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A CP-semigroup (or *quantum dynamical semigroup*) is a semigroup $\phi = \{\phi_t : t \geq 0\}$ of normal completely positive linear maps on $\mathcal{B}(H)$, H being a separable Hilbert space, which satisfies $\phi_t(1) = 1$ for all $t \geq 0$ and is continuous in the time parameter t the natural sense.

Let $\mathcal{D}$ be the natural domain of the generator L of ϕ, $\phi_t = \exp tL$, $t \geq 0$. Since the maps ϕ_t need not be multiplicative $\mathcal{D}$ is typically an operator space, but not an algebra. However, in this note we show that the set of operators

$$\mathcal{A} = \{A \in \mathcal{D} : A^*A \in \mathcal{D}, AA^* \in \mathcal{D}\}$$

is a $*$-subalgebra of $\mathcal{B}(H)$, indeed $\mathcal{A}$ is the largest self-adjoint algebra contained in $\mathcal{D}$. Examples are described for which the domain algebra $\mathcal{A}$ is, and is not, strongly dense in $\mathcal{B}(H)$.

1. Basic properties of $\mathcal{A}$.

Let $\phi = \{\phi_t : t \geq 0\}$ be a CP-semigroup as defined in the abstract. We first recall four characterizations of the domain of the generator of ϕ.

Lemma 1. *Let $A \in \mathcal{B}(H)$. The following are equivalent.*

(i) *The limit*

$$L(A) = \lim_{t \to 0+} \frac{1}{t}(\phi_t(A) - A)$$

exists relative to the strong-$*$ topology of $\mathcal{B}(H)$.

(ii) *The limit*

$$L(A) = \lim_{t \to 0+} \frac{1}{t}(\phi_t(A) - A)$$

exists relative to the weak operator topology of $\mathcal{B}(H)$.

(iii)

$$\sup_{t>0} \frac{1}{t}\|\phi_t(A) - A\| \leq M < \infty.$$

(iv) *There is a sequence $t_n \to 0+$ for which*

$$\sup_n \frac{1}{t_n}\|\phi_{t_n}(A) - A\| \leq M < \infty.$$

Proof. The implications (i) $\implies$ (ii) and (iii) $\implies$ (iv) are trivial, and (ii) $\implies$ (iii) is a straightforward consequence of the Banach-Steinhaus theorem.

Proof of (iv) $\implies$ (i). Since the unit ball of $\mathcal{B}(H)$ is weakly sequentially compact, the hypothesis (iv) implies that there is a sequence $t_n \to 0+$ such that

$$\frac{1}{t_n}(\phi_{t_n}(A) - A) \to T \in \mathcal{B}(H)$$

in the weak operator topology. We claim: for every $s > 0$,

$$\int_0^s \phi_\lambda(T)\, d\lambda = \phi_s(A) - A. \tag{1.1}$$

The integral on the left is interpreted as a weak integral; that is, for $\xi, \eta \in H$,

$$\int_0^s \langle \phi_\lambda(T)\xi, \eta\rangle \; d\lambda = \langle \phi_s(A)\xi, \eta\rangle - \langle A\xi, \eta\rangle .$$

To see that, fix $\lambda > 0$. Since ϕ_λ is weakly continuous on bounded sets in $\mathcal{B}(H)$ we have

$$\frac{1}{t_n}(\phi_{\lambda+t_n}(A) - \phi_\lambda(A)) = \phi_\lambda\left(\frac{1}{t_n}(\phi_{t_n}(A) - A)\right) \to \phi_\lambda(T)$$

in the weak operator topology, as $n \to \infty$. By the bounded convergence theorem, we find that for fixed $\xi, \eta \in H$,

$$\lim_{n\to\infty} \frac{1}{t_n}\left(\int_0^s \langle \phi_{\lambda+t_n}(A)\xi, \eta\rangle \; d\lambda - \int_0^s \langle \phi_\lambda(A)\xi, \eta\rangle \; d\lambda\right) = \int_0^s \langle \phi_\lambda(T)\xi, \eta\rangle \; d\lambda.$$

