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Many 3-manifolds can be represented as 2-fold branched coverings of links, but this representation is, in general, not unique. In the Seifert fibered case the problem is usually local: For example, if K is a Montesinos knot its 2-fold branched covering is Seifert fibered and there exists a complete system of local geometric modifications on K by which we can get every other Montesinos knot with the same 2-fold branched covering. On the other hand, if the 2-fold covering M of a knot is hyperbolic, the situation is globally determined by the structure of the isometry group of M. In this paper we develop a global approach for the case that M is hyperbolic and we study the orbifolds which are quotients of M by the action of a 2-group of isometries. This leads to a complete description of the geometry of the possible configurations of knots with the same 2-fold branched coverings. Moreover we are also able to settle the 2-component link case, which was still open, by finding an explicit bound on the number of inequivalent 2-component links which have the same hyperbolic 2-fold branched coverings.

1. Introduction.

Many 3-manifolds can be represented as 2-fold coverings of the 3-sphere S^3 branched over links ("2-fold branched coverings of links"), but this representation is, in general, not unique. Examples of non-uniqueness of the representation have been known for a long time ([**1**] and [**19**]); however a complete description of the general situation is not yet available (see Problems 3.25 and 1.22 in Kirby's list [**8**]).

As usual the two basic cases of the theory are the Seifert and the hyperbolic one.

The Seifert case is well understood and the problem is usually local.

The representation is unique for spherical Seifert fibered manifolds, because they have (up to conjugation) a unique involution with orbit space the 3-sphere S^3: This is proved in [**21**] for S^3, in [**6**] for lens spaces (which are 2-fold branched coverings of two-bridge links) and in [**7**], [**10**] for the other spherical manifolds.

In the nonspherical case the representation is highly not unique. By [3], [9] and [18] any involution on a Seifert manifold is standard, that is it is equivalent to a fiber-preserving one. If M is a Seifert fibered 2-fold branched covering of a link L, there are two possible situations: If the covering involution respects the orientation of the fibers, the link L is a Seifert link, that is its complement in S^3 admits a Seifert fibration by circles [4]; on the other hand if the covering involution reverses the orientation of the fibers, L is a Montesinos link, that is S^3 admits a Seifert fibration by circles and intervals such that L consists of all boundary points of the intervals [11], [5]. Both situations may occur simultaneously. Moreover the number of inequivalent Montesinos links which have the same 2-fold branched coverings may be arbitrarily large because a Seifert space does not change if we change the order of its exceptional fibers, but this permutation may affect the corresponding Montesinos branch sets. This phenomenon is local and well understood: If M is a Seifert fibered 2-fold branched covering of a Montesinos link L every other Montesinos link with the same 2-fold branched covering can be obtained by a sequence of elementary geometric modifications of L (mutations along Conway spheres [11], [20]).

The case that M is hyperbolic is quite different. By Thurston's Orbifold Theorem [2] any involution with nonempty fixed point set on a hyperbolic manifold M is standard, that is it is equivalent to an isometry. This implies that any link with 2-fold branched covering M is π-hyperbolic, that is S^3 admits a Riemannian metric of constant negative curvature which becomes singular folding with an angle π around the link.

The first difference with the Seifert case is that the number of inequivalent links which have the same hyperbolic 2-fold branched covering M is bounded by a constant C not depending on M. The estimate for C depends on the number of components of the link (by homological reasons two links with the same 2-fold branched coverings have the same number of components). It has been proved that $C \leq 9$ for knots [13] and that $C \leq 5$ for links which have at least three components [15]; for the most difficult case of 2-component links no explicit bound was known before.

A second major difference between the Seifert and the hyperbolic case is how inequivalent links with the same 2-fold branched coverings are related. We have recalled above that, if M is a Seifert fibered 2-fold covering of a Montesinos knot K, there exists a complete system of local geometric modifications on K by which we can get every other Montesinos knot with the same 2-fold branched coverings. But if M is hyperbolic there is no analogous system of *local* geometric modifications: Indeed the arguments of [13] and [15] make clear that the hyperbolic situation is *globally* determined by the structure of the isometry group of M.

In this paper we develop a global approach for the case that M is hyperbolic and we study the orbifolds which are quotients of M by the action of a 2-group of isometries. This leads to a new proof of the main Theorem of [**13**] and also to a complete description of the geometry of the possible configurations of knots with the same 2-fold branched coverings. More important we are able to settle the 2-component link case, which was still open, by finding the explicit bound *nine* on the number of inequivalent 2-component links which have the same hyperbolic 2-fold branched coverings.

The key result of the paper is (for 2-fold branched covering of a link we mean that every meridian of the link corresponds to a generator of the covering group):

Theorem 1. *Let M be the hyperbolic 2-fold branched covering of a link with one or two components. For any (finite) 2-group S of orientation-preserving isometries of M which contains the covering involution of the link, the singularity graph of the quotient orbifold M/S is combinatorially equivalent to one of the twelve graphs* IA, … , IIID *(Figure 1).*

By Mostow's Rigidity Theorem the number of inequivalent π-hyperbolic knots, respectively 2-component links, with the same 2-fold branched covering M is bounded by the number of the conjugacy classes of non-free involutions in the orientation-preserving isometry group of M. So, as a consequence of Theorem 1, by simply counting the number of edges and loops (at most nine) of the twelve graphs IA, … , IIID, we get the following:

Theorem 2. *There are at most nine different π-hyperbolic knots with the same 2-fold branched coverings.*

The word "different" in Theorem 2 must be made precise. In this paper we shall always work in the category of oriented manifolds and orientation-preserving diffeomorphisms. So two knots K and K' in S^3 are *equivalent* if and only if there is an orientation-preserving diffeomorphism of S^3 which carries K onto K'. This is equivalent to say that K and K' are ambient isotopic.

Thurston's Orbifold Theorem [**2**] and Theorem 2 imply the purely topological result that there are at most nine different simple Conway-irreducible knots (that is: Knots with no pairwise incompressible embedded 2-spheres and such that every embedded incompressible 2-torus is boundary parallel) with the same 2-fold branched coverings.

