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# NON-COMMUTATIVE CLARKSON INEQUALITIES FOR UNITARILY INVARIANT NORMS

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**It is shown that if $A$ and $B$ are operators on a separable complex Hilbert space and if $|||\cdot|||$ is any unitarily invariant norm, then**

$$\begin{aligned}2|||\,|A|^p+|B|^p\,||| &\le |||\,|A+B|^p+|A-B|^p\,|||\\ &\le 2^{p-1}|||\,|A|^p+|B|^p\,|||\end{aligned}$$

**for $2\le p<\infty$, and**

$$\begin{aligned}2^{p-1}|||\,|A|^p+|B|^p\,||| &\le |||\,|A+B|^p+|A-B|^p\,|||\\ &\le 2|||\,|A|^p+|B|^p\,|||\end{aligned}$$

**for $0<p\le 2$. These inequalities are natural generalizations of some of the classical Clarkson inequalities for the Schatten $p$-norms. Generalizations of these inequalities to larger classes of functions including the power functions are also obtained.**

## 1. Introduction.

The classical Clarkson inequalities for the Schatten $p$-norms of Hilbert space operators assert that

$$2\left(\|A\|_p^p+\|B\|_p^p\right)\le\|A+B\|_p^p+\|A-B\|_p^p\le 2^{p-1}\left(\|A\|_p^p+\|B\|_p^p\right)\tag{1}$$

for $2\le p<\infty$,

$$2^{p-1}\left(\|A\|_p^p+\|B\|_p^p\right)\le\|A+B\|_p^p+\|A-B\|_p^p\le 2\left(\|A\|_p^p+\|B\|_p^p\right)\tag{2}$$

for $0<p\le 2$,

$$2\left(\|A\|_p^p+\|B\|_p^p\right)^{q/p}\le\|A+B\|_p^q+\|A-B\|_p^q\tag{3}$$

for $2\le p<\infty$; $\frac{1}{p}+\frac{1}{q}=1$, and

$$\|A+B\|_p^q+\|A-B\|_p^q\le 2\left(\|A\|_p^p+\|B\|_p^p\right)^{q/p}\tag{4}$$

for $1<p\le 2$; $\frac{1}{p}+\frac{1}{q}=1$.

These inequalities, which can be found in [**11**], are non-commutative versions of the celebrated Clarkson inequalities for the classical sequence spaces. These inequalities have useful applications in operator theory and in mathematical physics (see, e.g., [**2**], [**5**], [**7**], [**10**], [**12**], and references therein). In particular, the uniform convexity of the Schatten $p$-classes, for $1 < p < \infty$, is an immediate consequence of the inequalities (1) and (4). For a comprehensive account of the Clarkson inequalities, the reader is referred to [**8**].

Extensions, with proof simplification, of the inequalities (1) and (2) (for $1 \leq p \leq 2$) to wider classes of unitarily invariant norms including the Schatten $p$-norms have been given in [**4**]. This has been achieved by formulating these inequalities in terms of direct sums of operators.

In this paper we give pretty natural generalizations of the inequalities (1) and (2) to all unitarily invariant norms. In fact, our new inequalities seem natural enough and applicable to be widely useful.

Let $B(H)$ denote the $C^*$-algebra of all bounded linear operators on a separable complex Hilbert space $H$. If $A$ is a compact operator in $B(H)$, then the singular values of $A$ are, by definition, the eigenvalues of the positive operator $|A| = (A^*A)^{1/2}$ enumerated as $s_1(A) \geq s_2(A) \geq \cdots \geq 0$.

Recall that, with the exception of the usual operator norm, which is defined on all of $B(H)$, each unitarily invariant norm is a symmetric gauge function of the singular values and is defined on a norm ideal contained in the ideal of compact operators. For the sake of brevity, we will make no explicit mention of this norm ideal. Thus, when we talk of $|||A|||$, we are assuming that $A$ belongs to the norm ideal associated with $|||\cdot|||$.