Writing

$$\int_0^s f(\lambda + t_n)\, d\lambda - \int_0^s f(\lambda)\, d\lambda = \int_s^{s+t_n} f(\lambda)\, d\lambda - \int_0^{t_n} f(\lambda)\, d\lambda,$$

the left side of the preceding formula becomes

$$\lim_{n\to\infty}\left(\frac{1}{t_n}\int_s^{s+t_n} \langle \phi_\lambda(A)\xi, \eta\rangle \; d\lambda - \frac{1}{t_n}\int_0^{t_n} \langle \phi_\lambda(A)\xi, \eta\rangle \; d\lambda\right)$$

which, because of continuity of ϕ in the time parameter, is $\langle \phi_s(A)\xi, \eta\rangle - \langle A\xi, \eta\rangle$, as asserted in (1.1).

To prove the strong-$*$ convergence asserted in (i), fix $\xi \in H$ and use (1.1) to write

$$\begin{aligned}\left\|\frac{1}{s}(\phi_s(A)\xi - A\xi) - T\xi\right\| &= \frac{1}{s}\left\|\int_0^s \phi_\lambda(T)\xi\, d\lambda - \int_0^s T\xi\, d\lambda\right\| \\ &\le \frac{1}{s}\int_0^s \|\phi_\lambda(T)\xi - T\xi\|\, d\lambda \le \left(\frac{1}{s}\int_0^s \|\phi_\lambda(T)\xi - T\xi\|^2\, d\lambda\right)^{1/2}.\end{aligned}$$

The integrand of the last term expands as follows

$$\begin{aligned}\|\phi_\lambda(T)\xi - T\xi\|^2 &= \langle \phi_\lambda(T)^*\phi_\lambda(T)\xi, \xi\rangle - 2\Re\langle\phi_\lambda(T)\xi, T\xi\rangle + \|T\xi\|^2 \\ &\le \langle \phi_\lambda(T^*T)\xi, \xi\rangle - 2\Re\langle\phi_\lambda(T)\xi, T\xi\rangle + \|T\xi\|^2,\end{aligned}$$

the last inequality by the Schwarz inequality for unital CP maps. Since $\phi_\lambda(T^*T)$ (resp. $\phi_\lambda(T)$) tends weakly to T^*T (resp. T) as $\lambda \to 0+$, it follows that

$$\limsup_{s\to 0+} \frac{1}{s}\int_0^s \|\phi_\lambda(T)\xi - T\xi\|^2\, d\lambda \le \langle T^*T\xi, \xi\rangle - 2\langle T\xi, T\xi\rangle + \|T\xi\|^2 = 0,$$

and we conclude that $\frac{1}{s}(\phi_s(A) - A)$ tends strongly to T as $s \to 0+$.

Similarly, $\frac{1}{s}(\phi_s(A) - A)^* = \frac{1}{s}(\phi_s(A^*) - A^*)$ tends strongly to T^*. □

Definition. Let $\mathcal{D}$ be the set of all operators $A \in \mathcal{B}(H)$ for which the four conditions of Lemma 1 are satisfied. $L : \mathcal{D} \to \mathcal{B}(H)$ denotes the generator of ϕ,

$$L(A) = \lim_{t\to 0+} \frac{1}{t}(\phi_t(A) - A), \qquad A \in \mathcal{D}.$$

It is obvious that $\mathcal{D}$ is a self-adjoint linear subspace of $\mathcal{B}(H)$, that $L(A^*) = L(A)^*$ for $A \in \mathcal{D}$, and a standard argument shows that $\mathcal{D}$ is dense in $\mathcal{B}(H)$ in the σ-strong operator topology.