It is not completely clear if the bound 'nine' in Theorem 2 is best possible. In [**14**] explicit examples of four different π-hyperbolic knots in S^3 with the same 2-fold branched coverings are constructed. Recently the author has obtained an example of six different π-hyperbolic knots in S^3 with the same 2-fold branched coverings (unpublished). There is some evidence that this

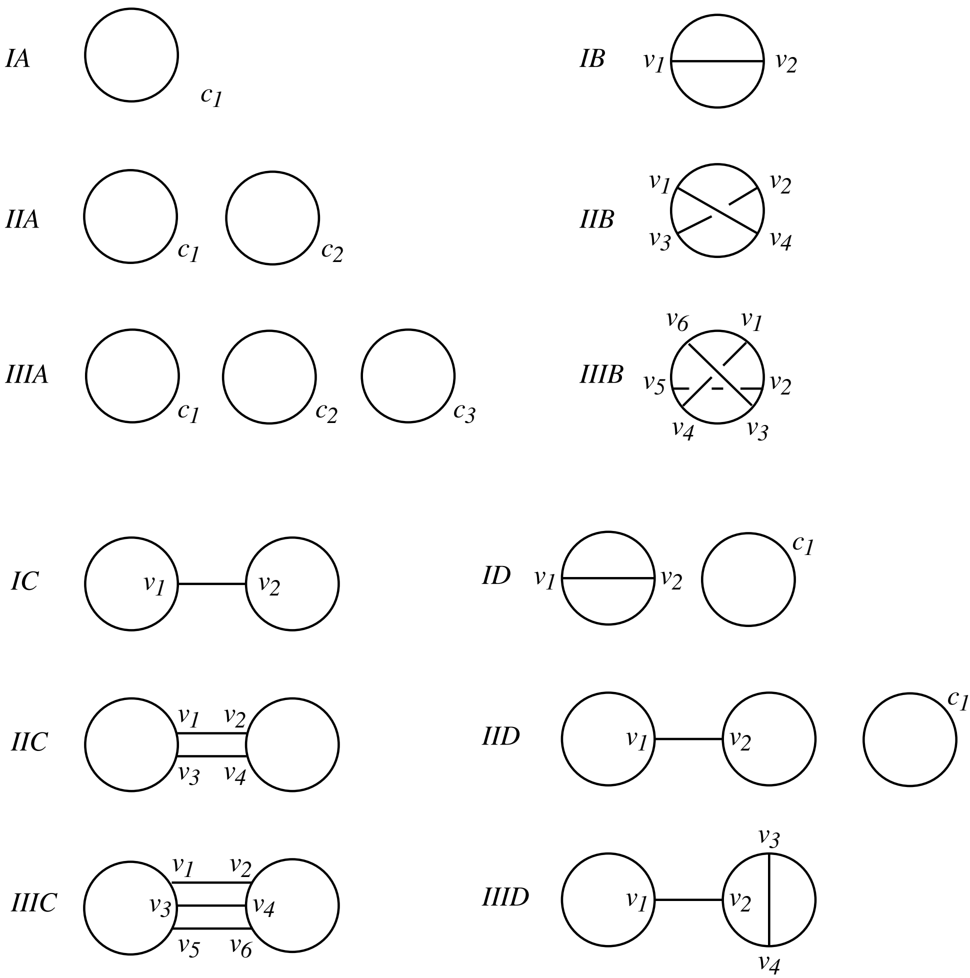

Figure 1

last construction can be possibly generalized to give an example with nine different knots, but computations are still in progress.

The estimate in Theorem 2 had already been obtained in [**13**] by abstract group-theoretical methods; the advantage here is that Theorem 1 describes also the possible configurations of knots with the same 2-fold branched coverings. Moreover if we turn to the 2-component link case, the algebraic methods of [**13**] and [**15**] become too involved; but, from a geometrical point of view, the situation is analogous to the knot case. Indeed Theorem 1 applies simultaneously also to the 2-component link case:

Theorem 3. *There are at most nine different π-hyperbolic 2-component links with the same 2-fold branched coverings.*

Finally a Corollary of Theorem 2, Theorem 3 and [**15**, Theorem 1] is the following explicit bound which does not depend on the number of components of the link:

Corollary. *There are at most nine different π-hyperbolic links with the same 2-fold branched coverings.*

2. Proof of Theorems 2 and 3.

By Mostow's Rigidity Theorem, the number of inequivalent π-hyperbolic knots, respectively 2-component links, with the same 2-fold branched covering M is bounded above by the number of the conjugacy classes of non-free involutions in the orientation-preserving isometry group of M. So it is also bounded above by the number of conjugacy classes of non-free involutions in a Sylow 2-subgroup S of the orientation-preserving isometry group of M.

The projection of the fixed point set of an involution of S to the quotient orbifold M/S contains one edge or one loop of the singularity graph of M/S. Moreover if two involutions of S are not conjugate, the projections of their fixed point sets have no common interior points. The thesis follows from Theorem 1 by counting the number of edges and loops (at most nine) of the twelve graphs IA,... ,IIID.

3. The family $\mathcal{F}$ of quotient orbifolds.

In this section we associate a family $\mathcal{F}$ of 3-orbifolds to each pair (M, S) where M is a hyperbolic 2-fold branched covering of a link L with one or two components and S a (finite) 2-group of orientation-preserving isometries of M containing the covering involution. Each orbifold of the family $\mathcal{F}$ will be a quotient M/H of M for some subgroup H of S; however we will not include in $\mathcal{F}$ all the quotient orbifolds of M (we are now concerned only with elements of S which have nonempty fixed point sets). We conclude the section by proving the most important properties of $\mathcal{F}$.

The following elementary algebraic result on 2-groups is crucial in the construction of $\mathcal{F}$:

Proposition 1 ([**17**, page 88, Theorem 1.6])**.** *If H is a proper subgroup of a finite 2-group S, then the normalizer $N_S H$ is strictly larger than H.*

Construction of $\mathcal{F}$.

We define $\mathcal{F}$ as a disjoint union of subfamilies $\mathcal{F}_1, \dots, \mathcal{F}_n$.

The first subfamily $\mathcal{F}_1$ of $\mathcal{F}$ is the set $\{\mathcal{O}(L)\}$ where $\mathcal{O}(L)$ is the orbifold with underlying topological space S^3 and singular set L with singular index two.

To construct $\mathcal{F}_2$, let $S(L)$ be the group of isometries of $\mathcal{O}(L)$ such that their lifts to M are elements of S (we briefly say: "Isometries which lift to S"). By classical Smith theory for finite group actions on S^3, if the fixed point set of an involution of $S(L)$ is nonempty, then it is connected. The subfamily $\mathcal{F}_2$ is the set of all the 3-orbifolds (up to isometry) which are quotients $\mathcal{O}(L)/u$ for some involution u of $S(L)$ which has nonempty fixed point set (in $\mathcal{O}(L)$). In particular, if $S(L)$ acts freely on M, the subfamily $\mathcal{F}_2$ is empty. By the positive solution of the Smith conjecture [**12**] the underlying topological space of any orbifold of $\mathcal{F}_2$ is S^3. Note that any orbifold of $\mathcal{F}_2$ is also a quotient M/H for some subgroup H of S which contains h; moreover H is generated by elements which have nonempty fixed point sets in M (if v is an involution of $S(L)$ with nonempty fixed point set, then at least one lift of v to M has also nonempty fixed point set).

