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In this paper we give a classification of simple stable singularities of Lagrange projections of the first open Whitney umbrella, the simplest singularity of Lagrange varieties. Our classification extends the ADE-classification, due to Arnold, of simple stable singularities of Lagrange projections of smooth Lagrange submanifolds. We also prove a criterion of equivalence of stable Lagrange projections of open Whitney umbrellas which is analogous to Mather's fundamental theorem on stable map-germs.

0. Introduction.

The systematic investigation of singularities of Lagrange mappings started in 1972 with V.I. Arnold's paper [1]. A Lagrange mapping is the projection of a Lagrange variety in a cotangent fibration onto its base. In the paper [1] it was discovered that singularities of Lagrange mappings of nonsingular Lagrange varieties are classified by degenerations of critical points of smooth functions and the discrete part of their classification is indexed by Coxeter's groups A_μ, D_μ, E_μ. This is the so-called ADE-classification of simple stable singularities of Lagrange mappings.

However, in applications singular Lagrange varieties appear. Among them open swallow tails and open Whitney umbrellas are very frequently encountered. Open swallow tails occur in the so-called obstacle problem about singularities of the distance on Riemannian manifold with boundary [2], [9], [20], [16]. Concerning open Whitney umbrellas see [8], [9], [13], [15]. They appear naturally in various situations; for instance, as singularities of the generalized Cauchy problems [9], [11], singularities of Riemannian invariants [18], and singularities of tangent developables [19], [20], [14] as the Legendre counterpart. In [9] the discrete part of local classification of Lagrange mappings of open swallow tails is carried out and some singularities of Lagrange mappings of open Whitney umbrellas are found. See also [22].

In this paper, we give the full discrete part of local classification of Lagrange mappings of the open Whitney umbrella of type one, or the first open Whitney umbrella. More accurately, we classify simple stable singularities of mappings of the first open Whitney umbrella. This problem was inspired by the classification problem of the composition of an isotropic mapping and a cotangent fibration. (A smooth mapping is called *isotropic* if the pull-back of the symplectic structure is equal to zero.) If the isotropic mapping is an immersion then the discrete part of local classification of the above compositions is the ADE-classification. Otherwise, the isotropic mapping can have singularities, first of all, open Whitney umbrellas [**8**]. Therefore, our classification gives the answer for simple stable compositions of the first open Whitney umbrella and a cotangent fibration.

We recall, in the ADE-classification and its generalizations, that stable mappings of Lagrange manifolds are classified by means of families of functions – generating families [**3**], [**25**]. Also in our problem, namely, the classification problem of stable projections of Lagrange varieties under Lagrange fibrations, the usage of generating families remains a powerful method, however in a different manner. Namely, we fix a Lagrange variety, while Lagrange fibrations are taken arbitrarily. A Lagrange fibration are regarded as a family of Lagrange submanifolds, and each Lagrange submanifold has a generating function. Thus we describe the Lagrange fibration by means of the family of the generating functions of the Lagrangian fibers, that is, the generating family of the Lagrange fibration. Then we describe stability and simplicity of Lagrange fibrations with respect to symplectic diffeomorphisms preserving the Lagrange variety, by means of the generating families. Similar methods are applied to other problems [**23**], [**24**].

First we recall the notions needed to state our main result.

Definition. A smooth fibration $\pi : E \to Y$ is called *Lagrange* if its space E is a symplectic manifold and the fibers of π are Lagrange submanifolds of E.

For example, the cotangent fibration of a smooth manifold is Lagrange.

We consider the classification problem of singularities on a Lagrange variety under various Lagrange projections. A subset of a symplectic manifold of dimension $2n$ is called a *Lagrange variety* if it has a stratification with maximal dimension n such that the symplectic form vanishes on each stratum.

Definition. Let $\Lambda \subset E$ be a Lagrange variety. Two Lagrange fibrations $\pi, \pi' : E \to Y$ are called Λ-*equivalent* and denoted by $\pi \sim_\Lambda \pi'$ if $\pi' \circ \tau = \sigma \circ \pi$ where σ is a diffeomorphism of Y and τ is a symplectic diffeomorphism of E which preserves Λ.

There are two basic notions for the classification problem – stability and simplicity.

Definition. A germ of Lagrange fibration $\pi : (E, z_0) \to Y$ is called *stable* with respect to Λ-equivalence, or Λ-*stable*, if for any sufficiently small Lagrange perturbation of any its representative $\widetilde{\pi} : U \to Y$, $z_0 \in U \subset E$

there exists a point $z \in U$ such that the germ of the perturbation at z is Λ-equivalent to the original germ π. (See [**9**], [**15**].)

Definition. A germ of Lagrange fibration $\pi : (E, z_0) \to Y$ is called *simple* with respect to Λ-equivalence, or Λ-*simple*, if there exists its representative $\widetilde{\pi} : U \to Y$, $z_0 \in U \subset E$ such that the number of Λ-equivalence classes of the germs $(U, z) \to Y$ for all $z \in U$ of all sufficiently small Lagrange perturbations of $\widetilde{\pi}$ is finite.

Let Λ_1 be the first open Whitney umbrella given by the following parametric form:

$$p_1 = x_2 t, \quad p_2 = t^3/3, \quad p_3 = \cdots = p_n = 0, \quad x_1 = t^2/2, \tag{1}$$

where t is the parameter, $(p, x) = (p_1, \ldots, p_n, x_1, \ldots, x_n)$ are local coordinates in E such that $\omega = dp \wedge dx$, and $n \geq 2$. See [**8**], [**9**], [**13**], [**15**].

Then Λ_1 is an n-dimensional algebraic Lagrange variety (Lemma 3), whose singular locus $\Sigma(\Lambda_1) = \{p = 0, x_1 = 0, x_2 = 0\}$ is a nonsingular submanifold of E of dimension $n - 2$.

Remark that, at a regular point $z_0 \in \Lambda_1$, there exists a system of symplectic coordinates (p, x) around (E, z_0) such that Λ_1 is defined by $\{p_1 = 0, \ldots, p_n = 0\}$.

Now a Lagrange fibration is given by a family of Lagrange submanifolds. It is well-known that a smooth Lagrange submanifold is locally given by $x_I = w_{p_I}(p_I, x_J), p_J = -w_{x_J}(p_I, x_J)$, where I, J is a decomposition of $\{1, 2, \ldots, n\}$ and $w(p_I, x_J)$ is a smooth function.

Definition. Let $W : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a smooth function of p_I, x_J, and $y = (y_1, \ldots, y_n)$, which satisfies the condition of nondegeneracy:

$$\det \left\| \begin{array}{c} W_{p_I y} \\ W_{x_J y} \end{array} \right\| \neq 0$$

where $I \cap J = \emptyset$, $I \cup J = \{1, \ldots, n\}$. Then W is called a *generating family* of the Lagrange fibration $\pi : (p, x) \mapsto y$ whose symplectic structure and fibers are given by the formulas

$$\omega = dp \wedge dx, \quad \pi^{-1}(y) = \{x_I = W_{p_I}(p_I, x_J, y),\ p_J = -W_{x_J}(p_I, x_J, y)\}.$$

(Locally π is a fibration in consequence of nondegeneracy of W.)

For example, the natural Lagrange projections $(p, x) \mapsto p$ and $(p, x) \mapsto x$ are given by the generating families $-xy$ and py respectively.

Then the main result of this paper is the following:

Theorem 1. *Let us consider a germ of a Lagrange fibration at a point of the first open Whitney umbrella Λ_1. If our germ is simple and stable with respect to Λ_1-equivalence then it is Λ_1-equivalent to the germ at the origin of the fibration defined by one of the following generating families:*

Classification of simple stable projections at regular points $\Lambda_1 = \{p_1 = \cdots = p_n = 0\}$.

A_1) $W(p_1, \ldots, p_n, y) = y_1 p_1 + \cdots + y_n p_n$;

$A_m^\pm$) $W(x_1, p_2, \ldots, p_n, y) = \pm x_1^{m+1} + y_2 x_1^{m-1} + \cdots + y_{m-1} x_1^2 - y_1 x_1 + y_2 p_2 + \cdots + y_n p_n$, *where* $n \geq m-1 \geq 1$ *and* $A_m^+ \sim_{\Lambda_1} A_m^-$ *if* m *is even;*

$D_m^\pm$) $W(x_1, x_2, p_3, \ldots, p_n, y) = x_1^2 x_2 \pm x_2^{m-1} + y_3 x_2^{m-2} + \cdots + y_{m-1} x_2^2 - y_1 x_1 - y_2 x_2 + y_3 p_3 + \cdots + y_n p_n$, *where* $n \geq m-1 \geq 3$;

$E_6^\pm$) $W(x_1, x_2, p_3, \ldots, p_n, y) = x_1^3 \pm x_2^4 + y_3 x_1 x_2^2 + y_4 x_1 x_2 + y_5 x_2^2 - y_1 x_1 - y_2 x_2 + y_3 p_3 + \cdots + y_n p_n$, *where* $n \geq 5$;

E_7) $W(x_1, x_2, p_3, \ldots, p_n, y) = x_1^3 + x_1 x_2^3 + y_3 x_1 x_2^2 + y_4 x_1 x_2 + y_5 x_2^3 + y_6 x_2^2 - y_1 x_1 - y_2 x_2 + y_3 p_3 + \cdots + y_n p_n$, *where* $n \geq 6$;

E_8) $W(x_1, x_2, p_3, \ldots, p_n, y) = x_1^3 + x_2^5 + y_3 x_1 x_2^3 + y_4 x_1 x_2^2 + y_5 x_1 x_2 + y_6 x_2^3 + y_7 x_2^2 - y_1 x_1 - y_2 x_2 + y_3 p_3 + \cdots + y_n p_n$, *where* $n \geq 7$.

Classification of simple stable projections at singular points.

S_3) $W(p, y) = y_1 p_1 + \cdots + y_n p_n$, *where* $n \geq 2$;

$S_m^\pm$) $W(x_1, p_2, \ldots, p_n, y) = \pm e_{m+1} + y_3 e_4 + \cdots + y_{m-1} e_m + y_1 x_1 + y_2 p_2 + \cdots + y_n p_n$, *where* $e_{2k} = x_1^k$, $e_{2k+1} = x_1^{k-1} p_2$, $n \geq m-1 \geq 3$, *and* $S_m^+ \sim_{\Lambda_1} S_m^-$ *if* m *is even;*

T_5) $W(p_1, x_2, p_3, \ldots, p_n, y) = x_2^3 + y_3 p_1 x_2 + y_4 x_2^2 + y_1 p_1 + y_2 x_2 + y_3 p_3 + \cdots + y_n p_n$, *where* $n \geq 4$;

U_6) $W(x_1, x_2, p_3, \ldots, p_n, y) = x_2^3 + y_3 x_1^2 + y_4 x_1 x_2 + y_5 x_2^2 + y_1 x_1 + y_2 x_2 + y_3 p_3 + \cdots + y_n p_n$, *where* $n \geq 5$;

V_6) $W(p_1, p_2, x_3, p_4, \ldots, p_n, y) = x_3^3 + y_4 p_1 x_3 + y_5 p_2 x_3 + y_1 p_1 + y_2 p_2 + y_3 x_3 + y_4 p_4 + \cdots + y_n p_n$, *where* $n \geq 5$.