Lemma 2. *For every operator $A \in \mathcal{D}$ we have*

$$\|L(A)\| = \sup_{t>0} \frac{1}{t}\|\phi_t(A) - A\|.$$

Proof. The inequality $\le$ is clear from the fact that $L(A)$ is the weak limit of operators $\frac{1}{t}(\phi_t(A) - A)$ near $t = 0+$, i.e.,

$$\|L(A)\| \le \limsup_{t\to 0+} \frac{1}{t}\|\phi_t(A) - A\| \le \sup_{t>0} \frac{1}{t}\|\phi_t(A) - A\|.$$

For $\ge$, set $T = L(A)$. Using (1.1), we can write for every $t > 0$

$$\frac{1}{t}\|\phi_t(A) - A\| = \frac{1}{t}\left\|\int_0^t \phi_\lambda(T)\, d\lambda\right\| \le \frac{1}{t}\int_0^t \|\phi_\lambda(T)\|\, d\lambda \le \|T\|,$$

since $\|\phi_\lambda\| \le 1$ for every $\lambda \ge 0$. □

Theorem A. *$\mathcal{A} = \{A \in \mathcal{D} : A^*A \in \mathcal{D}, AA^* \in \mathcal{D}\}$ is a $*$-subalgebra of $\mathcal{B}(H)$.*

Proof. $\mathcal{A}$ is obviously a self-adjoint set of operators. We have to show that $\mathcal{A}$ is a vector space satisfying $\mathcal{A}\cdot\mathcal{A} \subseteq \mathcal{A}$.

Fix $t > 0$. By Stinespring's theorem we can write

$$\phi_t(X) = V_t^*\pi_t(X)V_t, \qquad X \in \mathcal{B}(H) \tag{1.2}$$

where V_t is an isometry from H into some other Hilbert space H_t and where $\pi_t : \mathcal{B}(H) \to \mathcal{B}(H_t)$ is a *normal* $*$-homomorphism of von Neumann algebras. $P_t = V_t V_t^*$ is a self-ajoint projection in $\mathcal{B}(H_t)$.

For $t > 0$ we will consider the seminorms p_t, q_t defined on $\mathcal{B}(H)$ as follows

$$\begin{aligned} p_t(X) &= t^{-1}\|\phi_t(X) - X\|, \\ q_t(X) &= t^{-1/2}\|P_t\pi_t(X) - \pi_t(X)P_t\|, \qquad X \in \mathcal{B}(H). \end{aligned}$$

Lemma 3. *For every operator $X \in \mathcal{B}(H)$ we have the following characterizations.*

(i) *$X \in \mathcal{D}$ iff*

$$\sup_{t>0} p_t(X) < \infty,$$

and in that case $\|L(X)\| = \sup_{t>0} p_t(X)$.

(ii) *$X \in \mathcal{A}$ iff both $\sup_{t>0} p_t(X)$ and $\sup_{t>0} q_t(X)$ are finite, and in that case*

$$\max(\|\sigma_L(dX^*\,dX)\|^{1/2}, \|\sigma_L(dX\,dX^*)\|^{1/2}) \leq \limsup_{t\to 0+} q_t(X),$$

where $\sigma_L(dX^*\,dX)$ and $\sigma_L(dX\,dX^*)$ are the operators in $\mathcal{B}(H)$ defined by

$$\begin{aligned} \sigma_L(dX^*\,dX) &= L(X^*X) - X^*L(X) - L(X^*)X, \\ \sigma_L(dX\,dX^*) &= L(XX^*) - XL(X^*) - L(X)X^*. \end{aligned}$$

Remark. The second assertion of Lemma 3 requires clarification. By definition, an operator X belongs to $\mathcal{A}$ iff all four operators X, X^*, X^*X, XX^* belong to the domain of the generator L of $\phi = \{\phi_t : t \geq 0\}$. In that case both operators $\sigma_L(dX^*\,dX)$ and $\sigma_L(dX\,dX^*)$ are well-defined by the above formulas. The "symbol" map σ_L is discussed more fully in [**2**].