The construction of $\mathcal{F}_3$ is analogous to $\mathcal{F}_2$. For any orbifold $\mathcal{O} \in \mathcal{F}_2$, let $S(\mathcal{O})$ be the group of the isometries of $\mathcal{O}$ which lift to S. The subfamily $\mathcal{F}_3$ is the set of all the 3-orbifolds which are quotients $\mathcal{O}/u'$ of an orbifold $\mathcal{O} \in \mathcal{F}_2$ for some involution u' of $S(\mathcal{O})$ with nonempty fixed point set in $\mathcal{O}$ (if any). By the positive solution of the Smith conjecture the underlying topological space of any orbifold of $\mathcal{F}_3$ is S^3. Again any orbifold of $\mathcal{F}_3$ is also a quotient M/H' for some subgroup H' of S which contains h and it is generated by elements with nonempty fixed point sets (in M).

Iteratively, the subfamily $\mathcal{F}_4$ is the set of quotients of orbifolds of the subfamily $\mathcal{F}_3$ by involutions which have nonempty fixed point sets and lift to S.

Since S is a finite group, after a finite number of steps the construction ends. We denote by $\mathcal{F}_n$ the last nonempty subfamily we get, by $\mathcal{F}$ the disjoint union $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_n$.

We now turn to describe some properties of the family $\mathcal{F}$ (Propositions 2 and 3) which we need for the proof of Theorem 1.

We recall what is already evident from the construction above:

- the underlying topological space of each orbifold of $\mathcal{F}$ is S^3 (positive solution of the Smith conjecture);

- each orbifold of $\mathcal{F}_i$ for $1 \leq i \leq n$ is a quotient M/H of M for some subgroup H of S of order 2^i which contains h and is generated by elements with nonempty fixed point sets (in M).

We first characterize the last subfamily $\mathcal{F}_n$ of $\mathcal{F}$ by the following Proposition.

Proposition 2. *The last subfamily $\mathcal{F}_n$ contains exactly one orbifold: This orbifold is the quotient M/H of M where H is the subgroup of S which is generated by all the elements with nonempty fixed point sets.*

Proof. Let $\mathcal{O} \in \mathcal{F}_n$ be any orbifold. By construction $\mathcal{O}$ is a quotient $= M/H$ of M for some subgroup H of S which has order 2^n and is generated by elements with nonempty fixed point sets. To prove the thesis it is enough to show that any element of $S - H$ acts freely on M: It follows that H is the unique subgroup of S of order 2^n which is generated by elements with nonempty fixed point sets.

Suppose, ad absurdum, that there exists an element g of $S - H$ which has nonempty fixed point set in M. In particular H is a proper subgroup of S. By Proposition 1 the normalizer $N_S H$ of H in S is larger than H and the factor group $N_S H/H$ projects to a 2-group of isometries of $\mathcal{O}$. Hence there exists at least one isometry u of $\mathcal{O}$ which has order two and lifts to S. Since $\mathcal{F}_n$ is the last subfamily of $\mathcal{F}$, the involution u acts freely on $\mathcal{O}$. The quotient $\mathcal{O}_1 := \mathcal{O}/u$ is a hyperbolic 3-orbifold and it is also the quotient M/H_1 of M for a subgroup H_1 of S containing H as a normal subgroup of index two.

Note that g is not an element of H_1 because if $g \in H_1$ the fixed point set of g in M projects to (a subset of) the fixed point set of u in $\mathcal{O}$ which is impossible, since u acts freely. So again H_1, which does not contain g, is a proper subgroup of S and, by Proposition 1, there exists one isometry of $\mathcal{O}_1$ which is an involution and lifts to S. If every involution of $\mathcal{O}_1$ which lifts to S acts freely on $\mathcal{O}_1$ we can construct a further quotient $\mathcal{O}_2$ of $\mathcal{O}_1$ by any involution which lifts to S.

More generally, for some $m \geq 1$, we can construct iteratively a hierarchy $\mathcal{O}_1, \dots \mathcal{O}_m$ of orbifolds and a sequence $H_1, \dots H_m$ of corresponding subgroups of S such that every orbifold $\mathcal{O}_i$ is the quotient of $\mathcal{O}_{i-1}$ by an involution acting freely on $\mathcal{O}_{i-1}$; every group H_i contains H_{i-1} as a normal subgroup of index two. Since we know that $S - H$ contains at least one element g which has nonempty fixed point set, after finitely many steps we must find an orbifold, say $\mathcal{O}_m$, which admits an involution v which has nonempty fixed point set (in $\mathcal{O}_m$) and lifts to S. By construction $\mathcal{O}_m$ is the quotient M/H_m for the corresponding subgroup H_m of S. A lift $\widetilde{v}$ of v to S has nonempty fixed point set and it lies in the normalizer $N_S H_m$ of H_m in S; so $\widetilde{v}$ also normalizes the subgroup H of H_m because H is the subgroup of H_m generated by all the elements of H_m with nonempty fixed point set and thus $\widetilde{v}$ descends to an involution of $\mathcal{O}$ with nonempty fixed point set. This is impossible because $\mathcal{F}_n$ is the last subfamily of $\mathcal{F}$.

This finishes the proof.

We conclude this section by showing that the procedure described at the beginning of the paragraph can be reversed. Indeed we have constructed $\mathcal{F}$ 'up-bottom' starting from $\mathcal{F}_1$ and taking quotients by involutions until the last set $\mathcal{F}_n$ has been reached; but it is also possible to go back 'bottom-up' from $\mathcal{F}_n$ to $\mathcal{F}_1$, taking 2-fold branched coverings at each step. This reverse construction is made precise in Proposition 3. Before that we need to introduce the notion of first homology group and 2-fold branched covering of an orbifold.

The first homology group of an orbifold.

The first homology group $H_1(\mathcal{O})$ of a 3-orbifold $\mathcal{O}$ is the abelianization of the orbifold fundamental group $\pi_1(\mathcal{O})$ of $\mathcal{O}$. In our case the underlying topological space of $\mathcal{O}$ is S^3, each component of its singularity graph Γ is a knot or a trivalent graph and the singularity order at each point of the edges of Γ is a power of two (all vertices are of dihedral type).

Starting with a planar projection of the graph, one sees that $\pi_1\mathcal{O}$ admits a (Wirtinger) presentation of the form:

$$\pi_1\mathcal{O} = \langle x_1, \dots , x_n | \, r_1 = 1, \dots , r_m = 1; x_1^{i_1} = \dots = x_n^{i_n} = 1\rangle$$

where each x_j represents a loop around an arc contained either in some edge or in some knot of Γ and i_j its singularity order (a power of two). There are two possible types of relations r_j. The first type corresponds to the vertices of Γ and has the form $x_j {x_k}^e {x_s}^d$ with $e = +1$ or -1 and $d = +1$ or -1; since each vertex is of dihedral type at least two among the three elements x_j, x_k and x_s have order two. This type of relations involve only loops around edges of Γ and not around components which are knots. The second type of relations corresponds to the double points of Γ in the Wirtinger projection and has the form $x_j {x_k}^e {x_s}^{-1} {x_k}^{-e}$ with $e = +1$ or -1. When abelianizing, relations of the first type imply that $x_j^2 = x_k^2 = x_s^2 = 1$ and get the form $x_j x_k x_s$. Relations of the second type get the form $x_j {x_s}^{-1}$ where x_j and x_s correspond to two loops around the same edge or the same knot of Γ.

