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K-GROUPS AND CLASSIFICATION OF SIMPLE QUOTIENTS OF GROUP C*-ALGEBRAS OF CERTAIN DISCRETE 5-DIMENSIONAL NILPOTENT GROUPS

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The K-groups, the range of trace on K_0, and isomorphism classifications are obtained for simple infinite dimensional quotient C*-algebras of the group C*-algebras of six lattice subgroups, corresponding to each of the six non-isomorphic 5-dimensional connected, simply connected, nilpotent Lie groups. Connes' non-commutative geometry involving cyclic cocycles and the Chern character play a key role in the proofs.

1. Introduction.

It is known that there are only six non-isomorphic 5-dimensional connected, simply connected, nilpotent Lie groups. These groups are denoted by $G_{5,j} (j = 1, \dots, 6)$, and were studied in great detail in Nielsen's paper [**10**]. In [**9**], Milnes and the author have studied a natural lattice subgroup $H_{5,j}$ of $G_{5,j}$. These subgroups are higher dimensional analogues of the well-known discrete Heisenberg group H_3, (but with more complicated multiplication rules inherited from $G_{5,j}$). The main result of [**9**] is an identification of all the simple infinite dimensional quotient C*-algebras of the group C*-algebra $C^*(H_{5,j})$ – more specifically, they consist, respectively, of the 'primary' algebras $A_\theta^{5,1}$, $A_{\theta,\varphi}^{5,2}$, $A_\theta^{5,3}$, $A_{\theta,\varphi}^{5,4}$, $A_\theta^{5,5}$, $A_\theta^{5,6}$, (where θ, φ are irrational and are independent in the $5,2$ and $5,4$ cases), other simple C*-algebras isomorphic to matrix algebras over irrational rotation algebras (of any size and any irrational parameter), and a few more which are expressed as crossed products by the integers.

The objective of this paper is to find the K-groups, the range of the trace on K_0, and obtain a classification for the 'primary' simple quotient C*-algebras amongst themselves. Since each of these algebras is isomorphic to a crossed product by the integers, one uses the Pimsner-Voiculescu six term exact sequence [**13**] to compute their K-groups and Pimsner's Theorem on the tracial range [**12**]. For the algebras $A_\theta^{5,1} \cong A_\theta \otimes A_\theta$, $A_{\theta,\varphi}^{5,2}$, $A_\theta^{5,3}$, $A_{\theta,\varphi}^{5,4}$, application of the Pimsner-Voiculescu exact sequence is not hard. This is done briefly in Section 2, and included for comparison and completion.

For the algebras $A_\theta^{5,5}$ and $A_\theta^{5,6}$, however, the application of the Pimsner-Voiculescu sequence is not so straightforward, as the action of the underlying automorphism (of the crossed product) on K_* requires some careful work. More specifically, in order to calculate this action we make use of Connes' non-commutative geometry involving cyclic cocycles and the Connes Chern character [3] in order to decipher K-group elements. This is dealt with in Sections 3 and 4, which are the main parts of the paper. In summary, we obtain the following result on the K-groups and range of the trace on K_0:

Algebra	A_θ^3	A_θ^4	$A_\theta^{5,1}$	$A_{\theta,\varphi}^{5,2}$	$A_\theta^{5,3}$	$A_{\theta,\varphi}^{5,4}$	$A_\theta^{5,5}$	$A_\theta^{5,6}$
$K_0 = K_1 =$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^8$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^4$
$\tau_* K_0 = \mathbb{Z}+$	$\mathbb{Z}\theta$	$\mathbb{Z}\theta$	$\mathbb{Z}\theta + \mathbb{Z}\theta^2$	$\mathbb{Z}\theta + \mathbb{Z}\varphi$	$\mathbb{Z}\theta + \mathbb{Z}\theta^2$	$\mathbb{Z}\theta + \mathbb{Z}\varphi$	$\mathbb{Z}\theta$	$\mathbb{Z}\theta + \mathbb{Z}\theta^2$

Here, we have included the well-known result for the irrational rotation algebra $A_\theta = A_\theta^3$ ([13] and [14]), as well as for the Heisenberg C*-algebra A_θ^4 studied by Packer in [11] (where it is referred to as *class 2*). (According to the convention adopted in [8] and [9], the superscripts on A_θ^3, A_θ^4 indicate the dimensions of the discrete nilpotent groups for which these are simple infinite dimensional quotients of the associate group C*-algebra – namely, the discrete Heisenberg group H_3 and the discrete group H_4 introduced in [8], respectively.)

The determination of the simple infinite dimensional quotients arising from 6-dimensional discrete nilpotent groups $H_{6,j}$ has been done by Milnes in [6] and [7] for $H_{6,4}$ and $H_{6,10}$ and by Junghenn and Milnes in [5] for $H_{6,7}$. It is known that there are exactly twenty-four non-isomorphic 6-dimensional connected, simply connected, nilpotent Lie groups $G_{6,j}$ $(j = 1, \ldots, 24)$ (see Nielsen [10]), each of which contains a natural lattice subgroup $H_{6,j}$. In the 7-dimensional case, there are by contrast uncountably many non-isomorphic such Lie groups.

Notation. Throughout the paper we shall adopt Connes' and Rieffel's convention and write

$$e(t) \;:=\; e^{2\pi i t}.$$

Briefly, recall Pimsner's procedure [12] for finding the range of the trace in the case of crossed products by the integers. Let A be a C*-algebra (for us unital) and σ an automorphism of A. Let $\boldsymbol{\tau}$ be a trace state on $A \rtimes_\sigma \mathbb{Z}$ and use it also to denote its restriction to A. Let $q : \mathbb{R} \to \mathbb{R}/\boldsymbol{\tau}_* K_0(A)$ denote the quotient map. For an element $[u]$ in $\ker(\sigma_* - id_*) \subseteq K_1(A)$, where u is in $U_n(A)$ (the $n \times n$ unitary matrices in A), consider the element of $\mathbb{R}/\boldsymbol{\tau}_* K_0(A)$, called the "determinant" of $[u]$, given by

$$\Delta[u] = q\left(\frac{1}{2\pi i}\int_a^b (\boldsymbol{\tau}\otimes \mathrm{Tr})(\dot{\xi}(t)\xi(t)^{-1})dt\right)$$

where $\xi : [a,b] \to U_n(A)$ is a piecewise continuously differentiable path such that $\xi(a) = 1$ and $\xi(b) = \sigma(u)u^{-1}$. Pimsner's result [12] (Theorem 3) is that the following is a short exact sequence:

$$0 \longrightarrow \boldsymbol{\tau}_* K_0(A) \xrightarrow{\ \iota\ } \boldsymbol{\tau}_* K_0(A \rtimes_\sigma \mathbb{Z}) \xrightarrow{\ q\ } \Delta(\ker(\sigma_* - \mathrm{id}_*)) \longrightarrow 0$$

where ι is the canonical inclusion (as subgroups of $\mathbb{R}$) and q is the restriction of the canonical map q.

2. The C*-algebra $A_\theta^{5,k}$ for $k = 1, 2, 3, 4$.

2.1. The C*-algebra $A_\theta^{5,1}$. Let us first look at the C*-algebra $A_\theta^{5,1}$ generated by unitaries U, V, W, X satisfying

$$UV = \lambda VU, \qquad WX = \lambda XW, \qquad UW = WU,$$
$$UX = XU, \qquad VW = WV, \qquad VX = XV, \tag{2.1}$$

where $\lambda = e(\theta)$ and θ is irrational (as in [9], Section 1). It is clear that it is isomorphic to the simple C*-algebra $A_\theta \otimes A_\theta$. We prefer to view it, however, as the crossed product $(A_\theta \otimes C(\mathbb{T})) \rtimes_\sigma \mathbb{Z}$ where A_θ is generated by U, V, $C(\mathbb{T})$ by W, and $\sigma = \mathrm{Ad}_X$. So σ fixes U, V and $\sigma(W) = \lambda W$. Since σ is homotopic to the identity automorphism (in the sense of [1], 5.2.2), the Pimsner-Voiculescu sequence yields that $K_j(A_\theta^{5,1})$ is isomorphic to $K_j(A_\theta \otimes C(\mathbb{T}^2))$ $(j = 0, 1)$, which is isomorphic to $\mathbb{Z}^8$. (One can also use the Künneth Theorem [16] to get $K_j(A_\theta \otimes A_\theta) = \mathbb{Z}^8$.) From Pimsner's range of trace formula, one needs to know the generators of the kernel of $id_* - \sigma_*$ in $K_1(A_\theta \otimes C(\mathbb{T}))$. But $id_* - \sigma_* = 0$. It is easy to show that a basis for $K_1(A_\theta \otimes C(\mathbb{T}))$ consists of the following set $\{[V], [U], [W], [\xi]\}$ where $\xi = (1-e)\otimes 1 + e \otimes W$ and e is a Powers-Rieffel projection in A_θ of trace θ. This follows from the short exact sequence