Proof of Lemma 3. The assertion (i) follows from Lemmas 1 and 2 above. In order to prove (ii) we require the following more concrete expression for the seminorm q_t,

$$(1.3)\quad q_t(X) = \max\left(\left\|\frac{1}{t}(\phi_t(X^*X) - \phi_t(X)^*\phi_t(X))\right\|^{1/2}, \left\|\frac{1}{t}(\phi_t(XX^*) - \phi_t(X)^*\phi_t(X^*))\right\|^{1/2}\right).$$

To prove (1.3) we decompose the commutator $\pi_t(X)P_t - P_t\pi_t(X)$ into a sum

$$\pi_t(X)P_t - P_t\pi_t(X) = (\mathbf{1} - P_t)\pi_t(X)P_t - P_t\pi_t(X)(\mathbf{1} - P_t).$$

Since the first term $(\mathbf{1}-P_t)\pi_t(X)P_t$ has initial space in P_tH_t and final space in $(\mathbf{1}-P_t)H_t$, and the second term has the opposite property, it follows that

$$\|\pi_t(X)P_t - P_t\pi_t(X)\| = \max(\|(\mathbf{1}-P_t)\pi_t(X)P_t\|, \|P_t\pi_t(X)(\mathbf{1}-P_t)\|).$$

We have

$$\begin{aligned}\|(\mathbf{1}-P_t)\pi_t(X)P_t\|^2 &= \|V_t^*\pi_t(X^*)(\mathbf{1}-P_t)\pi_t(X)V_t\| \\ &= \|V_t^*\pi_t(X^*X)V_t - V_t^*\pi_t(X^*)V_tV_t^*\pi_t(X)V_t\| \\ &= \|\phi_t(X^*X) - \phi_t(X)^*\phi_t(X)\|.\end{aligned}$$

Similarly,

$$\begin{aligned}\|P_t\pi_t(X)(\mathbf{1}-P_t)\|^2 &= \|V_t^*\pi_t(X)(\mathbf{1}-P_t)\pi_t(X^*)V_t\| \\ &= \|\phi_t(XX^*) - \phi_t(X)^*\phi_t(X^*)\|,\end{aligned}$$

and formula (1.3) follows from these two expressions.

Now if $X \in \mathcal{A}$ then all four operators X, X^*, X^*X, XX^* belong to $\mathcal{D}$, hence all four limits

$$\begin{aligned}&\lim_{t\to 0+}\frac{1}{t}(\phi_t(X^*X) - X^*X) = L(X^*X),\\ &\lim_{t\to 0+}\frac{1}{t}(\phi_t(XX^*) - X^*X) = L(XX^*),\\ &\lim_{t\to 0+}\frac{1}{t}(\phi_t(X) - X) = L(X),\\ &\lim_{t\to 0+}\frac{1}{t}(\phi_t(X^*) - X^*) = L(X^*)\end{aligned}$$

exist relative to the strong operator topology. Writing

$$\begin{gathered}\phi_t(X^*X) - \phi_t(X)^*\phi_t(X) = \\ (\phi_t(X^*X) - X^*X) - X^*(\phi_t(X) - X) - (\phi_t(X^*) - X^*)\phi_t(X)\end{gathered} \tag{1.4}$$

and using strong continuity of multiplication on bounded sets, we find that the limit

$$\begin{aligned}\lim_{t\to 0+}\frac{1}{t}(\phi_t(X^*X) - \phi_t(X^*)\phi_t(X)) &= L(X^*X) - X^*L(X) - L(X^*)X \\ &= \sigma_L(dX^*\,dX)\end{aligned}$$

exists relative to the strong operator topology.

In the same way we deduce the existence of the strong limit

$$\begin{aligned}\lim_{t\to 0+}\frac{1}{t}(\phi_t(XX^*) - \phi_t(X)\phi_t(X^*)) &= L(XX^*) - XL(X^*) - L(X)X^* \\ &= \sigma_L(dX\,dX^*).\end{aligned}$$

It follows that for every $X \in \mathcal{A}$ the seminorms $q_t(X)$ are bounded for $t > 0$, and for such X we have

$$\max(\|\sigma_L(dX^*\, dX)\|^{1/2}, \|\sigma_L(dX\, dX^*)\|^{1/2}) \leq \limsup_{t \to 0+} q_t(X).$$