This shows that, if the singularity graph of $\mathcal{O}$ has q edges, p vertices and s knots, its first homology group $H_1(\mathcal{O})$ is the abelianization of a group which admits a presentation of the form:

$$\langle m_1, \dots , m_q, n_1, \dots , n_s | \, R_1 = 1, \dots , R_p = 1; \quad m_1^2 = \dots = m_q^2 = n_1^{k_1} = \dots = n_s^{k_s} = 1\rangle$$

where m_i is a small meridian around the i-th edge, n_j is a small meridian around the j-th knot with singularity order k_j and R_s is the abelianized relation at the sth-vertex.

2-fold branched covering of $\mathcal{O}$ along a cycle.

Let $\mathcal{O}$ be an orbifold of $\mathcal{F}$, Γ its singularity graph and c a subgraph of Γ which is either a cycle of edges or a knot. If there exists a map $\psi : H_1(\mathcal{O}) \longrightarrow \mathbb{Z}_2$ which sends a small meridian around each edge contained in c, respectively around the knot c, to the generator of $\mathbb{Z}_2$ and all the other small meridians around edges of Γ to the trivial element, we call *2-fold branched covering of $\mathcal{O}$ along c* the covering of $\mathcal{O}$ defined by ψ.

We are now ready to state Proposition 3 (recall that h is the covering involution of the covering $M \to \mathcal{O}(L)$):

Proposition 3. *Let $\mathcal{O}$ be an orbifold of the subfamily $\mathcal{F}_i$ for $i \geq 2$ and $\widetilde{\mathcal{O}}$ the 2-fold branched covering of $\mathcal{O}$ along a cycle of edges or a loop c. If c contains no interior points of the projection to $\mathcal{O}$ of the fixed point set of h, then $\widetilde{\mathcal{O}}$ is an orbifold of the subfamily $\mathcal{F}_{i-1}$.*

Proof. We first prove that the covering $\mathcal{O}(L) \to \mathcal{O}$ factors through $\widetilde{\mathcal{O}}$. Up to identification of $\pi_1\mathcal{O}(L)$ and $\pi_1\widetilde{\mathcal{O}}$ with subgroups of $\pi_1\mathcal{O}$, this is equivalent to prove that $\pi_1\mathcal{O}(L)$ is a subgroup of $\pi_1\widetilde{\mathcal{O}}$. Since $\pi_1\widetilde{\mathcal{O}}$ has index two in $\pi_1\mathcal{O}$ the intersection $\pi_1\widetilde{\mathcal{O}} \cap \pi_1\mathcal{O}(L)$ has index at most two in $\pi_1\mathcal{O}(L)$. So it is enough to prove that the case that $\pi_1\widetilde{\mathcal{O}} \cap \pi_1\mathcal{O}(L)$ has index two in $\pi_1\mathcal{O}(L)$ is impossible.

Suppose, ad absurdum, that $\pi_1\widetilde{\mathcal{O}} \cap \pi_1\mathcal{O}(L)$ is the fundamental group $\pi_1 N$ of an orbifold N which is a 2-fold covering of $\mathcal{O}(L)$. The 2-fold coverings of $\mathcal{O}(L)$ are branched and classified by epimorphisms of $H_1\mathcal{O}(L)$ onto $\mathbb{Z}_2$. If L is connected, $H_1\mathcal{O}(L) \cong \mathbb{Z}_2$ and $\mathcal{O}(L)$ has a unique 2-fold covering, $M_0 := M$, with covering involution $h_0 := h$. If L has two components, $H_1\mathcal{O}(L) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathcal{O}(L)$ has three distinct 2-fold coverings, namely: The 3-manifold $M_0 = M$ with covering involution $h_0 = h$, the 2-fold covering M_1 branched along the first component of L with covering involution, say h_1, the 2-fold covering M_2 branched along the second component of L with covering involution, say h_2. So for some i, N is homeomorphic to M_i and the involution h_i projects to a nontrivial involution of $\pi_1\widetilde{\mathcal{O}}$. Thus the quotient orbifold $\widetilde{\mathcal{O}}/h_i$ is homeomorphic to $\mathcal{O}$ and the projection of the fixed point set of h_i to $\mathcal{O}$ is a subset of the branching set of the covering $\widetilde{\mathcal{O}} \longrightarrow \mathcal{O}$. But this is impossible, because we have supposed that the branching set c contains no interior points of the projection of the fixed point set of h to $\mathcal{O}$.

We have thus proved that the covering $\mathcal{O}(L) \to \mathcal{O}$ factors through $\widetilde{\mathcal{O}}$.

Finally we have to show that $\widetilde{\mathcal{O}}$ is an element of $\mathcal{F}$ (note that what we have proved above implies that $\widetilde{\mathcal{O}}$ is a quotient of $\mathcal{O}(L)$, but it is not yet clear at this point if we can get $\widetilde{\mathcal{O}}$ by only quotienting by involutions with nonempty fixed point sets as it is the case when we construct the orbifolds of $\mathcal{F}$).

To fix notation let $\widetilde{\mathcal{O}} = M/\widetilde{H}$ for some subgroup $\widetilde{H}$ of S containing h. To the pair $(M, \widetilde{H})$ we can associate a family $\mathcal{E}$ of hyperbolic 3-orbifolds which is the disjoint union of subfamilies $\mathcal{E}_1, \dots, \mathcal{E}_m$. Each orbifold of $\mathcal{E}_j$ is the quotient of M by a subgroup of $\widetilde{H}$ of order 2^j and it is also an element of $\mathcal{F}_i$ (indeed now we are using only involutions with nonempty fixed point sets). So by Proposition 2, the last subfamily $\mathcal{E}_m$ of $\mathcal{E}$ contains a unique orbifold, say $\mathcal{O}' = M/H'$ where H' is the subgroup of $\widetilde{H}$ generated by all the elements with nonempty fixed point sets. To conclude the proof, it is enough to show that $\widetilde{\mathcal{O}} = \mathcal{O}'$, because $\mathcal{O}'$, hence $\widetilde{\mathcal{O}}$ is an element of $\mathcal{F}$ (in particular of $\mathcal{F}_{i-1}$ because it is the quotient of M by a subgroup of S of order 2^{i-1}).

