# Pacific Journal of Mathematics


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Volume 203 No. 1 March 2002

PACIFIC JOURNAL OF MATHEMATICS
Vol. 203, No. 1, 2002

# ON BOUNDARY AVOIDING SELECTIONS AND SOME EXTENSION THEOREMS

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**A theorem of Marc Frantz about controlled continuous extensions of functions inspired us to prove a general result concerning boundary avoiding continuous selections into Banach spaces, which has Frantz' theorem as a corollary. In addition, with relatively simple means we improve upon some other results of Frantz involving extensions of products and of disjoint families of functions.**

## 1. Introduction.

The following two extension theorems are presented in Frantz [**3**]. Let $I$ denote the interval $[0,1]$.

**Theorem 1.** *Let $X$ be a normal space, let $A$ be a closed subset of $X$, and let $Y_0, Y_1$ be disjoint closed $G_\delta$-subsets of $X$. If $f : A \to I$ is a continuous function such that for $i = 0,1$, $f^{-1}(i) = Y_i \cap A$ then there exists a continuous extension $\hat{f} : X \to I$ of $f$ with $\hat{f}^{-1}(i) = Y_i$ for $i = 0,1$.*

**Theorem 2.** *Let $X$ be a compact metric space and let $A$ be a closed subset of $X$. If $f : A \to \mathbb{R}$, $g : A \to [0,\infty)$, and $h : X \to \mathbb{R}$ are continuous functions such that $f \cdot g = h|A$ and $g^{-1}(0) \subset f^{-1}(0)$ then there are continuous extensions $\hat{f} : X \to \mathbb{R}$ and $\hat{g} : X \to [0,\infty)$ of $f$ and $g$ with $\hat{f} \cdot \hat{g} = h$.*

We present a general result (Theorem 4) about boundary avoiding continuous selections that has Theorem 1 as a corollary. We also give a very simple argument that shows that Theorem 2 is valid without any restrictions on the domain $X$ other than the necessary normality (see Corollary 8). In addition, with Corollary 12 and Example 3 we sharpen a result in [**3**] concerning the extension of pairwise disjoint collections of functions.

All spaces in this paper are assumed to be Tychonoff.

## 2. Boundary avoiding continuous selections.

If $Y$ is a set then $2^Y = \mathcal{P}(Y) \setminus \{\emptyset\}$. Let $X$ and $Y$ be topological spaces and let $\varphi : X \to 2^Y$ be a set-valued function. If $A \subset Y$ then we put $\varphi^{-1}[A] = \{x \in X : \varphi(x) \cap A \neq \emptyset\}$. The function $\varphi$ is called *lower semi-continuous* (LSC for short) if for each open set $O$ in $Y$ the set $\varphi^{-1}[O]$ is open in $X$. A function $f : X \to Y$ is called a *selection* of $\varphi$ if $f(x) \in \varphi(x)$ for every $x \in X$. If $Y$ is a metric space then we call $\varphi$ *bounded* if there is an $M > 0$ such that the diameter of every $\varphi(x)$ is less than $M$.

Let $(B, \|\cdot\|)$ be a Banach space and let $\varepsilon > 0$. Let $U_\varepsilon$ denote the open $\varepsilon$-ball $\{y \in B : \|y\| < \varepsilon\}$. If $C$ is a subset of $B$ then $\operatorname{int} C$ denotes the interior of $C$ in $B$ and if $\varepsilon > 0$ then we put

$$\operatorname{int}_\varepsilon C = \{y \in B : y + U_\varepsilon \subset C\}.$$

Note that $\operatorname{int}_\varepsilon C$ is always closed and that if $C$ is convex then so is $\operatorname{int}_\varepsilon C$.

A space $X$ is called *countably paracompact* if every countable open cover of the space has a locally finite open refinement that covers the space. For normal spaces this property is equivalent to the property that for every increasing sequence $U_1 \subset U_2 \subset \dots$ of open sets with $\bigcup_{i=1}^\infty U_i = X$ there exist a sequence $F_1, F_2, \dots$ of closed sets such that $F_i \subset U_i$ for $i \in \mathbb{N}$ and $\bigcup_{i=1}^\infty F_i = X$, see [**2**, Corollary 5.2.2]. Spaces that are normal but not countably paracompact are known as *Dowker spaces*, see Rudin [**6**].