Generic Lagrange fibration is stable with respect to Λ_1-equivalence if $n \leq 4$. If $n > 4$, there exists a Lagrange fibration such that any its sufficient small perturbation has an unstable germ.

Remark. The change $(t, p_2, x_2) \mapsto -(t, p_2, x_2)$ shows us that $S_m^+ \sim_{\Lambda_1} S_m^-$ if m is even.

We examine examples from the list of Theorem 1.

(1) The singularity S_4, $n = 3$. The generating family is given by

$$W = x_1 p_2 + y_3 x_1^2 + y_1 x_1 + y_2 p_2 + y_3 p_3,$$

and the Lagrange fibration is given by

$$\left\{ \begin{array}{lcl} y_1 & = & -p_1 - p_2 - 2x_1 x_3, \\ y_2 & = & -x_1 + x_2, \\ y_3 & = & x_3. \end{array} \right.$$

The composition with the parametrization of the open Whitney umbrella is given by

$$\begin{cases} y_1 = -x_2 t - \dfrac{t^3}{3} - x_3 t^2, \\ y_2 = -\dfrac{t^2}{2} + x_2, \\ y_3 = x_3. \end{cases}$$

The caustic of the singularity S_4 is a surface with a cuspidal edge (Figure 1). The edge consists of A_3-points and the S_4-point (the origin). The tangent line to the edge at the origin consists of S_3-points and the origin. In fact, the caustic is the *tangent developable* of the cuspidal edge, consisting of the tangent lines to the cuspidal edge.

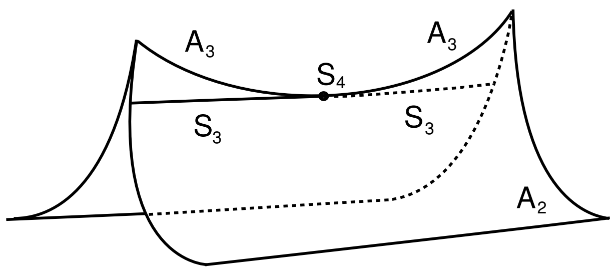

Figure 1. S_4-caustic.

(2) The singularity T_5, $n = 4$. The generating family is given by

$$W(p_1, x_2, p_3, p_4, y) = x_2^3 + y_3 p_1 x_2 + y_4 x_2^2 + y_1 p_1 + y_2 x_2 + y_3 p_3 + y_4 p_4.$$

Then the Lagrange fibration is given by

$$\begin{cases} y_1 = x_1 - x_2 x_3, \\ y_2 = -p_2 - 3x_2^2 - x_3 p_1 - 2x_2 x_4, \\ y_3 = x_3, \\ y_4 = x_4, \end{cases}$$

and the composition with the parametrization of Λ_1 is given by

$$\begin{cases} y_1 = \dfrac{t^2}{2} - x_2 x_3, \\ y_2 = -\dfrac{t^3}{3} - 3x_2^2 - x_2 x_3 t - 2x_2 x_4, \\ y_3 = x_3, \\ y_4 = x_4. \end{cases}$$

This provides the example of stable projection of corank two at the singular point in the smallest dimension.

Recall that, in the general singularity theory of mappings, the classification of C^∞-stable map-germs is reduced to their classification up to the contact equivalence [17] IV. Also recall that, in Lagrange singularity theory, the Lagrange classification of Lagrange stable immersion-germs is reduced to the classification of function-germs up to the right equivalence. Naturally, in our classification problem, we need an analogous result to establish the actual classification.

Definition. Let $\pi : (E, z_0) \to (Y, y_0)$ be a germ of Lagrange fibration. We call the germ of Lagrange submanifold $(\pi^{-1}(y_0), z_0) \subset E$ the *central* fiber of the germ π.

Definition. Let $\Lambda \subset E$ be a Lagrange variety. Two germs of Lagrange submanifolds $L, L' \subset E$ are called Λ-*equivalent* and denoted by $L \sim_\Lambda L'$ if they are the same with respect to a symplectic diffeomorphism preserving Λ, namely if there exists a symplectic diffeomorphism $\tau : E \to E$ such that $\tau(\Lambda) = \Lambda$ and that $\tau(L) = L'$.

Definition. The germs L and L' are called *formally* Λ-*equivalent* and denoted by $j^\infty(L) \sim_\Lambda j^\infty(L')$ if their ∞-jets are the same with respect to a symplectic diffeomorphism preserving Λ. More accurately, $j^\infty(L) \sim_\Lambda j^\infty(L')$ if there exist parametrizations $i, i' : (\mathbb{R}^n, 0) \to E$ of L, L' respectively, and a symplectic diffeomorphism $\tau : E \to E$ preserving Λ such that $j^\infty i'(0) = j^\infty(\tau \circ i)(0)$.

We say that $\Lambda \subset E$ is an open Whitney umbrella if Λ is the image of an open Whitney umbrella $f_{n,k} : (\mathbb{R}^n, 0) \to (E, 0)$ in the sense of [13]. We have open Whitney umbrellas $\Lambda_k, 0 \leq k \leq [n/2]$; Λ_0 is a Lagrange submanifold and Λ_1 is the first open Whitney umbrella already introduced.

Then we prove and use in this paper the following:

Theorem 2. *Let $\Lambda \subset E$ be an open Whitney umbrella. Then two Λ-stable germs of Lagrange fibrations $\pi, \pi' : E \to Y$ are Λ-equivalent if and only if their central fibers are Λ-equivalent. Moreover, π and π' are Λ-equivalent if and only if their central fibers are formally Λ-equivalent.*

Consider the case when Λ is a Lagrange submanifold ($k = 0$). Then we may take symplectic coordinates (p, x) with $\Lambda = \{x = 0\}$. Recall the fundamental theorem of Lagrange singularity theory [3]: Two germs of Lagrange submanifolds in $T^*\mathbb{R}^n$ are Lagrange equivalent for the canonical projection $T^*\mathbb{R}^n \to \mathbb{R}^n$ if and only if their generating families are stably R^+-equivalent. Moreover, a Lagrange submanifold is Lagrange stable if and only if its generating family $F(q, x)$, q being the inner variables, is an R^+-versal deformation of $F(q, 0)$ [3]. So, two Lagrange stable Lagrange submanifold are Lagrange equivalent if their generating families are deformations of stably R^+-equivalent function germs. Besides, we recall the notion of "contact equivalence" for Lagrange manifolds due to Golubitsky and Guillemin [**10**]. Then two germs L, L' of Lagrange submanifolds are contact equivalent via a symplectic diffeomorphism in the sense [**10**] if and only if L, L' are Λ-equivalent in our sense. Let L, L' be the graphs of the differentials of function germs $h(x), h'(x)$ which have an order ≥ 3. Then the germs L and L' are Λ-equivalent if and only if h and h' are right equivalent ([**10**], Prop. 4.2.). In general L, L' are Λ-equivalent if and only if $F(q, 0)$ and $F'(q', 0)$ are stably R^+-equivalent, for the generating families $F(q, x), F(q', x)$ of L, L' respectively. Therefore, Theorem 2 is a quite natural generalization of the fundamental theorem of Lagrange singularity theory. Also it is a Lagrange counterpart of the Mather's theorem "two $\mathcal{A}$-stable mappings are $\mathcal{A}$-equivalent if and only if they are $\mathcal{K}$-equivalent" [**17**] in ordinary singularity theory of stable mappings. See also [**15**] §6.

In the next section we give the formal classification of central fibers of simple stable projections of the first open Whitney umbrella Λ_1 up to Λ_1-preserving symplectic diffeomorphisms (Theorem 3). Theorem 3 follows from Theorem 4 which is reduced to technically key Lemma 1. In §2, we prepare Lemmas needed in the following section. In particular, we give explicit equations defining the first open Whitney umbrella. Then Lemma 1 is proved in §3. Theorems 2 and 1 are proved in §4. In §5 we describe relations between our study and simple stable compositions.

For the proof of Theorem 1 we use explicit equations defining the first open Whitney umbrella. In order to carry out classifications of simple stable projections of general open Whitney umbrellas applying the method used in the present paper, we need to get explicit equations for them. That problem is left open.

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1. Normal forms of fibers.

We start to prove Theorem 1 with finding formal normal forms for separate fibers which pass through singular points of the first open Whitney umbrella Λ_1 with respect to symplectic diffeomorphisms preserving Λ_1 itself.

Definition. A germ of Lagrange submanifold $L \subset E$ is called *simple* with respect to Λ-equivalence, or Λ-*simple*, if the number of Λ-equivalence classes of all germs of the kind $\tau(L)$ is finite, where $\tau : E \to E$ is any sufficiently small symplectic perturbation of the identity diffeomorphism.

In particular, a Λ-stable germ of Lagrange fibration is called *simple* with respect to Λ-equivalence, or Λ-*simple*, if there exists its representative such that the number of Λ-equivalence classes of all its germs is finite.

In coordinates (p, x) such that $\omega = dp \wedge dx$, any Lagrange submanifold is locally given by at least one of the 2^n *generating functions* $w(p_I, x_J)$ by the formulas:

$$x_I = w_{p_I}(p_I, x_J), \quad p_J = -w_{x_J}(p_I, x_J)$$

where $I \cap J = \emptyset$ and $I \cup J = \{1, \dots, n\}$.

Theorem 3. *Let us consider a germ of a Lagrange submanifold at a singular point of the first open Whitney umbrella Λ_1. If our germ is Λ_1-simple then it is formally Λ_1-equivalent to the germ at the origin of the Lagrange submanifold defined by one of the following generating functions:*

S_3) $w(p_1, \dots, p_n) = 0$;

$S_m^{\pm}$) $w(x_1, p_2, \dots, p_n) = \pm e_{m+1}$ *where* $e_{2k} = x_1^k$, $e_{2k+1} = x_1^{k-1} p_2$, $m \geq 4$, $S_m^+ \sim_{\Lambda_1} S_m^-$ *if* m *is even;*

T_5) $w(p_1, x_2, p_3, \dots, p_n) = x_2^3$;

U_6) $w(x_1, x_2, p_3, \dots, p_n) = x_2^3$;

V_6) $w(p_1, p_2, x_3, p_4, \dots, p_n) = x_3^3$ *where* $n \geq 3$.