If $H' = \widetilde{H}$, then $\widetilde{\mathcal{O}} = \mathcal{O}'$ and the proof is complete. We shall show that the case that H' is a proper subgroup of $\widetilde{H}$ is impossible. If H' is a proper subgroup of $\widetilde{H}$, the normalizer $N_{\widetilde{H}}H'$ is larger than H' by Proposition 1 and the factor group $N_{\widetilde{H}}H'/H'$ projects to a 2-group of isometries of $\mathcal{O}'$. Hence there exists at least one involution u of $\mathcal{O}'$ which lifts to $\widetilde{H}$. The involution u acts freely on $\mathcal{O}'$, because H' is the subgroup of $\widetilde{H}$ generated by all the elements with nonempty fixed point sets. The quotient $\mathcal{O}_1$ of $\mathcal{O}'$ by u is a hyperbolic 3-orbifold which is also the quotient of M by a 2-subgroup H_1 of $\widetilde{H}$ containing H' as a subgroup of index two.

If H_1 is a proper subgroup of $\widetilde{H}$, again by Proposition 1, there exists at least one involution v of $\mathcal{O}_1$ which lifts to $\widetilde{H}$. So we can construct a quotient $\mathcal{O}_2 := \mathcal{O}_1/v$ of $\mathcal{O}_1$ by v and v acts freely, because H' contains all the elements of $\widetilde{H}$ with nonempty fixed point sets.

Iteratively, for some $m \geq 1$ we can construct a hierarchy $\mathcal{O}_1, \dots \mathcal{O}_m$ of orbifolds and a sequence $H_1, \dots H_m$ of corresponding subgroups of $\widetilde{H}$ such that each orbifold $\mathcal{O}_i$ is the quotient of $\mathcal{O}_{i-1}$ by an involution acting freely on $\mathcal{O}_i$ and each H_i contains H_{i-1} as a subgroup of index two. After finitely many steps our procedure stops because we have got the quotient orbifold $\widetilde{\mathcal{O}} = M/\widetilde{H}$ of M by the full group of isometries $\widetilde{H}$.

Since $m \geq 1$, the orbifold $\widetilde{\mathcal{O}}$ admits a free regular 2-fold covering, which is impossible because the underlying topological space of $\widetilde{\mathcal{O}}$, which is the 2-fold branched covering of S^3 branched along a knot, is a $\mathbb{Z}_2$-homology 3-sphere [**16**, Sublemma 15.4].

This finishes the proof of Proposition 3.

4. The singularity graphs.

In this section we explain which combinatorial modifications may take place on the singularity graph of an orbifold $\mathcal{O}$ of $\mathcal{F}_i$ when passing to a quotient of

$\mathcal{O}$ in $\mathcal{F}_{i+1}$. Indeed this section may be skipped without affecting the proof of Theorem 1. We have included it in the paper just to give some intuition why one considers graphs of Type IA, ... , IIID as natural candidates for the singularity graphs of the orbifolds of $\mathcal{F}$.

First of all we need a generalized definition of graph which includes loops, because, in general, the singularity graph of an orbifold $\mathcal{O} \in \mathcal{F}$ is a union of trivalent graphs and disjoint knots (we use the term 'loop' in the combinatorial setting, 'knot' in the topological one). A graph Γ is a set $(V(\Gamma), E(\Gamma), c_1, \dots c_r)$ for some nonnegative integer r, where the vertex-set $V(\Gamma)$ is a finite set of elements called vertices, the edge-set $E(\Gamma)$ is a finite set of ordered pairs of distinct elements of $V(\Gamma)$ called edges and $c_1, \dots c_r$ is a finite set of disjoint loops.

A graph is called *admissible* if it is one of the twelve graphs IA, ... , IIID, *inadmissible* in any other case. So our notation for the twelve admissible graphs (see Figure 1) is:

IA one loop:

$$(V(\Gamma) = \emptyset, E(\Gamma) = \emptyset, c_1)$$

IIA two loops:

$$(V(\Gamma) = \emptyset, E(\Gamma) = \emptyset, c_1, c_2)$$

IIIA three loops:

$$(V(\Gamma) = \emptyset, E(\Gamma) = \emptyset, c_1, c_2, c_3)$$

IB theta-graph:

$(V(\Gamma) = \{v_1, v_2\}, E(\Gamma) = \{(v_1, v_2), (v_1, v_2)', (v_1, v_2)''$ and three reverse edges$\})$

IIB tetrahedral graph:

$(V(\Gamma) = \{v_1, v_2, v_3, v_4\}, E(\Gamma) = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_3), (v_2, v_4), (v_3, v_4)$ and six reverse edges$\})$

IIIB Kuratowski graph:

$(V(\Gamma) = \{v_1, v_2, v_3, v_4, v_5, v_6\}, E(\Gamma) = \{(v_1, v_2), (v_1, v_4), (v_1, v_6), (v_2, v_3), (v_2, v_5), (v_3, v_4), (v_3, v_6), (v_4, v_5), (v_5, v_6)$ and nine reverse edges$\})$

IC pince-nez graph:

$(V(\Gamma) = \{v_1, v_2\}, E(\Gamma) = \{(v_1, v_1), (v_1, v_2), (v_2, v_2)$ and $(v_2, v_1)\})$

IIC $(V(\Gamma) = \{v_1, v_2, v_3, v_4\}, E(\Gamma) = \{(v_1, v_3), (v_1, v_3)', (v_1, v_2), (v_2, v_4), (v_2, v_4)', (v_3, v_4)$ and six reverse edges$\})$

IIIC $(V(\Gamma) = \{v_1, v_2, v_3, v_4, v_5, v_6\}, E(\Gamma) = \{(v_1, v_2),\ (v_1, v_3),\ (v_1, v_5),\ (v_2, v_4),\ (v_2, v_6),\ (v_3, v_4),\ (v_3, v_5),\ (v_4, v_6),\ (v_5, v_6)$ and nine reverse edges$\})$

ID a theta-graph and a loop:
$(V(\Gamma) = \{v_1, v_2\}, E(\Gamma) = \{(v_1, v_2), (v_1, v_2)', (v_1, v_2)''$ and three reverse edges$\}, c_1)$

IID a pince-nez and a loop:
$(V(\Gamma) = \{v_1, v_2\}, E(\Gamma) = \{(v_1, v_1),\ (v_1, v_2),\ (v_2, v_2)$ and $(v_2, v_1)\}, c_1)$

IIID $(V(\Gamma) = \{v_1, v_2, v_3, v_4\}, E(\Gamma) = \{(v_1, v_1),\ (v_1, v_2),\ (v_2, v_3), (v_2, v_4), (v_3, v_4), (v_3, v_4)',\ (v_2, v_1), (v_3, v_2), (v_4, v_2), (v_4, v_3)$ and $(v_4, v_3)'\})$.

Our point here is to understand which combinatorial modifications occur on the singularity graph of an orbifold of $\mathcal{F}_i$ when passing to $\mathcal{F}_{i+1}$. We are concerned only with the combinatorial structure of the graph, so we forget singularity orders at the various points.

