > 0$ such that all the zeros of ϕ lies inside $B(\frac{R_0}{2})$. For each R, $||\partial v_R|| > 0$, and let $w_R = \log ||\partial v_R||$. Then

$$\Delta w_R = J_R(u_R).$$

We have

$$\begin{aligned}\int_{\partial B(R_0)} \frac{\partial w_R}{\partial r} &= \int_{B(R_0)} \Delta w_R \\ &= \int_{B(R_0)} J_R.\end{aligned}$$

Since on $\partial B(R_0)$, $||\partial v||^2 = |\phi| > 0$, let $R \to \infty$, we have

$$\int_{\partial B(R_0)} \frac{\partial w}{\partial r} = \int_{B(R_0)} J.$$

However, $w = \frac{1}{2}\log|\phi|$, and the degree of ϕ is at least 1, moreover, all zeros of ϕ lie inside $B(R_0)$, we conclude that

$$\int_{B(R_0)} J > 0. \tag{1.5}$$

Hence $J > 0$ somewhere, and (1.5) is true for some $R_0 > 0$. This implies that there is $\delta > 0$ such that if R is large then

$$\int_{B(R_0)} J_R \geq \delta. \tag{1.6}$$

Apply Theorem 7.1 in [**J**] to each map v_R, we conclude that for any R_1, there is $\epsilon > 0$, such that

$$J_R(v) \geq \epsilon > 0$$

in $B(R_1)$ provided R is large enough. This implies $J(v) > 0$ everywhere and v is a diffeomorphism onto its image.

Since $d(v(z), u(z))$ is uniformly bounded and subharmonic, $d(v(z), u(z))$ is a constant function. It is easy to see that $v(0) = u(0)$, and so $u \equiv v$. On the other hand, since $||\partial u||^2 \geq |\phi|$ and ϕ is a polynomial, we see that $||\partial u||^2 dz^2$ is complete. By the result of [**HTTW**], the image of u is a ideal polygon of n sides and so the image of u is the polygon spanned by the $a_k' s$, and we have the following:

Theorem 1.5. *Let $n \geq 3$, and let $a_k = e^{\frac{(2k+1)\pi\sqrt{-1}}{n}}$, $k = 0, \ldots, n-1$. Then there is a harmonic diffeomorphism u from $\mathbb{R}^2$ into $\mathbb{H}^2$ whose image is the ideal polygon spanned by the a_k's. Moreover, u satisfies*

$$u(e^{2\sqrt{-1}\theta_k}\bar{z}) = e^{2\sqrt{-1}\theta_k}\overline{u(z)}$$

and

$$u(e^{i\theta_1}z) = e^{\sqrt{-1}\theta_1}u(z).$$

In case of $n = 4$, we can do more. Let a_k, $1 \leq k \leq 4$ be four points on the unit circle, such that they are the vertices of a rectangle which is symmetric with respect to the real and imaginary axes.

Proposition 1.6. *There is a harmonic diffeomorphism from $\mathbb{R}^2$ into $\mathbb{H}^2$ whose image is the ideal polygon spanned by the a_k's. Moreover, u satisfies*

$$u(\bar{z}) = \overline{u(z)},$$

and

$$u(-\bar{z}) = -\overline{u(z)}.$$

The proof is similar to the proof of Theorem 1.5. We should remark that for any four points on the unit circle, there is a conformal map of the unit disk, which carries these four points to some a_k's satisfying the condition of Proposition 1.6.

2. Harmonic maps from $\mathbb{R}^2$ into $\mathbb{H}^m$.

In this section, we will use the harmonic maps constructed in §1 to obtain harmonic maps from $\mathbb{R}^2 = \mathbb{C}$ into $\mathbb{H}^m$, which are nontrivial in the sense that the image of each of the maps is not contained in any nontrivial totally geodesic submanifold in $\mathbb{H}^m$. We always use the Poincaré unit ball model for $\mathbb{H}^m$. Namely, $\mathbb{H}^m$ is identified with the unit ball $\mathbb{B}^m$ in $\mathbb{R}^m$ with the Poincaré metric, and the geometric boundary $\partial\mathbb{H}^m$ is identified with the unit sphere $\mathbb{S}^{m-1}$. For any set A in $\mathbb{H}^m \cup \partial\mathbb{H}^m$, we denote $\overline{A}$ to be the closure of A in $\mathbb{H}^m \cup \partial\mathbb{H}^m$, and denote the convex hull of A by $\mathrm{Con}\,(A)$. We will use the following fact: Suppose A is a close set in $\mathbb{H}^m \cup \partial\mathbb{H}^m$, then $\overline{\mathrm{Con}\,(A)} \cap \partial\mathbb{H}^m = A \cap \partial\mathbb{H}^m$.

Let $n \geq 4$ be an even number. Let $\theta_k = \frac{2k\pi}{n}$ and let W_k be the wedge in $\mathbb{R}^2$ defined by $\theta_k \leq \theta \leq \theta_{k+1}$ in polar coordinates. Note that $\theta_k = k\theta_1$. By Theorem 1.5, we can find a harmonic diffeomorphism u from $\mathbb{C} = \mathbb{R}^2$ into $\mathbb{H}^2$, such that:

(a) In the Poincaré disk model of $\mathbb{H}^2$, if we write
$$u(z) = (u^1(z), u^2(z)),$$
then $u^1(z) = 0$ on $\Im(z) = 0$, where $\Im(z)$ is the imaginary part of z;

(b) $\overline{u(\mathbb{R}^2)} \cap \partial\mathbb{H}^2$ does not contain the points $(0, \pm 1)$.

From (a) and (b), we have

(c) $\sup_{z\in\mathbb{R}^2,\ \Im(z)=0} d(u(z), 0) < \infty$.

From (b), we also have:

(b′) If $(a^1, a^2) \in \overline{u(\mathbb{R}^2)} \cap \partial\mathbb{H}^2$, then $a^1 \neq 0$.

We are going to use u to construct a harmonic map from $\mathbb{R}^2$ into $\mathbb{H}^3$. Identify $\mathbb{H}^2$ with $\{(v^1, v^2, v^3) \in \mathbb{H}^3 |\ v^2 = 0\}$. Then $u : \mathbb{R}^2 \to \mathbb{H}^2 \subset \mathbb{H}^3$ is also harmonic, and

$$u(z) = (u^1(z), 0, u^2(z)). \tag{2.1}$$

Define a harmonic map v from W_0 into $\mathbb{H}^3$ in the following way, see Figure 3. Let
$$\Psi : \{z \in \mathbb{C}|\ \Im(z) > 0\} \to \text{ interior of } W_0,$$
be a conformal diffeomorphism, $\Psi(\{\Im(z) = 0\}) = \partial W_0$ and Ψ is homeomorphism between $\Im(z) \geq 0$ and W_0 . Let $v(z) = u \circ \Psi^{-1}(z)$. Then v is a harmonic map from W_0 into $\mathbb{H}^3$, such that:

(i) $v(z) = (v^1(z), 0, v^3(z))$;

(ii) $v(z) = (0, 0, v^3(z))$ for $z \in \partial W_0$;

(iii) $\sup_{z\in\partial W_0} d(v(z), 0) < \infty$;

(iv) v is continuous up to the boundary of W_0

(v) suppose $(a^1, a^2, a^3) \in \overline{v(W_0)} \cap \partial\mathbb{H}^3$, $a^1 \neq 0$.

Property (v) follows from property (b′) of u and the fact that if $(a^1,a^2,a^3)\in \overline{v(W_0)}\cap\partial\mathbb{H}^3$ then $a^2=0$ and $(a^1,a^3)\in\overline{u(\mathbb{R}^2)}\cap\partial\mathbb{H}^2$.

Define g as follows, see Figure 4. Let us write any point $v=(v^1,v^2,v^3)$ of $\mathbb{H}^3$ in the form $(v^1+\sqrt{-1}v^2,v^3)$. Let $g(z)=v(z)$ for $z\in W_0$. Suppose we have defined $g=(g^1,g^2,g^3)=(g^1+\sqrt{-1}g^2,g^3)$ on W_k, $0\le k<n-1$, then for $z\in W_{k+1}$, let

$$\begin{aligned} g(z)&=\Big(e^{2\sqrt{-1}(\theta_{k+1}-\frac{\pi}{n})}(g^1-\sqrt{-1}g^2)(\hat z),g^3(\hat z)\Big)\\ &=\Big(e^{2\sqrt{-1}(k+\frac12)\theta_1}(g^1-\sqrt{-1}g^2)(\hat z),g^3(\hat z)\Big)\end{aligned}\tag{2.2}$$

here $\hat z=e^{2\sqrt{-1}\theta_{k+1}}\bar z$ which is in W_k. Here we simply ‘reflect’ g along the ray $\theta=\theta_{k+1}$ in the domain, and $\theta=\theta_{k+1}-\frac{\pi}{n}=(k+\frac12)\theta_1$ in the target. Then g is harmonic on the interior of each W_k. Suppose n is even, then g^3 is a well-defined and continuous function on $\mathbb{R}^2$, and since $g=(0,0,g^3)$ on ∂W_k for all k, g is well-defined and continuous.