Non-simple germs occur in families of Lagrange submanifolds depending generically on at least 5 *parameters. In generic* 4*-parametric families such germ do not occur.*

Remark. The change $(t, p_2, x_2) \mapsto -(t, p_2, x_2)$ shows us that $S_m^+ \sim_{\Lambda_1} S_m^-$ if m is even.

Proof. Theorem 3 follows from the following Theorem 4. □

Definition. Let $\Lambda \subset E$ be a Lagrange variety. Two germs w, w' of generating functions are called *formally* Λ*-equivalent* and denoted by $w \sim_\Lambda w'$ if the corresponding germs of Lagrange submanifolds are formally Λ-equivalent.

Theorem 4.

I. *A germ of a Lagrange submanifold at a singular point of the first open Whitney umbrella Λ_1 is Λ_1-equivalent to the germ at the origin of the Lagrange submanifold defined by a generating function $w(p_I, x_J)$ such that $w_{p_I}(0) = w_{x_J}(0) = w_{x_J x_J}(0) = 0$ and one of the following conditions is satisfied:*

1) $J = \emptyset$ $(c = 2)$;
2) $J = \{1\}$ $(c = 3)$;
3) $J = \{2\}$, $w_{p_1 x_2}(0) = 0$, $w_{x_2 x_2 x_2}(0) \neq 0$ $(c = 4)$;
4) $J = \{2\}$, $w_{p_1 x_2}(0) = w_{x_2 x_2 x_2}(0) = 0$ $(c = 5)$;
5) $J = \{3\}$, $w_{p_1 x_3}(0) = w_{p_2 x_3}(0) = 0$, $w_{x_3 x_3 x_3}(0) \neq 0$ $(c = 5)$;
6) $J = \{3\}$, $w_{p_1 x_3}(0) = w_{p_2 x_3}(0) = w_{x_3 x_3 x_3}(0) = 0$ $(c = 6)$;
7) $J = \{1, 2\}$, $w_{x_2 x_2 x_2}(0) \neq 0$ $(c = 5)$;
8) $J = \{1, 2\}$, $w_{x_2 x_2 x_2}(0) = 0$ $(c = 6)$;
9) $J = \{1, 3\}$, $w_{p_2 x_3}(0) = 0$ $(c = 6)$;
10) $J = \{2, 3\}$, $w_{p_1 x_2}(0) = w_{p_1 x_3}(0) = 0$ $(c = 7)$;
11) $J = \{3, 4\}$, $w_{p_1 x_3}(0) = w_{p_1 x_4}(0) = w_{p_2 x_3}(0) = w_{p_2 x_4}(0) = 0$ $(c = 9)$;
12) $\# J \geq 3$ $(c = 8)$.

Such germs occur in families of Lagrange submanifolds depending generically on at least c parameters. Case 8 is adjacent to Case 4; Cases 10 and 11 are adjacent to Case 9.

II. *In the above cases:*

1) $w \sim_{\Lambda_1} 0$;

2) *w is not* Λ_1*-simple or* $\exists\ m \geq 4$ *such that* $w \sim_{\Lambda_1} \pm e_{m+1}$ *where* $e_{2k} = x_1^k$, $e_{2k+1} = x_1^{k-1}p_2$;

3) $w \sim_{\Lambda_1} x_2^3$;

7) $w \sim_{\Lambda_1} x_2^3$;

5) $w \sim_{\Lambda_1} x_3^3$;

4,6,8-12) *w is not* Λ_1*-simple.*

The non-Λ_1*-simple germs from Case 2 have infinite codimension. They are adjacent to* $S_m^{\pm}$*.*

Proof. I. The singularities of the first open Whitney umbrella Λ_1 are defined by the equations $p_1 = \cdots = p_n = x_1 = x_2 = 0$. After a shift we get $w_{p_I}(0) = w_{x_J}(0) = 0$. If some principal minor $\det \|w_{x_{J'}x_{J'}}(0)\| \neq 0$ where $J' \subset J$, then we can change $I \mapsto I \cup J'$ and $J \mapsto J \setminus J'$. Therefore, we assume $w_{x_J x_J}(0) = 0$. After renumbering among $p_3, \dots, p_n$ we reach one (but not only one) of the following cases: $J = \emptyset$, $J = \{1\}$, $J = \{2\}$, $J = \{3\}$, $J = \{1,2\}$, $J = \{1,3\}$, $J = \{2,3\}$, $J = \{3,4\}$, or $\#J \geq 3$. In order to eliminate some of these cases let us note that we can replace $i \mapsto j$ in I and $j \mapsto i$ in J if $w_{p_i x_j}(0) \neq 0$ where $i \in I$ and $j \in J$.

The singularities of the first open Whitney umbrella Λ_1 form a submanifold of codimension $n+2$. So, germs passing through singularities of Λ_1 occur in families of Lagrange submanifolds depending generically on at least 2 parameters. This is the case $J = 0$. The other cases require the extra number of parameters which is equal to the quantity of conditions for the second and third derivatives.

II. It is sufficient to prove Cases 1, 2, 3, 7 for $n = 2$ and Case 5 for $n = 3$. This follows from the equivalence $w \sim_{\Lambda_1} w_0$ where $w_0(p_I, x_J) = w|_{p_{I''}=0}$, $I' = I \cap \{1,2\}$, and $I'' = I \cap \{3, \dots, n\}$. The equivalence is performed by the symplectic diffeomorphism

$$(p_I, p_J, x_I, x_J) \mapsto (p_I, p_J + \widetilde{w}_{x_J}, x_I - \widetilde{w}_{p_I}, x_J)$$

where $\widetilde{w} = w - w_0$. This diffeomorphism preserves Λ_1 because it shifts the plane $p_{I''} = 0$ along only $x_{I''}$ (preserving p_I, p_J, $x_{I'}$, and x_J) that follows from the equality $\widetilde{w}|_{p_{I''}=0} = 0$.

The following infinite chains

$$1_2 \Rightarrow 1_3 \Rightarrow \dots,$$

$$2_5 \Rightarrow 2_6 \Rightarrow \dots \quad \text{or} \quad 2_5 \Rightarrow 2_6 \Rightarrow \cdots \Rightarrow 2_{m+1} \Rightarrow 2_{m+1}^{m+2} \Rightarrow 2_{m+1}^{m+3} \Rightarrow \dots,$$

$$3_6 \Rightarrow 3_6^7 \Rightarrow 3_6^8 \Rightarrow \dots,$$

$$5_6 \Rightarrow 5_6^7 \Rightarrow 5_6^8 \Rightarrow \dots,$$
$$7_3 \Rightarrow 7_3^4 \Rightarrow 7_3^5 \Rightarrow \dots$$

of propositions of the following Lemma 1 prove Cases 1, 2, 3, 5, 7 respectively.

The Propositions 4_*, 6_*, and 9_* of Lemma 1 imply the Propositions 4, 6, 8–11 of Theorem 4 because Case 8 is adjacent to Case 4 and Cases 10, 11 are adjacent to Case 9.

It remains to prove that Case 12, namely, $\#J \geq 3$ is not Λ_1-simple. Indeed, the tangent plane to the first open Whitney umbrella Λ_1 at the point

$$p_1 = x_2 t, \quad p_2 = t^3/3, \quad p_3 = \dots = p_n = 0, \quad x_1 = t^2/2$$

is defined by the equations:

$$t dp_1 - x_2 dx_1 - t^2 dx_2 = 0, \quad dp_2 - t dx_1 = 0, \quad dp_3 = \dots = dp_n = 0.$$

Along the curve

$$p_1 = t^3, \quad p_2 = t^3/3, \quad p_3 = \dots = p_n = 0,$$
$$x_1 = t^2/2, \quad x_2 = t^2, \quad x_3 = \dots = x_n = 0$$

our tangent plane is defined by the equations

$$dp_1 - t dx_1 - t dx_2 = 0, \quad dp_2 - t dx_1 = 0, \quad dp_3 = \dots = dp_n = 0$$

and tends to the plane $dp_1 = \dots = dp_n = 0$ as $t \to 0$. Therefore, the case $\#J \geq 3$ is adjacent to the class of ordered pairs of germs of smooth Lagrange submanifolds whose tangent planes have three-dimensional intersection. This class is not simple up to symplectic equivalence because, according to [**10**], it corresponds to the so-called P_8 class consisting of the germs of smooth functions at critical points of corank 3. Moreover, the symplectic equivalence of ordered pairs of Lagrange germs corresponds to the stable right equivalence of the germs of smooth functions (see [**10**]). But the P_8 class contains a continuous invariant up to stable right equivalence [**3**]. This invariant comes from linear equivalence of cubic forms of three variables. □

Lemma 1. *Let $\alpha_I = \deg p_I$ and $\beta_J = \deg x_J$ be positive integer quasidegrees and $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \dots$ be the corresponding quasihomogeneous filtration in the algebra of germs at* 0 *of smooth functions of p_I and x_J. Then*

$1_l)$ *$n = 2$, $J = \emptyset$, $\alpha_1 = \alpha_2 = 1$, $w_l \in \mathcal{A}_l$, $l \geq 2 \Rightarrow w_l \sim_{\Lambda_1} 0 \,(\mathrm{mod}\ \mathcal{A}_{l+1})$;*

$2_l)$ *$n = 2$, $J = \{1\}$, $\beta_1 = 2$, $\alpha_2 = 3$, $w_l \in \mathcal{A}_l$, $l \geq 4 \Rightarrow w_l \sim_{\Lambda_1} \pm e_l \,(\mathrm{mod}\ \mathcal{A}_{l+1})$ if $w_l \notin (9p_2^2 - 8x_1^3) + \mathcal{A}_{l+1}$, and $w_l \sim_{\Lambda_1} 0 \,(\mathrm{mod}\ \mathcal{A}_{l+1})$ if $w_l \in (9p_2^2 - 8x_1^3) + \mathcal{A}_{l+1}$;*

$2_l^d)$ *$n = 2$, $J = \{1\}$, $\beta_1 = 2$, $\alpha_2 = 3$, $w_d \in \mathcal{A}_d$, $d > l \geq 4 \Rightarrow \pm e_l + w_d \sim_{\Lambda_1} \pm e_l \,(\mathrm{mod}\ \mathcal{A}_{d+1})$;*

$3_6)$ *$n = 2$, $J = \{2\}$, $\alpha_1 = 3$, $\beta_2 = 2$, $w_6 \in \mathcal{A}_6$, $w_{6,x_2x_2x_2} \neq 0 \Rightarrow w_6 \sim_{\Lambda_1} x_2^3 \,(\mathrm{mod}\ \mathcal{A}_7)$;*

3_6^d) $n=2$, $J=\{2\}$, $\alpha_1=3$, $\beta_2=2$, $w_d\in\mathcal{A}_d$, $d>6\Rightarrow x_2^3+w_d\sim_{\Lambda_1} x_2^3$ (mod $\mathcal{A}_{d+1}$);