**Lemma 3.** *Let $X$ be a normal space, let $B$ be a Banach space, let $C$ be a convex subset of $B$, let $\varphi : X \to 2^C$ be* LSC *and bounded such that every $\varphi(x)$ is closed and convex in $B$, and let $F_1, F_2, \dots$ be a sequence of closed subsets of $X$ such that $F_n \subset \varphi^{-1}[\operatorname{int}_{1/n} C]$ for each $n \in \mathbb{N}$.*

(a) *If $B$ is separable and every $\varphi(x)$ is compact, or*
(b) *if $B$ is separable and $X$ is countably paracompact, or*
(c) *if $X$ is paracompact*

*then there is a continuous selection $f$ of $\varphi$ with $f(F_n) \subset \operatorname{int} C$ for each $n$.*

*Proof.* We may assume that $F_n \subset F_{n+1}$ for every $n$. Put $F_0 = \emptyset$ and $A = X \setminus \bigcup_{n=1}^\infty F_n$. Let $M > 1$ be an upper bound for the diameters of the $\varphi(x)$'s. For $n \in \mathbb{N}$ put $\delta(n) = 1/(Mn^2)$ and $C_n = \operatorname{int}_{\delta(n)} C$. We define a function $\psi : X \to 2^C$ as follows:

$$\psi(x) = \begin{cases} \varphi(x), & \text{if } x \in A; \\ \varphi(x) \cap C_n, & \text{if } x \in F_n \setminus F_{n-1}. \end{cases}$$

Note that since $\operatorname{int}_\varepsilon C$ is closed and convex, every $\psi(x)$ is closed (and in case (a) compact) and convex. If $\psi$ is LSC then according to Michael [**5**] it has a continuous selection $f$ which obviously has the property $f(F_n) \subset C_n \subset \operatorname{int} C$ for each $n \in \mathbb{N}$.

It remains to prove that $\psi$ is LSC. Let $O$ be open in $B$ and let $x \in \psi^{-1}[O]$. Select a vector $a \in \psi(x) \cap O$. In order to prove that $x$ is an interior point of $\psi^{-1}[O]$ we distinguish two cases:

*Case* I. $x \notin A$. Let $n \in \mathbb{N}$ be such that $x \in F_n \setminus F_{n-1}$. So $\psi(x) = \varphi(x) \cap C_n$ and $a \in \varphi(x) \cap C_n \cap O$. Since by assumption $F_n \subset \varphi^{-1}[\operatorname{int}_{1/n} C]$ we can find a vector $b \in \varphi(x) \cap \operatorname{int}_{1/n} C$. Since $1/n > \delta(n)$ we have $b \in \operatorname{int}_{1/n} C \subset \operatorname{int} C_n$

and hence $b + U_\varepsilon \subset C_n$ for some $\varepsilon > 0$. Let $t \in (0,1]$ and note that by the convexity of $C_n$ we have $a+t(b-a)+U_{t\varepsilon} \subset C_n$. Note that $a \in O$ so for some small enough $t \in (0,1]$ the vector $c = a+t(b-a)$ is in $O \cap \operatorname{int} C_n$. By convexity of $\varphi(x)$ we have $c \in \varphi(x)$. Define the open set $U = \varphi^{-1}[O \cap \operatorname{int} C_n] \setminus F_{n-1}$. Note that $x \in U$. If $y \in U$ then there is a $d \in O \cap \varphi(y) \cap \operatorname{int} C_n$. Since $y \notin F_{n-1}$ we have $\varphi(y) \cap C_n \subset \psi(y)$ and hence $d \in O \cap \psi(y)$. Conclusion: $y \in \psi^{-1}[O]$ and $U \subset \psi^{-1}[O]$.