Lemma 2.1. *Suppose n is even, and n is not a multiple of* 4*. Then the map g defined above satisfies:*

- (i) $g(z)=(e^{2\sqrt{-1}(k-\frac12)\theta_1}(g^1-\sqrt{-1}g^2)(\hat z),g^3(\hat z))$*, where* $\hat z=e^{2\sqrt{-1}\theta_k}\bar z$*, for all z and* $0\le k\le n-1$*;*
- (ii) $\sup_{z\in\partial W_k}d(g(z),0)<\infty$*, for* $0\le k\le n-1$*; and*
- (iii) *suppose* $(a^1,a^2,a^3)\in\overline{g(\mathbb{R}^2)}\cap\partial\mathbb{H}^3$*, then* $a^1\ne0$*, and* $\arg(a^1+\sqrt{-1}a^2)=\theta_k$ *or* $\theta_k+\pi$*, for some* $0\le k\le n-1$*.*

Proof. Let $z_0\in W_0$, define z_s inductively by

$$z_{s+1}=e^{2\sqrt{-1}(s+1)\theta_1}\bar z_s,$$

for $s=0,\dots,n-1$. Then $z_s\in W_s$. Suppose $s=2l$, then

$$z_s=e^{2\sqrt{-1}l\theta_1}z_0\tag{2.3}$$

and

$$\begin{aligned} g(z_s)&=(e^{2\sqrt{-1}l\theta_1}(g^1+\sqrt{-1}g^2)(z_0),g^3(z_0))\\ &=(e^{\sqrt{-1}s\theta_1}(g^1+\sqrt{-1}g^2)(z_0),g^3(z_0)).\end{aligned}\tag{2.4}$$

If $s=2l+1$, then

$$\begin{aligned} z_s&=\hat z_{2l}\\ &=e^{2\sqrt{-1}s\theta_1}\bar z_{2l}\\ &=e^{2\sqrt{-1}(l+1)\theta_1}\bar z_0,\end{aligned}\tag{2.5}$$

and

$$
\begin{aligned}
g(z_s) &= g(\hat{z}_{2l}) \\
&= \left(e^{2\sqrt{-1}(s-\frac{1}{2})\theta_1}(g^1 - \sqrt{-1}g^2)(z_{2l}), g^3(z_{2l})\right) \\
&= \left(e^{2\sqrt{-1}(l+\frac{1}{2})\theta_1}(g^1 - \sqrt{-1}g^2)(z_0), g^3(z_0)\right) \\
&= \left(e^{\sqrt{-1}s\theta_1}(g^1 - \sqrt{-1}g^2)(z_0), g^3(z_0)\right).
\end{aligned} \tag{2.6}
$$

Hence $z_n = z_0$, and $z_{n-1} = \bar{z}_0$, because n is even, and

$$
g(\bar{z}_0) = (e^{-\sqrt{-1}\theta_1}(g^1 - \sqrt{-1}g^2)(z_0), g^3(z_0)). \tag{2.7}
$$

Now suppose $z = \rho e^{i\alpha}$ where $\theta_m \le \alpha < \theta_{m+1}$ for some $0 \le m \le n-1$. Then there exists $z_0 = \rho e^{i\alpha_0}$ with $0 \le \alpha_0 < \theta_1$, such that $z_m = z$. If $m = 2p$, then

$$
\begin{aligned}
\hat{z} &= e^{2\sqrt{-1}\theta_k}\bar{z} \\
&= e^{2\sqrt{-1}(k-p)\theta_1}\bar{z}_0.
\end{aligned}
$$

Without loss of generality, we may assume that $0 \le 2(k-p) \le n-1$. If $k - p = 0$, then, apply (2.4) to $g(\bar{z}_0) = g(\hat{z})$ and (2.7) to $g(\hat{z})$, we have

$$
\begin{aligned}
g(z) &= \left(e^{2\sqrt{-1}p\theta_1}(g^1 + \sqrt{-1}g^2)(z_0), g^3(z_0)\right) \\
&= \left(e^{2\sqrt{-1}k\theta_1}(g^1 + \sqrt{-1}g^2)(z_0), g^3(z_0)\right) \\
&= \left(e^{2\sqrt{-1}(k-\frac{1}{2})\theta_1}(g^1 - \sqrt{-1}g^2)(\bar{z}_0), g^3(\bar{z}_0)\right) \\
&= \left(e^{2\sqrt{-1}(k-\frac{1}{2})\theta_1}(g^1 - \sqrt{-1}g^2)(\hat{z}), g^3(\hat{z})\right).
\end{aligned}
$$

So (i) is true in this case. Suppose $k - p = l + 1$, with $l \ge 0$, then we can apply (2.5) and (2.6)

$$
\begin{aligned}
g(z) &= \left(e^{2\sqrt{-1}p\theta_1}(g^1(z_0) + \sqrt{-1}g^2(z_0)), g^3(z_0)\right) \\
&= \left(e^{2\sqrt{-1}(p+k-p-1+\frac{1}{2})\theta_1}(g^1(\hat{z}) - \sqrt{-1}g^2(\hat{z})), g^3(\hat{z})\right) \\
&= \left(e^{2\sqrt{-1}(k-\frac{1}{2})\theta_1}(g^1(\hat{z}) - \sqrt{-1}g^2(\hat{z})), g^3(\hat{z})\right).
\end{aligned}
$$

Then (i) is still true. The case that $m = 2p + 1$ can be proved similarly. The proof of (i) is completed. (ii) can be derived from the definition of g and property (iii) of v. To prove (iii), let $(a^1, a^2, a^3) \in \overline{g(\mathbb{R}^2)} \cap \partial\mathbb{H}^3$, then $(a^1, a^2, a^3) \in \overline{g(W_k)} \cap \partial\mathbb{H}^3$, for some $0 \le k \le n-1$. Since $g = v$ on W_0, by the definition of v and property (v) of v, if $(a^1, a^2, a^3) \in \overline{g(W_0)} \cap \partial\mathbb{H}^3$, then $a^2 = 0$, and $a^1 \ne 0$. In particular, $\arg(a^1 + \sqrt{-1}a^2) = \theta_0 = 0$ or π. Now suppose $(a^1, a^2, a^3) \in \overline{g(W_k)} \cap \partial\mathbb{H}^3$, for $1 \le k \le n-1$, then by (2.4), and (2.6), there is $(b^1, 0, b^3) \in \overline{g(W_0)} \cap \partial\mathbb{H}^3$, such that

$$a^1 + \sqrt{-1}a^2 = e^{\sqrt{-1}\theta_k} b^1.$$

Since n is not a multiple of 4, $e^{\sqrt{-1}\theta_k} \neq \pm i$, and since $b^1 \neq 0$ and is real, we have $a^1 \neq 0$. Moreover, $\arg(a^1 + \sqrt{-1}a^2) = \theta_k$ or $\theta_k + \pi$.

Theorem 2.2. *Let n and $g(z)$ be as Lemma* 2.1*. There exists a harmonic map h from $\mathbb{R}^2$ into $\mathbb{H}^3$, such that*

$$\sup_{z\in\mathbb{C}} d(h(z), g(z)) < \infty.$$

Moreover:

(a) *In the Poincaré ball model of $\mathbb{H}^3$, if $\Im(z) = 0$ and if we let*
$$h(z) = (h^1(z), h^2(z), h^3(z)),$$
then $\arg(h^1 + \sqrt{-1}h^2)(z) = -\frac{1}{2}\theta_1$ *or* $\pi - \frac{1}{2}\theta_1$*;*

(b) *suppose $(a^1, a^2, a^3) \in \overline{h(\mathbb{R}^2)} \cap \partial\mathbb{H}^3$, then $a^1 \neq 0$; and*

(c) $\sup_{z\in\mathbb{R}^2,\ \Im(z)=0} d(h(z), 0) < \infty$.

If, in addition, $\overline{u(\{\Im(z) \ge 0\})} \cap \partial\mathbb{H}^2$ is not contained in any straight line in the plane, then $\overline{h(\{\Im(z) \ge 0\})} \cap \partial\mathbb{H}^3$ is not contained in any hyperplane in $\mathbb{R}^3$. In particular, the image of h is not contained in any totally geodesic submanifold of dimension 2 *in $\mathbb{H}^3$.*

Proof. For any $R > 0$, let B_R be the disk of radius R with center at the origin in $\mathbb{R}^2$. Let h_R be the harmonic map from B_R into $\mathbb{H}^3$, such that $h_R = g$ on ∂B_R. If we write $h_R = (h_R^1, h_R^2, h_R^3) = (h_R^1 + \sqrt{-1}h_R^2, h_R^3)$, then by the uniqueness of harmonic maps and Lemma 2.1, we have

$$h_R(z) = \left(e^{2\sqrt{-1}(k-\frac{1}{2})\theta_1}(h_R^1 - \sqrt{-1}h_R^2)(\hat{z}), h_R^3(\hat{z})\right) \tag{2.8}$$

for any $z \in B_R$, where $\hat{z} = e^{2\sqrt{-1}\theta_k}\bar{z}$, $0 \le k \le n-1$. We want to show that there exists a constant C_1 independent of R such that

$$d(h_R(z), g(z)) \le C_1 \tag{2.9}$$

for all $z \in B_R$. Obviously, we only have to prove that (2.9) is true for all $z \in W_k \cap B_R$, for all $0 \le k \le n-1$. Let us consider W_0 for example. $\partial(W_0 \cap B_R)$ is the union of $W_0 \cap \partial B_R$, $\{\theta = 0\} \cap B_R$, and $\{\theta = \theta_1\} \cap B_R$. On $W_0 \cap \partial B_R$, $h_R = g$. On the other hand, for $z \in \{\theta = 0\} \cap B_R$, we have $z = \bar{z}$, and so by (2.8) with $k = 0$,

$$\begin{aligned} h_R(z) &= (e^{-\sqrt{-1}\theta_1}(h_R^1 - \sqrt{-1}h_R^2)(\bar{z}), h_R^3(\bar{z})) \\ &= (e^{-\sqrt{-1}\theta_1}(h_R^1 - \sqrt{-1}h_R^2)(z), h_R^3(z)). \end{aligned}$$