5_6) $n=3$, $J=\{3\}$, $\alpha_1=\alpha_2=3$, $\beta_3=2$, $w_6\in\mathcal{A}_6$, $w_{6,x_3x_3x_3}\neq 0\Rightarrow w_6\sim_{\Lambda_1} x_3^3$ (mod $\mathcal{A}_7$);

5_6^d) $n=3$, $J=\{3\}$, $\alpha_1=\alpha_2=3$, $\beta_3=2$, $w_d\in\mathcal{A}_d$, $d>6\Rightarrow x_3^3+w_d\sim_{\Lambda_1} x_3^3$ (mod $\mathcal{A}_{d+1}$);

7_l) $n=2$, $J=\{1,2\}$, $\beta_1=\beta_2=1$, $w_l\in\mathcal{A}_l$, $l\geq 3\Rightarrow w_l\sim_{\Lambda_1}\pm x_2^l$ (mod $\mathcal{A}_{l+1}$) *if* $w_l\notin(x_1)+\mathcal{A}_{l+1}$, *and* $w_l\sim_{\Lambda_1} 0$ (mod $\mathcal{A}_{l+1}$) *if* $w_l\in(x_1)+\mathcal{A}_{l+1}$;

7_l^d) $n=2$, $J=\{1,2\}$, $\beta_1=\beta_2=1$, $w_d\in\mathcal{A}_d$, $d>l\geq 3\Rightarrow\pm x_2^l+w_d\sim_{\Lambda_1}\pm x_2^l$ (mod $\mathcal{A}_{d+1}$);

4_*) $n=2$, $J=\{2\}$, $\alpha_1=2$, $\beta_2=1\Rightarrow\mathcal{A}_4/\mathcal{A}_5$ *contains a continuous invariant with respect to* Λ_1*-equivalence;*

6_*) $n=3$, $J=\{3\}$, $\alpha_1=\alpha_2=2$, $\beta_3=1\Rightarrow\mathcal{A}_4/\mathcal{A}_5$ *contains a continuous invariant with respect to* Λ_1*-equivalence;*

9_*) $n=3$, $J=\{1,3\}$, $\beta_1=1$, $\alpha_2=2$, $\beta_3=1$, $\mathcal{A}_3'=\{w_3\in\mathcal{A}_3\mid w_{3,p_2x_3}=0\}\Rightarrow\mathcal{A}_3'/\mathcal{A}_4$ *contains a continuous invariant with respect to* Λ_1*-equivalence.*

2. Hamiltonian vector fields.

It is well-known that a vector field which preserves the symplectic structure $\omega=dp\wedge dx$ is locally defined by its Hamiltonian H:

$$\dot{x}=H_p,\quad \dot{p}=-H_x.$$

Let $L(w)$ be the Lagrange submanifold given by a generating function $w(p_I,x_J)$:

$$L(w)=\{x_I=w_{p_I}(p_I,x_J),\ p_J=-w_{x_J}(p_I,x_J)\}\tag{2}$$

and $H(w)$ denote the derivative of the generating function when the Lagrange submanifold is perturbed by the vector field with a Hamiltonian H.

Lemma 2. $H(w)=H|_{L(w)}+\text{const}.$

Proof. Indeed, (p_I,x_J) are coordinates on $L(w)$ and for any two points $A,B\in L(w)$

$$w\Big|_A^B=w(B)-w(A)=\int_A^B\psi_I$$

along any path on $L(w)$ joining A and B, where $\psi_I=x_Idp_I-p_Jdx_J$. After differentiating along our Hamiltonian vector field we get

$$(H(w)+w_{p_I}\dot{p}_I+w_{x_J}\dot{x}_J)\Big|_A^B=\int_A^B H(\psi_I)$$

where $H(\psi_I)$ is the derivative of ψ_I along our Hamiltonian vector field. The Cartan formula implies that

$$\int_A^B H(\psi_I) = \int_A^B d(x_I\dot{p}_I - p_J\dot{x}_J) - \dot{p}dx + \dot{x}dp$$
$$= \int_A^B d(x_I\dot{p}_I - p_J\dot{x}_J + H) = (x_I\dot{p}_I - p_J\dot{x}_J + H)\Big|_A^B.$$

Comparing the two last equalities and taking into account (2) we get $H(w)\Big|_A^B = H\Big|_A^B$. □

Open Whitney umbrellas $\Lambda = \Lambda_k = f_{n,k}(\mathbb{R}^n)$ are real algebraic sets in $\mathbb{R}^{2n}$. In fact, the complexification $f_{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}^{2n}$ of the parametrization $f : \mathbb{R}^n \to \mathbb{R}^{2n}$ is proper and one to one. Therefore $f_{\mathbb{C}}(\mathbb{C}^n)$ is a complex algebraic set in $\mathbb{C}^{2n}$, and $f_{\mathbb{C}}(\mathbb{C}^n) \cap \mathbb{R}^{2n} = f(\mathbb{R}^n)$ is a real algebraic set. However, for the explicit classification, we need, furthermore, the explicit equation of Λ.

Let $\mathcal{I}(\Lambda_1) \subset \mathbb{R}[p,x]$ be the ideal consisting of all polynomials which vanish on the first open Whitney umbrella Λ_1.

Lemma 3.
$\mathcal{I}(\Lambda_1) = (p_1^2 - 2x_1x_2^2, 3p_1p_2 - 4x_1^2x_2, 2p_1x_1 - 3p_2x_2, 9p_2^2 - 8x_1^3, p_3, \dots, p_n).$

Proof. It is sufficient to prove this when $n = 2$. Let $\mathcal{I}_t(\Lambda_1) = (t^3/3 - p_2, t^2/2 - x_1, tx_2 - p_1) \subset \mathbb{R}[t, p_1, p_2, x_1, x_2]$ be the ideal defining the parametric form (1) of the first open Whitney umbrella Λ_1 if $n = 2$. Then the nine polynomials $t^2 - 2x_1$, $tp_1 - 2x_1x_2$, $3tp_2 - 4x_1^2$, $2tx_1 - 3p_2$, $tx_2 - p_1$, $p_1^2 - 2x_1x_2^2$, $3p_1p_2 - 4x_1^2x_2$, $2p_1x_1 - 3p_2x_2$, $9p_2^2 - 8x_1^3$ form a Gröbner basis of the ideal $\mathcal{I}_t(\Lambda_1)$ with respect to the lexicographic order. Hence, the four last polynomials which do not depend on t generate $\mathcal{I}(\Lambda_1) \subset \mathbb{R}[p_1, p_2, x_1, x_2]$. □

3. Proof of Lemma 1.

In Cases 1_l, 2_l, 2_l^d, 3_6, 3_6^d, 5_6, 5_6^d, 7_l, 7_l^d we use the standard homotopy method. Namely, let ω_τ be a family of generating functions depending smoothly on a parameter τ and H_τ be a smooth family of Hamiltonians satisfying the homological equation

$$H_\tau(\omega_\tau) + \partial_\tau\omega_\tau \equiv 0$$

on a segment $[0, 1]$. Besides, the corresponding Hamiltonian vector fields v_{H_τ} are assumed to be tangent to the first open Whitney umbrella Λ_1 and to preserve the origin ($v_{H_\tau}(0) \equiv 0$). For the Hamiltonians H_τ these conditions mean $H_\tau|_{\Lambda_1} \equiv 0$ and $\partial_p H_\tau(0) \equiv \partial_x H_\tau(0) \equiv 0$ respectively. So, taking into account Lemma 2 we can rewrite the homological equation as

$$H_\tau|_{L(\omega_\tau)} + \partial_\tau \omega_\tau \equiv 0, \quad H_\tau \in \mathcal{H}(\Lambda_1) = \mathcal{E}_\tau \otimes \left(\mathfrak{m}^2 \cap \mathcal{I}'(\Lambda_1)\right)$$

where $\mathcal{E}_\tau$ is the algebra of smooth functions on the segment $[0,1]$, $\mathfrak{m}$ is the maximal ideal in the algebra of germs at 0 of smooth functions on E, and $\mathcal{I}'(\Lambda_1)$ is the ideal consisting of all germs which vanish on the first open Whitney umbrella Λ_1.

Now solving the Cauchy problem

$$\dot{g}_\tau(p,x) \equiv v_{H_\tau}(g_\tau(p,x)), \quad g_0(p,x) = (p,x)$$

with respect to a family of diffeomorphisms g_τ on the segment $[0,1]$ for small (p,x) we get the equivalence $\omega_0 \sim_{\Lambda_1} \omega_1$ performed by the local symplectic diffeomorphism g_1 preserving Λ_1 and 0.

Let $\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \subset \mathcal{E}_\tau \otimes \mathcal{A}_0$ be the restriction of the ideal $\mathcal{H}(\Lambda_1)$ onto the family of Lagrange submanifolds given by the family $\omega_\tau(p_I, x_J) \in \mathcal{E}_\tau \otimes \mathcal{A}_0$ of generating functions and $\operatorname{Gr}\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \subset \mathcal{E}_\tau \otimes \mathcal{A}_0$ be the quasihomogeneous ideal generated by the principal quasihomogeneous parts of the germs from $\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)}$. Let us note that

$$\operatorname{Gr}\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \cap (\mathcal{E}_\tau \otimes \mathcal{A}_d) = \mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \cap (\mathcal{E}_\tau \otimes \mathcal{A}_d) \pmod{\mathcal{E}_\tau \otimes \mathcal{A}_{d+1}}.$$

Therefore, in the considered cases, the homological equation

$$H_\tau|_{L(\omega_\tau)} + \partial_\tau \omega_\tau \equiv 0 \pmod{\mathcal{E}_\tau \otimes \mathcal{A}_{d+1}}, \quad H_\tau \in \mathcal{H}(\Lambda_1)$$

is solvable because

$$\partial_\tau \omega_\tau \in \operatorname{Gr}\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \cap (\mathcal{E}_\tau \otimes \mathcal{A}_d)$$

that is shown below in each case.

1_l) Let $\omega_\tau = \tau w_l$. According to Lemma 3,

$$p_1^2 - 2x_1x_2^2\Big|_{L(\omega_\tau)},\ 3p_1p_2 - 4x_1^2x_2\Big|_{L(\omega_\tau)},\ 9p_2^2 - 8x_1^3\Big|_{L(\omega_\tau)} \in \mathcal{H}(\Lambda_1)\Big|_{L(\omega_\tau)}.$$

Since $L(\omega_\tau) = \{x_1 = x_2 = 0 \,(\operatorname{mod} \mathcal{E}_\tau \otimes \mathcal{A}_1)\}$, the principal parts of these polynomials are p_1^2, $3p_1p_2$, $9p_2^2$. Therefore, $\operatorname{Gr}\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \supset (p_1^2, p_1p_2, p_2^2) = \mathcal{E}_\tau \otimes \mathcal{A}_2$.