*Case* II. $x \in A$. Let $n \in \mathbb{N}$ be such that $a + U_{2/n} \subset O$. Define the open set $U = \varphi^{-1}[a + U_{1/n}] \setminus F_n$. Since $x \in A$ we have $x \in U$. Let $y \in U$ and select $b \in \varphi(y)$ such that $\|b - a\| < 1/n$. If $y \in A$ then $\psi(y) = \varphi(y)$ and obviously $y \in \psi^{-1}[O]$. So we may assume that $y \in F_m \setminus F_{m-1}$ for some $m > n$. Since $F_m \subset \varphi^{-1}[\operatorname{int}_{1/m} C]$ we can find a vector $c \in \varphi(y) \cap \operatorname{int}_{1/m} C$. So $c + U_{1/m} \subset C$ and $b \in \varphi(y) \subset C$. Put $t = 1/(Mm)$ and note that by the convexity of $C$ we have $b + t(c-b) + U_{t/m} \subset C$. So $d = b + t(c-b)$ is in $\operatorname{int}_{t/m} C = C_m$. Note that since $b$ and $c$ are in $\varphi(y)$ we have $\|c - b\| \le M$ and hence $\|d - b\| \le tM = 1/m < 1/n$. Also, by convexity of $\varphi(y)$ we have $d \in \varphi(y)$. So the distance between $d$ and $a$ is less than $2/n$ and hence $d \in O \cap \varphi(y) \cap C_m = O \cap \psi(y)$. Conclusion: $y \in \psi^{-1}[O]$ and $U \subset \psi^{-1}[O]$. $\square$

**Theorem 4.** *The following statements are equivalent:*

1. *$X$ is a normal and countably paracompact space.*
2. *For every separable Banach space $B$, every convex subset $C$ of $B$, every* LSC *function $\varphi : X \to 2^C$ such that each $\varphi(x)$ is* compact *and convex in $B$, and every $A \subset \varphi^{-1}[\operatorname{int} C]$ that is an $F_\sigma$-subset of $X$ there exists a continuous selection $f$ of $\varphi$ with $A \subset f^{-1}(\operatorname{int} C) \subset \varphi^{-1}[\operatorname{int} C]$.*
3. *For every separable Banach space $B$, every convex subset $C$ of $B$, every* LSC *function $\varphi : X \to 2^C$ such that each $\varphi(x)$ is* closed *and convex in $B$, and every $A \subset \varphi^{-1}[\operatorname{int} C]$ that is an $F_\sigma$-subset of $X$ there exists a continuous selection $f$ of $\varphi$ with $A \subset f^{-1}(\operatorname{int} C) \subset \varphi^{-1}[\operatorname{int} C]$.*

**Theorem 5.** *The following statements are equivalent:*

1. *$X$ is a paracompact space.*
2. *For every Banach space $B$, every convex subset $C$ of $B$, every* LSC *function $\varphi : X \to 2^C$ such that each $\varphi(x)$ is closed and convex in $B$, and every $A \subset \varphi^{-1}[\operatorname{int} C]$ that is an $F_\sigma$-subset of $X$ there exists a continuous selection $f$ of $\varphi$ with $A \subset f^{-1}(\operatorname{int} C) \subset \varphi^{-1}[\operatorname{int} C]$.*

*Proof.* We will prove both theorems at the same time. Note first that if we substitute $C = B$ then we have Michael's selection theorems so if (2) is valid then $X$ is normal in Theorem 4 and paracompact in Theorem 5. Note that the implication (3) $\Rightarrow$ (2) in Theorem 4 is trivial.

In order to prove that condition (2) in Theorem 4 implies that $X$ is countably paracompact we consider an countable, monotone open cover

$U_1 \subset U_2 \subset \cdots$ of $X$. Put $U_0 = \emptyset$ and define the LSC function $\varphi : X \to 2^I$ by

$$\varphi(x) = [0, 1/n] \quad \text{if } x \in U_n \setminus U_{n-1} \text{ for some } n \in \mathbb{N}.$$

Let $B = \mathbb{R}$, $C = I$, and $A = X = \varphi^{-1}[\operatorname{int} C]$. According to condition (2) there is a continuous function $f : X \to (0,1)$ such that $f(X \setminus U_n) \subset [0, 1/(n+1)]$ for each $n \in \mathbb{N}$. Then $F_n = f^{-1}([1/n, 1])$, $n \in \mathbb{N}$, is the closed cover of $X$ that proves countable paracompactness.