Hence $h_R(z) \in \Pi$ where Π is the plane $(v^1, v,^2, v^3) \in \mathbb{H}^3$, such that $\arg(v^1 + \sqrt{-1}v^2) = -\frac{1}{2}\theta_1$ or $\pi - \frac{1}{2}\theta_1$. Similarly, if z is in $\{\theta = \pi\} \cap B_R$, then $h_R(z)$ is also in Π. On the other hand, it is well-known that $u_R(B_R)$ is contained in the convex hull of $u_R(\partial B_R)$, which in turn is contained in the convex hull of $g(\mathbb{R}^2)$. Since $\overline{\mathrm{Con}\,(g(\mathbb{R}^2))} \cap \partial\mathbb{H}^3 = \overline{g(\mathbb{R}^2)} \cap \partial\mathbb{H}^3$, by Lemma 2.1 (iii) we conclude that if $(a^1, a^2, a^3) \in \overline{\mathrm{Con}\,(g(\mathbb{R}^2))} \cap \partial\mathbb{H}^3$, then $\arg(a^1 + \sqrt{-1}a^2) = \theta_k$ for some k, and $a^1 \neq 0$. However, by the definition of Π, if (a^1, a^2, a^3) is also in Π, then $\arg(a^1 + \sqrt{-1}a^2) = -\frac{1}{2}\theta_1$ or $\pi - \frac{1}{2}\theta_1$, which are not equal to θ_k modulo a multiple of 2π, because n is even. So

$$\Pi \cap \overline{\mathrm{Con}\,(g(\mathbb{R}^2))} \cap \partial\mathbb{H}^3 = \emptyset.$$

Since $h_R(z) \in \Pi$ for $z \in \{\theta = 0\} \cap B_R$, there exists a constant C_2 independent of R such that

$$d(h_R(z), 0) \leq C_2 \tag{2.10}$$

for $z \in \{\theta = 0\} \cap B_R$. By Lemma 2.1, there exists a constant C_3 independent of R such that for all $z \in \{\theta = 0\} \cap B_R$

$$d(g(z), 0) \leq C_3.$$

Combine this with (2.10), we have

$$d(h_R(z), g(z)) \leq C_2 + C_3$$

for all $z \in \{\theta = 0\} \cap B_R$. Similarly, one can prove that

$$d(h_R(z), g(z)) \leq C_4$$

for some constant C_4 independent of R, for all $z \in \{\theta = \theta_1\} \cap B_R$. Since g is harmonic on W_0, $d(h_R(z), g(z))$ is subharmonic on W_0. By the maximum principle, (2.9) is true on W_0. Similarly, (2.9) is true on W_k, for all k. By (2.9), passing to a subsequence if necessary, let $R \to \infty$, h_R converge to a harmonic map h from $\mathbb{R}^2$ to $\mathbb{H}^3$, such that

$$\sup_{z \in \mathbb{R}^2} d(h(z), g(z)) \leq C_1$$

for some constant C_1. In particular, $\overline{h(\mathbb{R}^2)} \cap \partial\mathbb{H}^3 = \overline{g(\mathbb{R}^2)} \cap \partial\mathbb{H}^3$. From this and Lemma 2.1, (b) follows. (c) follows from (2.9) and the property (ii) of g in Lemma 2.1. Since each h_R satisfies (a), so does h. It is well-known that a totally geodesic submanifold M is contained in a sphere or a hyperplane which intersects $\mathbb{S}^2$ orthogonally, see [**Sk**] for example. This implies that $\overline{M} \cap \partial\mathbb{H}^3$ is contained in a hyperplane. Hence, to prove the last statement, let us suppose $\overline{u(\{\Im(z) \geq 0\})} \cap \partial\mathbb{H}^2$ is not contained in any straight line in the plane, then it is sufficient to prove that the intersection of the closure of the image of h with $\partial\mathbb{H}^3$ is not contained in a hyperplane. By the construction of g, $g(W_0)$ consists of those points $(u^1(z), 0, u^2(z))$ with $\Im(z) > 0$. So

$$\overline{g(W_0)} \cap \partial\mathbb{H}^3 = \left\{ (v^1, 0, v^3) |\ (v^1, v^3) \in \overline{u(\{\Im(z) \geq 0\})} \cap \partial\mathbb{H}^2 \right\},$$

and the smallest affine subspace of $\mathbb{R}^3$ which contains $\overline{g(W_0)} \cap \partial\mathbb{H}^3$ is the subspace defined by $v^2 = 0$. By the definition of g,

$$\overline{g(W_1)} \cap \partial\mathbb{H}^3 = \left\{ (e^{\sqrt{-1}\theta_1}(v^1 - \sqrt{-1}v^2), v^3) |\ (v^1, v^2, v^3) \in \overline{g(W_0)} \cap \partial\mathbb{H}^3 \right\}.$$

Since $\theta_1 = \frac{2\pi}{n}$, $\overline{g(W_1)} \cap \partial\mathbb{H}^3$ is not contained in the subspace $v^2 = 0$. Since $W_0 \cup W_1$ is contained in $\Im(z) \geq 0$, we conclude that $\overline{g(\{\Im(z) \geq 0\})} \cap \partial\mathbb{H}^3$ is not contained in any hyperplane of $\mathbb{R}^3$. Using the fact that $d(h(z), g(z))$ is uniformly bounded from above, the same is true for h. From this, the last statement of the theorem follows.

By composing h with the isometry

$$(v^1 + \sqrt{-1}v^2, v^3) \to \left(e^{\frac{\sqrt{-1}}{2}(\theta_1 + \pi)}(v^1 + \sqrt{-1}v^2), v^3 \right)$$

on $\mathbb{H}^3$, we obtain a harmonic map u. Obviously, u also satisfies (c) of Theorem 2.2, (with h replaced by u). Also $u^1(z) = 0$ on $\Im(z) = 0$. Suppose $(a^1, a^2, a^3) \in \overline{u(\mathbb{R}^2)} \cap \partial\mathbb{H}^3$, then $a^1 + \sqrt{-1}a^2 = e^{\frac{\sqrt{-1}}{2}(\theta_1+\pi)}(b^1 + \sqrt{-1}b^2)$ for some $(b^1, b^2, b^3) \in \overline{h(\mathbb{R}^2)} \cap \partial\mathbb{H}^3$. From the proof we see that $b^1 + \sqrt{-1}b^2 = e^{\sqrt{-1}\theta_k}c$ for some $c \neq 0$, and for some $0 \leq k \leq n-1$. From this we conclude that $a^1 \neq 0$. Here we use the fact that n is even again.

We can proceed as before to use u to construct a harmonic map from $\mathbb{R}^2$ into $\mathbb{H}^4$. More precisely and more generally, suppose u is a harmonic map from $\mathbb{R}^2 \to \mathbb{H}^m$ for some $m \geq 2$, such that:

(a) In the Poincaré ball model of $\mathbb{H}^m$, if we write

$$u(z) = (u^1(z), u^2(z), \ldots, u^m(z)),$$

then $u^1(z) = 0$ on $\Im(z) = 0$;

(b) if $(a^1, \ldots, a^m) \in \overline{u(\mathbb{R}^2)} \cap \partial\mathbb{H}^m$ then $a^1 \neq 0$;

(c) $\sup_{z\in\mathbb{R}^2,\ \Im(z)=0} d(u(z), 0) < \infty$.

Let n be even, not divisible by 4, and defined θ_k, W_k, Ψ as before. Let $v(z) = u \circ \Psi^{-1}(z)$, for any $z \in W_0$. Define $g(z) = v(z)$ for any $z \in W_0$. Suppose we have already defined $g(z)$ on W_k, $0 \leq k \leq n-1$, then for any $z \in W_{k+1}$ define:

$$g(z) = \left(e^{2\sqrt{-1}(k+\frac{1}{2})\theta_1}(g^1 - \sqrt{-1}g^2)(\hat{z}), g^3(\hat{z}), \ldots, g^{m+1}(\hat{z}) \right)$$

here $\hat{z} = e^{2\sqrt{-1}\theta_{k+1}}\overline{z} \in W_k$. Using similar methods as in Theorem 2.2, we can prove:

Theorem 2.2′. *Let $g(z)$ be as above. There exists a harmonic map h from $\mathbb{R}^2$ into $\mathbb{H}^{m+1}$, such that*

$$\sup_{z\in\mathbb{C}} d(h(z), g(z)) < \infty.$$

Moreover:

(a) *In the Poincaré ball model of* $\mathbb{H}^{m+1}$*, if* $\Im(z) = 0$*, and if*

$$h(z) = (h^1(z), h^2(z), \ldots, h^{m+1}(z))$$

then $\arg(h^1 + \sqrt{-1}h^2)(z) = -\frac{1}{2}\theta_1$ *or* $\pi - \frac{1}{2}\theta_1$*;*

(b) *if* $(a^1, \ldots, a^{m+1}) \in \overline{h(\mathbb{R}^2)} \cap \partial\mathbb{H}^{m+1}$*, then* $a^1 \neq 0$*; and*

(c) $\sup_{z\in\mathbb{R}^2,\ \Im(z)=0} d(h(z), 0) < \infty$.

If, in addition, $\overline{u(\{\Im(z) \geq 0\})} \cap \partial\mathbb{H}^m$ *is not contained in any hyperplane in* $\mathbb{R}^m$*, then* $\overline{h(\{\Im(z) \geq 0\})} \cap \partial\mathbb{H}^{m+1}$ *is not contained in any hyperplane in* $\mathbb{R}^{m+1}$*. In particular, the image of* h *is not contained in any totally geodesic submanifold of dimension* m *in* $\mathbb{H}^{m+1}$*.*

Again by composing h with the isometry

$$(v^1 + \sqrt{-1}v^2, v^3, \ldots, v^{m+1}) \to (e^{\frac{\sqrt{-1}}{2}(\theta_1+\pi)}(v^1 + \sqrt{-1}v^2), v^3, \ldots, v^{m+1}),$$

we obtain a harmonic map from $\mathbb{R}^2$ into $\mathbb{H}^{m+1}$ satisfying required properties for the induction on construction.