2_l) Let $w_l = ae_l + \widetilde{w}_l \,(\operatorname{mod} \mathcal{A}_{l+1})$ where $\widetilde{w}_l$ is a quasihomogeneous element of the ideal $(9p_2^2 - 8x_1^3)$ such that $\deg \widetilde{w}_l = l$, $\omega_\tau = a(\tau)e_l + \tau\widetilde{w}_l$, $a(0) = \operatorname{sign}(a)$, $a(1) = a$. According to Lemma 3,

$$2p_1x_1 - 3p_2x_2\Big|_{L(\omega_\tau)},\ 9p_2^2 - 8x_1^3\Big|_{L(\omega_\tau)} \in \mathcal{H}(\Lambda_1)\Big|_{L(\omega_\tau)}.$$

Since $L(\omega_\tau) = \{p_1 = -\partial_{x_1}\omega_\tau,\ x_2 = \partial_{p_2}\omega_\tau\}$, the principal parts of this polynomials are $\mp l\omega_\tau$, $9p_2^2 - 8x_1^3$. Therefore, $\operatorname{Gr}\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \supset (a(\tau)e_l, 9p_2^2 - 8x_1^3) \ni \partial_\tau\omega_\tau$ if $a(\tau) \equiv 0$ or $a(\tau) \neq 0$ for $\forall \tau \in [0,1]$.

2^d_l) Let $\omega_\tau = \pm e_l + \tau w_d$. According to Lemma 3,

$$3p_1p_2 - 4x_1^2x_2\Big|_{L(\omega_\tau)},\ 2p_1x_1 - 3p_2x_2\Big|_{L(\omega_\tau)},\ 9p_2^2 - 8x_1^3\Big|_{L(\omega_\tau)} \in\ \mathcal{H}(\Lambda_1)\Big|_{L(\omega_\tau)}.$$

Since

$$L(\omega_\tau) = \{p_1 = \mp\partial_{x_1}e_l \pmod{\mathcal{A}_{l-1}},\ x_2 = \pm\partial_{p_2}e_l \pmod{\mathcal{A}_{l-2}}\},$$

the principal parts of these polynomials are

$$\mp(3p_2\partial_{x_1}e_l + 4x_1^2\partial_{p_2}e_l), \quad \mp(2x_1\partial_{x_1}e_l + 3p_2\partial_{p_2}e_l) = \mp le_l, \quad 9p_2^2 - 8x_1^3.$$

But

$$3p_2\partial_{x_1}e_l + 4x_1^2\partial_{p_2}e_l = \begin{cases} 3ke_{l+1} & \text{if} \quad e_l = x_1^k \\ (8k/3+4)e_{l+1} \pmod{(9p_2^2-8x_1^3)} & \text{if} \quad e_l = x_1^kp_2 \end{cases}$$

and $\operatorname{Gr}\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \supset (e_{l+1}, e_l, 9p_2^2 - 8x_1^3) \supset \mathcal{E}_\tau \otimes \mathcal{A}_l$.

3_6) Let $w_6 = ap_1^2 + bx_2^3 \pmod{\mathcal{A}_7}$, $\omega_\tau = a(\tau)p_1^2 + b(\tau)x_2^3$, $a(0) = 0$, $b(0) = \operatorname{sign}(b)$, $a(1) = a$, $b(1) = b$. According to Lemma 3,

$$p_1^2 - 2x_1x_2^2\Big|_{L(\omega_\tau)},\ 2p_1x_1 - 3p_2x_2\Big|_{L(\omega_\tau)} \in\ \mathcal{H}(\Lambda_1)\Big|_{L(\omega_\tau)}.$$

Since

$$L(\omega_\tau) = \left\{x_1 = 2a(\tau)p_1,\ p_2 = -3b(\tau)x_2^2\right\},$$

the principal parts of these polynomials are p_1^2, $4a(\tau)p_1^2 + 9b(\tau)x_2^3$. Therefore, $\operatorname{Gr}\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \supset (p_1^2, b(\tau)x_2^3) \ni \partial_\tau\omega_\tau$ if $b(\tau) \neq 0$ for $\forall\tau \in [0,1]$.

The change $(p_2, x_2) \mapsto -(p_2, x_2)$ shows us that $x_2^3 \sim_{\Lambda_1} -x_2^3$.

3^d_6) Let $\omega_\tau = x_2^3 + \tau w_d$. According to Lemma 3,

$$p_1^2 - 2x_1x_2^2\Big|_{L(\omega_\tau)},\ 3p_1p_2 - 4x_1^2x_2\Big|_{L(\omega_\tau)},\ 2p_1x_1 - 3p_2x_2\Big|_{L(\omega_\tau)} \in\ \mathcal{H}(\Lambda_1)\Big|_{L(\omega_\tau)}.$$

Since $L(\omega_\tau) = \left\{x_1 = 0 \pmod{\mathcal{A}_4},\ p_2 = -3x_2^2 \pmod{\mathcal{A}_5}\right\}$, the principal parts of these polynomials are p_1^2, $-9p_1x_2^2$, $9x_2^3$. Therefore, $\operatorname{Gr}\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \supset (p_1^2, p_1x_2^2, x_2^3) = \mathcal{E}_\tau \otimes \mathcal{A}_6$.

5_6) Let $w_6 = a_{11}p_1^2 + a_{12}p_1p_2 + a_{22}p_2^2 + bx_3^3 \pmod{\mathcal{A}_7}$, $\omega_\tau = a_{11}(\tau)p_1^2 + a_{12}(\tau)p_1p_2 + a_{22}(\tau)p_2^2 + b(\tau)x_3^3$, $a_{11}(0) = a_{12}(0) = a_{22}(0) = 0$, $b(0) = \operatorname{sign}(b)$, $a_{11}(1) = a_{11}$, $a_{12}(1) = a_{12}$, $a_{22}(1) = a_{22}$, $b(1) = b$. According to Lemma 3,

$$p_1^2 - 2x_1x_2^2\Big|_{L(\omega_\tau)}, \quad 3p_1p_2 - 4x_1^2x_2\Big|_{L(\omega_\tau)},$$
$$9p_2^2 - 8x_1^3\Big|_{L(\omega_\tau)}, \quad p_3x_3\Big|_{L(\omega_\tau)} \in\ \mathcal{H}(\Lambda_1)\Big|_{L(\omega_\tau)}.$$

Since $L(\omega_\tau) = \left\{x_1 = 2a_{11}(\tau)p_1 + a_{12}(\tau)p_2,\ x_2 = a_{12}(\tau)p_1 + 2a_{22}(\tau)p_2, p_3 = -3b(\tau)x_3^2\right\}$, the principal parts of these polynomials are p_1^2, $3p_1p_2$, $9p_2^2$, $-3b(\tau)x_3^3$. Therefore, $\operatorname{Gr}\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \supset (p_1^2, p_1p_2, p_2^2, b(\tau)x_3^3) \ni \partial_\tau\omega_\tau$ if $b(\tau) \neq 0$ for $\forall\tau \in [0,1]$.

The change $(p_3, x_3) \mapsto -(p_3, x_3)$ shows us that $x_3^3 \sim_{\Lambda_1} -x_3^3$.

5_6^d) Let $\omega_\tau = x_3^3 + \tau w_d$. According to Lemma 3,

$$p_1^2 - 2x_1x_2^2\Big|_{L(\omega_\tau)},\ 3p_1p_2 - 4x_1^2x_2\Big|_{L(\omega_\tau)},\ 9p_2^2 - 8x_1^3\Big|_{L(\omega_\tau)},$$
$$p_1p_3\Big|_{L(\omega_\tau)},\ p_2p_3\Big|_{L(\omega_\tau)},\ p_3x_3\Big|_{L(\omega_\tau)} \in \mathcal{H}(\Lambda_1)\Big|_{L(\omega_\tau)}.$$

Since $L(\omega_\tau) = \big\{x_1 = 0 \,(\mathrm{mod}\ \mathcal{A}_4),\ x_2 = 0\,(\mathrm{mod}\ \mathcal{A}_4), p_3 = -3x_3^2\,(\mathrm{mod}\ \mathcal{A}_5)\big\}$, the principal parts of these polynomials are p_1^2, $3p_1p_2$, $9p_2^2$, $-3p_1x_3^2$, $-3p_2x_3^2$, $-3x_3^3$. Therefore, $\mathrm{Gr}\,\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \supset (p_1^2, p_1p_2, p_2^2, p_1x_3^2, p_2x_3^2, x_3^3) = \mathcal{E}_\tau \otimes \mathcal{A}_6$.

7_l) Let $w_l = ax_2^l + x_1\widetilde{w}_{l-1}\,(\mathrm{mod}\ \mathcal{A}_{l+1})$ where $\widetilde{w}_{l-1}$ is a quasihomogeneous germ such that $\deg \widetilde{w}_{l-1} = l-1$, $\omega_\tau = a(\tau)x_2^l + \tau x_1\widetilde{w}_{l-1}$, $a(0) = \mathrm{sign}(a)$, $a(1) = a$. According to Lemma 3,

$$p_1^2 - 2x_1x_2^2\Big|_{L(\omega_\tau)},\ 3p_1p_2 - 4x_1^2x_2\Big|_{L(\omega_\tau)},$$
$$2p_1x_1 - 3p_2x_2\Big|_{L(\omega_\tau)},\ 9p_2^2 - 8x_1^3\Big|_{L(\omega_\tau)} \in \mathcal{H}(\Lambda_1)\Big|_{L(\omega_\tau)}.$$

Since $L(\omega_\tau) = \{p_1 = -\partial_{x_1}\omega_\tau,\ p_2 = -\partial_{x_2}\omega_\tau\}$, the principal parts of these polynomials are $-2x_1x_2^2$, $-4x_1^2x_2$, $-2x_1\partial_{x_1}\omega_\tau + 3la(\tau)x_2^l + 3\tau x_1x_2\partial_{x_2}\widetilde{w}_{l-1}$, $-8x_1^3$. Therefore, $\mathrm{Gr}\,\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \supset (x_1^3, x_1^2x_2, x_1x_2^2, a(\tau)x_2^l) \ni \partial_\tau\omega_\tau$ if $a(\tau) \equiv 0$ or $a(\tau) \neq 0$ for $\forall\tau \in [0,1]$.

