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Let X' be a complex affine algebraic threefold with $H_3(X') = 0$ which is a UFD and whose invertible functions are constants. Let Z be a Zariski open subset of X' which has a morphism $p : Z \to U$ into a curve U such that all fibers of p are isomorphic to $\mathbf{C}^2$. We prove that X' is isomorphic to $\mathbf{C}^3$ iff none of irreducible components of $X' \setminus Z$ has non-isolated singularities. Furthermore, if X' is $\mathbf{C}^3$ then p extends to a polynomial on $\mathbf{C}^3$ which is linear in a suitable coordinate system. This implies the fact formulated in the title of the paper.

1. Introduction.

A nonconstant polynomial on $\mathbf{C}^n$ is a variable if it is linear in a suitable polynomial coordinate system on $\mathbf{C}^n$. In 1961 Gutwirth [**Gu**] proved the following fact which was later reproved by Nagata [**Na**]: Every polynomial $p \in \mathbf{C}^{[2]}$ whose general fibers are isomorphic to $\mathbf{C}$ is a variable. In 1974-1975 Abhyankar, Moh, and Suzuki showed that a much stronger fact holds: Every irreducible polynomial $p \in \mathbf{C}^{[2]}$ with $p^{-1}(0) \simeq \mathbf{C}$ is variable [**AbMo**], [**Su**]. The Embedding conjecture formulated by Abhyankar and Sathaye [**Sa1**] suggests that the similar fact holds in higher dimensions:

Every irreducible polynomial $p \in \mathbf{C}^{[n]}$ with $p^{-1}(0) \simeq \mathbf{C}^{n-1}$ is a variable.

It seems that in the full generality the positive answer to the Embedding conjecture is not feasible in the near future but there is some progress for $n = 3$. In this dimension A. Sathaye, D. Wright, and P. Russell proved some special cases of this conjecture ([**Sa1**], [**Wr**], [**RuSa**], see also [**KaZa1**]). Then M. Koras and P. Russell proved the Linearization conjecture for $n = 3$ [**KoRu2**], [**KaKoM-LRu**] which implies the following theorem: If p is an irreducible polynomial on $\mathbf{C}^3$ such that it is quasi-invariant with respect to a regular $\mathbf{C}^*$-action on $\mathbf{C}^3$ and its zero fiber is isomorphic to $\mathbf{C}^2$, then p is a variable.[1] This paper and paper [**KaZa2**] contain another step in the direction of the Embedding conjecture – we prove the analogue of the

[1] In fact, P. Russell indicated to the author that the "hard-case" of the Linearization conjecture is equivalent to this theorem. This equivalence can be extracted from [**KoRu1**].

Gutwirth theorem in dimension 3, i.e., every polynomial with general $\mathbf{C}^2$–fibers is a variable. It is worth mentioning that a special case of this theorem (when additionally all fibers are UFDs and the generic fiber is a plane) follows from more general results of Miyanishi [**Miy1**] and Sathaye [**Sa2**]. In fact, in our paper the analogue of the Gutwirth theorem in dimension 3 is also a consequence of the following more general result.

Main Theorem. *Let X' be an affine algebraic variety of dimension 3 such that*

(0) *X' is a UFD and all invertible functions on X' are constants;*
(1) *X' is smooth and $H_3(X') = 0$;*
(2) *there exists a Zariski open subset Z of X' and a morphism $p : Z \to U$ into a curve U whose fibers are isomorphic to $\mathbf{C}^2$;*
(3) *each irreducible component of $X' \setminus Z$ has at most isolated singularities.*

Then U is isomorphic to a Zariski open subset of $\mathbf{C}$ and p can be extended to a regular function on X'. Furthermore, X' is isomorphic to $\mathbf{C}^3$ and p is a variable.

The same conclusion remains true if we replace (1) *and* (3) *by*

(1′) *the Euler characteristic of X' is $e(X') = 1$;*
(3′) *each irreducible component of $X' \setminus Z$ is a UFD.*

In the case when conditions (1) *and* (2) *hold but* (3′) *does not, X' is an exotic algebraic structure on $\mathbf{C}^3$ (that is, X' is diffeomorphic to $\mathbf{R}^6$ as a real manifold but not isomorphic to $\mathbf{C}^3$) with a nontrivial Makar-Limanov invariant.*

The Makar-Limanov invariant was introduced in [**M-L1**], [**KaM-L1**] (see also [**KaM-L2**], [**Za**], and [**De**]). For a reduced irreducible affine algebraic variety X' this invariant is the subalgebra $\mathrm{ML}(X')$ of the algebra of regular functions $\mathbf{C}[X']$ on X' that consists of all functions which are invariant under any regular $\mathbf{C}_+$-action on X'. If $\mathrm{ML}(X') \simeq \mathbf{C}$ then we call it trivial. This is so, for instance, when $X' \simeq \mathbf{C}^n$.

The proof of the Main Theorem consists of three Lemmas.

Lemma I (cf. [**Miy1**])**.** *Let X' be an affine algebraic variety of dimension 3 which satisfies assumption* (0)*,* (1)*, and* (3) *from the Main Theorem and*

(2′) *there exists a Zariski open subset Z of X' which is a $\mathbf{C}^2$-cylinder over a curve U (i.e., Z is isomorphic to the $\mathbf{C}^2 \times U$).*

Then U can be viewed as a subset of $\mathbf{C}$, X' is isomorphic to $\mathbf{C}^3$, and the natural projection $Z \to U$ can be extended to a regular function on X' which is a variable.

Miyanishi's theorem (which can be also proved by the technique we develop below) claims the same fact with assumptions (1) and (3) replaced by (1′) and (3′). The idea of the proof of Lemma I is as follows. Let $\sigma : X' \to X$

be an affine modification. The restriction σ to the complement of the exceptional divisor E of X' is an isomorphism between $X' \setminus E$ and $X \setminus D$ where D is a divisor of X. We show that under the assumption of Lemma I X' is an affine modification of $X = \mathbf{C}^3$ and the divisor D is the union of a finite number of parallel affine planes in $\mathbf{C}^3$. Then the problem is reduced to the case when D consists of one plane only. We consider the set of so-called basic modification which preserve normality, contractibility, and for which $C_0 = \sigma(E)$ is closed in X and E is naturally isomorphic to $\mathbf{C}^k \times C_0$. One of the central facts (Theorem 3.1) says that σ is the composition $\sigma_1 \circ \cdots \circ \sigma_m$ where each $\sigma_i : X_i \to X_{i-1}$ ($X' = X_m$ and $X = X_0$) is a basic modification. If $m = 1$ and C_0 is either a point or a straight line in the plane D then it is easy to check that $X \simeq \mathbf{C}^3$ and the other statements of the Lemma I hold. When $m > 1$, using the control over topology, one can show that the center of σ_1 is either a point or a curve in D which is isomorphic to $\mathbf{C}$. If the center is such a curve then it can be viewed as a straight line by [**AbMo**, **Su**] whence X_1 is isomorphic to $\mathbf{C}^3$. Now the induction by m yields Lemma I.