Remark 2.1. (i) By the result in §1, it is easy to see that there are many harmonic maps u from $\mathbb{R}^2$ into $\mathbb{H}^2$, which satisfy the conditions in Theorem 2.2.

(ii) If we begin with a harmonic map u constructed in §1, and obtain harmonic maps inductively using Theorem 2.2, and 2.2′, then the harmonic maps will be of polynomial growth, and the closure of the image of each of the maps intersects the geometric boundary of the hyperbolic space at finitely many points. This is related to the results in [**LW**].

3. Harmonic maps from $\mathbb{R}^m$ into $\mathbb{H}^m$.

In this section, we will use methods similar to those in §1 and §2 to construct nontrivial harmonic maps from $\mathbb{R}^m$ into $\mathbb{H}^m$, $m \geq 3$. First let us write $\mathbb{R}^m = \mathbb{R}^2 \times \mathbb{R}^{m-2}$. As in the previous section, let $n \geq 4$ be an even integer, $\theta_k = \frac{2k\pi}{n}$, $0 \leq k \leq n-1$, and let W_k be the wedge in $\mathbb{R}^2$ defined by $\theta_k \leq \theta \leq \theta_{k+1}$ in polar coordinates. Let $\Omega_k = W_k \times [0,\infty)^{m-2}$ which consists of points $(x^1, x^2, \ldots, x^m)$ with $(x^1, x^2) \in W_k$ and $x^j \geq 0$, for $3 \leq j \leq m$. We use the Poincaré unit ball model for $\mathbb{H}^m$ as before. Define a harmonic function f by

$$f(x^1, x^2, x^3, \ldots, x^m) = r^{\frac{n}{2}} \sin\left(\frac{n}{2}\theta\right) x^3 \cdots x^m,$$

on Ω_k, $k = 0, 1, \ldots, n-1$, where $x^1 + \sqrt{-1}x^2 = re^{\sqrt{-1}\theta}$. Let $\gamma : [0,\infty) \mapsto \mathbb{H}^m$ be the geodesic parametrized by arc length, such that $\gamma(0) = 0$,

$$\gamma(t) = (\gamma^1(t), 0, \gamma^3(t), \ldots, \gamma^m(t))$$

$\gamma^i(t) \geq 0$, and $\lim_{t\to\infty} \gamma^m(t) = ((m-1)^{-\frac{1}{2}}, 0, (m-1)^{-\frac{1}{2}}, \dots, (m-1)^{-\frac{1}{2}})$. Define $v : \Omega_0 \mapsto \mathbb{H}^m$ by

$$v(x^1, \dots, x^m) = \gamma(f(x^1, \dots, x^m)).$$

By the definition of f, we see that v maps the boundary of Ω_0 to the origin 0 in $\mathbb{H}^m$. Let us write $v = (v^1, v^2, v^3, \dots, v^m)$ as $(v^1 + \sqrt{-1}v^2, v^3, \dots, v^m)$. Suppose we have already defined v on Ω_k for any $0 \leq k < n-1$, then, as before, for any $(x^1, x^2, \dots, x^m)$ in Ω_{k+1}, we set:

$$v(x^1, x^2, x^3, \dots, x^m) = (e^{2\theta_1\sqrt{-1}(k+1/2)}(v^1 - \sqrt{-1}v^2)(\hat{x}), v^3(\hat{x}), \dots, v^m(\hat{x})),$$

where $\hat{x} = (e^{2\sqrt{-1}\theta_{k+1}}(x^1 - \sqrt{-1}x^2), x^3, \dots, x^m)$ which is in $W_k \times [0,\infty)^{m-2}$. Thus, we have defined v on $\mathbb{R}^2 \times [0,\infty)^{m-2}$. Now, we can define $g : \mathbb{R}^m \to \mathbb{H}^m$ by setting:

$$g(x^1, x^2, x^3, \dots, x^m) = (v^1(\widetilde{x}), v^2(\widetilde{x}), \epsilon_3 v^3(\widetilde{x}), \dots, \epsilon_m v^m(\widetilde{x})),$$

for any $x \in \mathbb{R}^m$, where $\widetilde{x} = (x^1, x^2, |x^3|, \dots, |x^m|)$, and $\epsilon_i = \mathrm{sign}(x^i)$, $3 \leq i \leq m$, see Figure 5.

Lemma 3.1. *g is Lipschitz on $\mathbb{R}^m$, and is harmonic on the set* $\arg(x^1 + \sqrt{-1}x^2) \neq \theta_k$*, $0 \leq k \leq n-1$, $\tau^3 \cdots \tau^m \neq 0$. Moreover, if we write*

$$g = (g^1, g^2, g^3, \dots, g^m) = (g^1 + \sqrt{-1}g^2, g^3, \dots, g^m)$$

then:

(i)

$$g(x^1, x^2, \dots, x^m) = (e^{2\sqrt{-1}(k+\frac{1}{2})\theta_1}(g^1 - \sqrt{-1}g^2)(\hat{x}), g^3(\hat{x}), \dots, g^m(\hat{x}))$$

where $\hat{x} = (e^{2\sqrt{-1}\theta_{k+1}}(x^1 - \sqrt{-1}x^2), x^3 \dots, x^m)$;

(ii) *for $i \geq 3$*

$$g^i(x^1, x^2, x^3, \dots, -x^i, \dots, x^m) = -g^i(x^1, x^2, x^3, \dots, x^i, \dots, x^m);$$

and

(iii) *if $j \neq i$ with $i \geq 3$, then*

$$g^j(x^1, x^2, x^3, \dots, -x^i, \dots, x^m) = g^j(x^1, x^2, x^3, \dots, x^i, \dots, x^m).$$

Proof. The first statement of the lemma follows immediately from the definition of g, the fact that f is harmonic and that γ is a geodesic. The proof of (i) is similar to the proof of Lemma 2.1(i). (ii) and (iii) also follow immediately from the definition of g.

Theorem 3.2. *Let g be the map as above. Then there exists a harmonic map $u : \mathbb{R}^m \mapsto \mathbb{H}^m$ such that*

$$\sup_{x\in\mathbb{R}^m} d(u(x), g(x)) < \infty.$$

Moreover, u is nontrivial in the sense that:

(i) *The image of u is not contained in any totally geodesic submanifold of dimension $m-1$ in of $\mathbb{H}^m$; and*
(ii) *u cannot be decomposed as $u = F \circ G$, such that F is an isometry of $\mathbb{R}^m$, and $G = G(y^1, \dots, y^{m-1})$ which is independent of the last coordinate.*

Proof. Let B_R be the ball of radius R in $\mathbb{R}^m$ with center at the origin, and let u_R be the harmonic map from B_R to $\mathbb{H}^m$ with $u_R = g$ on ∂B_R. By Lemma 3.1, and the uniqueness of harmonic maps, if we write

$$u_R = (u_R^1, u_R^2, u_R^3, \dots, u_R^m) = (u_R^1 + \sqrt{-1}u_R^2, u_R^3, \dots, u_R^m)$$

then
(3.1)

$$u_R(x^1, x^2, \dots, x^m) = \left(e^{2\sqrt{-1}(k+\frac{1}{2})\theta_1}(u_R^1 - \sqrt{-1}u_R^2)(\hat{x}), u_R^3(\hat{x}), \dots, u_R^m(\hat{x})\right)$$

where $\hat{x} = (e^{2\sqrt{-1}\theta_{k+1}}(x^1 - \sqrt{-1}x^2), x^3 \dots, x^m)$; for $i \geq 3$

$$u_R^i(x^1, x^2, x^3, \dots, -x^i, \dots, x^m) = -u_R^i(x^1, x^2, x^3, \dots, x^i, \dots, x^m), \tag{3.2}$$

and if $j \neq i$, with $i \geq 3$,

$$u_R^j(x^1, x^2, x^3, \dots, -x^i, \dots, x^m) = u_R^j(x^1, x^2, x^3, \dots, x^i, \dots, x^m). \tag{3.3}$$

We want to prove that there is a constant C which is independent of R such that

$$\sup_{x \in B_R \cap \Omega_0} d(u_R(x), g(x)) \leq C. \tag{3.4}$$

Note that $\partial(B_R \cap \Omega_0) = (\partial B_R \cap \Omega_0) \cup (\partial\Omega_0 \cap B_R)$. On $\partial B_R \cap \Omega_0$, $u_R = g$. $\partial\Omega_0 \cap B_R$ consists of those points $(x^1, x^2, \dots, x^m) \in B_R$ such that $\arg(x^1 + \sqrt{-1}x^2) = \theta_0$ or θ_1. By (3.1), if $\arg(x^1 + \sqrt{-1}x^2) = 0$, then as in the proof of Theorem 2.2, we have $\arg(u_R^1(x) + \sqrt{-1}u_R^2(x)) = -\frac{1}{2}\theta_1$ or $\pi - \frac{1}{2}\theta_1$. By the definition of g, it is easy to see that if $(a^1, a^2, \dots, a^m) \in \overline{g(\mathbb{R}^m)} \cap \partial\mathbb{H}^m$, then there exists k, such that

$$a^1 + \sqrt{-1}a^2 = \frac{e^{i\theta_k}}{\sqrt{m-1}} \neq 0.$$

As in the proof of Theorem 2.2, we conclude that

$$\Pi \cap \overline{g(\mathbb{R}^m)} \cap \partial\mathbb{H}^m = \emptyset$$

where Π is the hyperplane $(v^1, v,^2, \dots, v^m)$, such that $\arg(v^1 + \sqrt{-1}v^2) = -\frac{1}{2}\theta_1$ or $\pi - \frac{1}{2}\theta_1$. Hence there is a constant C_1 which is independent of R such that

$$d(u_R(x), 0) \leq C_1$$

for all $x \in \partial\Omega_0 \cap B_R$ with $\arg(x^1 + \sqrt{-1}x^2) = \theta_0 = 0$. Note that for such x, $g(x) = 0$. Hence

$$d(u_R(x), g(x)) \leq C_1 \tag{3.5}$$

for all $x \in \partial\Omega_0 \cap B_R$ with $\arg(x^1 + \sqrt{-1}x^2) = \theta_0 = 0$. Similarly, one can show that (3.5) is true for $x \in \partial\Omega_0 \cap B_R$ with $\arg(x^1 + \sqrt{-1}x^2) = \theta_1$. By the maximum principle, we conclude that (3.4) is true. By Lemma 3.1 of g and (3.1)–(3.3), we see that

$$\sup_{x\in\mathbb{R}^m} d(u_R(x), g(x)) \leq C_1$$

for some constant C_1 which is independent of R. Passing to a subsequence if necessary, we can find a harmonic map u from $\mathbb{R}^m$ into $\mathbb{H}^m$ such that

$$\sup_{x\in\mathbb{R}^m} d(u(x), g(x)) \leq C_1. \tag{3.6}$$

From this we have

$$\overline{u(\mathbb{R}^m)} \cap \partial\mathbb{H}^m = \overline{g(\mathbb{R}^m)} \cap \partial\mathbb{H}^m.$$