Conversely, suppose we are given an operator $X \in \mathcal{D}$ for which the seminorms $q_t(X)$ are bounded for $t > 0$. We have to show that X^*X and XX^* belong to $\mathcal{D}$; since $\mathcal{D}$ is self-adjoint and the seminorms q_t are symmetric in that $q_t(X^*) = q_t(X)$, it is enough to show that X^*X belong to $\mathcal{D}$. (1.4) implies that for fixed $t > 0$,

$$\begin{aligned}\phi_t(X^*X) - X^*X = (&\phi_t(X^*X) - \phi_t(X^*)\phi_t(X)) \\ &+ X^*(\phi_t(X) - X) + (\phi_t(X^*) - X^*)\phi_t(X).\end{aligned} \tag{1.5}$$

Because of (1.3), the first term on the right of (1.5) is bounded in norm by $M_1 \cdot t$ where M_1 is a positive constant. Similarly, since X and X^* belong to $\mathcal{D}$ the second and third terms are bounded in norm by terms of the form $M_2 \cdot t$ and $M_3 \cdot t$ respectively, hence

$$\|\phi_t(X^*X) - X^*X\| \leq (M_1 + M_2 + M_3) \cdot t.$$

By Lemma 1, X^*X must belong to $\mathcal{D}$. □

Turning now to the proof of Theorem A, (or more properly, to the proof that $\mathcal{A}$ is an algebra), Lemma 3 tells us that $\mathcal{A}$ consists of all operators $X \in \mathcal{B}(H)$ for which

$$\sup_{t>0} p_t(X) < \infty, \quad \text{and} \quad \sup_{t>0} q_t(X) < \infty.$$

Since p_t and q_t are both seminorms, it follows that $\mathcal{A}$ is a complex vector space which is obviously closed under the $*$-operation.

To see that $\mathcal{A}$ is closed under multiplication, pick $X, Y \in \mathcal{A}$. According to Lemma 3, it is enough to show

$$\sup_{t>0} q_t(XY) < \infty \tag{1.6}$$

and

$$\sup_{t>0} p_t(XY) < \infty. \tag{1.7}$$

To prove (1.6) we claim that

$$q_t(XY) \leq q_t(X)\|Y\| + \|X\| q_t(Y). \tag{1.8}$$

Indeed, writing $[A, B]$ for the commutator $AB - BA$ we have

$$[P_t, \pi_t(XY)] = [P_t, \pi_t(X)]\pi_t(Y) + \pi_t(X)[P_t, \pi_t(Y)],$$

and hence

$$\begin{aligned} q_t(XY) &= t^{-1/2}\|[P_t, \pi_t(XY)]\| \\ &\leq t^{-1/2}\|[P_t, \pi_t(X)]\| \cdot \|\pi_t(Y)\| + \|\pi_t(X)\| \cdot t^{-1/2}\|[P_t, \pi_t(Y)]\|, \end{aligned}$$

from which (1.8) is evident.

Finally, consider Condition (1.7). By definition of $\mathcal{A}$, $A \in \mathcal{A}$ implies $A^*A \in \mathcal{D}$. Since $\mathcal{A}$ is now known to be a linear space we can assert that if $X, Y \in \mathcal{A}$ then for every $k = 0, 1, 2, 3$ we have $Y + i^k X \in \mathcal{A}$, hence $(Y + i^k X)^*(Y + i^k X) \in \mathcal{D}$ and by the polarization formula

$$X^*Y = \frac{1}{4}\sum_{k=0}^{3} i^k (Y + i^k X)^*(Y + i^k X),$$

X^*Y must also belong to $\mathcal{D}$. Since $\mathcal{A}^* = \mathcal{A}$, we can replace X^* with X to conclude that $XY \in \mathcal{D}$. (1.7) now follows from Lemma 3 (i). □

Corollary. *Let $\mathcal{D}$ be the domain of the generator of a CP-semigroup acting on $\mathcal{B}(H)$ and let A be a self-adjoint operator such that $A \in \mathcal{D}$ and $A^2 \in \mathcal{D}$. Then $p(A) \in \mathcal{D}$ for every polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$.*

2. Examples, Remarks.

We describe two classes of examples which are in a sense at opposite extremes. In the first class of examples of CP-semigroups $\phi = \{\phi_t : t \geq 0\}$, each ϕ_t leaves the C^*-algebra $\mathcal{K}$ of all compact operators invariant, $\phi_t(\mathcal{K}) \subseteq \mathcal{K}$, its domain algebra $\mathcal{A}$ is strongly dense in $\mathcal{B}(H)$, and its generator restricts to a *second order* differential operator on $\mathcal{A}$ in the sense of [**2**]. In the second class of examples, the individual maps satisfy $\phi_t(\mathcal{K}) \cap \mathcal{K} = \{0\}$ for $t > 0$, $\mathcal{A}$ is not strongly dense in $\mathcal{B}(H)$, and its generator is degenerate in the sense that it restricts to a *derivation* on $\mathcal{A}$.