Lemma II. *Let X' be an affine algebraic variety of dimension 3 which satisfies assumptions (0), (1), and (2′), but does not satisfy assumption (3′). Then X' is an exotic algebraic structure on $\mathbf{C}^3$ with a nontrivial Makar-Limanov invariant.*

Under the assumption of Lemma II X' is still an affine modification of $X = \mathbf{C}^3$, σ is still a composition of basic modifications, and one can reduce the problem to the case when D is a coordinate plane. It can be shown that C_0 is either a point or an irreducible contractible curve in D. The remarkable Lin-Zaidenberg theorem [**LiZa**] says that such a curve is given by $x^n = y^m$ in a suitable coordinate system where n and m are relatively prime. This allows us to present explicitly a system of polynomial equations in some Euclidean space $\mathbf{C}^N$ whose zero set is X'. Here we use the fact that basic modifications of Cohen-Macaulay varieties are Davis modifications which were introduced in [**KaZa1**] and which fit perfectly the aim of presenting explicitly the result of a modification as a closed affine subvariety of a Euclidean space. This explicit presentation of X' as a subvariety of $\mathbf{C}^N$ enables us to compute $\mathrm{ML}(X')$, using the technique from [**KaM-L1**], [**KaM-L2**]. If condition (3′) does not hold then $\mathrm{ML}(X') \neq \mathbf{C}$ whence $X' \neq \mathbf{C}^3$. We show also that X' is contractible whence it is diffeomorphic to $\mathbf{R}^6$ by the Dimca-Ramanujam theorem [**ChDi**] which concludes Lemma II.

The Main Theorem follows from Lemmas I, II, Miyanishi's theorem, and:

Lemma III ([**KaZa2**]). *Assumptions (2) and (2′) are equivalent.*

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[**KaZa1**]. Later M. Zaidenberg decided not to participate in the project due to other obligations and the author had to finish it alone. It is also our pleasure to thank I. Dolgachev, P. Russell, and J. Lipman whose consultations were very useful for the author.

2. Affine modifications.

2.1. Notation and terminology. In this subsection we present central definitions and notation which will be used in the rest of the paper. The ground field in this paper will always be the field of complex numbers $\mathbf{C}$.

Definition 2.1. Let X be a reduced irreducible affine algebraic variety, $A = \mathbf{C}[X]$ be its algebra of regular functions, I be an ideal in A, and $f \in I \setminus \{0\}$. By the affine modification of A with locus (I, f) we mean the algebra $A' := A[I/f]$ together with the natural embedding $A \hookrightarrow A'$. That is, if $b_0 = f, b_1, \ldots, b_s$ are generators of I then A' is the subalgebra of the field of fractions of A which is generated over A by the elements $b_1/f, \ldots, b_s/f$. It can be easily checked [**KaZa1**] that A' is also an affine domain, i.e., its spectrum X' is an affine algebraic variety and the natural embedding $A \hookrightarrow A'$ generates a morphism $\sigma : X' \to X$. Sometimes we refer to σ as an affine modification or we say that X' is an affine modification of X. The reduction D (resp. E) of the divisor $f^*(0) \subset X$ (resp. $(f \circ \sigma)^*(0) \subset X'$) will be called the divisor (resp. the exceptional divisor) of the modification. The (reduction of the) subvariety of X defined by I will be called the (reduced) center of the modification and $\sigma(E)$ will be called the geometrical center of modification.

Definition 2.2. A morphism $p : Y \to Z$ of algebraic varieties is called a $\mathbf{C}^s$-cylinder over Z if there exists an isomorphism $\varphi : Y \to \mathbf{C}^s \times Z$ so that $p \circ \varphi^{-1}$ is the projection to the second factor. Let $\sigma(E)$ be an algebraic variety of pure dimension where σ and E are from Definition 2.1. We say that σ is a cylindrical modification of rank s if $\sigma|_E : E \to \sigma(E)$ is a $\mathbf{C}^s$-cylinder where $s + 1 = \operatorname{codim}_X \sigma(E)$.

Definition 2.3. A sequence of generators $b_0, \ldots, b_s$ of an ideal I of $A = \mathbf{C}[X]$ is called semi-regular if the height of I is $s + 1$. If in addition $b_0 = f$ then the affine modification $A \hookrightarrow A'$ with locus (I, f) is called *semi-basic* of rank s. Furthermore, if C is the set of the common zeros of I in X this semi-regular sequence is called an almost complete intersection when every irreducible components G of C meets $\operatorname{reg} X$ and $G \cap \operatorname{reg} X$ contains a Zariski open subset which is a complete intersection given by $b_0 = \cdots = b_s = 0$. If, in addition, $b_0 = f$ then the affine modification $A \hookrightarrow A'$ is *basic* of rank s.

Let S be a multiplicative system of A and $S^{-1}A$ (resp. $S^{-1}A'$) the ring of fractions of A (resp. A') with respect to S. Every ideal I in A generates an ideal $S^{-1}I$ in $S^{-1}A$. The following fact is an immediate consequence of the definitions of affine modifications and rings of fractions.

Proposition 2.1. *In the notation above we have* $S^{-1}A' = (S^{-1}A)[S^{-1}I/f]$.

Definition 2.4. Suppose that B is a localization of an affine domain, J is an ideal in B and $g \in B \setminus \{0\}$. By the local modification of B with locus (J, g) we mean the algebra $B' := B[J/g]$ together with the natural embedding $B \hookrightarrow B'$. By Proposition 2.1 $B \hookrightarrow B'$ can be obtained by the operation of taking fractions of an affine modification $A \hookrightarrow A'$. We call $B \hookrightarrow B'$ semi-basic (resp. basic) if the affine modification $A \hookrightarrow A'$ can be chosen semi-basic (resp. basic).

Definition 2.5. Let A_M be the localization of an affine domain A at a maximal ideal M, and I_M be the extension of an ideal $I \subset A$ in A_M. An affine modification $A \hookrightarrow A'$ is locally semi-basic (resp. basic) if for every maximal ideal M that vanishes at a point of the geometrical center the local modification $A_M \hookrightarrow A_M[I_M/f] = S^{-1}A'$ is semi-basic (resp. basic) where $S = A \setminus M$.

Convention 2.1. The algebra of regular functions of an affine algebraic variety Y will be denoted by $\mathbf{C}[Y]$. Further in this paper X and X' are **always** reduced irreducible affine algebraic varieties, $A = \mathbf{C}[X]$, and $A' = \mathbf{C}[X']$. We suppose that the notation $A \hookrightarrow A'$ is fixed throughout the paper. It will **always** mean an affine modification with locus (I, f). The corresponding morphism is **always** denoted by $\sigma : X' \to X$. The divisor, the exceptional divisor, and the reduced center of the modification are **always** denoted by D, E, and C respectively.

We shall also use the following notation in the rest of this section: If Y is an affine algebraic variety and $B = \mathbf{C}[Y]$ then for every closed algebraic subvariety Z of Y the defining ideal of Z in B will be denoted by $\mathcal{I}_B(Z)$. For every ideal J in $\mathbf{C}[Y]$ we denote by $\mathcal{V}_Y(J)$ the zero set of this ideal in Y.

2.2. General facts about affine modifications. The ideal $K = \{a \in A \,|\, a/f \in A'\}$ in A is called the f-largest ideal of modification $A \hookrightarrow A'$. Clearly, $I \subset K$ and $A' = A[K/f]$. When A and A' are fixed we denote K by I_f. Nullstellensatz implies that the geometrical center of an affine modification is always contained in the reduced center, in the case of an f-largest ideal one can see that it implies more.