Let us now turn to proving that (1) implies (3) in Theorem 4 and that (1) implies (2) in Theorem 5. So assume that $X$ is normal and countably paracompact (respectively paracompact) and let $B$, $C$, $\varphi$, and $A$ be as in the hypotheses of condition (3) in Theorem 4 (respectively (2) in Theorem 5). With Michael we choose a continuous selection $g$ of $\varphi$ and we define a function $\psi : X \to 2^C$ by

$$\psi(x) = \varphi(x) \cap \{a \in B : \|a - g(x)\| \leq 1\}.$$

We intend to apply Lemma 3 to $\psi$. It is obvious that $\psi$ is bounded and LSC and that every $\psi(x)$ is convex and compact (respectively closed). We verify that $\psi^{-1}[\operatorname{int} C] = \varphi^{-1}[\operatorname{int} C]$ so that $A \subset \psi^{-1}[\operatorname{int} C]$. Let $x \in \varphi^{-1}[\operatorname{int} C]$. So there is a vector $a \in \varphi(x) \cap \operatorname{int} C$ and hence $a + U_\varepsilon \subset C$ for some $\varepsilon > 0$. Note that $g(x) \in \varphi(x) \subset C$ and pick a $t \in (0, 1]$ with $t\|a - g(x)\| \leq 1$. Let $b = g(x) + t(a - g(x)) \in \varphi(x)$ and note that $\|b - g(x)\| = t\|a - g(x)\| \leq 1$. By convexity of $C$ we have $b + U_{t\varepsilon} \subset C$ and hence $b \in \operatorname{int} C$. So $b \in \psi(x) \cap \operatorname{int} C$ and $x \in \psi^{-1}[\operatorname{int} C]$.

Since $A$ is by assumption an $F_\sigma$-set we may choose a sequence $H_1 \subset H_2 \subset \cdots$ of closed subsets of $X$ such that $\bigcup_{k=1}^\infty H_k = A$. For every $n \in \mathbb{N}$ consider the open set $U_n = \psi^{-1}[\operatorname{int}(\operatorname{int}_{1/n} C)]$ and note that the $U_n$'s cover $\psi^{-1}[\operatorname{int} C]$ and hence $A$. Since $X$ is countably paracompact, which is a closed hereditary property, we can find for each $k \in \mathbb{N}$ a closed covering $K_{k1} \subset K_{k2} \subset \cdots$ of $H_k$ such that $K_{kn} \subset U_n$ for each $n \in \mathbb{N}$. If we define $F_n = \bigcup_{k=1}^n K_{kn}$ then the $F_n$'s cover $A$. Note that for each $n \in \mathbb{N}$ we have $F_n \subset U_n \subset \psi^{-1}[\operatorname{int}_{1/n} C]$ so we may apply Lemma 3 to $\psi$ to obtain a continuous selection $f$ with the property $f(A) = \bigcup_{n=1}^\infty f(F_n) \subset \operatorname{int} C$. Since $\psi(x) \subset \varphi(x)$ for each $x \in X$, $f$ is also a selection of $\varphi$ and we trivially have $f^{-1}(\operatorname{int} C) \subset \varphi^{-1}[\operatorname{int} C]$. □

Theorem 1 now follows immediately from Theorem 4 with the slight flaw that Dowker spaces are not covered. To obtain the full strength of Theorem 1 we derive it from Lemma 3:

*Proof of Theorem* 1. Let $X$ be a normal space, let $A$ be a closed subset of $X$, let $Y_0, Y_1$ be disjoint closed $G_\delta$-subsets of $X$, and let $f : A \to I$ be a continuous function such that for $i = 0, 1$, $f^{-1}(i) = Y_i \cap A$. Choose a continuous extension $g : X \to I$ of $f$ such that $g(Y_i) \subset \{i\}$ for $i = 0, 1$. Put

$G = g^{-1}(\{0,1\})$ and let $H_2, H_3, \ldots$ be a sequence of closed subsets of $X$ such that $\bigcup_{n=2}^{\infty} H_n = X \setminus (Y_0 \cup Y_1)$. We define for $n \geq 2$ the closed sets

$$F_n = g^{-1}([1/n, 1-1/n]) \cup (H_n \cap G).$$

For the purpose of applying Lemma 3 the role of the Banach space $B$ is played by $\mathbb{R}$ and $C = I$ so $\operatorname{int}_{1/n} C = [1/n, 1-1/n]$. Define the obviously bounded LSC function $\varphi : X \to 2^I$ by

$$\varphi(x) = \begin{cases} \{g(x)\}, & \text{if } x \in A \cup Y_0 \cup Y_1; \\ I, & \text{otherwise.} \end{cases}$$