If $A$ is a compact operator in $B(H)$, let

$$\|A\|_p = \left( \sum_{j=1}^{\infty} s_j^p(A) \right)^{1/p} = (\operatorname{tr} |A|^p)^{1/p}$$

for $0 < p \leq \infty$, where tr is the usual trace functional. This defines the Schatten $p$-norm (quasinorm) for $1 \leq p \leq \infty$ ($0 < p < 1$), where by convention $\|A\|_\infty = s_1(A)$ is the usual operator norm of $A$.

Since $\|A\|_p^p = \big\| \, |A|^p \, \big\|_1 = \operatorname{tr} |A|^p$ for $0 < p < \infty$, our generalizations of the inequalities (1) and (2) will be much appreciated if we rewrite them as

$$(5) \quad 2\big\| \, |A|^p + |B|^p \, \big\|_1 \leq \big\| \, |A+B|^p + |A-B|^p \, \big\|_1 \leq 2^{p-1}\big\| \, |A|^p + |B|^p \, \big\|_1$$

for $2 \leq p < \infty$, and

$$(6) \quad 2^{p-1}\big\| \, |A|^p + |B|^p \, \big\|_1 \leq \big\| \, |A+B|^p + |A-B|^p \, \big\|_1 \leq 2\big\| \, |A|^p + |B|^p \, \big\|_1$$

for $0 < p \leq 2$.

In Section 2 of this paper, we will show that the trace norm $\| \cdot \|_1$ in (5) and (6) can be replaced by any unitarily invariant norm $|||\cdot|||$, and that the power functions $f(t) = t^p$ can be replaced by more general classes of functions.

## 2. Main results.

To achieve our goal of generalizing the inequalities (1) and (2), we need the following two lemmas. The first lemma is a well-known result that can be proved by using the spectral theorem and Jensen's inequality. The inequalities in this lemma are of the Peierls-Bogoliubov type (see, e.g., [**3**, p. 281] or [**12**, pp. 101-102]).

**Lemma 1.** *Let $A$ be a positive operator in $B(H)$.*

(a) *If $g$ is a convex function on $[0,\infty)$, then*

$$g(\langle Ax, x\rangle) \leq \langle g(A)x, x\rangle \tag{7}$$

*for every unit vector $x$ in $H$.*

(b) *If $h$ is a concave function on $[0,\infty)$, then*

$$\langle h(A)x, x\rangle \leq h(\langle Ax, x\rangle) \tag{8}$$

*for every unit vector $x$ in $H$.*

The second lemma, which is due to Ando and Zhan [**1**], contains norm inequalities comparing $f(A+B)$ and $f(A)+f(B)$ for certain functions $f$ (see, also [**6**]).

In this lemma and in the sequel, $|||\cdot|||$ designates any unitarily invariant norm.

**Lemma 2.** *Let $A$ and $B$ be positive operators in $B(H)$.*

(a) *If $g$ is an increasing function on $[0,\infty)$ such that $g(0)=0$, $\lim_{t\to\infty} g(t) = \infty$, and $g^{-1}$ is an operator monotone function, then*

$$|||g(A)+g(B)||| \leq |||g(A+B)|||\,. \tag{9}$$

(b) *If $h$ is a nonnegative operator monotone function on $[0,\infty)$, then*

$$|||h(A+B)||| \leq |||h(A)+h(B)|||\,. \tag{10}$$

Now we are in a position to present our main results. The first result is a considerable generalization of the inequalities (1).

**Theorem 1.** *Let $A$ and $B$ be operators in $B(H)$ and let $f$ be an increasing function on $[0,\infty)$ such that $f(0)=0$, $\lim_{t\to\infty} f(t)=\infty$, and the inverse function of $g(t)=f(\sqrt{t})$ is operator monotone. Then*

$$\begin{aligned} 2\,|||f(|A|)+f(|B|)||| &\leq |||f(|A+B|)+f(|A-B|)||| \\ &\leq \frac{1}{2}\,|||f(2|A|)+f(2|B|)|||\,. \end{aligned} \tag{11}$$