Proposition 2.2. *Let* $A \hookrightarrow A'$ *be an affine modification such that* $I = I_f$. *Then the reduced center of the modification is the closure of the geometrical one.*

Proposition 2.3. *Let $g \in A \setminus \{0\}$ and $f = g^n$ for a natural n. Suppose that $\mathcal{I}_{A'}(E)$ coincides with the principal ideal in A' generated by g. Then $(\mathcal{I}_A(C))^n \subset I_f$ (in particular, for $n = 1$ we have $\mathcal{I}_A(C) = I_f$). Furthermore, for every ideal J in A which is contained in $\mathcal{I}_A(C)$ the algebra $A_1 := A[J/g]$ is contained in A'.*

Proof. Note that for every $a \in (\mathcal{I}_A(C))^n$ we have $a \in (\mathcal{I}_{A'}(E))^n$ whence $a/f \in A'$. Thus $a \in I_f$ which is the first statement. This implies that $g^{n-1}J \subset I_f$. Hence $A_1 = A[g^{n-1}J/f] \subset A[I_f/f] = A'$. □

Corollary 2.1 (cf. [**WaWe**, Prop. 1.2])**.** *Suppose that $J = \mathcal{I}_A(C)$ in Proposition 2.3. Then there exists an ideal K_1 in A_1 such that $A' = A_1[K_1/g^{n-1}]$. That is, $A_1 \hookrightarrow A'$ may be viewed as an affine modification with locus (K_1, g^{n-1}).*

Proof. Let $b_0 = g^n, b_1, \ldots, b_s$ be generators of I. Note that $b_i/g \in A_1$ for every i. The ideal K_1 in A_1 generated by $g^{n-1}, b_1/g, \ldots, b_s/g$ is the desired ideal. □

Proposition 2.4. *Let B be a UFD, J be an ideal in B, $g \in J \setminus 0$ be irreducible in B, $f = g^n$, $B' = B[J/f]$, and $B' \neq B[1/f]$. Then g is irreducible in B'.*

Proof. Let $g^k = a'b'$ where $a' = a/f^l, b' = b/f^m, a \in J^l$, and $b \in J^m$. Hence $g^{k+nl+nm} = ab$ in B. Since B is a UFD we have $a = ug^s$ and $b = vg^r$ where $s + r = k + nl + nm$ and u, v are units. If $s < nl$ then $a' = u/g^{nl-s}$ whence $1/f \in B'$ which contradicts $B' \neq B[1/f]$. Thus $s \geq nl$ and, similarly, $r \geq nm$. Hence $a' = ug^{s-nl}$ and $b' = vg^{r-nm}$ are in B whence g is irreducible in B'. □

Proposition 2.5. *Let $A \hookrightarrow A'$ be an affine modification and $\forall k > 0$ each divisor $g \in A$ of f^k is not a unit in A' (i.e., $(g \circ \sigma)^{-1}(0)$ is not empty). Then the units of A' and A are the same.*

Proof. Since A' is a subalgebra of $A[1/f]$ its units are also units of $A[1/f]$. The units of the last algebra are the products of irreducible divisors of f^k and the units of A. By the assumption these divisors are not invertible functions on X' whence the units of A' coincide with the units of A. □

Proposition 2.6. *Let I_j be an ideal in A for $j = 1, \ldots, k$, and let $f_j \in I_j \setminus \{0\}$. Suppose that $f = f_1 \cdots f_k$ and $I = (f/f_1)I_1 + \cdots + (f/f_k)I_k$. Let $A_j = A[I_j/f_j]$ and let $\delta_j : X_j \to X$ be the morphism of affine algebraic varieties associated with the affine modification $A \hookrightarrow A_j$ with locus (I_j, f_j). Suppose that E_j is the exceptional divisor of this modification. These morphisms define the affine variety $Y = X_1 \times_X X_2 \times_X \cdots \times_X X_k$ and its subvariety $Y^* = (X_1 \setminus E_1) \times_X \cdots \times_X (X_k \setminus E_k)$.*

(1) *X' is isomorphic to the closure $\bar{Y}^*$ of Y^* in Y and under this isomorphism σ coincides with the restriction of the natural projection $\tau : Y \to X$ to $\bar{Y}^*$.*
(2) *If $\forall j \neq l$ f_j and f_l have no common zeros on X then $X' \simeq Y$.*

Proof. Let $D_j = f_j^{-1}(0)$. Then $D = \bigcup_{j=1}^k D_j$. As $\delta_j|_{X_j \setminus E_j}$ is an isomorphism between $X_j \setminus E_j$ and $X \setminus D_j$ we see that Y^* is isomorphic to $X \setminus D$. Thus $B := \mathbf{C}[\bar{Y}^*]$ is a subalgebra in the field of fractions of A. The natural projection $\bar{Y}^* \to X_j$ enables us to treat A_j as a subalgebra of B. Note that $A_1, \dots, A_k$ generate B and A' as $I = (f/f_1)I_1 + \ldots + (f/f_k)I_k$. Hence $A' = B$ which yields (1). For (2) it suffices to prove that Y is irreducible. Assume that Y has an irreducible component $Y_1 \neq \bar{Y}^*$. Note that $\tau(Y_1) \subset D$ as $\tau^{-1}|_{X \setminus D}$ is an isomorphism between $X \setminus D$ and Y^*. We can suppose that $\tau(Y_1) \subset D_1$. Put $T = \bigcup_{j=2}^k D_j$ and consider $\theta : Y \setminus \tau^{-1}(T) \to X \setminus T$ where θ is the restriction of τ. As for $j \geq 2$ the restriction of δ_j to $X_j \setminus \delta_j^{-1}(T)$ is an isomorphism between this variety and $X \setminus T$ we see that $Y \setminus \tau^{-1}(T)$ is isomorphic to $X_1 \setminus \delta_1^{-1}(T)$ and θ coincides with the restriction of δ_1 to $X_1 \setminus \delta_1^{-1}(T)$ under this isomorphism. Thus $\delta_1^{-1}(X \setminus T) \simeq \tau^{-1}(X \setminus T)$. As T does not meet D_1, $\tau^{-1}(X \setminus T)$ contains Y_1, i.e., it is not irreducible. But $\delta_1^{-1}(X \setminus T) \subset X_1$ is irreducible. Contradiction. $\square$

Remark 2.1. We shall need the coordinate interpretation of Proposition 2.6 (2). Suppose for simplicity that $k = 2$. Let X be a closed affine subvariety of $\mathbf{C}^n$ with a coordinate system $\bar{x}$ and let X_j be a closed affine subvariety of $\mathbf{C}^{n_j}$ with a coordinate system $(\bar{x}, \bar{z}_j)$. Suppose that X_j coincides with the zeros of a polynomial system of equations $P_j(\bar{x}, \bar{z}_j) = 0$ and σ_j can be identified with the restriction of the natural projection $\mathbf{C}^{n_j} \to \mathbf{C}^n$. Consider the space $\mathbf{C}^{n_1+n_2-n}$ with coordinates $(\bar{x}, \bar{z}_1, \bar{z}_2)$. Then Proposition 2.6 (2) implies that the zero set of the system $P_1(\bar{x}, \bar{z}_1) = P_2(\bar{x}, \bar{z}_2) = 0$ in this space is isomorphic to X'.

2.3. Semi-basic modifications.

Lemma 2.1. *Let Z be a closed reduced subvariety of X of codimension $s+1$ and let $J = \mathcal{I}_A(Z)$. Suppose that $f \in J \setminus \{0\}$.*

(1) *Then J contains a semi-regular sequence $\mathcal{L} = \{f = b_0, \dots, b_s\}$.*
(2) *Let this sequence generate an ideal J_1. If none of the irreducible components of Z and $f^{-1}(0)$ is contained in* $\operatorname{sing} X$ *then $\mathcal{L}$ can be chosen so that none of the irreducible components of $\mathcal{V}_X(J_1)$ is contained in* $\operatorname{sing} X$.
(3) *If (2) holds and the zero multiplicity of f at general points of each irreducible component of Z is 1, then one can choose $\mathcal{L}$ to be an almost complete intersection.*

(4) *There exists a finite-dimensional subspace S of J such that* (1)–(3) *hold when $b_1, \dots, b_s$ are general points of any finite-dimensional subspace of J containing S.*

The proof of the Lemma is straightforward. The first two statements is just an induction on s. For (3) and (4) consider generators $g_0 = f, g_1, \dots, g_r$ of J and a closed embedding $X \hookrightarrow \mathbf{C}^n$. Let S consist of elements of the form $\sum_{j=0}^{r} l_j g_j$ where each l_j is the restriction to X of a linear function on $\mathbf{C}^n$. It is easy to see that the Lemma holds with this choice of S.