$$0 \longrightarrow A_\theta \otimes C_0(\mathbb{T}) \xrightarrow{\ i\ } A_\theta \otimes C(\mathbb{T}) \xrightarrow{\ \varepsilon\ } A_\theta \longrightarrow 0$$

where $C_0(\mathbb{T})$ is the ideal of functions in $C(\mathbb{T})$ vanishing at 1, ε is evaluation at 1, and i is inclusion. Using the Bott periodicity isomorphism $s^0 : K_0(A_\theta) \to K_1(A_\theta \otimes C_0(\mathbb{T}))$ (as given by Connes [4]) one has $s^0[e] = [1 \otimes 1 + e \otimes (W - 1)] = [\xi]$, giving us the fourth basis element. For the range of trace, and since we already know that $\boldsymbol{\tau}_*(K_0(A_\theta)) = \mathbb{Z} + \mathbb{Z}\theta$, we need to compute the "determinant" of each basis element. From U, V, W we get determinants already in $\mathbb{Z} + \mathbb{Z}\theta$, since $\sigma(V)V^* = \sigma(U)U^* = 1$ and $\sigma(W)W^* = \lambda = e(\theta)$. For ξ one has

$$\sigma(\xi)\xi^* = ((1-e)\otimes 1 + \lambda e \otimes W)\cdot((1-e)\otimes 1 + e \otimes W^*) = ((1-e) + \lambda e)\otimes 1.$$

A path of unitaries connecting this element to the identity is simply $\eta_t = ((1-e)+e(t\theta)e)\otimes 1$ for $0 \le t \le 1$. Thus

$$\frac{1}{2\pi i}\int_0^1 \boldsymbol{\tau}(\dot\eta_t\eta_t^*)\,dt = \frac{1}{2\pi i}\int_0^1 2\pi i\theta\boldsymbol{\tau}(e)\,dt = \theta^2$$

since $\boldsymbol{\tau}(e)=\theta$. From Pimsner's trace formula one therefore obtains

$$\boldsymbol{\tau}_*(K_0(A_\theta^{5,1})) = \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2.$$

One now has the isomorphism classification for the algebras $A_\theta^{5,1}$ for non-quartic irrationals θ (i.e., those that are not zeros of an integral polynomial of degree at most four). Therefore, for two such non-quartic irrationals θ,θ', the algebras $A_\theta^{5,1}$ and $A_{\theta'}^{5,1}$ are isomorphic if and only if $\theta' = n\pm\theta$ for some integer n.

The C*-algebra $A_{\theta,\phi}^{5,2}$. The C*-algebra $A_{\theta,\phi}^{5,2}$ is generated by unitaries U,V,W satisfying

$$UV=\lambda VU, \qquad UW=\mu WU, \qquad VW=WV, \tag{2.2}$$

where $\mu = e(\phi)$ and $\lambda = e(\theta)$ are assumed to be independent elements of the abelian group $\mathbb{T}$, so that in fact the algebra is simple. (See [9], Section 2.) This algebra can be realized as the crossed product $C(\mathbb{T}^2)\rtimes_\gamma\mathbb{Z}$ where $C(\mathbb{T}^2)$ is generated by V,W and $\gamma(V)=\lambda V$, $\gamma(W)=\mu W$. Since this automorphism is homotopic to the identity, the Pimsner-Voiculescu sequence gives $K_j(A_{\theta,\phi}^{5,2}) = \mathbb{Z}^4$ since $K_j(C(\mathbb{T}^2)) = \mathbb{Z}^2$, for $j=0,1$. Since $K_1(C(\mathbb{T}^2))$ has basis $[V],[W]$ and since $\gamma(V)V^* = e(\theta)$ and $\gamma(W)W^*=e(\phi)$, one easily obtains the range of trace as

$$\boldsymbol{\tau}_*(K_0(A_{\theta,\phi}^{5,2})) = \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\phi.$$

(This does in fact hold also for rational θ,ϕ, but one must use the canonical trace on the crossed product.) The classification for the algebras $A_{\theta,\phi}^{5,2}$ now follows:

Proposition. *For independent irrationals θ,ϕ the C*-algebras $A_{\theta,\phi}^{5,2}$ and $A_{\theta',\phi'}^{5,2}$ are isomorphic if, and only if there exists $X\in \mathrm{GL}(2,\mathbb{Z})$ such that $[\theta'\ \phi'] = [\theta\ \phi]X$.*

Proof. Given $X = \left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]$ in $\mathrm{GL}(2,\mathbb{Z})$ the substitutions $V' = V^aW^b$, $W' = V^cW^d$ satisfy the relations

$$UV' = \lambda^a\mu^bV'U, \qquad UW' = \lambda^c\mu^dW'U, \qquad V'W'=W'V',$$

so that U,V',W', which already generate $A_{\theta,\phi}^{5,2}$, also generate $A_{a\theta+b\phi,c\theta+d\phi}^{5,2}$, hence these algebras are isomorphic. Conversely, if $A_{\theta,\phi}^{5,2}$ and $A_{\theta',\phi'}^{5,2}$ are isomorphic then by the above one has $\mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\phi = \mathbb{Z}+\mathbb{Z}\theta'+\mathbb{Z}\phi'$. Writing each of θ', ϕ' in terms of θ, ϕ (modulo $\mathbb{Z}$) and vice versa, and using their rational independence, one easily obtains a matrix X in $\mathrm{GL}(2, \mathbb{Z})$ such that $[\theta' \ \phi'] = [\theta \ \phi]X$. □

The C*-algebra $A_\theta^{5,3}$. The C*-algebra $A_\theta^{5,3}$ is generated by unitaries U, V, W, X satisfying

$$UV = XVU, \qquad UX = \lambda XU, \qquad VX = XV,$$
$$VW = \lambda WV, \qquad UW = WU, \qquad WX = XW. \tag{2.3}$$

where $\lambda = e(\theta)$ and θ is irrational. (See [9], Section 3.) One can view this algebra as the crossed product $A_\theta^4 \rtimes_\nu \mathbb{Z}$, where A_θ^4 is the Heisenberg C*-algebra generated by the unitaries U, V, X satisfying the three relations in the first line of (2.3), and $\nu(X) = X$, $\nu(U) = U$, $\nu(V) = \bar{\lambda}V$. Since this automorphism is also homotopic to the identity, and since $K_j(A_\theta^4) = \mathbb{Z}^3$, the Pimsner-Voiculescu exact sequence immediately gives $K_j(A_\theta^{5,3}) = \mathbb{Z}^6$ for $j = 0, 1$. To find the range of the trace on K_0 using Pimsner's Theorem we will need to do the following. The Pimsner-Voiculescu exact sequence applied to A_θ^4, viewed as the crossed product $A_\theta \rtimes_\sigma \mathbb{Z}$ (where A_θ is generated by U, X and $\sigma = \mathrm{Ad}_V$) is

$$\begin{array}{ccccc} K_0(A_\theta) & \xrightarrow{id_* - \sigma_* = 0} & K_0(A_\theta) & \xrightarrow{i_*} & K_0(A_\theta^4) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(A_\theta^4) & \xleftarrow{i_*} & K_1(A_\theta) & \xleftarrow{id_* - \sigma_*} & K_1(A_\theta). \end{array}$$

Recall from Lemma 1.2 of [13] that the group $K_1(A_\theta \rtimes_\sigma \mathbb{Z})$ is generated by classes of unitaries of the form

$$(1 \otimes I_n - F) + Fx(V^{-1} \otimes I_n)F,$$

where F is a projection in $M_n(A)$ and $x \in M_n(A)$. (Here, V is the canonical unitary of the crossed product: $\sigma(a) = VaV^{-1}$.) In addition, from page 102 of [13], the connecting homomorphism $\delta_1 : K_1(A \rtimes_\sigma \mathbb{Z}) \to K_0(A)$ is given on classes of such unitaries by

$$\delta_1[(1 \otimes I_n - F) + Fx(V^{-1} \otimes I_n)F] \ = \ [F]. \tag{2.4}$$

Lemma. *A basis for $K_1(A_\theta^4)$ is $\{[U],\ [V],\ [\xi]\}$ where*

$$\xi \ := \ (1 - e) + ew^{-1}V^{-1}e$$

and $e \in A_\theta = C^*(X, U)$ is a Powers-Rieffel projection of trace θ and w is a unitary in A_θ such that $wew^{-1} = V^{-1}eV$.