7_l^d) Let $\omega_\tau = \pm x_2^l + \tau w_d$. According to Lemma 3,

$$p_1^2 - 2x_1x_2^2\Big|_{L(\omega_\tau)},\ 3p_1p_2 - 4x_1^2x_2\Big|_{L(\omega_\tau)},$$
$$2p_1x_1 - 3p_2x_2\Big|_{L(\omega_\tau)},\ 9p_2^2 - 8x_1^3\Big|_{L(\omega_\tau)} \in \mathcal{H}(\Lambda_1)\Big|_{L(\omega_\tau)}.$$

Since $L(\omega_\tau) = \Big\{p_1 = 0\,(\mathrm{mod}\ \mathcal{A}_l),\ p_2 = \mp lx_2^{l-1}\,(\mathrm{mod}\ \mathcal{A}_l)\Big\}$, the principal parts of these polynomials are $-2x_1x_2^2$, $-4x_1^2x_2$, $\pm 3lx_2^l$, $-8x_1^3$. Therefore,

$$\mathrm{Gr}\,\mathcal{H}(\Lambda_1)|_{L(\omega_\tau)} \supset (x_1^3, x_1^2x_2, x_1x_2^2, x_2^l) \supset \mathcal{E}_\tau \otimes \mathcal{A}_l.$$

In Cases 4_*, 6_*, and 9_* we consider the Lie algebra of germs of Hamiltonian vector fields which are tangent to the first open Whitney umbrella Λ_1 and preserve 0. For the Hamiltonians H these conditions mean $H|_{\Lambda_1} = 0$ and $\partial_p H(0) = \partial_x H(0) = 0$ respectively. So, our Lie algebra is the ideal $\mathfrak{m}^2 \cap \mathcal{I}'(\Lambda_1)$ where $\mathfrak{m}$ is the maximal ideal in the algebra of germs at 0 of smooth functions on E and $\mathcal{I}'(\Lambda_1)$ is the ideal consisting of all germs which vanish on the first open Whitney umbrella Λ_1.

Let $\mathcal{B} = \mathcal{A}_4/\mathcal{A}_5$ in Cases 4_*, 6_* and $\mathcal{B} = \mathcal{A}'_3/\mathcal{A}_4$ in Case 9_*. In all these cases our Lie algebra $\mathfrak{m}^2 \cap \mathcal{I}'(\Lambda_1)$ acts on $\mathcal{B}$ by the formula from Lemma 2: $H(\omega) = H|_{L(\omega)}$. But it turns out that, for any $\omega \in \mathcal{B}$,

$$\dim \mathfrak{m}^2 \cap \mathcal{I}'(\Lambda_1)|_{L(\omega)} < \dim \mathcal{B}.$$

It remains to check these inequalities.

4_*) Let $\omega = ap_1^2 + bp_1x_2^2 + cx_2^4 \in \mathcal{B}$ where $\mathcal{B} = \mathcal{A}_4/\mathcal{A}_5$. Then

$$L(\omega) = \left\{x_1 = 2ap_1 + bx_2^2,\ p_2 = -2bp_1x_2 - 4cx_2^3\right\},$$

$$p_1^2 - 2x_1x_2^2\Big|_{L(\omega)} \in \mathcal{B}, \quad 2p_1x_1 - 3p_2x_2\Big|_{L(\omega)} \in \mathcal{B},$$

$$3p_1p_2 - 4x_1^2x_2\Big|_{L(\omega)} = 9p_2^2 - 8x_1^3\Big|_{L(\omega)} = 0 \pmod{\mathcal{A}_5}.$$

Therefore, according to Lemma 3,

$$\dim \mathfrak{m}^2 \cap \mathcal{I}'(\Lambda_1)|_{L(\omega)} \leq 2$$

but $\dim \mathcal{B} = 3$.

6_*) Let $\omega = a_1p_1^2 + a_2p_1p_2 + a_3p_2^2 + b_1p_1x_3^2 + b_2p_2x_3^2 + cx_3^4 \in \mathcal{B}$ where $\mathcal{B} = \mathcal{A}_4/\mathcal{A}_5$. Then

$$L(\omega) = \big\{x_1 = 2a_1p_1 + a_2p_2 + b_1x_3^2,\ x_2 = a_2p_1 + 2a_3p_2 + b_2x_3^2,$$
$$p_3 = -2b_1p_1x_3 - 2b_2p_2x_3 - 4cx_2^3\big\},$$

$$p_1^2 - 2x_1x_2^2\Big|_{L(\omega)} = p_1^2 \pmod{\mathcal{A}_5} \in \mathcal{B},$$

$$3p_1p_2 - 4x_1^2x_2\Big|_{L(\omega)} = 3p_1p_2 \pmod{\mathcal{A}_5} \in \mathcal{B},$$

$$2p_1x_1 - 3p_2x_2\Big|_{L(\omega)} \in \mathcal{B},\ \ 9p_2^2 - 8x_1^3\Big|_{L(\omega)} = 9p_2^2 \pmod{\mathcal{A}_5} \in \mathcal{B},\ \ p_3x_3\Big|_{L(\omega)} \in \mathcal{B},$$

$$p_1p_3\Big|_{L(\omega)} = p_2p_3\Big|_{L(\omega)} = p_3^2\Big|_{L(\omega)} = x_1p_3\Big|_{L(\omega)} = x_2p_3\Big|_{L(\omega)} = 0 \pmod{\mathcal{A}_5}.$$

Therefore, according to Lemma 3,

$$\dim \mathfrak{m}^2 \cap \mathcal{I}'(\Lambda_1)|_{L(\omega)} \leq 5$$

but $\dim \mathcal{B} = 6$.

9_*) Let $\omega = ax_1p_2 + b_1x_1^3 + b_2x_1^2x_3 + b_3x_1x_3^2 + b_4x_3^3 \in \mathcal{B}$ where $\mathcal{B} = \mathcal{A}'_3/\mathcal{A}_4$. Then

$$L(\omega) = \Big\{p_1 = -ap_2 - 3b_1x_1^2 - 2b_2x_1x_3 - b_3x_3^2,\ x_2 = ax_1,$$
$$p_3 = -b_2x_1^2 - 2b_3x_1x_3 - 3b_4x_3^2\Big\},$$

$$p_1^2 - 2x_1x_2^2\Big|_{L(\omega)} = -2a^2x_1^3 \pmod{\mathcal{A}_4} \in \mathcal{B},$$

$$3p_1p_2 - 4x_1^2x_2\Big|_{L(\omega)} = -4ax_1^3 \pmod{\mathcal{A}_4} \in \mathcal{B},$$

$$2p_1x_1 - 3p_2x_2\Big|_{L(\omega)} = -5ax_1p_2 - 6b_1x_1^3 - 4b_2x_1^2x_3 - 2b_3x_1x_3^2 \in \mathcal{B},$$

$$9p_2^2 - 8x_1^3\Big|_{L(\omega)} = -8x_1^3 \pmod{\mathcal{A}_4} \in \mathcal{B},$$

$$x_1p_3\Big|_{L(\omega)} = -b_2x_1^3 - 2b_3x_1^2x_3 - 3b_4x_1x_3^2 \in \mathcal{B}, \quad x_2p_3\Big|_{L(\omega)} = ax_1p_3\Big|_{L(\omega)} \in \mathcal{B},$$

$$p_3x_3\Big|_{L(\omega)} = -b_2x_1^2x_3 - 2b_3x_1x_3^2 - 3b_4x_3^3 \in \mathcal{B},$$

$$p_1p_3\Big|_{L(\omega)} = p_2p_3\Big|_{L(\omega)} = p_3^2\Big|_{L(\omega)} = 0 \pmod{\mathcal{A}_4}.$$

Therefore, according to Lemma 3,

$$\dim \mathfrak{m}^2 \cap \mathcal{I}'(\Lambda_1)|_{L(\omega)} \leq 4$$

but $\dim \mathcal{B} = 5$.

4. Stable Lagrange mappings.

In this Section we prove Theorems 1 and 2. Theorem 1 follows from Theorem 2, the proved Theorem 3, and the following Lemma 4.

Definition. Let $\Lambda \subset E$ be a Lagrange variety. The germ at 0 of the Lagrange fibration given by a generating family $W(p_I, x_J, y)$ such that $w(p_I, x_J) = W(p_I, x_J, 0)$ is called Λ-*versal* if

$$\mathcal{I}'(\Lambda)|_{L(w)} + \langle W_y|_{y=0}, 1\rangle_{\mathbb{R}} = \mathcal{A}_0$$

where $\mathcal{I}'(\Lambda)$ is the ideal consisting of the germs of all functions on E which vanish on the Lagrange variety Λ, $L(w) \subset E$ is the Lagrange submanifold defined by the generating function w, and $\mathcal{A}_0$ is the algebra of germs at 0 of smooth functions of p_I and x_J.

Remark. This is nothing but the Givental' versality [**9**] for the Lagrange mapping $\Lambda \subset E \to Y$ when the Lagrange fibration is defined by the generating family W. Also, in the case Λ is an open Whitney umbrella, the Givental' versality condition is equivalent to that the parametrization of Λ is Lagrange stable with respect to the Lagrange fibration in the sense of [**13**], [**15**] (Theorem 2 from [**13**], page 216).

Lemma 4. *The germs of Lagrange fibrations from Theorem 1 are stable with respect to Λ_1-equivalence.*

Proof. Let $\alpha_I = \deg p_I$ and $\beta_J = \deg x_J$ be positive integer quasidegrees, $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \dots$ the corresponding quasihomogeneous filtration in the algebra of germs at 0 of smooth functions of p_I and x_J, and $\operatorname{Gr}\mathcal{I}'(\Lambda_1)|_{L(\omega_\tau)} \subset \mathcal{A}_0$ the quasihomogeneous ideal generated by the principal quasihomogeneous parts of the germs from $\mathcal{I}'(\Lambda_1)|_{L(\omega_\tau)}$. Let us note that

$$\operatorname{Gr}\mathcal{I}'(\Lambda_1)\big|_{L(\omega_\tau)} \cap \mathcal{A}_d = \mathcal{I}'(\Lambda_1)\big|_{L(\omega_\tau)} \cap \mathcal{A}_d \pmod{\mathcal{A}_{d+1}}.$$

By analogy with the cases 1_l, 2_l^d, 3_6^d, 7_l^d, 5_6^d from the proof of Lemma 1 we get

S_3) $w(p_1, p_2) = 0$, $\alpha_1 = \alpha_2 = 1 \Rightarrow \operatorname{Gr}\mathcal{I}'(\Lambda_1)|_{L(w)} \supset (p_1^2, p_1p_2, p_2^2) = \mathcal{A}_2$;
$S_m^\pm$) $w(x_1, p_2) = \pm e_{m+1}$, $\beta_1 = 2$, $\alpha_2 = 3 \Rightarrow \operatorname{Gr}\mathcal{I}'(\Lambda_1)|_{L(w)} \supset (e_{m+1}, e_{m+2}, 9p_2^2 - 8x_1^3) \supset \mathcal{A}_{m+1}$;
T_5) $w(p_1, x_2) = x_2^3$, $\beta_1 = 3$, $\alpha_2 = 2 \Rightarrow \operatorname{Gr}\mathcal{I}'(\Lambda_1)|_{L(w)} \supset (p_1^2, p_1x_2^2, x_2^3) = \mathcal{A}_6$;
U_6) $w(x_1, x_2) = \pm x_2^{m-3}$, $\beta_1 = \beta_2 = 1 \Rightarrow \operatorname{Gr}\mathcal{I}'(\Lambda_1)|_{L(w)} \supset (x_1^3, x_1^2x_2, x_1x_2^2, x_2^{m-3}) \supset \mathcal{A}_{m-3}$;
V_6) $w(p_1, p_2, x_3) = x_3^3$, $\alpha_1 = \alpha_2 = 3$, $\beta_3 = 2 \Rightarrow \operatorname{Gr}\mathcal{I}'(\Lambda_1)|_{L(w)} \supset (p_1^2, p_1p_2, p_2^2, x_3^2) \supset \mathcal{A}_6$.