Proposition 2.7. *Suppose that $A \hookrightarrow A'$ is a semi-basic modification of rank $s > 0$. Then it is a cylindrical modification of rank s. Furthermore, the reduced and geometrical centers of this modification coincide.*

Proof. Let J_0 be the maximal ideal in $\mathbf{C}^{[s+1]} = \mathbf{C}[x_0, x_1, \dots, x_s]$ that vanishes at the origin o in $\mathbf{C}^{s+1}$. Put $B_0 = \mathbf{C}^{[s+1]}[J_0/x_0]$ and consider the modification $\mathbf{C}^{[s+1]} \hookrightarrow B_0$ with locus (J_0, x_0). Then B_0 is isomorphic to $\mathbf{C}[x_0, y_1, \dots, y_s]$ and $x_i = x_0 y_i$ for $i = 1, \dots, s$. That is, $Z_0 := \operatorname{spec} B_0$ may be viewed as the subvariety of $\mathbf{C}^{2s+1}$ (whose coordinates are $x_0, x_1, \dots, x_s$, $y_1, \dots, y_s$) given by the system of equations $x_i - x_0 y_i = 0$, $i \geq 1$. Let $\rho : \mathbf{C}^{2s+1} \to \mathbf{C}^{s+1}$ be the natural projection to the first $s+1$ coordinates. Our modification is nothing but the restriction of ρ to Z_0. Its reduced and geometrical centers are o and the exceptional divisor $E_0 = \rho^{-1}(o) \simeq \mathbf{C}^s$. Put $Z = \mathbf{C}^{s+1} \times X$, $B = \mathbf{C}[Z]$, and $J = J_0 B$. Consider the modification $B \hookrightarrow B'$ with locus (J, x_0). By [**KaZa1**, Cor. 2.1] we see that $Z' := \operatorname{spec} B' = Z_0 \times X$ and the above modification is the restriction δ to $Z' \subset \mathbf{C}^{2s+1} \times X$ of the natural projection $(\rho, \mathrm{id}) : \mathbf{C}^{2s+1} \times X \to \mathbf{C}^{s+1} \times X = Z$. In particular, its reduced and geometrical centers are $C^0 = o \times X$ and the exceptional divisor $E^0 = E_0 \times X$. Let $b_0 = f, b_1, \dots, b_s$ be a semi-regular system of generators of I. Consider the embedding $i : X \hookrightarrow Z$ given by the system of equations $x_j - b_j = 0$, $j = 0, \dots, s$. The restriction of J to X coincides with I. By [**KaZa1**, Cor. 2.1] we have the commutative diagram

$$
\begin{array}{ccc}
X' & \overset{i'}{\hookrightarrow} & Z' \\
\Big\downarrow \sigma & & \Big\downarrow \delta \\
X & \overset{i}{\hookrightarrow} & Z
\end{array}
$$

where $i' : X' \hookrightarrow Z'$ is a closed embedding. The reduced center of σ is $C = C^0 \cap i(X)$, and it is of codimension $s+1$ in X as σ is semi-basic. As $\operatorname{codim}_{X'} E = 1$ each fiber F of $\sigma|_E : E \to \sigma(E) \subset C$ must be at least of dimension s. But F is contained in a fiber $F^0 \simeq \mathbf{C}^s$ of $\delta|_{E^0} : E^0 \to C^0$. Hence $\dim F = s$ and $\sigma(E)$ is dense in C. Furthermore, as i' is a closed embedding $F = F^0$ and $\sigma(E) = C$ whence E is a $\mathbf{C}^s$-cylinder over C and the reduced and geometrical centers of σ coincide. $\square$

2.4. Davis modifications.

Theorem 2.1 ([**Da**], see also [**Ei**, Ex. 17.14])**.** *Let $f = b_0, b_1, \ldots, b_s$ be generators of an ideal J in a Noetherian domain B. Consider the surjective homomorphism*

$$\beta : B^{[s]} = B[y_1, \ldots, y_s] \longrightarrow\!\!\!\!\!\rightarrow B[J/f] = B[b_1/f, \ldots, b_s/f] \simeq B'$$

where $y_1, \ldots, y_s$ are independent variables and $\beta(y_i) = b_i/f,\ i = 1, \ldots, s$. Denote by J' the ideal of $B^{[s]}$ generated by the elements $L_1, \ldots, L_s \in \ker\beta$ where $L_i = fy_i - b_i$. Then $\ker\beta$ coincides with J' iff J' is a prime ideal. The latter is true, for instance, if the system of generators $b_0 = f,\ b_1, \ldots, b_s$ of the ideal J is regular.

Definition 2.6. Let B be (a localization of) an affine domain. When J' from Theorem 2.1 is prime the (local) affine modification $B \hookrightarrow B'$ with locus (J, f) is called Davis, and $b_0 = f, b_1, \ldots, b_s$ is its representative system of generators.

Remark 2.2. It is easy to see that in the case of a nonempty reduced center every (local) affine Davis modification is automatically semi-basic.

Proposition 2.8. *Let $A \hookrightarrow A'$ be an affine modification, $b_0 = f, b_1, \ldots, b_s$ be a system of generators of I, M be a maximal ideal in A, and A_M, I_M, S be as in Definition 2.5, i.e., $A_M \hookrightarrow S^{-1}A'$ is the local modification with locus (I_M, f). Let for every maximal ideal M this local modification be Davis and $b_0, \ldots, b_s$ be a representative system of generators. Then $A \hookrightarrow A'$ is a Davis modification.*

Proof. Let $Y = \mathbf{C}^s \times X$, i.e., $\mathbf{C}[Y] = A^{[s]} = A[y_1, \ldots, y_s]$. Let I' be the ideal in $A^{[s]}$ generated by $L_i = y_i f - b_i$, $i = 1, \ldots, s$ and $Y_1 = \mathcal{V}_Y(I')$. Show that I' is prime, i.e., Y_1 is reduced irreducible. Choose a maximal ideal M' in $A^{[s]}$ which vanishes at $x' \in Y_1$. Let x be the image of x' in X under the natural projection and let M be the maximal ideal of A that vanishes at x. Then $A \setminus M \subset A^{[s]} \setminus M'$ and $A^{[s]}_{M'}$ is a further localization of $S^{-1}A^{[s]}$. Since $A_M \hookrightarrow S^{-1}A'$ is a Davis modification and $b_0, \ldots, b_s$ is a representative system of generators of this modification, the ideal $S^{-1}I'$ is prime in $S^{-1}A^{[s]}$ whence the localization $I'_{M'}$ of this ideal is also prime, i.e., the germ of Y_1 at x' is reduced irreducible. It remains to show that Y_1 is connected. As $E_1 = Y_1 \cap f^{-1}(0) \simeq \mathbf{C}^s \times C$ and the localizations of our modification are Davis the codimension of irreducible component of C in X must be $s+1$ by Remark 2.2. Hence $\dim E_1 = \dim X - 1$ unless E_1 is empty. By construction $Y_1 \setminus E_1$ is isomorphic to $X \setminus D$ and, therefore, irreducible. As $\operatorname{codim}_Y Y_1 = s$

the numbers of irreducible components of Y_1 and $Y_1 \setminus E_1$ are the same. Thus Y_1 is irreducible. □

Proposition 2.9.