The set on the right hand side contains all points of the form

$$\frac{1}{\sqrt{m-1}}(\cos\theta_k, \sin\theta_k, a^3, \dots, a^m)$$

for some k, where a^j is either $+1$ or -1, for $3 \leq j \leq m$. Hence the set cannot be contained in any hyperplane in $\mathbb{R}^m$. We conclude that $u(\mathbb{R}^m)$ is not contained in any totally geodesic submanifold of dimension $m-1$ in of $\mathbb{H}^m$. This proves (i). To prove (ii), we may assume that F is a linear isomorphism and it is sufficient to show that for any $(m-1)$ dimensional subspace $\mathbb{P}$ of $\mathbb{R}^m$, $\overline{u(\mathbb{P})} \cap \partial\mathbb{H}^m \neq \overline{u(\mathbb{R}^m)} \cap \partial\mathbb{H}^m$. By (3.6), it is sufficient to show that

$$\overline{g(\mathbb{P})} \cap \partial\mathbb{H}^m \neq \overline{g(\mathbb{R}^m)} \cap \partial\mathbb{H}^m. \tag{3.7}$$

Since $\mathbb{P}$ is a proper subspace, there is some fixed ϵ_i which is either $+1$ or-1, $3 \leq i \leq m$, and there is some k such that if

$$\Omega = \{(x^1, \dots, x^m) |\ (x^1, x^2, \epsilon_3 x^3, \dots, \epsilon_3 x^m) \in \Omega_k\}$$

then $\mathbb{P}$ will not intersect the interior of Ω. By the definition of g, we see that (3.7) is true.

Again the harmonic map u in the theorem is of polynomial growth, and the closure of its image intersects the geometric boundary at $n \times 2^{m-2}$ points.

4. Asymptotic behaviors of harmonic diffeomorphisms from $\mathbb{R}^2$ into $\mathbb{H}^2$.

In this section, we will discuss the asymptotic behavior of a harmonic diffeomorphism u from $\mathbb{R}^2$ into $\mathbb{H}^2$ with Hopf differential ϕdz^2 such that ϕ is a polynomial. It was proved in [**HTTW**] that the image of such a map is an ideal polygon. However, it is unclear how to determine the exact positions of the vertices of the polygon in terms of ϕ. On the other hand, it is also proved that each horizontal ray of ϕdz^2 is mapped under u into a curve which is asymptotically a geodesic ray in $\mathbb{H}^2$. In this section we want to show that the behavior of a harmonic diffeomorphism with Hopf differential $z^n dz^2$ is rather typical, in the sense that the image of the harmonic map along each ray in certain directions will tend to infinity at a rate depending only on n and the direction of the ray. While the results may have interest in their own right, they will be applied in the next section to study the relation between the Hopf differential and the image of a harmonic map from $\mathbb{R}^2$ to $\mathbb{H}^2$.

To fix notations, let u be an orientation preserving harmonic diffeomorphism from $\mathbb{R}^2$ into $\mathbb{H}^2$, so that its Hopf differential is of the form ϕdz^2 where

$$\phi(z) = z^n + \sum_{j=1}^{n} a_j z^{n-j} = z^n(1+h)$$

and

$$h(z) = \sum_{j=1}^{n} a_j z^{-j}.$$

Lemma 4.1. *Let θ be such that $\cos((\frac{n}{2}+1)\theta) \neq 0$, and let $L(T,\theta)$ be the length of $u(te^{i\theta})$, $0 \le t \le T$. We have:*

(a) *If $n = 2m$, then as $T \to \infty$*

$$\begin{aligned}\frac{1}{2}L(T,\theta) = \Bigg| & \frac{T^{m+1}}{m+1}\cos\left(\left(\frac{n}{2}+1\right)\theta\right) \\ & + \sum_{j=1}^{m} \frac{T^{m-j+1}}{m-j+1}\binom{\frac{1}{2}}{j}\Re\left(e^{\sqrt{-1}(m+1-j)\theta}c_j\right) \\ & + \binom{\frac{1}{2}}{m+1}\log T \cdot \Re(c_{m+1})\Bigg| + O(1).\end{aligned}$$

(b) *If $n = 2m+1$, then as $T \to \infty$*

$$\begin{aligned}\frac{1}{2}L(T,\theta) = \Bigg| & \frac{T^{m+\frac{3}{2}}}{m+\frac{3}{2}}\cos\left(\left(\frac{n}{2}+1\right)\theta\right) \\ & + \sum_{j=1}^{m+1} \frac{T^{m-j+\frac{3}{2}}}{m-j+\frac{3}{2}}\binom{\frac{1}{2}}{j}\Re\left(e^{\sqrt{-1}(m-j+\frac{3}{2})\theta}c_j\right)\Bigg| + O(1).\end{aligned}$$

Here for each $1 \le j \le n$, c_j are functions of $a_1, \dots, a_j$. $\Re(z)$ is the real part of the complex number z.

Proof. Since there exists $R > 0$ such that $\phi(z) \neq 0$ outside $|z| < R$, by deleting a half line, we can choose a branch of $\sqrt{\phi}$ on $|z| > R$. We may assume that $te^{\sqrt{-1}\theta}$ is not on the deleted half line. Let $\xi + i\eta = w =$

$\int^z \sqrt{\phi(\zeta)}d\zeta$. Then locally, w is a complex coordinates of $\mathbb{R}^2$. The pull-back metric of $\mathbb{H}^2$ under u is

$$(e+2)d\xi^2 + (e-2)d\eta^2,$$

where e is energy density of u with respect to the metric $|dw|^2 = |\phi||dz|^2$. Let $z = te^{i\theta}$, then

$$\frac{dw}{dt} = \frac{dw}{dz}\frac{dz}{dt} = e^{i\theta}\sqrt{\phi(te^{\sqrt{-1}\theta})}. \tag{4.1}$$

By [**Hn**], there is a constant $C_1 > 0$ such that

$$0 \le e(z) - 2 \le \exp(-C_1|z|). \tag{4.2}$$

Also $|h| < 1$ when $|z|$ is large,

$$\begin{aligned} \left|\frac{d\xi}{dt}\right| &= \left|\Re\left\{e^{\sqrt{-1}\theta}\sqrt{\phi(te^{\sqrt{-1}\theta})}\right\}\right| \\ &= \left|\Re\left\{t^{\frac{n}{2}}e^{\sqrt{-1}(\frac{n}{2}+1)\theta}\left(1 + \sum_{j=1}^{\infty}\binom{\frac{1}{2}}{j}h^j\right)\right\}\right| \\ &= t^{\frac{n}{2}}\left|\left(\cos\left(\left(\frac{n}{2}+1\right)\theta\right) + \sum_{j=1}^{n}t^{-j}\binom{\frac{1}{2}}{j}\Re\left(e^{\sqrt{-1}(\frac{n}{2}+1-j)\theta}c_j\right)\right.\right. \\ &\qquad \left.\left. + O(t^{-n-1})\right)\right| \end{aligned} \tag{4.3}$$

as $t \to \infty$. Since $\cos((\frac{n}{2}+1)\theta) \neq 0$, we have

$$\begin{aligned} (e+2)\left|\frac{d\xi}{dt}\right|^2 + (e-2)\left|\frac{d\eta}{dt}\right|^2 &= 4\left|\frac{d\xi}{dt}\right|^2 + (e-2)\left|\frac{dw}{dt}\right|^2 \\ &= 4\left|\frac{d\xi}{dt}\right|^2 + O(\exp(-C_2 t)). \end{aligned}$$

Hence there exists $t_0 > 0$ such that if $t > t_0$,

$$\begin{aligned} L(t,\theta) &= \left|\int_{t_0}^{t}\sqrt{(e+2)\left|\frac{d\xi}{dt}\right|^2 + (e-2)\left|\frac{d\eta}{dt}\right|^2}\right| + O(1) \\ &= \int_{t_0}^{t} 2\left|\frac{d\xi}{dt}\right| + O(1). \end{aligned}$$

Using (4.3), the lemma follows.

Remark 4.1. As one can see from the proof, even if $\cos((\frac{n}{2}+1)\theta) = 0$, we still have $\lim_{t\to\infty} L(t,\theta) = \infty$, provided that one of the coefficients of the term T^{m-j+1} or $\log T$ is not zero for the case $n = 2m$. The situation for $n = 2m+1$ is similar.