Proof. From the above exact sequence we see that δ_1 is surjective and thus $K_1(A_\theta^4)$ contains elements that are mapped by δ_1 to $[1]$ and $[e]$. Applying (2.4) with $F = 1$ one has $\delta_1[V^{-1}] = [1]$. To find an element that δ_1 maps to

$[e]$, note that $V^{-1}eV$ is a projection in A_θ whose trace is θ, so by Rieffel's Cancellation Theorem [**15**] there exists a unitary w in A_θ such that $wew^{-1} = V^{-1}eV$. Now it is straightforward to see that $(1-e)+ew^{-1}V^{-1}e$ is a unitary in A_θ^4 (with inverse $(1-e)+eVwe$). Therefore, one has

$$\delta_1[(1-e)+ew^{-1}V^{-1}e] \;=\; [e].$$

Finally, on $K_1(A_\theta)$, $id_* - \alpha_*$ maps $[X]$ to zero and $[U]$ to $[X]$, hence $[U]$ is the third basis element. □

First, since $\nu(V)V^{-1} = \bar\lambda = e(-\theta)$, Pimsner's Theorem gives us θ in the range (which is already contained in the range of the trace on $K_0(A_\theta^4)$). Since e is in $A_\theta = C^*(X,U)$, which is fixed by ν, one obtains for ξ (since Vw commutes with e)

$$\nu(\xi)\xi^{-1} = ((1-e)+\lambda w^{-1}V^{-1}e)\cdot((1-e)+Vwe) = (1-e)+\lambda e.$$

This is exactly the same situation we had for the algebra $A_\theta^{5,1}$ which yielded θ^2 in the trace range. Therefore one concludes in the same manner that $\boldsymbol{\tau}_*(K_0(A_\theta^{5,3})) = \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2$ which also yields the same type of classification statement as for the case of the algebra $A_\theta^{5,1}$ above (namely, for non-quartic irrationals θ).

The C*-algebra $A_{\theta,\phi}^{5,4}$. The C*-algebra $A_{\theta,\phi}^{5,4}$ is generated by unitaries U, V, W satisfying

$$(2.5)\qquad WV = UVW, \qquad WU = \lambda UW, \qquad VU = \mu UV,$$

where $\mu = e(\phi)$ and $\lambda = e(\theta)$ are assumed to be independent. This algebra is Packer's Heisenberg C*-algebra of class 3 [**11**]. As shown in [**9**], Section 4, this algebra is simple with a unique trace state. As in [**9**], we can view $A_{\theta,\phi}^{5,4}$ as the crossed product $A_\phi \rtimes_\sigma \mathbb{Z}$, where A_ϕ is generated by U, V, and σ is the "Anzai" automorphism $\sigma(U) = \lambda U$, $\sigma(V) = UV$. Since σ_* induces the identity map on $K_0(A_\phi)$, and since on $K_1(A_\phi) = \mathbb{Z}[U]+\mathbb{Z}[V]$ one has $(id_* - \sigma_*)[U] = 0$, $(id_* - \sigma_*)[V] = -[U]$, the Pimsner-Voiculescu exact sequence gives $K_0(A_{\theta,\phi}^{5,4}) = K_1(A_{\theta,\phi}^{5,4}) = \mathbb{Z}^3$. Now Pimsner's machine states that the range of trace is obtained from that of $\boldsymbol{\tau}_*K_0(A_\phi) = \mathbb{Z}+\mathbb{Z}\phi$ and from the class $[U]$. But $\sigma(U)U^* = \lambda = e(\theta)$ so that one has

$$\boldsymbol{\tau}_*K_0(A_{\theta,\phi}^{5,4}) = \mathbb{Z}+\mathbb{Z}\phi+\mathbb{Z}\theta.$$

The classification for underlying algebras is the same as for the algebras $A_{\theta,\phi}^{5,2}$ above.

Proposition. *For independent irrationals θ, ϕ the C*-algebras $A_{\theta,\phi}^{5,4}$ and $A_{\theta',\phi'}^{5,4}$ are isomorphic if, and only if there exists $X \in \mathrm{GL}(2,\mathbb{Z})$ such that $[\theta'\ \phi'] = [\theta\ \phi]X$.*

Proof. Since the group $\mathrm{GL}(2,\mathbb{Z})$ is generated by the matrices $\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, it suffices to check that $A^{5,4}_{\phi,\theta}$ and $A^{5,4}_{\theta,\theta+\phi}$ are isomorphic to $A^{5,4}_{\theta,\phi}$. To get the algebra $A^{5,4}_{\phi,\theta}$, one lets $W' = V^*,\ \ V' = W^*,\ \ U' = \mu\lambda U^*$ so that the universal relations (2.5) are checked for W', V', U' in place of W, V, U, respectively, and with μ and λ switched. To get the algebra $A^{5,4}_{\theta,\theta+\phi}$, in the same manner one lets $W' = W,\ \ V' = WV,\ \ U' = \lambda U$ which satisfy (2.5) with λ remaining the same and μ replaced by $\lambda\mu$. Conversely, if $A^{5,4}_{\theta,\phi}$ and $A^{5,4}_{\theta',\phi'}$ are isomorphic then exactly as in the proof of previous proposition one shows that there is a matrix X in $\mathrm{GL}(2,\mathbb{Z})$ such that $[\theta'\ \phi'] = [\theta\ \phi]X$. $\square$

3. The C*-algebra $A^{5,5}_\theta$.

Let us view the commutative 3-torus $C(\mathbb{T}^3)$ as generated by its three canonical unitaries X, W, V, where $X(r,s,t) = e(r),\ W(r,s,t) = e(s),\ V(r,s,t) = e(t)$. The C*-algebra $A^{5,5}_\theta$ can be viewed as the crossed product $C(\mathbb{T}^3) \rtimes_\sigma \mathbb{Z}$ where

$$\sigma(X) = \lambda X, \qquad \sigma(W) = XW, \qquad \sigma(V) = WV. \tag{3.1}$$

(where $\lambda = e(\theta)$) which was introduced in [9] (Section 5). When θ is irrational, $A^{5,5}_\theta$ is the unique C*-algebra generated by unitaries X, W, V, U satisfying the relations

$$\begin{aligned} UV &= WVU, \qquad & UW &= XWU, \qquad & UX &= \lambda XU, \\ VW &= WV, \qquad & VX &= XV, \qquad & WX &= XW. \end{aligned} \tag{3.2}$$

We shall prove the following result.

Theorem 3.1. *For any θ (rational or irrational) one has*

$$K_0(A^{5,5}_\theta) = \mathbb{Z}^4, \qquad K_1(A^{5,5}_\theta) = \mathbb{Z}^4.$$

If θ is irrational, and if $\boldsymbol{\tau}$ is the unique trace state on $A^{5,5}_\theta$, then $\boldsymbol{\tau}_* K_0(A^{5,5}_\theta) = \mathbb{Z} + \mathbb{Z}\theta$. (This yields the usual isomorphism classification for irrational θ upon noting that $A^{5,5}_\theta \cong A^{5,5}_{1-\theta}$.)