As a warming up example, start with an orbifold $\mathcal{O}(K)$ of $\mathcal{F}_1$, with K a knot, and let u be an involution of $\mathcal{O}(K)$ with connected fixed point set. Either the fixed point set of u intersects K into two points and u acts as a reflection on K or the fixed point set of u is disjoint from K and u acts as a rotation on K (the fixed point set of u is distinct from K by the positive solution of the Smith conjecture). Correspondingly, the graph of the quotient orbifold $\mathcal{O}(K)/u$ is of Type IB or IIA. Thus, in the case of a knot, the singular set of any orbifold of the subfamily $\mathcal{F}_2$ is combinatorially a theta-graph or a set of two disjoint loops.

Passing to $\mathcal{F}_3$ we construct the quotients of the orbifolds of $\mathcal{F}_2$. For example, consider an orbifold $\mathcal{O}$ of $\mathcal{F}_2$ with singularity graph of Type IB. An involution v of $\mathcal{O}$ with connected fixed point set may act in the following ways (up to combinatorial equivalence):

i)

$$v_1 \to v_1 \quad v_2 \to v_2$$

$$\begin{array}{cc} (v_1, v_2) \to (v_1, v_2) & (v_2, v_1) \to (v_2, v_1) \\ (v_1, v_2)' \to (v_1, v_2)'' & (v_2, v_1)' \to (v_2, v_1)'' \\ (v_1, v_2)'' \to (v_1, v_2)' & (v_2, v_1)'' \to (v_2, v_1)' \end{array}$$

ii)

$$v_1 \to v_2 \quad v_2 \to v_1$$

$$\begin{aligned} (v_1, v_2) \to (v_2, v_1) &\quad (v_2, v_1) \to (v_1, v_2) \\ (v_1, v_2)' \to (v_2, v_1)' &\quad (v_2, v_1)' \to (v_1, v_2)' \\ (v_1, v_2)'' \to (v_2, v_1)'' &\quad (v_2, v_1)'' \to (v_1, v_2)'' \end{aligned}$$

iii)

$$v_1 \to v_2 \quad v_2 \to v_1$$

$$\begin{aligned} (v_1, v_2) \to (v_2, v_1) &\quad (v_2, v_1) \to (v_1, v_2) \\ (v_1, v_2)' \to (v_2, v_1)'' &\quad (v_2, v_1)' \to (v_1, v_2)'' \\ (v_1, v_2)'' \to (v_2, v_1)' &\quad (v_2, v_1)'' \to (v_1, v_2)'. \end{aligned}$$

The singularity graph of the corresponding quotient orbifold is, respectively, of Type IB, IIB or IC.

By a routine, not unpleasant, exercise in combinatorial theory, it is easy to show that the twelve graphs IA, ... , IIID appear after a few steps. This is a purely combinatorial operation, which gives many graphs which are inadmissible. But not all graphs we get combinatorially are singularity graphs of some orbifolds in some family $\mathcal{F}$. The proof of Theorem 1 in Section 5 will make clear that topological obstructions exclude graphs which are inadmissible.

5. Proof of Theorem 1.

Let M be a hyperbolic 2-fold branched covering of a link with one or two components, S a 2-group of orientation-preserving isometries of M containing the covering involution and $\mathcal{F}$ the family of orbifolds associated to M and S (see Section 3). To prove Theorem 1 we show that inadmissible graphs can not occur as singularity graphs of the orbifolds of $\mathcal{F}$.

This implies the thesis that the singularity graph of the quotient orbifold M/S is of Type IA, ... , IIID. In fact, by Proposition 2, the last subfamily $\mathcal{F}_n$ of $\mathcal{F}$ contains the quotient orbifold M/H where H is the subgroup of S which is generated by all the elements with nonempty fixed point sets. By Proposition 1 the quotient orbifold M/S is the final output of a hierarchy of quotients, starting with M/H and quotienting by involutions at each step. Since any element of $S - H$ acts freely on M the quotienting involutions act also freely at each step and it is easy to check that, since the singularity graph of M/H is admissible, the singularity graph we get at each step is also admissible.

To prove that the singularity graphs of the orbifolds of $\mathcal{F}$ are admissible, we proceed by contradiction: Let i be the minimal index such that $\mathcal{F}_i$ contains an orbifold $\mathcal{O}$ with inadmissible singularity graph. Minimality of i implies that $\mathcal{O}$ is a quotient of an orbifold of $\mathcal{F}_{i-1}$ which has admissible singularity graph. Combinatorially the graph of $\mathcal{O}$ is obtained, when passing from $\mathcal{F}_{i-1}$ to $\mathcal{F}_i$, by a modification of one of the graphs IA, … , IIID as explained in Section 4.

Again it is a routine exercise to check that the graphs we get combinatorially, when passing from $\mathcal{F}_{i-1}$ to $\mathcal{F}_i$, from the graphs IA, … , IIID are the following:

IA: $\rightarrow$ IA, IIA, IB
IIA: $\rightarrow$ IIA, IIIA, IIB, IIC, ID
IIIA: $\rightarrow$ IIIA, IIIB, IIIC, ID, X1, X2, X3, X4, X5, X6, X7
IB: $\rightarrow$ IB, IIB, IC
IIB: $\rightarrow$ IIB, IIC
IIIB: $\rightarrow$ IIC, IIIC
IC: $\rightarrow$ IIB, IC, IIC
IIC: $\rightarrow$ IIIB, IIC, IIIC, ID, IID, IIID
IIIC: $\rightarrow$ IIIB, IIIC, IIID
ID: $\rightarrow$ IIC, IIIC, ID, IID, IIID, X2, X4
IID: $\rightarrow$ IIIB, IIIC, IID, IIID, X2, X3, X4, X5, X7
IIID: $\rightarrow$ IIIC, IIID, X2.