Therefore, the Nakayama lemma implies that

S_3) $\mathcal{I}'(\Lambda_1)|_{L(w)} \supset (p_1^2, p_1p_2, p_2^2)$;
$S_m^\pm$) $\mathcal{I}'(\Lambda_1)|_{L(w)} \supset (e_{m+1}, e_{m+2}, 9p_2^2 - 8x_1^3)$;
T_5) $\mathcal{I}'(\Lambda_1)|_{L(w)} \supset (p_1^2, p_1x_2^2, x_2^3)$;
U_6) $\mathcal{I}'(\Lambda_1)|_{L(w)} \supset (x_1^3, x_1^2x_2, x_1x_2^2, x_2^{m-3})$;
V_6) $\mathcal{I}'(\Lambda_1)|_{L(w)} \supset (p_1^2, p_1p_2, p_2^2, x_3^2)$.

Hence, in the case of an arbitrary n:

S_3) $\mathcal{I}'(\Lambda_1)|_{L(w)} \supset (p_1^2, p_1p_2, p_2^2, p_3, \dots, p_n)$;
$S_m^\pm$) $\mathcal{I}'(\Lambda_1)|_{L(w)} \supset (e_{m+1}, e_{m+2}, 9p_2^2 - 8x_1^3, p_3, \dots, p_n)$;
T_5) $\mathcal{I}'(\Lambda_1)|_{L(w)} \supset (p_1^2, p_1x_2^2, x_2^3, p_3, \dots, p_n)$;
U_6) $\mathcal{I}'(\Lambda_1)|_{L(w)} \supset (x_1^3, x_1^2x_2, x_1x_2^2, x_2^{m-3}, p_3, \dots, p_n)$;
V_6) $\mathcal{I}'(\Lambda_1)|_{L(w)} \supset (p_1^2, p_1p_2, p_2^2, x_3^2, p_4, \dots, p_n)$.

So, our germs of Lagrange fibrations are Λ_1-versal. According to Theorem 3 from [**9**], they are stable with respect to Λ_1-equivalence. □

Proof of Theorem 1. The classification at regular points is just a rewrite of Arnold's theorem. The classification at singular points follows from Theorem 3, Lemma 4, and Theorem 2. □

Now we are going to prove Theorem 2.

Lemma 5. *Let r be a nonnegative integer or $r = \infty$. Let $i : (\mathbb{R}^n, 0) \to (E, 0)$, $i' : (\mathbb{R}^n, 0) \to (E, 0)$ be germs of Lagrange immersions with $j^r i(0) =$*

$j^r i'(0)$. *Then there exists a germ of symplectic diffeomorphism* τ *such that* $i' = \tau \circ i$, *and that* $j^r\tau(0) = j^r\mathrm{id}_E(0)$.

Proof. It is sufficient to show in the case i is the inclusion of the zero-section $\mathbb{R}^n \to T^*\mathbb{R}^n$ and $i' : \mathbb{R}^n \to T^*\mathbb{R}^n$ is defined as the graph of dh for a function $h : \mathbb{R}^n \to \mathbb{R}$ with $\mathrm{ord}_0 h > r + 1$. Then τ may be defined by $(p, x) \mapsto (p + dh(x), x)$. □

For a germ of symplectic manifold $(E, 0)$ and a germ of Lagrange submanifold $(L, 0) \subset (E, 0)$ at a base point 0, we denote by $\mathrm{Sp}(E, L)$ the group consisting of germs of symplectic diffeomorphisms $(E, 0) \to (E, 0)$ preserving L. Take a Lagrange fibration $\pi : (E, 0) \to (Y, 0)$ having L as the central fiber: $\pi^{-1}(0) = L$. We denote by $\mathrm{Lag}(E, \pi)$ the group consisting of π-fiber preserving symplectic diffeomorphism-germs $(E, 0) \to (E, 0)$. Notice that $\mathrm{Lag}(E, \pi) \subset \mathrm{Sp}(E, L)$.

Lemma 6. $\mathrm{Lag}(E, \pi)$ *is a deformation retract of* $\mathrm{Sp}(E, L)$. *More exactly, there exists a mapping* $D : \mathrm{Sp}(E, L) \times [0, 1] \to \mathrm{Sp}(E, L)$ *with the properties:*

(1) $D(\tau, 0) = \tau$, $D(\tau, 1) \in \mathrm{Lag}(E, \pi)$, $(\tau \in \mathrm{Sp}(E, L))$.
(2) $D(\tau, t) = \tau, (\tau \in \mathrm{Lag}(E, \pi), t \in [0, 1])$.
(3) $D(\tau, \cdot) : E \times [0, 1] \to E$ *is smooth on* $E \times [0, 1]$ *for each* $\tau \in \mathrm{Sp}(E, L)$ *and continuous on a compact neighborhood of* $0 \times [0, 1]$ *in* $E \times [0, 1]$ *with respect to* C^∞*-topology, when* τ *is considered as a variable.*
(4) $j^1(D(\tau, t)|_L)(0) = j^1\mathrm{id}_L(0)$, $(\tau \in \mathrm{Sp}(E, L), 0 \le t \le 1)$.

In particular, each element of $\mathrm{Sp}(E, L)$ *is connected to an element of* $\mathrm{Lag}(E, \pi)$ *by a smooth path, fixing the* 1*-jet of the restriction to* L*, within* $\mathrm{Sp}(E, L)$.

Proof. It suffices to show when $E = T^*\mathbb{R}^n$ with the canonical coordinates (p, x), $L = \{x = 0\}$ and $\pi : T^*\mathbb{R}^n \to Y = \mathbb{R}^n$ is the standard projection $\pi(p, x) = x$. Let $\tau \in \mathrm{Sp}(E, L)$. Set $\tau(p, x) = (P(p, x), X(p, x))$. Then $X(p, 0) = 0$. Remark that the Jacobi matrix $A = (\partial X/\partial x)(0, 0)$ is regular. Now consider the graph $\Gamma(\tau)$ of τ in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$, with coordinates $= (p, x; p', x')$. Then $\Gamma(\tau)$ is a Lagrange submanifold with respect to the symplectic form $\Omega = \sum_i dp'_i \wedge dx'_i - \sum_i dp_i \wedge dx_i$ of $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$. Consider the Lagrange projection $\Pi : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to \mathbb{R}^{2n}$ defined by $\Pi(p, x, p', x') = (p, x')$. Then $\Pi|_{\Gamma(\tau)} : \Gamma(\tau) \to \mathbb{R}^{2n}$ is a germ of diffeomorphism at 0. Also consider the projection $\Pi' : T^*\mathbb{R}^n \times T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ defined by $\Pi'(p, x; p', x') = (p, x)$, then $\Phi(\tau) = \Pi' \circ (\Pi|_{\Gamma(\tau)})^{-1} : \mathbb{R}^{2n} \to T^*\mathbb{R}^n$ is a germ of diffeomorphism at 0. If we set $\Phi(p, x') = (p, x(p, x'))$, then the condition that τ preserves L is interpreted to the equation $x(p, 0) = 0$. Moreover $\tau \in \mathrm{Lag}(E, \pi)$ if and only if $x(p, x')$ does not depend on p. Remark that $\left(\dfrac{\partial x}{\partial x'}\right)(0, 0)$ is equal to the inverse matrix of A.

Now take the *generating function* $H = H(p, x')$ of $\Gamma(\tau)$ for the 1-form $\theta = \sum_i p'_i dx'_i + \sum_i x_i dp_i$: We have $(\Pi|_{\Gamma(\tau)})^*(dH) = \theta|_{\Gamma(\tau)}$, and $x = \partial H/\partial p, p' = \partial H/\partial x'$. Then $(\partial H/\partial p)(p, 0) = 0$. Now we write $H(p, x') = h_0(x') + \sum_{i=1}^n h_i(x')p_i + I(p, x')$, $I(p, x')$ being of order ≥ 2 for p. Then $h_i(0) = 0, 1 \leq i \leq n$ and $(\partial I/\partial p)(p, 0) = 0$. Moreover τ belongs to $\mathrm{Lag}(E, \pi)$ if and only if the generating function $H = H(p, x')$ of $\Gamma(\tau)$ is an inhomogeneous linear function with respect to p, namely if $I(p, x') = 0$. Then we set $H_t(p, x') = h_0(x') + \sum_{i=1}^n h_i(x')p_i + (1 - t)I(p, x'), 0 \leq t \leq 1$. The restriction of Π' to the graph of dH_t in $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ is a diffeomorphism, and therefore dH_t defines a family of symplectic diffeomorphisms $\tau_t : (T^*\mathbb{R}^n, 0) \to (T^*\mathbb{R}^n, 0)$ in $\mathrm{Sp}(E, L)$. Then we set $D(\tau, t) = \tau_t$. The points (1), (2), (3) and (4) are clear. □

Now we set

$$\mathcal{F} := \{f : (\mathbb{R}^n, 0) \to (T^*\mathbb{R}^n, 0) \mid f \text{ is isotropic of corank} \leq 1\},$$

$$\mathrm{Sp}\text{-}\mathcal{A} := \{(\sigma, \tau) \mid \sigma \in \mathrm{Diff}(\mathbb{R}^n, 0), \tau \in \mathrm{Sp}(T^*\mathbb{R}^n, 0)\},$$

$$\begin{aligned}\mathrm{Lag}\text{-}\mathcal{A} : &= \{(\sigma, \tau) \mid \sigma \in \mathrm{Diff}(\mathbb{R}^n, 0), \tau \in \mathrm{Lag}(T^*\mathbb{R}^n, \pi)\} \\ &= \mathrm{Diff}(\mathbb{R}^n, 0) \times \mathrm{Lag}(T^*\mathbb{R}^n, \pi),\end{aligned}$$

group of ordinary Lagrange equivalence with respect to the canonical fibration $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$, $\pi(p, x) = x$, and set

$$\mathrm{Lag}\text{-}\mathcal{K} := \{(\sigma, \tau) \in \mathrm{Sp}\text{-}\mathcal{A} \mid \tau(L) = L\} = \mathrm{Diff}(\mathbb{R}^n, 0) \times \mathrm{Sp}(T^*\mathbb{R}^n, L),$$

where $L = \pi^{-1}(0)$, the central fiber. Then the group Lag-$\mathcal{A}$ (resp. Lag-$\mathcal{K}$) acts on $\mathcal{F}$ naturally: $(\sigma, \tau)f := \tau \circ f \circ \sigma^{-1}$. Moreover we recall that $J^r_I(n, 2n) := \{j^r f(0) \mid f \in \mathcal{F}\}$ is a submanifold of the ordinary jet space $J^r(n, 2n)$ [**12**]. Then Lag-$\mathcal{A}$ (resp. Lag-$\mathcal{K}$) acts on $J^r_I(n, 2n)$ naturally as well.