The Pimsner-Voiculescu exact sequence corresponding to the above crossed product is

$$\begin{array}{ccccc} K_0(C(\mathbb{T}^3)) & \xrightarrow{id_* - \sigma_*} & K_0(C(\mathbb{T}^3)) & \xrightarrow{i_*} & K_0(A^{5,5}_\theta) \\ {\scriptstyle\delta_1}\uparrow & & & & \downarrow{\scriptstyle\delta_0} \\ K_1(A^{5,5}_\theta) & \xleftarrow{i_*} & K_1(C(\mathbb{T}^3)) & \xleftarrow{id_* - \sigma_*} & K_1(C(\mathbb{T}^3)) \end{array} \tag{3.3}$$

and our goal is to compute $id_* - \sigma_*$ at the K_0 and K_1 levels.

The Connes Chern character on $K_*(C(\mathbb{T}^3))$. First we need to find concrete bases for the K-groups of $C(\mathbb{T}^3)$. It is already known that $K_0(C(\mathbb{T}^3)) = \mathbb{Z}^4$ and $K_1(C(\mathbb{T}^3)) = \mathbb{Z}^4$. Let B denote the Bott projection in $M_2(C(\mathbb{T}^2))$ given by

$$B = \begin{bmatrix} 1-f & g \\ \bar{g} & f \end{bmatrix}$$

where $f, g \in C(\mathbb{T}^2)$ are smooth functions satisfying

(3.4)
$$(\phi\#\mathrm{Tr})(B,B,B) = -6\phi(f,g,\bar{g}) = -\frac{6}{2\pi i}\iint_{\mathbb{T}^2} f[g_x\bar{g}_y - g_y\bar{g}_x]\,dxdy = 1,$$

where $g_x := \partial g/\partial x$, which is just the Connes pairing of $[B]$ with $[\phi]$ (this number being often called the 'twist' of B in the C*-literature), where ϕ is the fundamental cyclic cocycle on $\mathbb{T}^2$:

$$\phi(f^0, f^1, f^2) = \frac{1}{2\pi i}\iint_{\mathbb{T}^2} f^0[f^1_x f^2_y - f^1_y f^2_x]\,dxdy.$$

For $1 \le i < j \le 3$, let P_{ij} denote the Bott projection in $M_2(C(\mathbb{T}^3))$ in the variables i, j. More specifically,

$$P_{12}(r,s,t) = B(r,s), \qquad P_{13}(r,s,t) = B(r,t), \qquad P_{23}(r,s,t) = B(s,t).$$

Putting $b_{ij} = [P_{ij}] - [1]$ (the Bott elements), it is not hard to check that $\{[1], b_{12}, b_{13}, b_{23}\}$ is a basis for $K_0(C(\mathbb{T}^3))$. Now the (numerical) Connes Chern character ch_0 is the homomorphism

$$\mathrm{ch}_0 : K_0(C(\mathbb{T}^3)) \to \mathbb{Z}^4$$

given by

$$\mathrm{ch}_0(x) = (\boldsymbol{\tau}(x),\ \langle x, \phi_{12}\rangle,\ \langle x, \phi_{13}\rangle,\ \langle x, \phi_{23}\rangle)$$

where

$$\phi_{ij}(f^0, f^1, f^2) = \frac{1}{2\pi i}\iiint_{\mathbb{T}^3} f^0[f^1_i f^2_j - f^1_j f^2_i]\,dx_1dx_2dx_3$$

is a cyclic 2-cocycle on $C(\mathbb{T}^3)$ and $f_k := \partial f/\partial x_k$. (Henceforth, all triple integrals are over the 3-torus.) From (3.4) one gets

$$\langle [P_{ij}], [\phi_{k\ell}]\rangle = \delta_{i,k}\delta_{j,\ell}$$

which gives

$$\mathrm{ch}_0[1] = (1,0,0,0), \qquad \mathrm{ch}_0[b_{12}] = (0,1,0,0),$$
$$\mathrm{ch}_0[b_{13}] = (0,0,1,0), \qquad \mathrm{ch}_0[b_{23}] = (0,0,0,1),$$

so that ch_0 is injective on $K_0(C(\mathbb{T}^3))$.

Lemma 3.2. *One has the following action of σ_* on $K_0(C(\mathbb{T}^3))$:*

$$\sigma_*[1] = [1], \qquad \sigma_*(b_{12}) = b_{12},$$
$$\sigma_*(b_{13}) = b_{12} + b_{13}, \qquad \sigma_*(b_{23}) = b_{12} + b_{13} + b_{23}.$$

Proof. For simplicity consider the change of variables $(u, v, w) = (r+\theta, r+s, s+t)$, and note that by the chain rule one has

$$\begin{aligned}\tfrac{\partial}{\partial r} h(u,v,w) &= h_1(u,v,w) + h_2(u,v,w),\\ \tfrac{\partial}{\partial s} h(u,v,w) &= h_2(u,v,w) + h_3(u,v,w),\\ \tfrac{\partial}{\partial t} h(u,v,w) &= h_3(u,v,w),\end{aligned}$$

which can be simplified by writing

$$\tfrac{\partial}{\partial x_i} h(u,v,w) = h_i(u,v,w) + h_{i+1}(u,v,w)$$

where $h_4 = 0$, and $x_1 = r, x_2 = s, x_3 = t$. From this one gets

$$\tfrac{\partial}{\partial x_i} g(u,v,w) \tfrac{\partial}{\partial x_j} \bar{g}(u,v,w) = (g_i + g_{i+1})(\bar{g}_j + \bar{g}_{j+1})(u,v,w)$$

and

$$\begin{aligned}&\tfrac{\partial}{\partial x_i} g(u,v,w) \tfrac{\partial}{\partial x_j} \bar{g}(u,v,w) - \tfrac{\partial}{\partial x_j} g(u,v,w) \tfrac{\partial}{\partial x_i} \bar{g}(u,v,w)\\ &\qquad = [(g_i + g_{i+1})(\bar{g}_j + \bar{g}_{j+1}) - (g_j + g_{j+1})(\bar{g}_i + \bar{g}_{i+1})](u,v,w).\end{aligned}$$

Now if we write

$$P_{ij} = \begin{bmatrix} 1-f & g \\ \bar{g} & f \end{bmatrix}$$

where f, g depend only on the i, j coordinates $(i < j)$, then

$$\sigma(P_{ij}) = \begin{bmatrix} 1 - f(u,v,w) & g(u,v,w) \\ \bar{g}(u,v,w) & f(u,v,w) \end{bmatrix}$$

for which one has

$$\begin{aligned}\langle [\sigma(P_{ij})], [\phi_{k\ell}] \rangle &= (\phi_{k\ell} \# \mathrm{Tr})(\sigma(P_{ij}), \sigma(P_{ij}), \sigma(P_{ij}))\\ &= -6\phi_{k\ell}(f(u,v,w), g(u,v,w), \bar{g}(u,v,w))\\ &= -\frac{6}{2\pi i} \iiint f(u,v,w) \Big[\tfrac{\partial}{\partial x_k} g(u,v,w) \tfrac{\partial}{\partial x_\ell} \bar{g}(u,v,w)\\ &\qquad - \tfrac{\partial}{\partial x_\ell} g(u,v,w) \tfrac{\partial}{\partial x_k} \bar{g}(u,v,w) \Big]\, drdsdt\\ &= -\frac{6}{2\pi i} \iiint f(u,v,w) \Big[(g_k + g_{k+1})(\bar{g}_\ell + \bar{g}_{\ell+1})\\ &\qquad - (g_\ell + g_{\ell+1})(\bar{g}_k + \bar{g}_{k+1}) \Big](u,v,w)\, drdsdt.\end{aligned}$$