The seven inadmissible graphs X_j are (Figure 2):

X1 four loops:

$$(V(\Gamma)=\emptyset, E(\Gamma)=\emptyset, c_1, c_2, c_3, c_4)$$

X2

$(V(\Gamma) = \{v_1, v_2, v_3, v_4, v_5, v_6\}, E(\Gamma) = \{(v_1, v_2),\ (v_1, v_2)',\ (v_1, v_6),\ (v_2, v_3),\ (v_3, v_4),\ (v_3, v_5),\ (v_4, v_5),\ (v_4, v_6),\ (v_5, v_6)$ and nine reverse edges$\})$

X3

$(V(\Gamma) = \{v_1, v_2, v_3, v_4, v_5, v_6\}, E(\Gamma) = \{(v_1, v_2),\ (v_1, v_2)',\ (v_1, v_6),\ (v_2, v_3),\ (v_3, v_4),\ (v_3, v_4)',\ (v_4, v_5),\ (v_5, v_6),\ (v_5, v_6)'$ and nine reverse edges$\})$

X4 a tetrahedral graph and a loop:

$(V(\Gamma) = \{v_1, v_2, v_3, v_4\}, E(\Gamma) = \{(v_1, v_2),\ (v_1, v_3),\ (v_1, v_4),\ (v_2, v_3),\ (v_2, v_4), (v_3, v_4)$ and six reverse edges$)\}, c_1)$

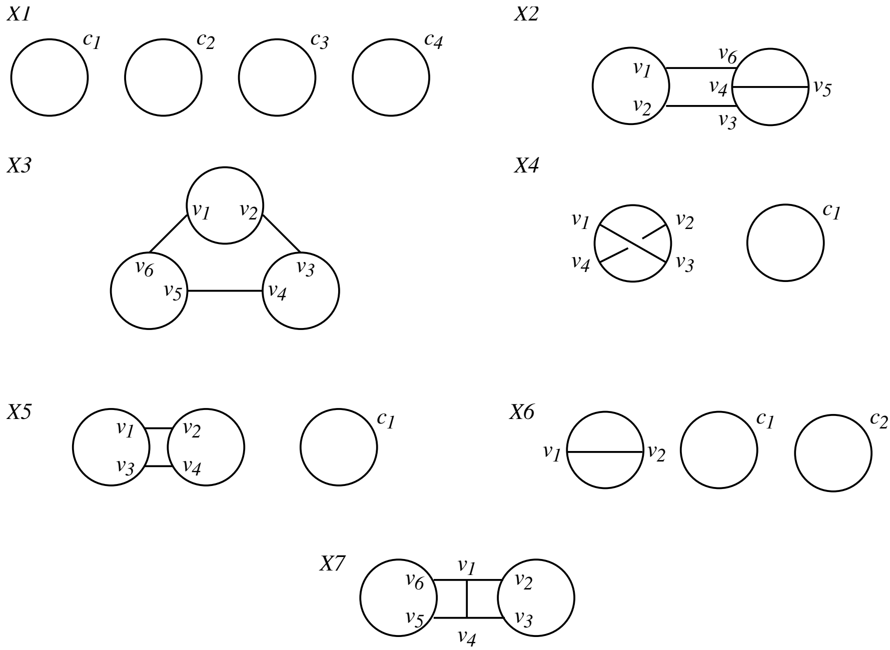

Figure 2

X5

$(V(\Gamma) = \{v_1, v_2, v_3, v_4\}, E(\Gamma) = \{(v_1, v_3),\ (v_1, v_3)',\ (v_1, v_2),\ (v_2, v_4),$ $(v_2, v_4)', (v_3, v_4)$ and six reverse edges$\}, c_1)$

X6 a theta-graph and two loops:

$(V(\Gamma) = \{v_1, v_2\}, E(\Gamma) = \{(v_1, v_2), (v_1, v_2)', (v_1, v_2)''$ and three reverse edges$\}, c_1, c_2)$

X7

$(V(\Gamma) = \{v_1, v_2, v_3, v_4, v_5, v_6\}, E(\Gamma) = \{(v_1, v_2),\ (v_2, v_3),\ (v_2, v_3)',$ $(v_3, v_4),\ (v_1, v_4),\ (v_4, v_5),\ (v_5, v_6),\ (v_5, v_6)',\ (v_6, v_1)$ and nine reverse edges$\})$.

Thus the singularity graph of $\mathcal{O}$ is combinatorially one of the seven graphs X_j.

The thesis now follows from the following Claim which contradicts minimality of i. The proof of the Claim occupies the rest of the section.

Claim. If $\mathcal{F}_i$ contains an orbifold with singularity graph of Type X_j, there exists an index k, $k < i$ and $k \geq 1$, such that $\mathcal{F}_k$ contains an orbifold with inadmissible singularity graph.

Proof of the Claim.

We first prove that if $\mathcal{O} \in \mathcal{F}_i$ has singularity graph X_j, there exists an orbifold $\widetilde{\mathcal{O}}$ of $\mathcal{F}_{i-1}$ such that its singularity graph is either inadmissible or of Type IIIA.

The orbifold $\widetilde{\mathcal{O}}$ is a 2-fold branched covering of $\mathcal{O}$ along a cycle which satisfies the hypotheses of Proposition 3. More precisely, we can always find a cycle of edges c or a loop in the singularity graph of $\mathcal{O}$ such that there exists the 2-fold branched covering $\widetilde{O}$ of $\mathcal{O}$ along c and c contains no interior points of the projection $F(h)$ of the fixed point set of h to $\mathcal{O}$. Here is the cycle to choose for the various graphs X_j (remember that $F(h)$ has at most two disjoint components and each component is an edge or a loop of the singularity graph of $\mathcal{O}$):

- X1 Set $c = c_1$ (up to renaming loops). The singularity graph of $\widetilde{\mathcal{O}}$ is a set of at least three disjoint loops, so inadmissible or of Type IIIA.
- X2 Whatever is the projection $F(h)$, we can always make one of the following choices: $c = (v_1, v_2) \cup (v_2, v_1)'$; $c = (v_4, v_5) \cup (v_5, v_6) \cup (v_6, v_4)$; $c = (v_3, v_4) \cup (v_4, v_6) \cup (v_6, v_5) \cup (v_5, v_3)$. In all cases the singularity graph of $\widetilde{\mathcal{O}}$ is inadmissible.
- X3 Set $c = (v_1, v_2) \cup (v_2, v_1)'$ (up to renaming vertices). The singularity graph of $\widetilde{\mathcal{O}}$ is inadmissible.
- X4 Whatever is the projection $F(h)$, we can make one of the following choices: $c = c_1$; $c = (v_1, v_2) \cup (v_2, v_3) \cup (v_3, v_4) \cup (v_4, v_5)$; if the singularity order of, say (v_1, v_3), is greater than two, set $c = (v_1, v_3) \cup (v_3, v_4) \cup (v_4, v_1)$. In the first case the singularity graph of $\widetilde{\mathcal{O}}$ is inadmissible, in the second case it is of Type IIIA, in the third case is inadmissible.
- X5 Either we can choose $c = c_1$ or $c = (v_1, v_3) \cup (v_3, v_1)'$ (up to renaming vertices). In all cases the singularity graph of $\widetilde{\mathcal{O}}$ is inadmissible.
- X6 Either we can choose $c = c_1$ or $c = (v_1, v_2) \cup (v_2, v_1)'$. In the first case the singularity graph of $\widetilde{\mathcal{O}}$ is inadmissible, in the second case it is of Type IIIA or inadmissible.
- X7 Either we can choose $c = (v_2, v_3) \cup (v_3, v_2)'$ or $c = (v_1, v_2) \cup (v_2, v_3) \cup (v_3, v_4) \cup (v_4, v_1)$. In all cases the singularity graph of $\widetilde{\mathcal{O}}$ is inadmissible.