If $x \in F_n$ then either $g(x) \in \varphi(x) \cap \operatorname{int}_{1/n} C$ or $x \in H_n \cap G$ which means that $x \notin Y_0 \cup Y_1$ and $g(x) \in \{0,1\}$. In the second case we have $x \notin A$ and $\varphi(x) = I$ which implies $1/2 \in \varphi(x) \cap \operatorname{int}_{1/n} C$. So in either case we may conclude that $F_n \subset \varphi^{-1}[\operatorname{int}_{1/n} C]$ for every $n \geq 2$. Observe that $\varphi$ satisfies all the hypotheses of Lemma 3 so there is a continuous selection $\hat{f}$ of $\varphi$ such that $\hat{f}(\bigcup_{n=2}^{\infty} F_n) \subset (0,1)$. Note that $\hat{f}$ extends $g$ (and $f$) so $f(Y_i) \subset \{i\}$. Let $x \in X \setminus (Y_0 \cup Y_1)$. If $g(x) \in (0,1)$ then $x$ is in some $g^{-1}([1/n, 1-1/n])$ and if $g(x) \in \{0,1\}$ then $x$ is in some $H_m \cap G$. So $x$ is in some $F_k$ and $\hat{f}(x) \in (0,1)$. We have shown that $\hat{f}^{-1}(i) = Y_i$ for $i = 0,1$. □

As to the question of whether it is necessary for $C$ to be convex in Theorems 4 and 5 note that if $C$ is any open set or any set with empty interior then (2) is always valid, the condition $A \subset f^{-1}(\operatorname{int} C)$ being trivially satisfied. According to [**1**, p. TVS II.14], if $C$ is a convex set with nonempty interior then $\operatorname{int} C$ is dense in $C$ and $\operatorname{int} \overline{C} = \operatorname{int} C$, which means that the content of Theorems 4 and 5 does not change if we add the requirement that $C$ be closed. These observations suggest that the theorems are primarily of interest if $C$ is a closed set with dense interior so let us consider that case. It is obvious that (2) is valid if $C$ is for instance a union of two disjoint convex and closed sets so also in this case convexity is not strictly necessary. However, convexity plays an important role: The following proposition implies that if $C$ is a closed set with a dense and connected interior such that condition (2) is valid then $C$ must be convex.

**Proposition 6.** *Let $B$ be a Banach space and let $C$ be a closed subset of $B$. If for every* LSC *function $\varphi : I \to 2^C$ such that each $\varphi(x)$ is compact and convex there is a continuous selection $f$ of $\varphi$ with $f^{-1}(\operatorname{int} C) = \varphi^{-1}[\operatorname{int} C]$ then each component of $\operatorname{int} C$ is convex.*

*Proof.* Let $O$ be a component of $\operatorname{int} C$ and let $a$ and $b$ be two distinct vectors in $O$. Consider $\langle a, b \rangle$, the line segment $\{a + t(b-a) : t \in I\}$ that connects $a$ and $b$. Since we are in a Banach space $O$ is open and arcwise connected. We can find an embedding $\alpha : I \to O$ such that $\alpha(0) = a$ and $\alpha(1) = b$. Let

$$s = \sup\{t \in I : \langle a, \alpha(t) \rangle \subset O\}.$$

Since $a$ has convex neighbourhoods in $O$ we know that $s > 0$. Put $c = \alpha(s)$ and note that $\langle a, c\rangle$ is contained in the closure of $O$ and hence in the closed set $C$. Define the LSC function $\varphi : I \to 2^C$ by

$$\varphi(t) = \begin{cases} \{a\}, & \text{if } t = 0; \\ \langle a, c\rangle, & \text{if } 0 < t < 1; \\ \{c\}, & \text{if } t = 1. \end{cases}$$

Let $f : I \to C$ be a continuous selection of $\varphi$ such that $f^{-1}(\operatorname{int} C) = \varphi^{-1}[\operatorname{int} C] = I$. Since $f(I) \subset \langle a, c\rangle$, $f(0) = a$, and $f(1) = c$ the function $f$ must be surjective onto $\langle a, c\rangle$. So $\langle a, c\rangle$ is a subset of $\operatorname{int} C$ and $O$. If $s = 1$ then $\langle a, b\rangle = \langle a, c\rangle \subset O$ and we are finished. Note that $\langle a, c\rangle$ must have a convex neighbourhood in $O$ so if $s < 1$ then there is an $\varepsilon > 0$ with $\langle a, \alpha(t)\rangle \subset O$ for all $t \in (s-\varepsilon, s+\varepsilon)$. This result contradicts the maximality of $s$. □

## 3. Extending products.

Put $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^- = (-\infty, 0]$.