Now since the transformation $(u,v,w) = (r+\theta, r+s, s+t)$ has Jacobian determinant 1, the change of variables formula gives

$$\begin{aligned}
&\langle [\sigma(P_{ij})], [\phi_{k\ell}]\rangle \\
&= -\frac{6}{2\pi i}\iiint f(r,s,t)\cdot\Big[(g_k+g_{k+1})(\bar g_\ell+\bar g_{\ell+1}) \\
&\qquad\qquad - (g_\ell+g_{\ell+1})(\bar g_k+\bar g_{k+1})\Big](r,s,t)\,drdsdt \\
&= \langle [P_{ij}],[\phi_{k\ell}]\rangle + \langle [P_{ij}],[\phi_{k,\ell+1}]\rangle + \langle [P_{ij}],[\phi_{k+1,\ell}]\rangle + \langle [P_{ij}],[\phi_{k+1,\ell+1}]\rangle \\
&= \delta_{i,k}\delta_{j,\ell} + \delta_{i,k}\delta_{j,\ell+1} + \delta_{i,k+1}\delta_{j,\ell} + \delta_{i,k+1}\delta_{j,\ell+1}
\end{aligned}$$

where $\phi_{3,\ell} = \phi_{k,4} = 0$. One thus gets

$$\begin{aligned}
&\langle [\sigma(P_{12})],[\phi_{12}]\rangle = 1, && \langle [\sigma(P_{12})],[\phi_{13}]\rangle = 0, && \langle [\sigma(P_{12})],[\phi_{23}]\rangle = 0,\\
&\langle [\sigma(P_{13})],[\phi_{12}]\rangle = 1, && \langle [\sigma(P_{13})],[\phi_{13}]\rangle = 1, && \langle [\sigma(P_{13})],[\phi_{23}]\rangle = 0,\\
&\langle [\sigma(P_{23})],[\phi_{12}]\rangle = 1, && \langle [\sigma(P_{23})],[\phi_{13}]\rangle = 1, && \langle [\sigma(P_{23})],[\phi_{23}]\rangle = 1,
\end{aligned}$$

which yields

$$\mathrm{ch}_0[\sigma(P_{12})] = (1,1,0,0), \qquad \mathrm{ch}_0[\sigma(P_{13})] = (1,1,1,0), \qquad \mathrm{ch}_0[\sigma(P_{23})] = (1,1,1,1)$$

and the injectivity of ch_0 thus yields the following equalities in $K_0(C(\mathbb{T}^3))$

$$\sigma_*(b_{12}) = b_{12}, \qquad \sigma_*(b_{13}) = b_{12} + b_{13}, \qquad \sigma_*(b_{23}) = b_{12} + b_{13} + b_{23}.$$

These give the desired result. □

We now turn our attention to K_1.

Lemma 3.3. *A basis for $K_1(C(\mathbb{T}^3))$ is $\{[X],[W],[V],[\xi]\}$, where $\xi = I_2 + (V-1)\otimes P_{12}$ is a unitary in $M_2(C(\mathbb{T}^3))$ and P_{12} is the Bott projection in the variables X, W.*

Proof. This immediately follows from the Künneth Theorem applied to the tensor product expansion of $K_1(C(\mathbb{T}^3)) = K_1(C(\mathbb{T}^2)\otimes C(\mathbb{T}))$ and using the individual generators of each factor. □

Lemma 3.4. *The action of σ on $K_1(C(\mathbb{T}^3))$ is given by*

$$\sigma_*[X] = [X], \quad \sigma_*[W] = [X]+[W], \quad \sigma_*[V] = [W]+[V], \quad \sigma_*[\xi] = [\xi]+[W].$$

Proof. The only nontrivial part is to show $\sigma_*[\xi] = [\xi]+[W]$ (the rest follow trivially from the definition of σ). From Lemma 3.2 one has $[\sigma(P_{12})] = [P_{12}]$, and since $C(\mathbb{T}^2)$ has the cancellation property, there is a unitary R in $M_2(C(\mathbb{T}^3))$ (which depends only on the first two variables) such that

$\sigma(P_{12}) = RP_{12}R^*$. Hence

$$\begin{aligned}\sigma_*[\xi] &= [I_2 + (WV-1)\otimes\sigma(P_{12})]\\ &= [I_2 + (WV-1)\otimes RP_{12}R^*]\\ &= [R(I_2 + (WV-1)\otimes P_{12})R^*]\\ &= [I_2 + (WV-1)\otimes P_{12}]\\ &= [I_2 + (W-1)\otimes P_{12}] + [I_2 + (V-1)\otimes P_{12}]\\ &= [\xi] + [I_2 + (W-1)\otimes P_{12}]\end{aligned}$$

and now we claim that $[I_2 + (W-1)\otimes P_{12}] = [W]$. It is enough to show that this equality holds in $K_1(C(\mathbb{T}^2))$ (since all concerned variables here are the first two – involving X, W). This is shown in the following remark. □

Remark. Let us view the 2-torus $\mathbb{T}^2$ as $\mathbb{T}\times[0,1]$ with the endpoints of the interval identified. Recall that the Bott projection in $M_2(C(\mathbb{T}^2))$ can be given by $P(x,s) = M(x,s)e_0M(x,s)^*$ for $0\le s\le 1$, where

$$M(x,s) = E^s\begin{bmatrix}\overline{x} & 0\\ 0 & 1\end{bmatrix}E^{-s},$$
$$E^s = \begin{bmatrix}\cos(\pi s/2) & -\sin(\pi s/2)\\ \sin(\pi s/2) & \cos(\pi s/2)\end{bmatrix},\quad e_0 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},$$

where $M:\mathbb{T}\times[0,1]\to U_2(\mathbb{C})$ is smooth (but clearly M is not in $M_2(C(\mathbb{T}^2))$), and E^s satisfies the usual exponential property. In our case above, the unitary V corresponds to x and $W(x,s) = e(s)$. The unitary $\eta := I_2 + (W-1)\otimes P$ can now be written as

$$\eta = I_2 + (W-1)\otimes Me_0M^* = M(I_2 + (W-1)\otimes e_0)M^* = M\begin{bmatrix}W & 0\\ 0 & 1\end{bmatrix}M^*.$$

Now an explicit path of unitaries $t\mapsto\eta_t$ in $M_2(C(\mathbb{T}^2))$ connecting η to $\begin{bmatrix}W & 0\\ 0 & 1\end{bmatrix}$ can be given by

$$\eta_t(x,s) = M(x,ts)\begin{bmatrix}e(s) & 0\\ 0 & 1\end{bmatrix}M(x,ts)^*. \tag{3.5}$$

(For each t one has $\eta_t(x,0) = \eta_t(x,1) = I_2$ so that $\eta_t\in M_2(C(\mathbb{T}^2))$.) It follows, in particular, that η and W give the same class in $K_1(C(\mathbb{T}^2))$. The explicit form of the unitary path (3.5) is used in the trace computation below.

In view of Lemmas 3.2 and 3.4 one obtains, from the Pimsner-Voiculescu exact sequence (3.3), the K_0 and K_1 groups of $A_\theta^{5,5}$ as stated in Theorem 3.1.

Tracial Range on $K_0(A_\theta^{5,5})$. To complete the proof of Theorem 3.1 we now use Pimsner's Theorem. In the present case, the quotient map is $q : \mathbb{R} \to \mathbb{R}/\boldsymbol{\tau}_*(K_0(C(\mathbb{T}^3))) = \mathbb{R}/\mathbb{Z}$, since the range of the canonical trace state $\boldsymbol{\tau}$ on $K_0(C(\mathbb{T}^3))$ is $\mathbb{Z}$. From Lemma 3.4 the kernel of $id_* - \sigma_*$ in $K_1(C(\mathbb{T}^3))$ is generated by the classes $[X]$ and $[\xi] - [V]$. For $[X]$, since $\sigma(X) = \lambda X$, one clearly has $\Delta[X] = q(\theta)$. For $[\xi]-[V]$, it suffices to show that $\Delta([\xi]-[V]) = 0$, and this will complete the proof that $\boldsymbol{\tau}_* K_0(A_\theta^{5,5}) = \mathbb{Z} + \mathbb{Z}\theta$. As in the proof of Lemma 3.4, we noted that

$$\sigma(\xi) = R\xi\big(I + (W-1)\otimes P_{12}\big)R^* = R\xi M W_1 M^* R^*$$

where R is a unitary in $M_2(C(\mathbb{T}^2))$ such that $\sigma(P_{12}) = RP_{12}R^*$, and $I + (W-1)\otimes P_{12} = MW_1M^*$ (in the notation of the above remark). Writing $V_1 = \left[\begin{smallmatrix} V & 0 \\ 0 & 1\end{smallmatrix}\right]$ and similarly for W_1, we have $[\xi] - [V] = [\xi V_1^*]$. Thus one has

$$\begin{aligned}
&\sigma\left(\begin{bmatrix}\xi V_1^* & \\ & I\end{bmatrix}\right)\begin{bmatrix}\xi V_1^* & \\ & I\end{bmatrix}^{-1} \\
&= \begin{bmatrix}\sigma(\xi) & \\ & I\end{bmatrix}\begin{bmatrix}V_1^*W_1^* & \\ & I\end{bmatrix}\begin{bmatrix}V_1 & \\ & I\end{bmatrix}\begin{bmatrix}\xi^* & \\ & I\end{bmatrix} \\
&= \begin{bmatrix}R & \\ & R^*\end{bmatrix}\begin{bmatrix}\xi & \\ & I\end{bmatrix}\begin{bmatrix}MW_1M^* & \\ & I\end{bmatrix}\begin{bmatrix}R^* & \\ & R\end{bmatrix}\begin{bmatrix}W_1^* & \\ & I\end{bmatrix}\begin{bmatrix}\xi^* & \\ & I\end{bmatrix}.
\end{aligned}$$