*Proof.* Since $g^{-1}$ is operator monotone, it follows that it is concave (see, e.g., [**3**, p. 120]). Since $g$ is increasing, the concavity of $g^{-1}$ implies that $g$

is convex. Now for any unit vector $x$ in $H$, we have

$$\begin{aligned}
&\langle (f(|A+B|)+f(|A-B|))x,x\rangle \\
&= \langle g(|A+B|^2)x,x\rangle + \langle g(|A-B|^2)x,x\rangle \\
&\geq g(\langle |A+B|^2x,x\rangle) + g(\langle |A-B|^2x,x\rangle) \quad \text{(by Lemma 1(a))} \\
&\geq 2g\left(\frac{\langle |A+B|^2x,x\rangle + \langle |A-B|^2x,x\rangle}{2}\right) \quad \text{(by the convexity of } g\text{)} \\
&= 2g(\langle (|A|^2+|B|^2)x,x\rangle).
\end{aligned}$$

Using the min-max principle (see, e.g., [**3**, p. 58] or [**9**, p. 25]) and the fact that $g$ is increasing, we see that

$$\begin{aligned}
s_j(f(|A+B|)+f(|A-B|)) &\geq 2g(s_j(|A|^2+|B|^2)) \\
&= 2s_j(g(|A|^2+|B|^2))
\end{aligned}$$

for $j=1,2,\ldots$. Since unitarily invariant norms are increasing with respect to singular values (see, e.g., [**3**, p. 52] or [**9**, p. 71]), it follows that

$$\begin{aligned}
&|||f(|A+B|)+f(|A-B|)||| \\
&\geq 2\left|\left|\left|g(|A|^2+|B|^2)\right|\right|\right| \\
&\geq 2\left|\left|\left|g(|A|^2)+g(|B|^2)\right|\right|\right| \quad \text{(by Lemma 2(a))} \\
&= 2\,|||f(|A|)+f(|B|)|||,
\end{aligned}$$

which proves the first inequality in (11). The second inequality in (11) follows from the first one by replacing $A$ and $B$ by $A+B$ and $A-B$, respectively.

Based on Lemmas 1(b) and 2(b), one can employ an argument similar to that used in the proof of Theorem 1 to derive the following generalization of the inequality (2).

**Theorem 2.** *Let $A$ and $B$ be operators in $B(H)$ and let $f$ be a nonnegative function on $[0,\infty)$ such that $h(t)=f(\sqrt{t})$ is operator monotone. Then*

$$\begin{aligned}
\frac{1}{2}\,|||f(2|A|)+f(2|B|)||| &\leq |||f(|A+B|)+f(|A-B|)||| \\
&\leq 2\,|||f(|A|)+f(|B|)|||.
\end{aligned} \tag{12}$$

Specializing Theorems 1 and 2 to the functions $f(t)=t^p$ $(2\leq p<\infty)$ and $f(t)=t^p$ $(0<p\leq 2)$, respectively, we obtain our promised natural generalizations of the inequalities (1) and (2).

**Corollary 1.** *Let $A$ and $B$ be operators in $B(H)$. Then*

$$2\,|||\,|A|^p+|B|^p\,||| \leq |||\,|A+B|^p+|A-B|^p\,||| \leq 2^{p-1}\,|||\,|A|^p+|B|^p\,||| \tag{13}$$

*for* $2 \le p < \infty$, *and*

$$2^{p-1} \, ||| \, |A|^p + |B|^p \, ||| \le ||| \, |A+B|^p + |A-B|^p \, ||| \le 2 \, ||| \, |A|^p + |B|^p \, ||| \tag{14}$$

*for* $0 < p \le 2$.

It should be observed here that the inequalities (5) and (6) (and so the inequalities (1) and (2)) follow from the inequalities (13) and (14) specialized to the trace norm.

It has been remarked in [**4**] that, although the inequalities (1) and (2) are usually proved separately, they follow from (3) and (4), respectively, simply by the convexity of the function $f(t) = t^{p/q}$ ($2 \le p < \infty$; $\frac{1}{p} + \frac{1}{q} = 1$) and the concavity of the function $f(t) = t^{p/q}$ ($1 < p \le 2$; $\frac{1}{p} + \frac{1}{q} = 1$). Thus, it would be desirable to find natural generalizations (perhaps along the lines of our generalizations of (1) and (2)) of the inequalities (3) and (4) to all unitarily invariant norms.