**Theorem 7.** *Let $X$ be a normal space and let $A$ be a closed subset of $X$. If $f : A \to \mathbb{R}^+$, $g : A \to \mathbb{R}^+$, and $h : X \to \mathbb{R}^+$ are continuous functions such that $f \cdot g = h|A$ then there are continuous extensions $\hat{f}, \hat{g} : X \to \mathbb{R}^+$ of $f$ and $g$ with $\hat{f} \cdot \hat{g} = h$. If in addition $g^{-1}(0) \subset f^{-1}(0)$ then it can be arranged that $\hat{g}^{-1}(0) \subset \hat{f}^{-1}(0)$.*

*Proof.* Let $\tilde{f}, \tilde{g} : X \to \mathbb{R}^+$ be Tietze extensions of $f$ and $g$. Define the obviously continuous functions $\hat{f}, \hat{g} : X \to \mathbb{R}^+$ by

$$\hat{f}(x) = \frac{\tilde{f}(x) - \tilde{g}(x) + \sqrt{(\tilde{f}(x) - \tilde{g}(x))^2 + 4h(x)}}{2}$$

and

$$\hat{g}(x) = \frac{\tilde{g}(x) - \tilde{f}(x) + \sqrt{(\tilde{f}(x) - \tilde{g}(x))^2 + 4h(x)}}{2}.$$

Some straightforward algebra shows that $\hat{f} \cdot \hat{g} = h$ and that whenever $\tilde{f}(x) \cdot \tilde{g}(x) = h(x)$ we have $\hat{f}(x) = \tilde{f}(x)$ and $\hat{g}(x) = \tilde{g}(x)$ which means that $\hat{f}$ and $\hat{g}$ are extensions of $f$ and $g$.

If we have $g^{-1}(0) \subset f^{-1}(0)$ or, equivalently, $f^{-1}(0) = h^{-1}(0) \cap A$ then we choose $\tilde{g}$ as above but we let $\tilde{f}$ be a Tietze extension of $f \cup (0|h^{-1}(0))$. We then define $\hat{f}$ and $\hat{g}$ as above. If $\hat{g}(x) = 0$ then $h(x) = \hat{f}(x) \cdot \hat{g}(x) = 0$ and hence $\tilde{f}(x) = 0$. Substitution of this information into the definition of $\hat{f}$ gives $\hat{f}(x) = -\tilde{g}(x) + \tilde{g}(x) = 0$ and we may conclude that $\hat{g}^{-1}(0) \subset \hat{f}^{-1}(0)$. □

The following result is Theorem 2 without the restrictions on the domain.

**Corollary 8.** *Let $X$ be a normal space and let $A$ be a closed subset of $X$. If $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}^+$, and $h : X \to \mathbb{R}$ are continuous functions such that $f \cdot g = h|A$ and $g^{-1}(0) \subset f^{-1}(0)$ then there are continuous extensions $\hat{f} : X \to \mathbb{R}$ and $\hat{g} : X \to \mathbb{R}^+$ of $f$ and $g$ with $\hat{f} \cdot \hat{g} = h$.*

*Proof.* Let $\tilde{f}, \tilde{g} : X \to \mathbb{R}^+$ be continuous extensions of $|f|$ and $g$ such that $\tilde{f} \cdot \tilde{g} = |h|$ and $\tilde{f}^{-1}(0) = h^{-1}(0)$. If we put $\hat{f} = (\tilde{f}|h^{-1}(\mathbb{R}^+)) \cup (-\tilde{f}|h^{-1}(\mathbb{R}^-))$ and $\hat{g} = \tilde{g}$ then $\hat{f}$ is continuous and $\hat{f} \cdot \hat{g} = h$. ☐

A natural question is how this corollary extends to the complex numbers. Let $\mathbb{C}^+$ stand for $\mathbb{C}$ with the negative real numbers removed.