Proposition 4.2. *With the notations and assumptions as in Lemma* 4.1, *we have*

$$\begin{aligned}\lim_{t\to\infty}\frac{d(o,u(te^{\sqrt{-1}\theta}))}{L(t,\theta)} &= \lim_{t\to\infty}\frac{(\frac{n}{2}+1)d(o,u(te^{\sqrt{-1}\theta}))}{t^{\frac{n}{2}+1}\cos((\frac{n}{2}+1)\theta)}\\ &= 1\end{aligned}$$

where o is a fixed point in $\mathbb{H}^2$.

Proof. Let $\gamma(t) = u(te^{\sqrt{-1}\theta})$, and let $w = \xi + i\eta$ be as in the proof of Lemma 4.1. In these coordinates, the geodesic curvature of $\gamma(t)$ is

(4.4)
$$\begin{aligned}\kappa(t) = (\dot{t})^3\sqrt{e^2-4}\Big[&\Gamma^2_{11}(\xi')^3 + (2\Gamma^2_{12}-\Gamma^1_{11})(\xi')^2\eta' - (2\Gamma^1_{12}-\Gamma^2_{22})\xi'(\eta')^2\\ &- \Gamma^1_{22}(\eta')^3 + \xi'\eta'' - \xi''\eta'\Big]\end{aligned}$$

where

$$\Gamma^1_{11} = \frac{1}{2}(e+2)^{-1}\frac{\partial e}{\partial \xi},\quad \Gamma^1_{12} = \frac{1}{2}(e+2)^{-1}\frac{\partial e}{\partial \eta},\quad \Gamma^1_{22} = \frac{1}{2}(e+2)^{-1}\frac{\partial e}{\partial \xi},$$

$$\Gamma^2_{11} = -\frac{1}{2}(e-2)^{-1}\frac{\partial e}{\partial \eta},\quad \Gamma^2_{12} = \frac{1}{2}(e-2)^{-1}\frac{\partial e}{\partial \xi},\quad \Gamma^2_{22} = \frac{1}{2}(e+2)^{-1}\frac{\partial e}{\partial \eta},$$

$\dot{t} = \frac{dt}{ds}$, s is the arc length of $\gamma(t)$ and e is the energy density of u with respect to the metric $|dw|^2 = |\phi|\,|dz|^2$. As in [**Hn**], we have

$$(e-2)^{-\frac{1}{2}}|\nabla e| \le C_1\exp(-C_2|z|) \tag{4.5}$$

for z large enough, and ∇ is the gradient with respect to the metric $|\phi||dz|^2 = |dw|^2$. Since $\cos((\frac{n}{2}+1)\theta) \neq 0$, by (4.3)

$$\frac{ds}{dt} = \sqrt{(e+2)\left|\frac{d\xi}{dt}\right|^2 + (e-2)\left|\frac{d\eta}{dt}\right|^2} = 2\left|\frac{d\xi}{dt}\right| + O(\exp(-Ct)). \tag{4.6}$$

Note that we also have

$$\left|\frac{dw}{dt}\right| \le C_2 t^{n/2}, \tag{4.7}$$

$$t^{-\frac{n}{2}}\left|\frac{d\xi}{dt}\right| = \left|\cos\left(\left(\frac{n}{2}+1\right)\theta\right)\right| + o(1). \tag{4.8}$$

$$\left|\frac{d^2w}{dt^2}\right| \le C_2 t^{n/2-1} \tag{4.9}$$

for some constants C_2, C_3. By (4.4)–(4.9), we have

$$|\kappa(t)| \le C_4\exp(-C_5 t) \tag{4.10}$$

for some positive constants C_4 and C_5. By (4.10) and Lemma 3.1 in [**HTTW**], given $\epsilon > 0$, there is $t_0 > 0$, and a geodesic line α passing through $\gamma(t_0)$ such that

$$d(\gamma(t), \alpha) \leq \epsilon \tag{4.11}$$

for all $t > t_0$. Let $f^2 d\rho^2 + d\tau^2$ be the Fermi coordinates with respect to α, so that $\tau = 0$ is the geodesic α, where $f = \cosh\tau$. Under this coordinates, $\gamma(t) = (\rho(t), \tau(t))$. By (4.10), we have at $\gamma(t)$

$$|\ddot{\tau} - ff_\tau(\dot{\rho})^2| \leq C_4 \exp(-C_5 t),$$

and so

$$\begin{aligned} |\ddot{\tau}| &\leq C_4 \exp(-C_5 t) + |ff_\tau(\dot{\rho})^2| \\ &\leq C_4 \exp(-C_5 t) + C_6\epsilon|f(\dot{\rho})^2| \\ &\leq C_4 \exp(-C_5 t) + C_6\epsilon|f^2(\dot{\rho})^2| \\ &\leq C_7\epsilon \end{aligned}$$

for some constants C_6, C_7, provided t_0 is large enough, where we have used the fact that $|\tau| \leq \epsilon$, $f = \cosh\tau$ and the fact that $f^2(\dot{\rho})^2 \leq 1$. Here and below, "$\cdot$" means differentiation with respect to arc length s and where "$\,'\,$" means differentiation with respect to t. Hence

$$\left|\frac{d}{dt}(\dot{\tau})\right| = \left|\ddot{\tau}\frac{ds}{dt}\right| \leq C_7\epsilon\left|\frac{ds}{dt}\right|. \tag{4.12}$$

Since $|\dot{\gamma}| = 1$, we have

$$f^2(\dot{\rho})^2 + (\dot{\tau})^2 = 1.$$

For any $T > t_0$, suppose $\tau'(T) = 0$, then

$$f^2(\dot{\rho})^2 = 1,$$

at T. Suppose $\tau'(T) \neq 0$, let us we assume $\tau'(T) > 0$, the case that $\tau'(T) < 0$ is similar. Let b be the supremum of c such that $\tau' > 0$ on $[T, T+c)$. Suppose $b < \infty$, then $\tau'(T+b) = 0$. By (4.12),

$$\left|\frac{d}{dt}[(\dot{\tau})^2]\right| = 2|\dot{\tau}|\left|\frac{d}{dt}(\dot{\tau})\right| \leq C_7\epsilon|\dot{\tau}|\left|\frac{ds}{dt}\right| = C_7\epsilon\frac{d\tau}{dt}$$

in $(T, T+b)$. Hence

$$\begin{aligned} (\dot{\tau})^2(T) - (\dot{\tau})^2(T+b) &\leq \int_T^{T+b} \left|\frac{d}{dt}[(\dot{\tau})^2]\right| dt \\ &\leq C_7\epsilon \int_T^{T+b} \frac{d\tau}{dt} \\ &\leq C_7\epsilon^2 \end{aligned}$$

where we have used the fact that $|\tau| \leq \epsilon$. Since $(\dot{\tau})^2(T+b) = 0$, we have

$$(\dot{\tau})^2(T) \leq C_7\epsilon^2$$

and so

$$f^2(\dot{\rho})^2 \geq 1 - C_7\epsilon^2$$

at T. If $b = \infty$, then we can choose $t_i \to \infty$ with $\tau'(t_i) \to 0$, and we obtain the same inequality. In particular, $f^2(\dot{\rho})^2(T)$ is not 0 for all $T > t_0$, provided t_0 is large enough. Without loss of generality, we may assume that $f\dot{\rho} > 0$ on $[t_0, \infty)$. For any $T > t_0$,

$$\begin{aligned}\rho(T) - \rho(t_0) &= \int_{t_0}^{T} \frac{d\rho}{dt} dt \\ &= \int_{t_0}^{T} \dot{\rho}\frac{ds}{dt} dt \\ &= \int_{t_0}^{T} f^{-1} f\dot{\rho}\frac{ds}{dt} dt \\ &\geq (1 - C_8\epsilon)(s(T) - s(t_0))\end{aligned}$$

for some constant C_8. So

$$d(o, u(Te^{i\theta})) \geq \rho(T) - \tau(T) \geq (1 - C_8\epsilon)(s(T) - s(t_0)) - \epsilon.$$

It is obvious that,

$$d(o, u(Te^{i\theta})) \leq s(T).$$

Note that $s(T) = L(T, \theta)$ in our previous notation and the lemma follows easily.

5. Hopf differentials and images of harmonic maps.

In [**HTTW**], it was proved that if u is a harmonic diffeomorphism from $\mathbb{R}^2$ into $\mathbb{H}^2$ with polynomial Hopf differential, then its image is an ideal polygon. In this section, we will use the analysis in §4 to study explicit relation in some special cases between the Hopf differential and the position of the vertices of the image of u.