We first recall the CP-semigroups of [**1**], including the heat flow of the CCR algebra. While for simplicity we confine the discussion to the case of one degree of freedom, the reader will note that everything carries over verbatim to the case of n degrees of freedom, $n = 1, 2, \ldots$.

Let $\{W_z : z \in \mathbb{R}^2\}$ be an irreducible Weyl system acting on a Hilbert space H. Thus, $z \in \mathbb{R}^2 \mapsto W_z$ is a strongly continuous mapping from $\mathbb{R}^2$ into the unitary operators on H which satisfies the canonical commutation relations in Weyl's form

$$W_{z_1} W_{z_2} = e^{i\omega(z_1, z_2)} W_{z_1 + z_2}, \qquad z_1, z_2 \in \mathbb{R}^2,$$

ω denoting the symplectic form on $\mathbb{R}^2$ given by

$$\omega((x, y), (x', y')) = \frac{1}{2}(x'y - xy').$$

Let $\{\mu_t : t \geq 0\}$ be a one-parameter family of probability measures on $\mathbb{R}^2$ which is a semigroup under the natural convolution of measures

$$\mu * \nu(S) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \chi_S(z + w)\, d\mu(z)\, d\nu(w),$$

which satisfies $\mu_0 = \delta_{(0,0)}$, and which is measurable in t in the natural sense. It is convenient to define the Fourier transform of a measure μ in terms of the symplectic form ω as follows,

$$\hat{\mu}(z) = \int_{\mathbb{R}^2} e^{i\omega(z,\zeta)}\, d\mu(\zeta), \qquad z \in \mathbb{R}^2.$$

Given such a semigroup of probability measures $\{\mu_t : t \geq 0\}$ there is a unique CP semigroup $\phi = \{\phi_t : t \geq 0\}$ acting on $\mathcal{B}(H)$ which satisfies

$$\phi_t(W_z) = \hat{\mu}_t(z) W_z, \qquad z \in \mathbb{R}^2, \quad t \geq 0$$

see [1], Proposition 1.7. Two cases of particular interest are

(CCR heat flow) $$\phi_t(W_z) = e^{-t|z|^2} W_z, \qquad t \geq 0$$

where $|(x, y)|$ denotes the Euclidean norm $(x^2 + y^2)^{1/2}$, and

(Cauchy flow) $$\phi_t(W_z) = e^{-t|z|} W_z, \qquad t \geq 0.$$

For both of these examples a straightforward estimate shows that for fixed $z \in \mathbb{R}^2$ there is a constant $M > 0$ such that

$$\|\phi_t(W_z) - W_z\| = |\hat{\mu}_t(z) - 1| \leq M \cdot t, \qquad t > 0$$

and hence $W_z \in \mathcal{D}$. Since W_z is unitary, $\mathbf{1} = W_z^* W_z = W_z W_z^*$ belongs to $\mathcal{D}$, and hence W_z belongs to the domain algebra $\mathcal{A}$ of ϕ for every $z \in \mathbb{R}^2$. We conclude that for these examples, the domain algebra is strongly dense in $\mathcal{B}(H)$.

Indeed, it can be seen that $\mathcal{A}$ contains a $*$-algebra of compact operators that is norm-dense in the algebra $\mathcal{K}$ of all compact operators. Unlike the examples to follow, these flows leave $\mathcal{K}$ invariant in the sense that $\phi_t(\mathcal{K}) \subseteq \mathcal{K}$ for all $t \geq 0$, and can therefore be considered as CP-semigroups which act on the separable C^*-algebra $\mathcal{K}$, rather than than as CP-semigroups acting on $\mathcal{B}(H)$.