**Corollary 9.** *Let $X$ be a normal space and let $A$ be a closed subset of $X$. If $f : A \to \mathbb{C}$, $g : A \to \mathbb{C}^+$, and $h : X \to \mathbb{C}$ are continuous functions such that $f \cdot g = h|A$ and $g^{-1}(0) \subset f^{-1}(0)$ then there are continuous extensions $\hat{f} : X \to \mathbb{C}$ and $\hat{g} : X \to \mathbb{C}^+$ of $f$ and $g$ with $\hat{f} \cdot \hat{g} = h$.*

*Proof.* Let $\tilde{f}, \tilde{g} : X \to \mathbb{R}^+$ be continuous extensions of $|f|$ and $|g|$ such that $\tilde{f} \cdot \tilde{g} = |h|$ and $\tilde{g}^{-1}(0) \subset \tilde{f}^{-1}(0)$. Put $O = \tilde{g}^{-1}((0,\infty))$ and $G_n = \tilde{g}^{-1}([1/n,\infty))$ for $n \in \mathbb{N}$. Since $g(A) \subset \mathbb{C}^+$ we can find a continuous function $\theta : A \cap O \to (-\pi,\pi)$ such that $g(x) = |g(x)|e^{i\theta(x)}$ for each $x \in A \cap O$. Let $\theta_1 : G_1 \to (-\pi,\pi)$ be a Tietze extension of $\theta|A \cap G_1$. Proceeding inductively, let $\theta_{n+1} : G_{n+1} \to (-\pi,\pi)$ be a Tietze extension of $\theta_n \cup (\theta|A \cap G_{n+1})$. Put $\tilde{\theta} = \bigcup_{n=1}^{\infty} \theta_n$ and note that since $O = \bigcup_{n=1}^{\infty} \operatorname{int} G_n$ we have that $\tilde{\theta} : O \to (-\pi,\pi)$ is a continuous extension of $\theta$.

Define for $x \in X$,

$$\hat{g}(x) = \begin{cases} \tilde{g}(x)e^{i\tilde{\theta}(x)}, & \text{if } x \in O; \\ 0, & \text{if } x \notin O, \end{cases}$$

and

$$\hat{f}(x) = \begin{cases} h(x)/\hat{g}(x), & \text{if } x \in O; \\ 0, & \text{if } x \notin O. \end{cases}$$

It is obvious that $\hat{f}$ and $\hat{g}$ extend $f$ and $g$, that $\hat{f} \cdot \hat{g} = h$, and that $\hat{f}$ and $\hat{g}$ are continuous at points in $O$. What remains is to verify the continuity at points in $X \setminus O$. Let $x \in X \setminus O$ and $y \in X$. Then $\tilde{g}(x) = \hat{g}(x) = \hat{f}(x) = 0$ and since $\tilde{g}^{-1}(0) \subset \tilde{f}^{-1}(0)$ we have also $\tilde{f}(x) = 0$. Note that $\hat{g}(y) = \hat{f}(y) = 0$ or $|\hat{g}(y) - \hat{g}(x)| = |\hat{g}(y)| = \tilde{g}(y) = |\tilde{g}(y) - \tilde{g}(x)|$ and $|\hat{f}(y) - \hat{f}(x)| = |\hat{f}(y)| = |h(y)|/\tilde{g}(y) = \tilde{f}(y) = |\tilde{f}(y) - \tilde{f}(x)|$. Since $\tilde{g}$ and $\tilde{f}$ are continuous we have that $\hat{g}$ and $\hat{f}$ are continuous at $x$. ☐

The two restrictions, $g(A) \subset \mathbb{C}^+$ and $g^{-1}(0) \subset f^{-1}(0)$, are essential as the following examples show. Let $D$ be the unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$. Choose $X = D$ and $A = \partial D = \{z \in \mathbb{C} : |z| = 1\}$.

**Example 1.** For $z \in \partial D$, $f(z) = z$ and $g(z) = 1/z$ and let $h$ be the constant function 1 on $D$. If $\hat{f}$ extends $f$ over $D$ then according to Brouwer $\hat{f}(z) = 0$ for some $z \in D$ which contradicts $\hat{f} \cdot \hat{g} = 1$.

**Example 2.** For $z \in \partial D$, let $f(z) = z$ and $g(z) = 0$ and for $z \in D$ let $h(z) = 1 - |z|$. If $\hat{f}$ extends $f$ over $D$ then $\hat{f}(z) = 0$ for some $z \in D \setminus \partial D$ which contradicts $\hat{f}(z) \cdot \hat{g}(z) = 1 - |z| > 0$.