To complete the proof we finally show that in the remaining case that $\mathcal{O}$ has a 2-fold branched covering $\widetilde{\mathcal{O}} \in \mathcal{F}_{i-1}$ with singularity graph of Type IIIA, there also exists an index k, $k < i$ and $k \geq 1$, such that $\mathcal{F}_k$ contains an orbifold with inadmissible singularity graph, again contradicting minimality of i. The rest of the section describes how to construct such an orbifold in the various cases.

In most cases the orbifold with inadmissible singularity graph is again a 2-fold branched covering of $\widetilde{\mathcal{O}}$ along a cycle which satisfies the hypotheses of Proposition 3. More precisely, we can always find a loop c_1 in the singularity graph $(V(\Gamma) = \emptyset, E(\Gamma) = \emptyset, c_1, c_2, c_3)$ of $\widetilde{\mathcal{O}}$ which contains no interior points of the projection of the fixed point set of h to $\widetilde{\mathcal{O}}$. Thus, by Proposition 3, the 2-fold branched covering $\widetilde{\mathcal{O}}'$ of $\widetilde{\mathcal{O}}$ along c_1 is an orbifold of $\mathcal{F}_{i-2}$.

Before going on we fix some notation. We denote by u, respectively by v, the covering involution of the covering $\widetilde{\mathcal{O}} \to \mathcal{O}$, respectively $\widetilde{\mathcal{O}}' \to \widetilde{\mathcal{O}}$. Since we assume that the singularity graph of $\mathcal{O}$ is of Type X_j, it can be easily checked that u fixes setwise each component c_i of the singularity graph of $\widetilde{\mathcal{O}}$. In particular u lifts to $\widetilde{\mathcal{O}}'$ and its two lifts u_1 and $u_2 := (u_1)v$ generate a dihedral group D of order four.

The fixed point set $F(u)$ of u in $\widetilde{\mathcal{O}}$ may intersect the singularity graph of $\widetilde{\mathcal{O}}$ in three different ways. We have to distinguish these three cases.

a) $F(u) \cap c_2 = F(u) \cap c_3 = \emptyset$.

In this case the singularity graph of $\widetilde{\mathcal{O}}'$ is inadmissible. Indeed the singularity graph of $\widetilde{\mathcal{O}}'$ contains the preimages $\widetilde{c}_2$ and $\widetilde{c}_3$ in $\widetilde{\mathcal{O}}'$ of c_2 and c_3. The preimage $\widetilde{c}_2$ can not be connected because, if $\widetilde{c}_2$ is connected, the group D would contain three distinct rotations of order two of $\widetilde{c}_2$ such that their fixed point sets do not intersect $\widetilde{c}_2$, which is impossible. So $\widetilde{c}_2$ has two components. An analogous argument holds for the preimage $\widetilde{c}_3$ of c_3 in $\widetilde{\mathcal{O}}'$ which also has two components.

b) $F(u) \cap c_2 = \emptyset$; $F(u) \cap c_3 \neq \emptyset$.

In this case either the singularity graph of $\widetilde{\mathcal{O}}'$ is inadmissible or there exists in $\mathcal{F}_{i-1}$ a quotient $\widetilde{\mathcal{O}}'/u_1$ of $\widetilde{\mathcal{O}}'$ with inadmissible singularity graph. By the same argument as in a), $\widetilde{c}_2$ has two components which are interchanged by the action of the covering involution v.

If $\widetilde{c}_3$ has also two components, the singularity graph of $\widetilde{\mathcal{O}}'$ is inadmissible.

If $\widetilde{c}_3$ is connected we show that the quotient $\widetilde{\mathcal{O}}'/u_1$ of $\widetilde{\mathcal{O}}'$ by a lift u_1 of u is an orbifold of $\mathcal{F}_{i-1}$ and its singularity graph is inadmissible. To prove this it is enough to look at the action induced by u_1 on $\widetilde{c}_2$ and $\widetilde{c}_3$. First of all, if $\widetilde{c}_3$ is connected, both u_1 and u_2 act as reflections on $\widetilde{c}_3$. Indeed an involution of $\widetilde{\mathcal{O}}'$ either acts freely or it has connected fixed point set because its underlying topological space is S^3. This implies that each of u_1 and u_2 has a connected fixed point set intersecting $\widetilde{c}_3$ into two distinct points (the preimage of the two intersection points $F(u) \cap c_3$ in $\widetilde{\mathcal{O}}$ consists of four distinct points of $\widetilde{c}_3$).

On the other hand u_2 is the product $u_2 = (u_1 v)$; so one between u_1 and u_2, say u_2, interchanges the two components of $\widetilde{c}_2$ as v, while u_1 fixes setwise each of the two components of $\widetilde{c}_2$ acting as a rotation on them. It follows now that the singularity graph of the quotient orbifold $\widetilde{\mathcal{O}}'/u_1$, which is an orbifold of $\mathcal{F}_{i-1}$, is inadmissible.

c) $F(u) \cap c_2 \neq \emptyset$; $F(u) \cap c_3 \neq \emptyset$.

In this case either the singularity graph of $\widetilde{\mathcal{O}}'$ is inadmissible or there exists in $\mathcal{F}_{i-1}$ a quotient $\widetilde{\mathcal{O}}'/u_1$ of $\widetilde{\mathcal{O}}'$ with inadmissible singularity graph.

If both $\widetilde{c}_2$ and $\widetilde{c}_3$ have two components, the singularity graph of $\widetilde{\mathcal{O}}'$ is inadmissible.

So assume for the following that $\widetilde{c}_3$ is connected; in this case arguing as in b), we find that both lifts u_1 and u_2 of u to $\widetilde{\mathcal{O}}'$ act as reflections on $\widetilde{c}_3$.

If $\widetilde{c}_2$ has two components, one between u_1 and u_2, say u_1, fixes setwise each of the two components acting as a reflection on them. Factoring by u_1 we find an orbifold in $\mathcal{F}_{i-1}$ with singularity graph which is either of Type IIIB or IIIC or inadmissible. The case that the singularity graph is of Type IIIB or IIIC is impossible because this orbifold is a 2-fold covering of $\widetilde{O}$ and it can not have a inadmissible singularity graph (see the list at the beginning of this section).

The only possible left case is that both $\widetilde{c}_2$ and $\widetilde{c}_3$ are connected. So the singularity graph of $\widetilde{\mathcal{O}}'$ has two components and both u_1 and u_2 acts as reflections on each component. This case is impossible because, in this case, the quotient $\widetilde{\mathcal{O}}'/D$ which is homeomorphic to $\widetilde{\mathcal{O}}$ has a singularity graph of Type IIIB or IIIC and not of Type X_j, a contradiction.

This finishes the proof of the Claim.

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