Let $f(t) = e^{t^2} - 1$. Then $f$ is increasing on $[0, \infty)$, $f(0) = 0$, $\lim_{t\to\infty} f(t) = \infty$, and the inverse of $g(t) = f(\sqrt{t}) = e^t - 1$ is the operator monotone function $g^{-1}(t) = \log(t+1)$.

Applying Theorem 1 to this special function, we have the following corollary.

**Corollary 2.** *Let $A$ and $B$ be operators in $B(H)$. Then*

$$\begin{aligned} 2 \left|\left|\left| e^{|A|^2} + e^{|B|^2} - 2I \right|\right|\right| &\le \left|\left|\left| e^{|A+B|^2} + e^{|A-B|^2} - 2I \right|\right|\right| \\ &\le \frac{1}{2} \left|\left|\left| e^{4|A|^2} + e^{4|B|^2} - 2I \right|\right|\right|. \end{aligned} \tag{15}$$

Now let $f(t) = \log(t+1)$. Then $h(t) = f(\sqrt{t}) = \log(\sqrt{t}+1)$ is operator monotone on $[0, \infty)$. So applying Theorem 2 to this function, we have the following corollary.

**Corollary 3.** *Let $A$ and $B$ be operators in $B(H)$. Then*

$$\begin{aligned} &\frac{1}{2} \, |||\log(2|A| + I) + \log(2|B| + I)||| \\ &\le |||\log(|A+B| + I) + \log(|A-B| + I)||| \\ &\le 2 \, |||\log(|A| + I) + \log(|B| + I)||| \, . \end{aligned} \tag{16}$$

The most basic unitarily invariant norms are the Ky Fan norms $\| \cdot \|_{(k)}$ defined as

$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A)$$

for $k = 1, 2, \ldots$. The Ky Fan dominance principle says that $|||A||| \le |||B|||$ for all unitarily invariant norms if and only if $\|A\|_{(k)} \le \|B\|_{(k)}$ for all $k = 1, 2, \ldots$ (see, e.g., [**3**, p. 93] or [**9**, p. 72]).

Utilizing the Ky Fan dominance principle, enables us to conclude the following finite-dimensional consequence of Corollary 2.

**Corollary 4.** *Let $A$ and $B$ be operators in $B(H)$, where $H$ is an $n$-dimensional Hilbert space. Then*

$$\begin{aligned} 2\left|\left|\left|e^{|A|^2}+e^{|B|^2}\right|\right|\right| &\leq \left|\left|\left|e^{|A+B|^2}+e^{|A-B|^2}+2I\right|\right|\right| \\ &\leq \frac{1}{2}\left|\left|\left|e^{4|A|^2}+e^{4|B|^2}+6I\right|\right|\right|. \end{aligned} \tag{17}$$

*Proof.* Applying Corollary 2 to the Ky Fan norms, we have

$$\begin{aligned} 2\left\|e^{|A|^2}+e^{|B|^2}-2I\right\|_{(k)} &\leq \left\|e^{|A+B|^2}+e^{|A-B|^2}-2I\right\|_{(k)} \\ &\leq \frac{1}{2}\left\|e^{4|A|^2}+e^{4|B|^2}-2I\right\|_{(k)} \end{aligned}$$

for $k=1,2,\ldots,n$. Thus,

$$\begin{aligned} 2\left\|e^{|A|^2}+e^{|B|^2}\right\|_{(k)}-4k &\leq \left\|e^{|A+B|^2}+e^{|A-B|^2}+2I\right\|_{(k)}-4k \\ &\leq \frac{1}{2}\left\|e^{4|A|^2}+e^{4|B|^2}+6I\right\|_{(k)}-4k \end{aligned}$$

for $k=1,2,\ldots,n$, from which we get

$$\begin{aligned} 2\left\|e^{|A|^2}+e^{|B|^2}\right\|_{(k)} &\leq \left\|e^{|A+B|^2}+e^{|A-B|^2}+2I\right\|_{(k)} \\ &\leq \frac{1}{2}\left\|e^{4|A|^2}+e^{4|B|^2}+6I\right\|_{(k)} \end{aligned}$$

for $k=1,2,\ldots,n$. Now the desired inequalities (17) follow by the Ky Fan dominance principle.

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