(1) *Let A be Cohen-Macaulay and $A \hookrightarrow A'$ be semi-basic. Then this modification is Davis.*

(2) *Suppose that $A \hookrightarrow A'$ is a Davis modification. Let A be Cohen-Macaulay. Then A' is also Cohen-Macaulay.*

Proof. (1) Let M be a maximal ideal in A. Then A_M is also Cohen-Macaulay [**Ma**, Th. 30]. In the local ring A_M every semi-regular sequence is regular [**Ma**, Th. 31]. Thus the modification $A_M \hookrightarrow A_M[I_M/f] = S^{-1}A'$ (where $S = A \setminus M$) is Davis by Theorem 2.1. Hence $A \hookrightarrow A'$ is Davis by Proposition 2.8. For (2) consider $L_1, \ldots, L_s \in A^{[s]}$ as in the proof of Proposition 2.8. Since A is Cohen-Macaulay $A^{[s]}$ is Cohen-Macaulay as well [**Ei**, Prop. 18.9]. The ideal I' generated by $L_1, \ldots, L_s$ has height s. Hence $A' \simeq A^{[s]}/I'$ is Cohen-Macaulay by [**Ei**, Prop. 18.13]. □

2.5. Basic modifications.

Remark 2.3. Let $A \hookrightarrow A'$ be a basic modification and $b_0 = f, \ldots, b_s$ be a system of generators of I which is an almost complete intersection. Note $b_0, \ldots, b_s$ may be viewed as elements of a local holomorphic coordinate system at a general point x of C. That is, in a neighborhood of x the basic modification is nothing but a usual (affine) monoidal transformation. This implies that every point $y \in \sigma^{-1}(x)$ is a smooth point of X' and the zero multiplicity of $f \circ \sigma$ at y is 1.

Remark 2.3 and [**KaZa1**, Th. 3.1 and Prop. 3.1] imply the following fact.

Proposition 2.10. *Let $A \hookrightarrow A'$ be a basic modification, C and, therefore, E be irreducible topological manifolds, and the natural embedding of C into D generate an isomorphism of the homology of C and D. Then σ generates isomorphisms of the fundamental groups and the homology groups of X and X'.*

Proposition 2.11. *Let $A \hookrightarrow A'$ be a basic modification. Suppose that A is normal and Cohen-Macaulay. Then A' is normal and Cohen-Macaulay.*

Proof. By Proposition 2.9 this modification is Davis and A' is Cohen-Macaulay. Note that if the singularities of X' is at least of codimension 2 then X' is normal by [**Ha**, Ch. 2, Prop. 8.23]. Since X is normal the codimension of $\sigma^{-1}(\operatorname{sing} X \setminus D) = \sigma^{-1}(\operatorname{sing} X \setminus C)$ in X' is at least 2 whence we can ignore this subvariety. Let C^0 be the subset of the reduced center C, at the points of which the gradients of a system of generators of I (which are an almost a complete intersection) are linearly independent. The codimension of $C \setminus C^0$

in C is at least 1. Since σ is cylindrical the codimension of $\sigma^{-1}(C \setminus C^0)$ in E is at least 1, and in X' it is at least 2, and we can ignore these points again. The other points of X' are smooth by Remark 2.3. □

2.6. Preliminary decomposition. We shall fix first notation for this subsection.

Convention 2.2.

(1) In this subsection A is normal Cohen-Macaulay. When we speak about the modification $A \hookrightarrow A'$ then $I = I_f$, $f = g^n$ where $g \in A$ generates $\mathcal{I}_A(D)$, E is nonempty irreducible and $\mathcal{I}_{A'}(E)$ is also generated by g.

(2) We consider affine domains $A_i = \mathbf{C}[X_i]$, $i \geq 0$ such that $A \hookrightarrow A_i \hookrightarrow A'$. These embeddings generate morphisms $\delta_i : X_i \to X$ and $\rho_i : X' \to X_i$ where $\sigma = \delta_i \circ \rho_i$. It is easy to see that there exist ideals I_i in A and K_i in A_i such that the locus of $A \hookrightarrow A_i$ is (I_i, g^{n_i}) and the locus of $A_i \hookrightarrow A'$ is (K_i, f). Hence the exceptional divisor E_i of δ_i coincides with the divisor D_i of ρ_i.

(3) We suppose that K_i is the f-largest ideal of ρ_i whence by Proposition 2.2 the closure of the geometrical center C_i of ρ_i coincides with its reduced center $\overline{C}_i$.

Lemma 2.2. *Let $A_1 \hookrightarrow A'$ be an affine modification as in Convention 2.2. Suppose that A_1 is normal and the closure of $C_1 = \rho_1(E)$ in X_1 is an irreducible component D_1^1 of D_1. Let E_0 be the Zariski open subset of E that consists of all points $x' \in E$ such that x' is a connected component of $\rho_1^{-1}(\rho_1(x'))$. Put $D_0 = \rho_1(E_0)$ and let D_1^2 be the union of irreducible components of D_1 different from D_1^1. Then*

(i) *$D_0 = D_1^1 \setminus D_1^2$ and $E_0 = \rho_1^{-1}(D_0)$;*

(ii) *the restriction of ρ_1 to $(X' \setminus E) \cup E^0$ is an isomorphism between this variety and $(X_1 \setminus D_1) \cup D_0$;*

(iii) *in particular, if $E = E_0$ (this is so, for instance, when D_1^2 does not meet D_1^1) then ρ_1 is an embedding, and if $D_1 = D_1^1$ then ρ_1 is an isomorphism.*

Proof. Put $x_1 = \rho_1(x')$ for $x' \in E_0$. As X_1 is normal x_1 cannot be a fundamental point of the birational map ρ_1^{-1} by the Zariski Main Theorem [**Ha**, Ch. 5, Th. 5.2]. That is, ρ_1^{-1} an embedding in a neighborhood of x_1 which proves (ii). Put $X_1^0 = (X_1 \setminus D_1) \cup (D_1^1 \setminus D_1^2)$. The complement to $(X_1 \setminus D_1) \cup D_0$ in X_1^0 is a constructive subset of codimension at least 2. Since X' is affine and X_1^0 is normal we can extend morphism ρ_1^{-1} to a morphism from X_1^0 to X' [**Dan**, Sect. 7.1]. This implies that $\rho_1^{-1}|_{X_1^0} : X_1^0 \to X'$ is an embedding whence $D_0 \supset D_1^1 \setminus D_1^2$. Assume that $x' \in E_0$ and $x_1 = \rho_1(x') \in D_1^1 \cap D_1^2$. As ρ_1^{-1} is an embedding in a neighborhood of x_1 we see that the exceptional divisor of ρ_1 must contain a component different from E. This contradiction yields (i). The last statement follows immediately from (i) and (ii). □

Definition 2.7. Let $A \hookrightarrow A'$ be an affine modification, $A \hookrightarrow A_1$ be a basic modification such that $A_1 \subset A'$, $h \in A \backslash \{0\}$ and $S = \{h^n \mid n \in \mathbf{N}\}$. Suppose that $(h \circ \sigma)^{-1}(0)$ does not contain E and $S^{-1}A_1 = S^{-1}A'$. Then we call $A \hookrightarrow A'$ a pseudo-basic modification (relative to $A \hookrightarrow A_1$).

Note that if the assumption of Lemma 2.2 holds and $A \hookrightarrow A_1$ is basic then it follows from this Lemma that $A \hookrightarrow A'$ is pseudo-basic.