Letting $t \mapsto \mathfrak{R}_t$ be the standard unitary path such that $\mathfrak{R}_0 = I$ and $\mathfrak{R}_1 = \left[\begin{smallmatrix} R & \\ & R^*\end{smallmatrix}\right]$ one considers the following path of unitaries in $M_4(C(\mathbb{T}^2))$

$$\gamma_t = \mathfrak{R}_t \begin{bmatrix}\xi & \\ & I\end{bmatrix}\begin{bmatrix}\eta_t & \\ & I\end{bmatrix}\mathfrak{R}_t^*\begin{bmatrix}W_1^* & \\ & I\end{bmatrix}\begin{bmatrix}\xi^* & \\ & I\end{bmatrix}$$

where η_t is the path defined by (3.5) such that $\eta_0 = W_1$, $\eta_1 = MW_1M^*$. The path γ_t clearly connects the above element to the identity. Now it is straightforward to see that $(\boldsymbol{\tau}\otimes \mathrm{Tr}_4)(\dot{\gamma}_t\gamma_t^*) = (\boldsymbol{\tau}\otimes\mathrm{Tr}_2)(\dot{\eta}_t\eta_t^*)$, since the fact that both $\mathfrak{R}_t$ and $\mathfrak{R}_t^*$ appear in γ_t leads to their cancellation under the trace. Since η_t has the form (3.5), one similarly obtains $(\boldsymbol{\tau}\otimes\mathrm{Tr}_2)(\dot{\eta}_t\eta_t^*) = 0$. Therefore, $\Delta([\xi V_1^*]) = 0$ which completes the proof of Theorem 3.1.

4. The C*-algebra $A_\theta^{5,6}$.

The C*-algebra $A_\theta^{5,6}$ can be characterized as the unique C*-algebra (when θ is irrational) generated by unitaries U, V, W, Z such that

$$\begin{aligned}
&ZV = \lambda VZ, \qquad ZU = V^{-1}UZ, \qquad ZW = WZ, \\
&UV = WVU, \qquad UW = \lambda WU, \qquad VW = WV.
\end{aligned} \tag{4.1}$$

(As in [9].) It will be convenient to present $A_\theta^{5,6}$ as the crossed product $(C(\mathbb{T})\otimes A_\theta)\rtimes_\nu \mathbb{Z}$, where $C(\mathbb{T})$ is generated by W, A_θ is generated by V, Z, and $\nu = \mathrm{Ad}_U := U(\ )U^*$ is the automorphism given by

$$\nu(W) = \lambda W, \qquad \nu(Z) = VZ, \qquad \nu(V) = W \otimes V = WV.$$

The aim of this section is to prove the following.

Theorem 4.1. *For any θ (rational or irrational) one has*

$$K_0(A_\theta^{5,6}) = \mathbb{Z}^4, \qquad K_1(A_\theta^{5,6}) = \mathbb{Z}^4.$$

If θ is irrational, and if $\boldsymbol{\tau}$ is the unique trace state on $A_\theta^{5,6}$, then $\boldsymbol{\tau}_* K_0(A_\theta^{5,6}) = \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2$. (This yields the isomorphism classification for non-quartic irrationals θ upon noting that $A_\theta^{5,6} \cong A_{1-\theta}^{5,6}$.)

Since $ZV = \lambda VZ$, let $p = V^*g + f + gV$ be a Powers-Rieffel projection of trace θ, where $f = f(Z),\ g = g(Z)$ are C^∞ real functions of Z satisfying the usual properties that would make p a projection. Among these, to be used below, are (after interpreting f, g as functions of period 1 on $\mathbb{R}$)

$$f(t+\theta) = 1 - f(t) \text{ for } 0 \le t \le 1-\theta, \tag{4.2}$$
$$g(t) - f(t)g(t) = g(t)f(t+\theta) \text{ for all } t,$$
$$\int_0^1 f(g^2)^\bullet = -\int_0^1 g^2 \dot{f}\,dt = \frac{1}{6},$$

where we may assume, with no loss of generality, that $\frac{1}{2} < \theta < 1$, and where the dot indicates usual differentiation of a real function. Recall that g is supported on $[\theta, 1]$ on which it is given by $(f - f^2)^{1/2}$ and $f = 1$ on $[1-\theta, \theta]$.

Lemma 4.2. *One has the following equalities in $K_0(C(\mathbb{T}) \otimes A_\theta)$ for irrational θ:*

$$\nu_*[p] = [p] + ([P_{W,V}] - [1]) + ([P_{W,Z}] - [1])$$
$$\nu_*[P_{W,Z}] = [P_{W,Z}] + [P_{W,V}] - [1].$$

Proof. We first compute the Connes Chern character for the algebra $C(\mathbb{T}) \otimes A_\theta$. Consider the canonical cyclic 1-cocycles φ_0 and $\varphi_j, j = 1, 2$, of $C(\mathbb{T})$ and A_θ, respectively, given by

$$\varphi_0(f^0, f^1) = \frac{1}{2\pi i}\int_0^1 f^0 \frac{d}{dt}(f^1)dt,$$
$$\varphi_1(x^0, x^1) = \frac{1}{2\pi i}\boldsymbol{\tau}(x^0 \delta_V(x^1)), \quad \varphi_2(x^0, x^1) = \frac{1}{2\pi i}\boldsymbol{\tau}(x^0 \delta_Z(x^1))$$

where δ_V, δ_Z are the canonical derivations of A_θ, and $\boldsymbol{\tau}$ is the canonical trace. Let ρ denote Connes' canonical cyclic 2-cocycle of A_θ

$$\rho(x^0, x^1, x^2) = \frac{1}{2\pi i}\ \boldsymbol{\tau}\left(x^0\left[\delta_V(x^1)\delta_Z(x^2) - \delta_Z(x^1)\delta_V(x^2)\right]\right).$$

The Connes Chern character now takes the form of the group homomorphism

$$\mathrm{ch}_0 : K_0(C(\mathbb{T}) \otimes A_\theta) \to (\mathbb{Z} + \mathbb{Z}\theta) \oplus \mathbb{Z}^3$$

given by taking the Connes pairings with the various cup products as

$$\mathrm{ch}_0(x) := (\boldsymbol{\tau}(x),\ \langle x, \varphi_0\#\varphi_1\rangle,\ \langle x, \varphi_0\#\varphi_2\rangle,\ \langle x, \boldsymbol{\tau}_0\#\rho\rangle)$$

where $\boldsymbol{\tau}_0$ is the canonical trace of $C(\mathbb{T})$. It is straightforward to check that it assumes the following values on the basis for $K_0(C(\mathbb{T})\otimes A_\theta)$ given by the classes $\{[1],\ [1\otimes p],\ [P_{W,V}],\ [P_{W,Z}]\}$:

$$\begin{aligned} \mathrm{ch}_0[1] &= (1,0,0,0) \\ \mathrm{ch}_0[1\otimes p] &= (\theta,0,0,1) \\ \mathrm{ch}_0[P_{W,V}] &= (1,1,0,0) \\ \mathrm{ch}_0[P_{W,Z}] &= (1,0,1,0). \end{aligned} \tag{4.3}$$

(This follows immediately from the multiplicative property of Connes' canonical pairing with respect to tensor products of algebras, see [**3**, III.3].) It is immediate that ch_0 is injective on K_0 (for any θ).