Theorem 5.1. *Let $\phi(z) = z^{2m} + az^{m-1}$, where a is a real number. Suppose u is an orientation preserving harmonic diffeomorphism from $\mathbb{R}^2$ to $\mathbb{H}^2$ with Hopf differential ϕdz^2. Then by composing an isometry of $\mathbb{H}^2$ if necessary, the image of u is a regular ideal polygon.*

Proof. Let $w = \log ||\partial u||$, where $||\partial u|| = \sigma|\frac{\partial u}{\partial z}|$ and $\sigma^2|du|^2$ is the metric on $\mathbb{H}^2$. Then w is the unique solution of

$$\Delta_0 w = e^{2w} - |\phi|^2 e^{-2w} \tag{5.1}$$

such that $e^{2w}|dz|^2$ is a complete metric on $\mathbb{R}^2$, see [**WA**]. Here Δ_0 is the Laplacian on $\mathbb{R}^2$ with respect to the standard metric $|dz|^2$. Observe that ϕ satisfies

$$\begin{cases} \phi(z) & = \overline{\phi(\bar{z})}, \quad \text{and} \\ \phi(e^{2\sqrt{-1}\theta}\bar{z}) & = e^{4m\sqrt{-1}\theta}\overline{\phi(z)} \end{cases} \tag{5.2}$$

where $\theta = \frac{\pi}{m+1}$. Identify $\mathbb{H}^2$ with the unit disk $\{u|\ |u| < 1\}$ in $\mathbb{C}$ with Poincaré metric $\sigma^2|du|^2$. Without loss of generality, we may assume that $u(0) = 0$. By Proposition 4.2, we know that if t is real, then $d(u(t), 0) \to \infty$, as $t \to \infty$. We may also assume that $u(t_k)$ tends to the point 1 on the boundary of $\mathbb{H}^2$ for some $t_k \to \infty$, with t_k to be real. Let $v(z) = \overline{u(\bar{z})}$. It is easy to see that v is also an orientation preserving harmonic diffeomorphism. Moreover, let $\zeta = \bar{z}$

$$\begin{aligned} \sigma(v(z)) \left|\frac{\partial v}{\partial z}\right|(z) &= \sigma(\overline{u(\bar{z})}) \left|\frac{\partial \bar{u}}{\partial \bar{\zeta}}\right|(\zeta) \\ &= \sigma(u(\zeta)) \left|\frac{\partial u}{\partial \zeta}\right|(\zeta) \\ &= e^{w(\bar{z})}. \end{aligned}$$

Hence if we let

$$\widetilde{w}(z) = \log\left(\sigma(v(z)) \left|\frac{\partial v}{\partial z}\right|(z)\right)$$

then $\widetilde{w}(z) = w(\bar{z})$. By (5.2), it is easy to see that $\widetilde{w}(z)$ also satisfies (5.1), such that $e^{2\widetilde{w}}|dz|^2$ is complete. By uniqueness, we have

$$w(\bar{z}) = \widetilde{w}(z) = w(z). \tag{5.3}$$

On the other hand,

$$\begin{aligned} \sigma^2(v(z))\frac{\partial v}{\partial z}(z)\frac{\partial \bar{v}}{\partial z}(z) &= \sigma^2(\overline{u(\bar{z})})\overline{\frac{\partial u(\zeta)}{\partial \zeta}}\frac{\partial u(\zeta)}{\partial \bar{\zeta}} \\ &= \overline{\sigma^2(u(\zeta))\frac{\partial u(\zeta)}{\partial \zeta}\frac{\partial \bar{u}(\zeta)}{\partial \zeta}} \\ &= \overline{\phi(\zeta)} \\ &= \overline{\phi(\bar{z})} \\ &= \phi(z). \end{aligned} \tag{5.4}$$

By (5.3), (5.4) and the result in [**TW**], $v = \iota \circ u$ for some orientation preserving isometry ι of $\mathbb{H}^2$. Note that $v(0) = \overline{u(0)} = 0$, and for real number t, $v(t) = \overline{u(t)}$. Since we have normalized u so that $u(t_k) \to 1$ as $t \to \infty$, we also have $v(t_k) \to 1$. So ι must be the identity map, and $v \equiv u$. That is to say

$$\overline{u(\bar{z})} = u(z). \tag{5.5}$$

In particular, $u(t)$ is real if t is real. Hence there is $0 < c < 1$ such that the set consisting of those real ξ with $c < \xi < 1$ is in the image of the positive real axis under the harmonic map u. Let $v_1(z) = e^{2\sqrt{-1}\theta}\overline{u(e^{2\sqrt{-1}\theta}\bar{z})}$. Using similar method and (5.2), one can show that $v_1(z) = \iota_1 \circ u(z)$ for some isometry of $\mathbb{H}^2$. Since $v_1(0) = 0 = u(0)$, we have $v_1(z) = e^{\sqrt{-1}\alpha}u(z)$ for some real number α. Hence

$$u(e^{2\sqrt{-1}\theta}\bar{z}) = e^{2\sqrt{-1}\beta}\overline{u(z)} \tag{5.6}$$

where $2\beta = 2\theta - \alpha$. We want to prove that $e^{\sqrt{-1}s\beta}$, $0 \le s \le 2m+1$ are distinct $(2m+2)$th roots of unity. Moreover the image of u is the ideal polygon spanned by the $e^{\sqrt{-1}s\beta}$, $0 \le s \le 2m+1$. This will conclude the proof of the theorem. First, we claim that for any real number t,

$$u(e^{\sqrt{-1}(s+2)\theta}t) = e^{2\sqrt{-1}\beta}u(e^{\sqrt{-1}s\theta}t) \tag{5.7}$$

for all integer $0 \le s \le 2m+1$. For $s = 0$, (5.7) follows from (5.6) and (5.5) by letting $z = t$. Suppose (5.7) is true for $0 \le s < 2m+1$. By (5.6) and (5.5)

$$\begin{aligned} u(e^{\sqrt{-1}(s+3)\theta}t) &= e^{2\sqrt{-1}\beta}\overline{u(e^{-\sqrt{-1}(s+1)\theta}t)} \\ &= e^{2\sqrt{-1}\beta}u(e^{\ i(s+1)\theta}t). \end{aligned}$$

Hence (5.7) is true. By (5.7), we have

$$u(e^{2\sqrt{-1}s\theta}t) = e^{2\sqrt{-1}s\beta}u(t) \tag{5.8}$$

for any integer s. Take $s = m+1$, we have

$$e^{2\sqrt{-1}(m+1)\beta} = 1.$$

By Proposition 4.2, for any $0 \le s \le 2m+1$, $d(u(te^{\sqrt{-1}s\theta}), 0) \to \infty$ as $t \to \infty$, t is real. Hence there exists $t_k \to \infty$, and real number b_s such that

$$u(t_k e^{\sqrt{-1}s\theta}) \to e^{\sqrt{-1}b_s},$$

for $0 \le s \le 2m+1$. Obviously, $b_0 = 0$, $b_s = \sqrt{-1}s\beta$ for s even by (5.8). On the other hand, by (5.6)

$$\begin{aligned} u(te^{\sqrt{-1}\theta}) &= u(e^{2\sqrt{-1}\theta} \cdot te^{-\sqrt{-1}\theta}) \\ &= e^{2\sqrt{-1}\beta}\overline{u(te^{\sqrt{-1}\theta})}. \end{aligned} \tag{5.9}$$

So we have

$$e^{\sqrt{-1}b_1} = e^{2\sqrt{-1}(\beta - b_1)},$$

and we may assume $b_1 = \beta$ be adding a multiple of 2π to 2β, which does not affect the previous arguments. By (5.7), we again have $b_s = \sqrt{-1}s\beta$ if s is odd. Hence $e^{\sqrt{-1}s\beta}$, $0 \le s \le 2m+1$, are in the closure of the image of u in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$. It remains to prove the $e^{\sqrt{-1}s\beta}$ are distinct. Suppose not, then $e^{\sqrt{-1}s\beta} = 1$ for some $0 < s \le 2m+1$. If s is even, then by (5.8), we have a contradiction, because u is one to one and $e^{\sqrt{-1}s\theta} \ne 1$. Suppose, s is odd. By (5.9), we have $u(te^{\sqrt{-1}\theta}) = \rho(t)e^{\sqrt{-1}\beta}$ where $\rho(t) > 0$. By (5.7),

$$u(te^{\sqrt{-1}s\theta}) = \rho(t)e^{\sqrt{-1}s\beta} = \rho(t).$$

Since $\rho(t_k) \to 1$ and $c < \xi < 1$ is in the image of the positive real axis under u, this contradicts the fact the u is one to one. The theorem follows from the fact that the image of u is a ideal polygon of $2m+2$ sides [**HTTW**].

Next we will discuss the Hopf differentials of the harmonic diffeormorphisms constructed in Proposition 1.6.

Proposition 5.2. *Let $u(z)$ be the harmonic diffeomorphism constructed in Proposition* 1.6. *Then there is a conformal map $z = z(\zeta)$ such that the Hopf differential of u with respect to ζ is of the form $(\zeta^2 + \sqrt{-1}\alpha)d\zeta^2$ where α is a real number.*

Proof. Let $u(z)$ be the harmonic diffeomorphism constructed in Proposition 1.6. Then $u(\bar{z}) = \overline{u(z)}$ and $u(-\bar{z}) = -\overline{u(z)}$. Let $\phi(z)dz^2$ be the Hopf differential of u, then

$$\phi(z) = \sigma^2(u(z))\frac{\partial u}{\partial z}\frac{\partial \bar{u}}{\partial z}.$$

It is easy to see that $\phi(\bar{z}) = \overline{\phi(z)}$ and $\phi(-\bar{z}) = \overline{\phi(z)}$. By the result of [**HTTW**], ϕ is a polynomial of degree 2, that is $\phi(z) = az^2 + bz + c$. Now $\phi(\bar{z}) = \overline{\phi(z)}$ implies that a, b and c are real. $\phi(-\bar{z}) = \overline{\phi(z)}$ implies that $b = 0$. Hence $\phi(z) = az^2 + c$, where a and c are real. Let β be any one of the fourth root of a, and let $\zeta = \beta z$, then

$$\begin{aligned}\phi(z)dz^2 &= (az^2 + c)dz^2\\ &= (a\beta^{-2}\zeta + c)\beta^{-2}d\zeta^2\\ &= (\zeta^2 + \sqrt{-1}\alpha)d\zeta^2\end{aligned}$$

where $\sqrt{-1}\alpha = c\beta^{-2}$. Suppose $a > 0$, then we may choose β to be a positive real number. Hence $\sqrt{-1}\alpha$ is real. By Remark 4.1, the length of the image under u of the half line $\zeta > 0$ is infinite. On the other hand, $\beta > 0$, $\zeta > 0$ implies $z = \beta^{-1}\zeta$ is real and positive. However, by the construction of u in Proposition 1.6, the image of $z > 0$ under u has finite length. This is a contradiction. So $a < 0$, and we may choose $\beta = |a|^{\frac{1}{4}}e^{\frac{\pi}{4}\sqrt{-1}}$. Then

$$\sqrt{-1}\alpha = c\beta^{-2} = c|a|^{\frac{1}{2}}e^{\frac{\pi}{2}\sqrt{-1}} = \sqrt{-1}c|a|^{\frac{1}{2}}.$$

This implies that α is real, because c is real.