Proposition 2.12. *Let Convention 2.2 hold, C be not contained in* $\mathrm{sing}\, X$*, $\mathrm{codim}_X C \geq 2$, and the zero multiplicity of g at general points of C is* 1*. Let*

$$A = A_0 \hookrightarrow \cdots \hookrightarrow A_{k-1} \hookrightarrow A_k,\ k \geq 0$$

be a strictly increasing sequence of affine domains such that $A_k \subset A'$, and $\forall i \leq k$

(i) *affine modification $A_{i-1} \hookrightarrow A_i$ is basic with locus (J_i, g) and of rank s_{i-1} where $s_{i-1} + 1 = \mathrm{codim}_{X_{i-1}} C_{i-1}$.*

(1) *Then $k \leq n$ (recall that $f = g^n$) and this sequence can be extended to a strictly increasing sequence of affine domains*

$$A_0 \hookrightarrow \cdots \hookrightarrow A_{m-1} \hookrightarrow A_m,\ k \leq m \leq n$$

so that (i) holds $\forall i \leq m$ and $A_{m-1} \hookrightarrow A'$ is pseudo-basic relative to $A_{m-1} \hookrightarrow A_m$.

(2) *Suppose that $\sigma_i : X_i \to X_{i-1}$ is the morphism associated with the affine modification $A_{i-1} \hookrightarrow A_i$. Then $\sigma_i(C_i) = C_{i-1}$ for $i \leq m-1$, and $\rho_{m-1}(E) = C_{m-1}$.*

(3) *Let the closure E_m^1 of $\rho_m(E)$ be a connected component of E_m (resp. E_m be irreducible). Then ρ_{m-1} is a locally basic (resp. basic) modification.*

Proof. Let us show (2) first. By Convention 2.2 the exceptional divisor of ρ_i is E whence $\rho_i(E) = C_i$. As $\sigma = \rho_0$ we have $\sigma(E) = C_0$. As $\sigma = \rho_i \circ \delta_i$ we see that $\delta_i(C_i) = C_0$. Hence $\sigma_i(C_i) = C_{i-1}$ since $\delta_i = \delta_{i-1} \circ \sigma_i$. For (1) note first that as A is normal Cohen-Macaulay so is each A_i by Proposition 2.11. Hence if $s_k = 0$ then we put $m = k$ and (1) follows from Lemma 2.2. Let $s_k > 0$. By Remark 2.3 $\sigma_1^{-1}(x) \subset \mathrm{reg}\, X_1$ for a general point $x \in C_0$. As $\sigma_1(C_1) = C_0$ we see that $\sigma_1^{-1}(x)$ contains general points of C_1 whence C_1 meets $\mathrm{reg}\, X_1$, and the zero multiplicity of $g \circ \delta_1$ at general points of C_1 is 1 by Remark 2.3. By induction the similar facts are true for C_k and $g \circ \delta_k$. By Lemma 2.1 and Proposition 2.3 we can choose a basic modification $A_k \hookrightarrow A_{k+1}$ with locus (J_{k+1}, g) such that its rank is s_k and $A_{k+1} \subset A'$. Thus we can extend our strictly increasing sequence of affine domains and we can always suppose that $k \geq 1$ in (1). There are two possibilities: Either this sequence is infinite or there exists m such that

$s_m = 0$ which implies (1). Show by induction that the first possibility does not hold and that $m \le n$. Assume first that (1) holds for $n-1>0$. Let $b_0 = g, b_1, \dots, b_s$ be a system of generators of J_1 which is an almost complete intersection. By Definition 2.3 there exists $h \in A$ such that $h^{-1}(0)$ does not contain C, $X \setminus h^{-1}(0)$ is smooth, $C \setminus h^{-1}(0)$ is a complete intersection in $X \setminus h^{-1}(0)$ given by $b_0 = \cdots = b_s = 0$. If $S = \{h^j \,|\, j \in \mathbf{N}\}$ then the affine modification $S^{-1}A \hookrightarrow S^{-1}A'$ satisfies the analogue of assumption of this proposition and, furthermore, $S^{-1}J_1$ is the defining ideal of $C \setminus h^{-1}(0)$ in $S^{-1}A$. By Proposition 2.1 the locus of $S^{-1}A \hookrightarrow S^{-1}A_1$ is $(S^{-1}J_1, g)$, and by Corollary 2.1 the locus of $S^{-1}A_1 \hookrightarrow S^{-1}A'$ can be chosen in the form (L_1, g^{n-1}). By the induction assumption the codimension of the reduced center of the modification $S^{-1}A_m \hookrightarrow S^{-1}A'$ is 1 for some $m \le n$. Hence the same is true for the reduced center of $A_m \hookrightarrow A'$, i.e., $s_m = 0$ which concludes this step of induction. The next step is for $n = 1$. By Proposition 2.3 in this case $S^{-1}J_1$ coincides with the g-largest ideal of the affine modification $S^{-1}A \hookrightarrow S^{-1}A'$. Hence $S^{-1}A_1 = S^{-1}A'$. As h is chosen so that $h^{-1}(0)$ does not contain C this implies (1) which concludes induction. Claim (3) is now a consequence of Lemma 2.2. □

Let C^*_{m-1} be the complement in C_{m-1} to the intersection of C_{m-1} with the other components of the reduced center of σ_m. Then the exceptional divisor of σ_m contains $E^*_m \simeq \mathbf{C}^{s_{m-1}} \times C^*_{m-1}$. By Lemma 2.2 under the assumption of Proposition 2.12 the restriction of ρ_m^{-1} to $(X_m \setminus E_m) \cup E^*_m$ is an embedding. Hence:

Corollary 2.2 (cf. [**Miy2**, Lemma 2.3])**.** *Under the assumption of Proposition 2.12 the exceptional divisor E contains a Zariski open cylinder $E^*_m \simeq \mathbf{C}^{s_{m-1}} \times C^*_{m-1}$ such that $\rho_{m-1}|_{E^*_m}$ is the projection to the second factor.*

3. The geometry of the exceptional divisor and the reduced center.

3.1. The exceptional divisor. In this section we shall strengthen Proposition 2.12 in the case when X is a threefold. Our main aim is to make $A_m = A'$.

Lemma 3.1. *Let X' be an affine threefold with $H_3(X') = 0$ and E be a closed irreducible surface in X' which admits a surjective morphism $\tau : E \to C_{m-1}$ into an irreducible curve C_{m-1}* [2] *such that for a Zariski open subset $C^*_{m-1} \subset C_{m-1}$ and $E^* = \tau^{-1}(C^*_{m-1})$ the morphism $\tau|_{E^*} : E^* \to C^*_{m-1}$ is a $\mathbf{C}$-cylinder and $L := E \setminus E^*$ is a curve. Let $H_2(X' \setminus E) = H_3(X' \setminus E) = 0$. Let $z \in C_{m-1} \setminus C^*_{m-1}$ and $\mathcal{C}^z$ be the punctured germ of C_{m-1} at z. Put $L^z = \tau^{-1}(z)$. Then there exists an isomorphism $H_0(\operatorname{reg} L) \simeq H_1(E^*)$ such*

[2] We denote this curve by C_{m-1} since it will play later the role of the geometrical center of the modification ρ_{m-1} from Proposition 2.12 and τ will be $\rho_{m-1}|_E$.

that for every germ $\mathcal{C}^z$ as above the restriction of this isomorphism generates an isomorphism $H_0(\operatorname{reg} L^z) \simeq H_1(\mathcal{C}^z)$ (i.e., the number of irreducible components of L^z is the same as the number of connected components of $\mathcal{C}^z$). Furthermore, the normalization of C_{m-1} is $\mathbf{C}$.