It is clear that $\boldsymbol{\tau}(\nu(p)) = \theta$. So to compute $\mathrm{ch}_0(\nu(p))$, we will have to calculate the above three 2-cocycles on $\nu(p)$. First, we show that $\langle[\nu(p)], \boldsymbol{\tau}_0\#\rho\rangle = 1$. Since $p = V^*g + f + gV$, where $f = f(Z), g = g(Z)$, one has $\nu(p) = V^*W^*G + F + GWV$, where $F = \nu(f) = f(VZ),\ G = \nu(g) = g(VZ)$, and hence

$$\begin{aligned} &(\boldsymbol{\tau}_0\#\rho)(\nu(p),\nu(p),\nu(p)) \\ &= (\boldsymbol{\tau}_0\#\rho)\big(W^*V^*G + F + WGV,\ W^*V^*G + F + WGV, \\ &\qquad\qquad\qquad\qquad W^*V^*G + F + WGV\big) \\ &= \rho(F,F,F) + 3\rho(F,GV,V^*G) + 3\rho(F,V^*G,GV). \end{aligned}$$

(In the expansion, the only possibly nonzero terms are ones of the form $(\boldsymbol{\tau}_0\#\rho)(W^a\ldots, W^b\ldots, W^c\ldots)$ where $a+b+c=0$.) First, it is easy to verify that

$$\nu^{-1}\delta_Z\nu = \delta_Z, \qquad \nu^{-1}\delta_V\nu = \delta_V + \delta_Z.$$

We thus see that $\rho(F,F,F) = \rho(f,f,f) = 0$ since $\delta_V(f) = 0$. Next, we have

$$\begin{aligned} &2\pi i\rho(F,GV,V^*G) \\ &= \boldsymbol{\tau}\left(F\left[\delta_V(GV)\delta_Z(V^*G) - \delta_Z(GV)\delta_V(V^*G)\right]\right) \\ &= \boldsymbol{\tau}\left(F\left[(\delta_V(G)V + 2\pi i GV)V^*\delta_Z(G) - \delta_Z(G)V(-2\pi i V^*G + V^*\delta_V(G))\right]\right) \\ &= \boldsymbol{\tau}\left(F\left[(\delta_V(G) + 2\pi i G)\delta_Z(G) - \delta_Z(G)(-2\pi i G + \delta_V(G))\right]\right) \\ &= \boldsymbol{\tau}\left(f\left[(\delta_V(g) + \delta_Z(g) + 2\pi i g)\delta_Z(g) - \delta_Z(g)(-2\pi i g + \delta_V(g) + \delta_Z(g))\right]\right) \\ &= 2\pi i\boldsymbol{\tau}(f[g\delta_Z(g) + \delta_Z(g)g]) = 2\pi i\boldsymbol{\tau}(f\delta_Z(g^2)). \end{aligned}$$

Similarly, one checks that

$$\rho(F,V^*G,GV) = -\boldsymbol{\tau}(VfV^*\delta_Z(g^2)).$$

Therefore, using the properties (4.2) one gets

$$(\boldsymbol{\tau}_0\#\rho)(\nu(p),\nu(p),\nu(p)) = 3\boldsymbol{\tau}((f - VfV^*)\delta_Z(g^2)) = 1.$$

Fix $j = 1, 2$ and for simplicity let $\psi = \varphi_0\#\varphi_j$. From the definition of the cup product it can easily be shown that

$$\begin{aligned}\psi(a^0 \otimes b^0, a^1 \otimes b^1, a^2 \otimes b^2) \\ = \varphi_0(a^2a^0, a^1)\varphi_j(b^0b^1, b^2) - \varphi_0(a^0a^1, a^2)\varphi_j(b^2b^0, b^1)\end{aligned} \tag{4.4}$$

for $a^k \in C(\mathbb{T})$ and $b^k \in A_\theta$. We want to calculate $\psi(\nu(p),\nu(p),\nu(p))$. From $p = V^*g + f + gV$ and $\nu(p) = W^* \otimes V^*G + 1 \otimes F + W \otimes GV$ and upon expanding the expression $\psi(\nu(p),\nu(p),\nu(p))$ we note that the only possibly nonzero terms are of the form $\psi(W^a \ldots, W^b \ldots, W^c \ldots)$ for $a + b + c = 0$. Hence using (4.4) and the cyclicity of ψ we get

$$\begin{aligned}
&\psi(W^* \otimes V^*G + 1 \otimes F + W \otimes GV,\ W^* \otimes V^*G + 1 \otimes F + W \otimes GV, \\
&\qquad\qquad W^* \otimes V^*G + 1 \otimes F + W \otimes GV) \\
&= \psi(W^* \otimes V^*G,\ 1 \otimes F,\ W \otimes GV) + \psi(W^* \otimes V^*G,\ W \otimes GV,\ 1 \otimes F) \\
&\quad + \psi(1 \otimes F,\ W^* \otimes V^*G,\ W \otimes GV) + \psi(1 \otimes F,\ 1 \otimes F,\ 1 \otimes F) \\
&\quad + \psi(1 \otimes F,\ W \otimes GV,\ W^* \otimes V^*G) \\
&\quad + \psi(W \otimes GV,\ W^* \otimes V^*G,\ 1 \otimes F) + \psi(W \otimes GV,\ 1 \otimes F,\ W^* \otimes V^*G) \\
&= 3\psi(W^* \otimes V^*G,\ 1 \otimes F,\ W \otimes GV) \\
&\quad + 3\psi(W^* \otimes V^*G,\ W \otimes GV,\ 1 \otimes F) + \psi(1 \otimes F,\ 1 \otimes F,\ 1 \otimes F).
\end{aligned}$$

Note that $\psi(1 \otimes F,\ 1 \otimes F,\ 1 \otimes F) = 0$ since $\varphi_0(x, 1) = 0$. Also, since $\varphi_0(W^*, W) = 1$ one has

$$\begin{aligned}
\psi(W^* \otimes V^*G,\ 1 \otimes F,\ W \otimes GV) &= -\varphi_0(W^*, W)\varphi_j(G^2, F) \\
&= -\varphi_j(G^2, F), \\
\psi(W^* \otimes V^*G,\ W \otimes GV,\ 1 \otimes F) &= \varphi_0(W^*, W)\varphi_j(V^*G^2V, F) \\
&= \varphi_j(V^*G^2V, F)
\end{aligned}$$

and hence

$$\langle \nu(p), \varphi_0\#\varphi_j\rangle = -3\varphi_j(G^2, F) + 3\varphi_j(V^*G^2V, F).$$

First, for $j = 1$, one has

$$2\pi i\varphi_1(G^2, F) = \boldsymbol{\tau}(\nu(g^2)\delta_V(\nu(f))) = \boldsymbol{\tau}(g^2\delta_Z(f)) = -\boldsymbol{\tau}(f\delta_Z(g^2))$$

and

$$\begin{aligned}
2\pi i\varphi_1(V^*G^2V, F) &= \boldsymbol{\tau}(V^*\nu(g^2)V\delta_V(\nu(f))) = \boldsymbol{\tau}(V^*g^2V\delta_Z(f)) \\
&= \boldsymbol{\tau}(g^2\delta_Z(VfV^*)) \\
&= -\boldsymbol{\tau}(VfV^*\delta_Z(g^2))
\end{aligned}$$

hence by (4.2)

$$\langle \nu(p), \varphi_0 \# \varphi_1 \rangle = \psi(\nu(p), \nu(p), \nu(p)) = \tfrac{3}{2\pi i} \boldsymbol{\tau}((f - VfV^*)\delta_Z(g^2)) = 1.$$

When $j = 2$, one similarly gets

$$\varphi_2(G^2, F) = -\tfrac{1}{2\pi i}\boldsymbol{\tau}(f\delta_Z(g^2)), \qquad \varphi_2(V^*G^2V, F) = -\tfrac{1}{2\pi i}\boldsymbol{\tau}(VfV^*\delta_Z(g^2))$$

and thus $\langle \nu(p), \varphi_0 \# \varphi_2 \rangle = 1$. Therefore, $\mathrm{ch}_0(\nu(p)) = (\theta, 1, 1, 1)$ from which one concludes the equality in the lemma. The proof of the second equality in the lemma follows in a similar way (in fact more like the proof of the third equality in Lemma 3.2 except with A_θ in place of $C(\mathbb{T}^2)$). □