## 4. Extending pairwise disjoint collections.

We call two functions $f, g : X \to \mathbb{R}$ *disjoint* if their product $f \cdot g$ is the zero function. Frantz [**3**] presents the following two propositions.

**Proposition 10.** *Let $A$ be a closed subset of a normal space $X$ and let the functions $f_1, f_2, \ldots, f_n : A \to \mathbb{R}$ be continuous and pairwise disjoint. Then there exist pairwise disjoint continuous extensions $\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n$ of the respective $f_i$ over all of $X$.*

**Proposition 11.** *Let $A$ be a closed subset of a metric space $X$ and let $\{f_\gamma : \gamma \in \Gamma\}$ be a set of continuous and pairwise disjoint functions from $A$ to $\mathbb{R}$. Then there exist a set $\{\hat{f}_\gamma : \gamma \in \Gamma\}$ of pairwise disjoint continuous functions from $X$ to $\mathbb{R}$ such that $\hat{f}_\gamma | A = f_\gamma$ for each $\gamma \in \Gamma$.*

Frantz states that Proposition 10 is also valid for countably infinite collections of functions but that the proof is rather technical and will be included in later work. We observe, however, that this result can easily be obtained as a corollary to Proposition 11.

**Corollary 12.** *Let $A$ be a closed subset of a normal space $X$ and let the functions $f_1, f_2, \ldots : A \to \mathbb{R}$ be continuous and pairwise disjoint. Then there exist pairwise disjoint continuous extensions $\hat{f}_1, \hat{f}_2, \ldots$ of the respective $f_i$ over all of $X$.*

*Proof.* Let $\widetilde{f_i} : X \to \mathbb{R}$ be a Tietze extension of $f_i$ for each $i \in \mathbb{N}$. Consider the metric space $\mathbb{R}^{\mathbb{N}}$ and let $\pi_i : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be the projection on the $i$-th coordinate. Define the map $F : X \to \mathbb{R}^{\mathbb{N}}$ by $\pi_i \circ F = \widetilde{f_i}$ for every $i \in \mathbb{N}$. Let $B$ stand for the closure of $F(A)$ in $\mathbb{R}^{\mathbb{N}}$. If $i \neq j$ then $\pi_i \cdot \pi_j | F(A)$ is the zero function and hence by continuity $\pi_i \cdot \pi_j | B$ is zero as well. So Proposition 11 implies that there are pairwise disjoint continuous extensions $g_i : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ of $\pi_i | B$, $i \in \mathbb{N}$. Then the functions $\hat{f}_i = g_i \circ F$ are as required. □

**Example 3.** It can be shown that Proposition 11 fails for any space $X$ that contains an uncountable product of nontrivial spaces, which answers a question raised in [**3**]. The same examples also show that Corollary 12 does not extend to families of functions with cardinality $\aleph_1$.

Let $X$ contain the space $Y = \prod_{\gamma \in \Gamma} Y_\gamma$, where $\Gamma$ is uncountable and every $Y_\gamma$ consists of at least two points. Let $\pi_\gamma : Y \to Y_\gamma$ be the projection. We may assume that every $Y_\gamma$ contains only two points, $a_\gamma$ and $b_\gamma$. Define for each $\gamma \in \Gamma$ a point $x_\gamma \in Y$ by $\pi_\gamma(x_\gamma) = b_\gamma$ and $\pi_\beta(x_\gamma) = a_\beta$ for $\beta \neq \gamma$ and note that $D = \{x_\gamma : \gamma \in \Gamma\}$ is a discrete space. Define $a \in Y$ by $\pi_\gamma(a) = a_\gamma$ for all $\gamma \in \Gamma$ and note that $A = D \cup \{a\}$ is the one-point compactification of $D$ and hence $A$ is closed in $X$. Define for $\gamma \in \Gamma$, $f_\gamma : A \to \mathbb{R}$ as the characteristic function of the singleton $\{x_\gamma\}$. So $\mathcal{F} = \{f_\gamma : \gamma \in \Gamma\}$ is an uncountable pairwise disjoint family of continuous functions. According to [**4**, Theorem 1.9] the Cantor cube $Y$ satisfies the countable chain condition which means that every pairwise disjoint collection of open sets in $Y$ is countable. So no continuous extension of the family $\mathcal{F}$ over $Y$ (and hence over $X$) is pairwise disjoint.

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Received May 5, 2000 and revised November 30, 2000.


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