Proof. Consider the following exact homology sequences of pairs:

$$\longrightarrow H_{j+1}(X') \to H_{j+1}(X', X' \setminus L) \to H_j(X' \setminus L) \\ \to H_j(X') \to H_j(X', X' \setminus L) \to$$

and

$$\ldots \longrightarrow H_j(X' \setminus E) \to H_j(X' \setminus L) \to H_j(X' \setminus L, X' \setminus E) \to H_{j-1}(X' \setminus E) \to .$$

Note that $H_4(X') = 0$ since X' is an affine algebraic variety [**Mil**, Th. 7.1]. Replace X' (resp. E, resp. L) with the complement to $\operatorname{sing} L$ in X' (resp. E, resp. L). Though X' is no more affine, this replacement does not affect $H_3(X'), H_4(X')$, and $H_i(X \setminus E)$, and the advantage is that L is smooth now. From the above sequences and Thom's isomorphisms (e.g., see [**Do**, Ch. 8, 11.21]) we have

$$H_0(L) \simeq H_4(X', X' \setminus L) \simeq H_3(X' \setminus L) \simeq H_3(X' \setminus L, X' \setminus E) \simeq H_1(E^*).$$

As $H_1(C_{m-1}^*) \simeq H_1(E^*)$ we have an isomorphism $H_0(L) \simeq H_1(C_{m-1}^*)$. Let L_i be an irreducible component of L (which is now a connected smooth component of L), and V be a tubular neighborhood of L_i in X'. Consider the germ $\mathcal{S}_i'$ of a smooth complex surface whose image under a natural retraction $V \to L_i$ is a point $z' \in L_i$, i.e., $\mathcal{S}_i$ is transversal to L_i at z'. We can suppose that $\mathcal{S}_i'$ is diffeomorphic to a ball and its boundary $\partial \mathcal{S}_i'$ in X' is diffeomorphic to a three-sphere which meets E^* transversally along a smooth real curve γ_i. Let $[\mathcal{S}_i] \in H_4(V, V \setminus L_i)$ be generated by $\mathcal{S}_i$. Thom's isomorphism $H_4(V, V \setminus L_i) \to H_0(L_i)$ sends $[\mathcal{S}_i]$ to the positive generator of $H_0(L_i)$ [**FoFu**, Ch. 4, Sect. 30, p. 262]. As Thom's isomorphisms are functorial under open embeddings [**Do**, Ch. 8, 11.5], isomorphism $H_4(X', X' \setminus L) \to H_0(L)$ sends $[\mathcal{S}_i]$ to the positive generator of $H_0(L_i)$. Isomorphism $H_4(X', X' \setminus L) \simeq H_3(X' \setminus L)$ sends $[\mathcal{S}_i']$ to the element $[\partial \mathcal{S}_i'] \in H_3(X' \setminus L)$ generated by $\partial \mathcal{S}_i'$. Let T_i be a small tubular neighborhood of γ_i in $\partial \mathcal{S}_i'$. By the excision theorem T_i generates an element $[T_i]$ of $H_3(X' \setminus L, X' \setminus E)$ which coincides with $[\partial \mathcal{S}_i']$. Thus the constructed isomorphism $H_0(L) \simeq H_3(X' \setminus L, X' \setminus E)$ sends the positive generator of $H_0(L_i)$ to $[T_i]$. The same argument as above implies that under isomorphism $H_3(X' \setminus L, X' \setminus E) \to H_1(E^*)$ the cycle $[T_i]$ goes to the element $[\gamma_i] \in H_1(E^*) \simeq H_1(C_{m-1}^*)$ generated by γ_i. Note that $H_1(C_{m-1}^*) = \oplus_{z \in C_{m-1} \setminus C_{m-1}^*} H_1(\mathcal{C}^z) \oplus N$ where the group N is not trivial provided that either C_{m-1} is of positive genus or C_{m-1} has more than one puncture. As $\tau(z') = z$ and γ_i is contained in a small neighborhood of z', the image of the generator of $H_0(L_i)$ under isomorphism $H_0(L) \simeq H_1(C_{m-1}^*)$ is contained in $H_1(\mathcal{C}^z)$. Hence the image of $H_0(L)$ is contained in $\oplus_{z\in C_{m-1}\setminus C^*_{m-1}} H_1(\mathcal{C}^z)$. Thus N is trivial and $H_0(L^z) \simeq H_1(\mathcal{C}^z)$ which is the desired conclusion. $\square$

Remark 3.1. If E is a UFD then there is no need to assume in Lemma 3.1 that X' is smooth and $H_3(X') = H_2(X' \setminus E) = H_3(X' \setminus E) = 0$. One can show that the fibers of τ are irreducible whence the Euler characteristics $e(E) = e(C_{m-1}) \leq 1$. Thus in order to make C_{m-1} contractible one need $e(E) = 1$.

The proof of Lemma 3.1 implies more. Fix $z \in C_{m-1} \setminus C^*_{m-1}$. Let $\mathcal{C}^j$, $j = 1, \ldots, k$ be the irreducible components of $\mathcal{C}^z$, i.e., $\mathcal{C}^j$ corresponds to a generator α_j of $H_1(\mathcal{C}^z)$. Each irreducible component L_i of $\tau^{-1}(z)$ corresponds to a generator β_i of $H_0(L^z)$. By Lemma 3.1 the image of β_i under isomorphism $H_0(L^z) \simeq H_1(\mathcal{C}^z)$ is $\sum_j m_i^j \alpha_j$. One can extract from Lemma 3.1 a way to compute these coefficients m_i^j.

Lemma 3.2. *Let the notation above hold and $\tau = \rho_{m-1}|_E$ where $\rho_{m-1} : X' \to X_{m-1}$ is the same as in Proposition 2.12. Let $\mathcal{S}_i'$ be the Euclidean germ of a smooth algebraic surface transversal to L_i at a smooth point $z' \in L_i$, and $\mathcal{S}_i = \rho_{m-1}(\mathcal{S}_i')$. Then for every point $x \in \mathcal{C}^j$ the germ of $\mathcal{S}_i$ at x consists of m_i^j smooth branches which meet the divisor D_{m-1} of modification ρ_{m-1} transversally along the germ of $\mathcal{C}^j$ at x.*

Proof. Suppose that $\mathcal{S}_i'$ meets $\tau^{-1}(\mathcal{C}^j)$ along the germ Γ_i^j of a curve. As $\mathcal{S}_i'$ is transversal to L_i, it is transversal to E at every $x' \in \Gamma_i^j$. Let B' be the germ of $\mathcal{S}_i'$ at x', $x = \rho_{m-1}(x')$, and $B = \rho_{m-1}(B')$. As ρ_{m-1} is pseudo-basic it is basic in a neighborhood of x', i.e., it can be viewed as a monoidal transformation at x, by Remark 2.3. Hence B is smooth and transversal to D_{m-1} at x. It remains to show that the number of such branches B is m_i^j which is equivalent to the fact that the mapping $\tau|_{\Gamma_i^j} : \Gamma_i^j \to \mathcal{C}^j$ is m_i^j-sheeted. One can suppose that $\mathcal{S}_i'$ is the same as in the proof of Lemma 3.1. The boundary of Γ_i^j may be viewed as a smooth real curve γ_i^j and $\bigcup_j \gamma_i^j = \gamma_i$ where $\gamma_i = \partial\mathcal{S}_i' \cap E$. Let $\mathcal{E}^z = \tau^{-1}(\mathcal{C}^z)$. It was shown in the proof of Lemma 3.1 that the image of β_i under the isomorphism $H_0(L^z) \simeq H_1(\mathcal{E}^z)$ is $[\gamma_i] = \sum_j [\gamma_i^j]$ where $[\gamma_i^j]$ is the cycle in $H_1(\mathcal{E}^z)$ generated by γ_i^j. Then the image of $[\gamma_i^j]$ under the isomorphism $H_1(\mathcal{E}^z) \simeq H_1(\mathcal{C}^z)$ coincides with $m_i^j \alpha_j$ where m_i^j is the winding number of $\tau(\gamma_i^j)$ in $\mathcal{C}^j$ around z. As γ_i^j is the boundary of the punctured disc Γ_i^j this implies that $\tau|_{\Gamma_i^j} : \Gamma_i^j \to \mathcal{C}^j$ is m_i^j-sheeted. $\square$

3.2. The reduced center. We shall describe some condition under which the reduced center of an affine modification is a complete intersection.