Since

$$K_1(C(\mathbb{T}) \otimes A_\theta) = [K_1(C(\mathbb{T})) \otimes K_0(A_\theta)] \oplus [K_0(C(\mathbb{T})) \otimes K_1(A_\theta)] = \mathbb{Z}^2 \oplus \mathbb{Z}^2$$

it is easily seen that it has as basis the four elements $[W], [Z], [V], [\zeta]$, where $\zeta := W \otimes p + (1-p)$. From Lemma 4.2 one has

$$[\nu(p)] + [p_0] + [p_0] = [p] + [P] + [Q]$$

where $P = P_{W,V}$, $Q = P_{W,Z}$, and $p_0 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Therefore, there exist integers m, n and an invertible matrix w over $C(\mathbb{T}) \otimes A_\theta$ such that

$$\nu(p) \oplus p_0 \oplus p_0 \oplus e_0 \;=\; w(p \oplus P \oplus Q \oplus e_0)w^{-1}$$

where $e_0 = I_n \oplus O_m$. By suitably enlarging m one could assume that w is connectable to the identity by a smooth path of invertibles (upon replacing w by $w \oplus w^{-1}$). So let $t \mapsto w_t$ be such a path with $w_0 = I, w_1 = w$. Let

$$p' = p \oplus p_0 \oplus p_0 \oplus e_0, \qquad \text{and} \qquad \zeta' = W \otimes p' + (I - p')$$

so that $\nu(p') = w(p \oplus P \oplus Q \oplus e_0)w^{-1}$ and it is easily seen that $[\nu(\zeta')] = [\zeta']$ in K_1 of $C(\mathbb{T}) \otimes A_\theta$. Now since $[\zeta'] = [\zeta] + (n+2)[W]$, one gets $\nu_*[\zeta] = [\zeta]$. It now follows that on $K_1(C(\mathbb{T}) \otimes A_\theta)$ one has

$$\ker(\nu_* - id_*) = \mathbb{Z}[W] + \mathbb{Z}[\zeta], \qquad \mathrm{Im}(\nu_* - id_*) = \mathbb{Z}[W] + \mathbb{Z}[V].$$

In view of the basis in (4.3) and the second equality in Lemma 4.2, on $K_0(C(\mathbb{T}) \otimes A_\theta)$ one has

$$\begin{aligned} \ker(\nu_* - id_*) &= \mathbb{Z}[1] + \mathbb{Z}([P_{W,V}] - [1]), \\ \mathrm{Im}(\nu_* - id_*) &= \mathbb{Z}([P_{W,V}] - [1]) + \mathbb{Z}([P_{W,Z}] - [1]). \end{aligned}$$

The Pimsner-Voiculescu exact sequence for $A_\theta^{5,6} = C \rtimes_\nu \mathbb{Z}$, where $C := C(\mathbb{T}) \otimes A_\theta$:

$$\begin{array}{ccccc} K_0(C) & \xrightarrow{id_* - \nu_*} & K_0(C) & \xrightarrow{i_*} & K_0(A_\theta^{5,6}) \\ \uparrow & & & & \downarrow \\ K_1(A_\theta^{5,6}) & \xleftarrow{i_*} & K_1(C) & \xleftarrow{id_* - \nu_*} & K_1(C) \end{array}$$

now immediately yields $K_0(A_\theta^{5,6}) = K_1(A_\theta^{5,6}) = \mathbb{Z}^4$, as stated in Theorem 4.1.

It remains to obtain the range of the trace on K_0. For convenience, let us use the notation

$$\begin{bmatrix} X \\ Y \\ \vdots \end{bmatrix} := X \oplus Y \oplus \cdots$$

for block diagonal matrices. One then has (since W is central in $C(\mathbb{T}) \otimes A_\theta$)

(4.5)

$$\begin{aligned}
\nu(\zeta')\zeta'^{-1} &= \big(\lambda W \otimes w(p \oplus P \oplus Q \oplus e_0)w^{-1} + (I - w(p \oplus P \oplus Q \oplus e_0)w^{-1})\big) \\
&\quad \cdot \big(W^{-1} \otimes (p \oplus p_0 \oplus p_0 \oplus e_0) + (I - p \oplus p_0 \oplus p_0 \oplus e_0)\big) \\
&= w \begin{bmatrix} \lambda W \otimes p + (1-p) \\ \lambda W \otimes P + (I-P) \\ \lambda W \otimes Q + (I-Q) \\ \lambda W \otimes e_0 + (I - e_0) \end{bmatrix} w^{-1} \begin{bmatrix} W^{-1} \otimes p + (1-p) \\ W^{-1} \otimes P + (I-P) \\ W^{-1} \otimes Q + (I-Q) \\ W^{-1} \otimes e_0 + (I-e_0) \end{bmatrix}.
\end{aligned}$$

For $0 \leq t \leq 1$, let $t \mapsto a_t$ be a smooth path of invertibles in $C^*(W,V) \cong C(\mathbb{T}^2)$ such that $a_0 = \left[\begin{smallmatrix} W & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $a_1 = \lambda W \otimes P + (I-P)$, let $t \mapsto b_t$ be a smooth path of invertibles in $C^*(W,Z) \cong C(\mathbb{T}^2)$ such that $b_0 = \left[\begin{smallmatrix} W & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $b_1 = \lambda W \otimes Q + (I-Q)$, and let $t \mapsto c_t$ be a smooth path of invertibles in $C^*(W) \cong C(\mathbb{T})$ such that $c_0 = \left[\begin{smallmatrix} W & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $c_1 = \lambda W \otimes e_0 + (I - e_0)$. Let

$$\eta_t := \begin{bmatrix} e(\theta t) W \otimes p + (1-p) \\ a_t \\ b_t \\ c_t \end{bmatrix}$$

and consider the smooth path

$$\gamma_t := (w_t \eta_t w_t^{-1} \eta_t^{-1}) \cdot \eta_t \xi$$

where ξ is the right-most matrix in (4.5). Clearly, $\gamma_0 = I$ and $\gamma_1 = \nu(\zeta')\zeta'^{-1}$. Now $v_t := w_t \eta_t w_t^{-1} \eta_t^{-1}$ being a commutator, one obtains (under the trace) $(\boldsymbol{\tau} \otimes \mathrm{Tr})(\dot{v}_t v_t^{-1}) = 0$. Hence

$$\begin{aligned}
(\boldsymbol{\tau} \otimes \mathrm{Tr})(\dot{\gamma}_t \gamma_t^{-1}) &= (\boldsymbol{\tau} \otimes \mathrm{Tr})(\dot{\eta}_t \eta_t^{-1}) \\
&= 2\pi i \theta \boldsymbol{\tau}(p) + (\boldsymbol{\tau} \otimes \mathrm{Tr})(\dot{a}_t a_t^{-1}) + (\boldsymbol{\tau} \otimes \mathrm{Tr})(\dot{b}_t b_t^{-1}) \\
&\quad + (\boldsymbol{\tau} \otimes \mathrm{Tr})(\dot{c}_t c_t^{-1})
\end{aligned}$$

and since $\boldsymbol{\tau}(p) = \theta$, ones gets

$$\begin{aligned}\Delta[\zeta'] &= q\biggl(\frac{1}{2\pi i}\int_0^1 (\boldsymbol{\tau}\otimes \mathrm{Tr})(\dot{\gamma}_t\gamma_t^{-1})dt\biggr)\\ &= q\biggl(\frac{1}{2\pi i}\int_0^1 \Bigl[2\pi i\theta^2 + (\boldsymbol{\tau}\otimes \mathrm{Tr})(\dot{a}_t a_t^{-1}) + (\boldsymbol{\tau}\otimes \mathrm{Tr})(\dot{b}_t b_t^{-1})\\ &\qquad\qquad + (\boldsymbol{\tau}\otimes \mathrm{Tr})(\dot{c}_t c_t^{-1})\Bigr]\, dt\biggr)\\ &= q(\theta^2)\end{aligned}$$

since the last three integrals are integers (as a_t, b_t, c_t are paths of invertibles in matrix algebras over $C(\mathbb{T}^2)$). Now as $q : \mathbb{R} \to \mathbb{R}/(\mathbb{Z}+\mathbb{Z}\theta)$, one deduces that $\boldsymbol{\tau}_* K_0(A_\theta^{5,6}) = \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2$.

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