We now describe a class of examples of CP semigroups whose domain algebras are *not* strongly dense in $\mathcal{B}(H)$. The referee has kindly pointed out that there are previously known examples of singular Markov semigroups in the literature which exhibit a similar phenomenon ([15]). Consequently, we have omitted proofs of the results below. The examples we describe here are inspired by a class of CP semigroups that have emerged in recent work of Robert Powers, to whom we are indebted for useful discussions.

Let $H = L^2(0, \infty)$ and let $U = \{U_t : t \geq 0\}$ be the semigroup of isometries $U_t\xi(x) = \xi(x - t)$ for $x \geq t$, $U_t\xi(x) = 0$ for $0 \leq x < t$. Fix a real number

$\alpha > 0$, and let f be the unit vector in $L^2(0,\infty)$ obtained by normalizing the exponential function $u(x) = e^{-\alpha x}$, $x \geq 0$. One has $U_t^* f = e^{-\alpha t} f$ for every $t \geq 0$, hence the vector state $\omega(A) = \langle Af, f\rangle$ satisfies $\omega(U_t A U_t^*) = e^{-2\alpha t}\omega(A)$, $A \in \mathcal{B}(H)$.

We consider the family of unit-preserving normal completely positive maps $\phi = \{\phi_t : t \geq 0\}$ defined on $\mathcal{B}(H)$ by

$$\phi_t(A) = \omega(A)E_t + U_t A U_t^*, \qquad t \geq 0,$$

where $E_t = \mathbf{1} - U_t U_t^*$ is the projection on the subspace $L^2(0,t) \subseteq L^2(0,\infty)$. Since

$$\omega(E_t) = \omega(\mathbf{1}) - \omega(U_t U_t^*) = 1 - e^{-2\alpha t},$$

it follows that $\omega(\phi_t(A)) = \omega(A)$ for every A. A routine computation now shows that ϕ satisfies the semigroup property $\phi_s \circ \phi_t = \phi_{s+t}$, hence ϕ is a CP semigroup.

Let $\mathcal{D}$ be the domain of the generator of ϕ and let $\mathcal{A}$ be the domain algebra

$$\mathcal{A} = \{A \in \mathcal{D} : A^*A \in \mathcal{D}, AA^* \in \mathcal{D}\}.$$

Theorem A implies that $\mathcal{A}$ is a unital $*$-algebra, and its strong closure is described as follows.

Proposition. *The strong closure of $\mathcal{A}$ consists of all operators $B \in \mathcal{B}(H)$ such that B commutes with the rank-one projection $f \otimes \bar{f}$.*

Thus the strong closure $\mathcal{A}^-$ of $\mathcal{A}$ has the form $\mathcal{B}(H_0) \oplus \mathbb{C}$ where $H_0 \subseteq H$ is a subspace of codimension one in H. The following consequence is easily deduced from the Proposition; it implies that these examples are "almost" E_0-semigroups in the sense that there is an E_0-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on $\mathcal{B}(H_0)$ such that ϕ_t acts as follows on $\mathcal{A}^-$,

$$\phi_t(B \oplus \lambda) = \alpha_t(B) \oplus \lambda, \qquad B \in \mathcal{B}(H_0), \quad \lambda \in \mathbb{C}.$$

Corollary. *Let $\overline{\mathcal{A}}$ be the strong closure of $\mathcal{A}$. Then $\phi_t(\overline{\mathcal{A}}) \subseteq \overline{\mathcal{A}}$ for every $t \geq 0$ and ϕ restricts to a semigroup of $*$-endomorphisms of this von Neumann algebra.*

The Corollary implies that the semigroup ϕ is degenerate in the sense that its generator is essentially a derivation, not a true "second order" noncommutative differential operator. Whether or not this degeneracy is related to the non-density of the domain algebra $\mathcal{A}$ is an interesting question about which we as yet have very little information.

In particular, we do not know how small the domain algebra can be. For example, does there exist a CP semigroup whose domain algebra is just the scalars $\mathbb{C} \cdot \mathbf{1}$?

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