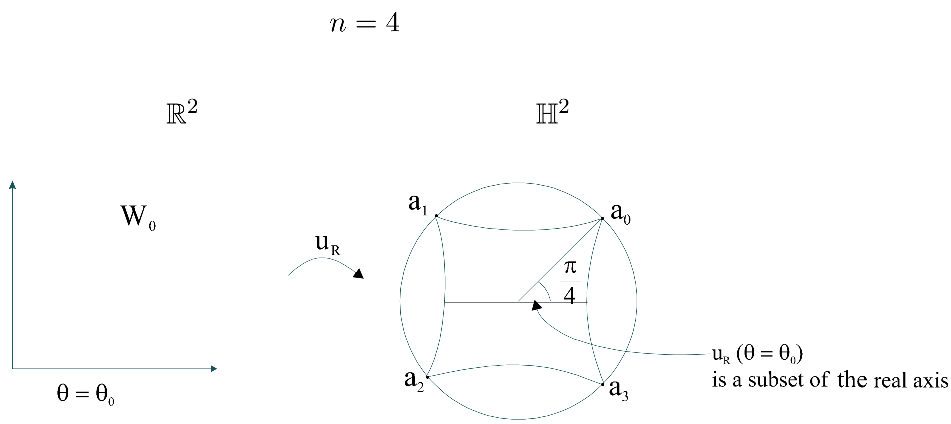

Figure 1

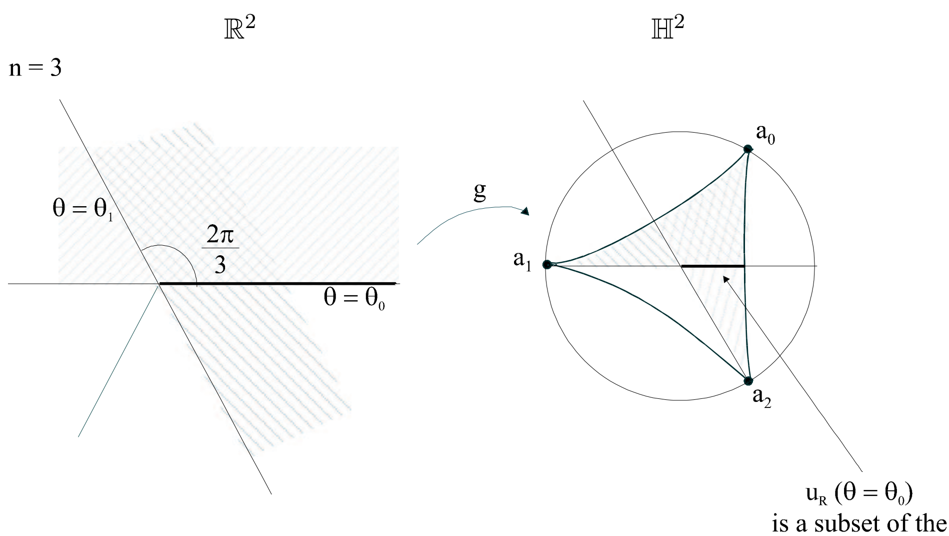

Figure 2

$n = 6$

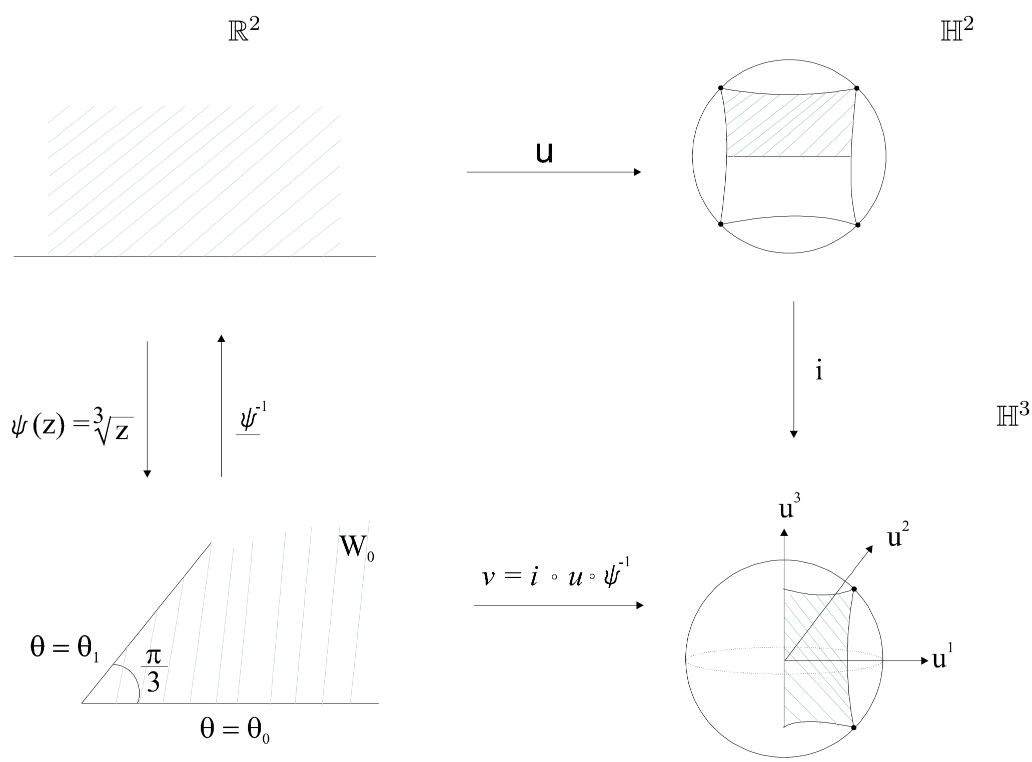
$\mathbb{R}^2$
$\mathbb{H}^2$
u
i
$\mathbb{H}^3$
$\psi(z) = \sqrt[3]{z}$
ψ^{-1}
W_0
$v = i \circ u \circ \psi^{-1}$
u^3
u^2
u^1
$\theta = \theta_1$
$\frac{\pi}{3}$
$\theta = \theta_0$

Figure 3

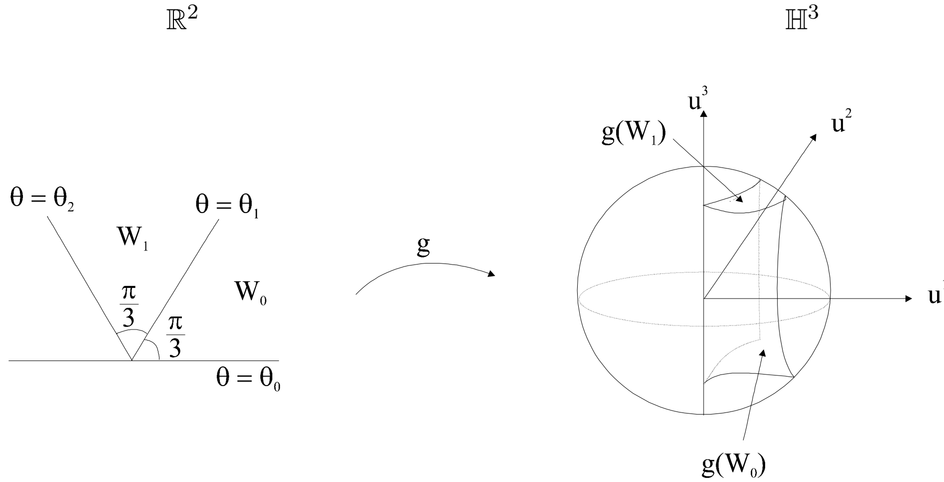

Figure 4

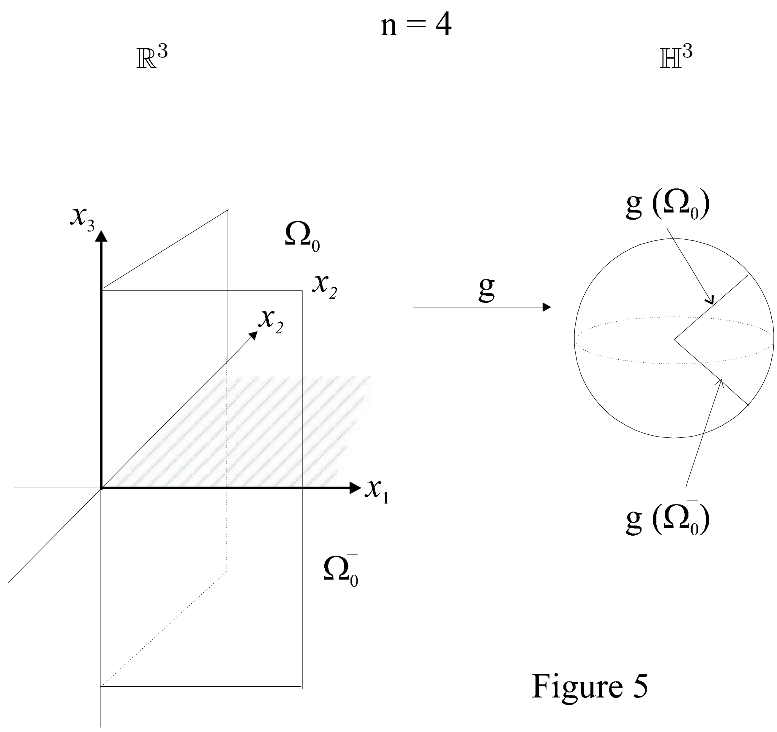

Figure 5

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