Lemma 7. *Let $f, f' \in \mathcal{F}$. If f, f' are Lagrange stable and* Lag-$\mathcal{K}$*-equivalent, then they are* Lag-$\mathcal{A}$*-equivalent.*

Proof. Since f, f' are Lag-$\mathcal{K}$-equivalent, there is a $(\sigma, \tau) \in$ Lag-$\mathcal{K}$ with $f' = \tau \circ f \circ \sigma^{-1}$. By Lemma 6, there is a smooth path $\tau_t \in \mathrm{Sp}(E, L), 0 \leq t \leq 1$, with $\tau_0 = \tau, \tau_1 \in \mathrm{Lag}(E, \pi)$ and $j^1(\tau_t|_L)(0) = j^1(\mathrm{id}|_L)(0)$. We set $f_t := \tau_t \circ f$. Then we have $f_0 = f' \circ \sigma$.

Remark that $f_t, 0 \leq t \leq 1$ are Lagrange stable ([**13**], Theorem 1.2, [**15**], Theorem 1.1). Therefore $f_t, (0 \leq t \leq 1)$ are finitely Lag-$\mathcal{A}$-determined ([**15**], Theorem 1.3), so we may discuss on an isotropic jet space $J^r_I(n, 2n)$ of sufficiently high order. (This argument is analogous to the ordinary one. See [**21**].)

The vector field $v = \left.\dfrac{\partial f_t}{\partial t}\right|_{t=t_0} : (\mathbb{R}^n, 0) \to T(T^*\mathbb{R}^n)$ along f_{t_0} belongs to the tangent space $tf_0(m_n V_n) + wf_0(m_n V I_{2n})$ at f_{t_0} to the Lag-$\mathcal{A}$-orbit, for each $t_0 \in [0,1]$. (See [**13**], [**15**] for the notations.) In fact there exists $\eta \in m_n V I_{2n}$ such that $v - \eta \circ f_{t_0}$ has null generating function. So we have $\xi \in V_n$ with $tf(\xi) = v - \eta \circ f_{t_0}$ (cf. [**13**], Lemma 4.3). Then ξ is tangent to the stratum through 0, with respect to the stratification of $\mathbb{R}^n$, by the types of open Whitney umbrellas. Since $tf(\xi)$ must vanish at 0, and f is immersive along each stratum, we see v vanishes at 0. Thus, by Mather's Lemma ([**17**] IV, pp. 534-535), we see that $j^r f_t(0), (0 \le t \le 1)$ belong to the single Lag-$\mathcal{A}$-orbit in $J^r_I(n, 2n)$. In particular $j^r f_0(0)$ and $j^r f_1(0)$ belong to the same Lag-$\mathcal{A}$ orbit. By the determinacy, we have f_0 and f_1 are Lag-$\mathcal{A}$-equivalent. Since f_0 and f' are Lag-$\mathcal{A}$-equivalent, and f_1 and f are Lag-$\mathcal{A}$-equivalent, we see that f and f' are Lag-$\mathcal{A}$-equivalent. □

Proof of Theorem 2. Let $\pi, \pi' : (E, 0) \to (Y, 0)$ be Λ-stable Lagrange projections. Set $L = \pi^{-1}(0)$ and $L' = \pi'^{-1}(0)$, and assume L and L' are formally Λ-equivalent. By taking symplectic coordinates, we may assume $E = T^*\mathbb{R}^n$ and $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$ is the standard fibration.

Then $j^\infty i(0) = j^\infty(\tau \circ i')(0)$, for some parametrizations i of L and i' of L' and for a symplectic diffeomorphism τ preserving Λ. By Lemma 5, there exists a symplectic diffeomorphism τ' with $j^\infty \tau'(0) = j^\infty \mathrm{id}(0)$ and $i = \tau' \circ \tau \circ i'$. Remark that τ' needs not preserve Λ. Set $f' = \tau' \circ f$ for a parametrization (open Whitney umbrella) $f : (\mathbb{R}^n, 0) \to (E, 0)$ of Λ. Then f' is symplectically equivalent to f [**13**], and $j^\infty f'(0) = j^\infty f(0)$. Moreover (f, π) is Lagrange stable (Theorem 1.2 of [**13**]). Then as shown in [**15**] Theorem 1.3, f is finitely Lag-$\mathcal{A}$-determined, and therefore (f', π) and (f, π) are Lagrange equivalent by (σ, τ''), namely, $\tau'' \circ f' = f \circ \sigma$ and $\tau'' \in \mathrm{Lag}(E, \pi)$. Set $\tau_1 = \tau'' \circ \tau' \circ \tau$. Then $\tau_1(\Lambda) = \Lambda$ and $\tau_1(L') = L$. Therefore L and L' are Λ-equivalent.

Now take a symplectic diffeomorphism $\tau_2 : (E, 0) \to (E, 0)$ such that $\pi = \pi' \circ \tau_2$ [**3**]. Set $\widetilde{f} = \tau_2^{-1} \circ f$. Since π is Λ-stable, we see that f is Lagrange stable with respect to π. So is $\widetilde{f}$, since π' is Λ-stable. Since $\tau_1 \circ \tau_2$ maps L to L and $f(\mathbb{R}^n)$ to $\widetilde{f}(\mathbb{R}^n)$, f and $\widetilde{f}$ are Lag-$\mathcal{K}$-equivalent, by Lemma 8 below. Thus, by Lemma 7, f and $\widetilde{f}$ are Lag-$\mathcal{A}$-equivalent: For a $(\sigma, \tau_3) \in$ Lag-$\mathcal{A}$, $\widetilde{f} = \tau_3 \circ f \circ \sigma^{-1}$. Then $\tau_2 \circ \tau_3$ preserves Λ and $\pi' \circ \tau_2 \circ \tau_3 = \pi$. Therefore π and π' are Λ-equivalent. □

5. Simple stable compositions.

Here we remark that our result is applied to the classification problem for compositions of an isotropic mapping and a Lagrange fibration [**13**], when the isotropic mapping is the first open Whitney umbrella. For this, first recall the notion of C^∞ normalization [**6**], [**7**] in the special case we need:

A map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is called a C^∞ *normalization* if f is C^∞-right-left equivalent to an analytic map-germ $f' : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ which is a normalization of the image.

Recall that a map-germ $f : (\mathbb{R}^n, 0) \to (E, 0)$ is called an open Whitney umbrella of type k if f is symplectically equivalent to a polynomial map-germ $f_{n,k} : (\mathbb{R}^n, 0) \to (T^*\mathbb{R}^n, 0)$ explicitly given in [**13**], namely $f \circ \sigma = \tau \circ f_{n,k}$ for a diffeomorphism $\sigma : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and a symplectic diffeomorphism $\tau : (E, 0) \to (E, 0)$.

Remark that the normal form $f_{n,k}$ of open Whitney umbrellas is an analytic normalization of the image, and therefore any open Whitney umbrella is a C^∞-normalization. The following lemma is a special case of Theorem 1.11 of [**7**]:

Lemma 8. *Let $f : (\mathbb{R}^n, 0) \to (E, 0)$ be a C^∞ normalization. Denote by $f(\mathbb{R}^n)$ the well-defined germ of the image of f. If a germ of diffeomorphism $h : (E, 0) \to (E, 0)$ preserves $f(\mathbb{R}^n)$, namely if $h(f(\mathbb{R}^n)) = f(\mathbb{R}^n)$, then there exists a germ of diffeomorphism $\sigma : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $h \circ f = f \circ \sigma$.*

Then we have:

Lemma 9. *Let $f, f' : (\mathbb{R}^n, 0) \to (E, 0)$ be open Whitney umbrellas of same type k, and $\pi, \pi' : (E, 0) \to (Y, 0)$ Lagrange fibrations. Then the following conditions are equivalent:*

(1) *There exist a diffeomorphism $\sigma : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, a symplectic diffeomorphism $\tau : (E, 0) \to (E, 0)$ and a diffeomorphism $\bar\tau : (Y, 0) \to (Y, 0)$ with $f' \circ \sigma = \tau \circ f$ and $\pi' \circ \tau = \bar\tau \circ \pi$. (Lagrange equivalence of (f, π) and (f', π'), in the sense of* [**15**].)

(2) *There exists a symplectic diffeomorphism $\tau : (E, 0) \to (E, 0)$ and a diffeomorphism $\bar\tau : (Y, 0) \to (Y, 0)$ satisfying $\tau(f(\mathbb{R}^n)) = f'(\mathbb{R}^n)$ and $\pi' \circ \tau = \bar\tau \circ \pi$.*

Proof. The implication (1) $\Rightarrow$ (2) is straightforward. (2) $\Rightarrow$ (1): First assume $\pi' = \pi$. Since both f and f' are symplectically equivalent to the parametrization $f_{n,k}$, there exist diffeomorphisms $\sigma', \sigma'' : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and symplectic diffeomorphisms $\tau', \tau'' : (E, 0) \to (E, 0)$ such that $f \circ \sigma' = \tau' \circ f_{n,k}$ and $f' \circ \sigma'' = \tau'' \circ f_{n,k}$. We set $\Lambda_k = f_{n,k}(\mathbb{R}^n)$. Then we see that $\tau''^{-1} \circ \tau \circ \tau'(\Lambda_k) = \Lambda_k$. By Lemma 8, there exists a diffeomorphism $\sigma''' : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\tau''^{-1} \circ \tau \circ \tau' \circ f_{n,k} = f_{n,k} \circ \sigma'''$. Then we have $\tau \circ f = f' \circ \sigma'' \circ \sigma''' \circ \sigma'^{-1}$.

In general case, apply the above argument to f and $T \circ f'$, for a symplectic diffeomorphism $T : (E, 0) \to (E, 0)$ satisfying $\pi' = \pi \circ T$ ([**3**]). □

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