Lemma 3.3. *Let C be an affine reduced irreducible curve, v be a coordinate on the first factor of $D_1 = \mathbf{C} \times C$, and $\theta : D_1 \to C$ be the natural projection. Let o be a singular point of C, $\mathcal{V}$ be the germ of C at o, $\mathcal{D}_1 = \theta^{-1}(\mathcal{V})$, and $F_{\mathcal{V}}$ (resp. $\mathcal{O}_{\mathcal{V}}$) be the ring of complex-valued (resp. holomorphic) functions on $\mathcal{V}$.*

(1) *Suppose that a function $h \in F_{\mathcal{V}}[v]$ is holomorphic everywhere in $\mathcal{D}_1$ except for a finite number of points. Then h is holomorphic in $\mathcal{D}_1$.*
(2) *Let $h \in F_{\mathcal{V}}[v]$ be holomorphic in $\mathcal{D}_1$, $h^{-1}(0)$ not contain $\theta^{-1}(o)$, and the zero multiplicity of h at general points of $h^{-1}(0)$ be n. Then $h^{1/n}$ is holomorphic.*
(3) *Let C_1 be a reduced irreducible curve in D_1 so that projection $\theta|_{C_1} : C_1 \to C$ is finite and for each singular point $o \in C$ there exist $\mathcal{V}$ and $\mathcal{D}_1$ as in (1) for which the defining ideal of $C_1 \cap \mathcal{D}_1$ in $\mathcal{O}_{\mathcal{V}}[v]$ is principal. Then the defining ideal of C_1 in $\mathbf{C}[D_1]$ is generated by a function $b \in \mathbf{C}[D_1]$ which is a monic polynomial in v.*

Proof. The argument is of local analytic nature whence it is enough to consider the case when C is contractible. Let $\nu_0 : \mathbf{C} \simeq C^\nu \to C$ be a normalization, t be a coordinate on C^ν, and $\nu = (\nu_0, \mathrm{id}) : \mathbf{C}^2 \simeq \mathbf{C} \times C^\nu \to D_1$, i.e., (v,t) is a coordinate system on $\mathbf{C}^2$. Note that $\gamma = h \circ \nu$ is of form $r_k(t)v^k + r_{k-1}(t)v^{k-1} + \cdots + r_0(t)$. One can suppose that o is the origin of $\mathbf{C}^n \supset C$. As h is holomorphic everywhere on $\mathcal{D}_1$ except for a finite number of points implies that for every fixed $v = v_0$ except for a finite number of values, $\gamma(v_0, t)$ is contained in the ring of convergent power series of the coordinate functions $x_1(t), \dots, x_n(t)$ of ν_0. Hence each $r_i(t)$ belongs to this ring whence h is holomorphic in $\mathcal{D}_1$ which is (1). For (2) it suffices to note that the function $h^{1/n}$ is holomorphic everywhere in $\mathcal{D}_1$ except for possibly points from the finite set $h^{-1}(0) \cap \theta^{-1}(o)$. Let $C_1^\nu = \nu^{-1}(C_1)$ be the zero fiber of an irreducible polynomial $\beta(v,t)$ on $\mathbf{C}^2$. The projection of C_1^ν to the t-axis is finite as $\theta|_{C_1}$ is finite whence $\beta(v,t)$ is monic in v. The function $b = \beta \circ \nu^{-1}$ is rational on D_1, and in order to show that it is regular, it suffices to show that b is holomorphic at each point of D_1 (e.g., see [**Ka1**]). Let o be a singular point of C. It is enough to check that b is holomorphic at the points of $\theta^{-1}(o)$. Let O be the ring of germs of analytic functions at $o \in \mathbf{C}^n$ and h be the generator of the defining ideal of $C_1 \cap \mathcal{D}_1$ in $\mathcal{O}_{\mathcal{V}}[v]$. By Cartan's theorems (e.g., see [**GuRo**, Ch. 8A, Th. 18]) we can extend each coefficient of h to a holomorphic function in a neighborhood of $o \in \mathbf{C}^n$ whence we can treat h as an element of $O[v]$. Applying the Weierstrass Preparation Theorem [**Rem**, Ch. 1, Th. 1.4] one can show that $h = \omega e$ where $\omega \in O[v]$ is a monic polynomial and $e \in O[v]$ is invertible on $\mathcal{D}_1$. Thus $\omega|_{\mathcal{D}_1}$ generates the same ideal as h whence we can suppose that h is a monic polynomial in v. Thus $\gamma = h \circ \nu$ is monic as a polynomial in v over the ring of germs of analytic functions at $\nu_0^{-1}(o) \subset \mathbf{C}$. Note that $\gamma = \beta\alpha$

where α does not vanish since the zero multiplicity of γ and β at general points C_1^ν is 1. Hence α is is constant on each line parallel to the v-axis. This constant is 1 since both γ and β are monic (look at the quotient γ/β as v approaches ∞ along any of these lines). Thus $\beta = \gamma$ whence b coincides with h in $\mathcal{D}_1$ and, therefore, b is holomorphic. □

Definition 3.1. We say that X is a locally analytic UFD of for every $x \in X$ the ring of germs of holomorphic functions on X at x is a UFD.

Proposition 3.1. *Let the assumptions of Convention 2.2 and Proposition 2.12 hold. Suppose that* $\dim X = 3$, $m \geq 2$ *where* m *is from Proposition 2.12,* X' *is smooth, and* $H_3(X') = H_2(X' \setminus E) = H_3(X' \setminus E) = 0$. *Suppose that for* $i = 1, \dots, r$ *the divisor* D_i *is naturally isomorphic to* $\mathbf{C} \times C_{i-1}$ *(i.e.,* C_{i-1} *is a curve), the natural projection* $\sigma_i|_{C_i} : C_i \to C_{i-1}$ *is finite,* $D_0\,(= D)$ *is smooth. Let* X *be a locally analytic UFD. Then the defining ideal of* C_r *in* $\mathbf{C}[D_r]$ *is principal.*

Proof. Let z, L, L^z be as in Lemma 3.1 and let L_i be an irreducible component of L^z. Suppose that for each $z \in C_{m-1} \setminus C_{m-1}^*$ the objects $\mathcal{S}_i', \mathcal{S}_i, \mathcal{C}^j, m_i^j$ are the same as in Lemma 3.2. As these objects depend on z, the notation, say, $m_i^j(z)$ has an obvious meaning. By Lemma 3.1 the matrix $(m_i^j(z))$ is invertible whence there exists a vector $\bar{v}(z)$ with integer entries $v^i(z)$ such that each entry of the vector $(m_i^j(z))\bar{v}(z)$ is equal to 1. Consider the Euclidean germ $\mathcal{T}' = \sum_{z \in C_{m-1} \setminus C_{m-1}^*} \sum_{L_i \subset L^z} v^i(z)\mathcal{S}_i'(z)$ of a divisor and its strict transforms $\mathcal{T}_l = \rho_l(\mathcal{T}')$. By Lemma 3.2 the germ of $\mathcal{T}_{m-1}$ at any point of $\mathcal{C}^j$ consists of smooth branches transversal to D_{m-1} and the sum of multiplicities of these branches is 1. For $i \leq m-1$ put $\gamma_i = \rho_i \circ \rho_{m-1}^{-1} = \sigma_{i+1} \circ \dots \circ \sigma_{m-1}$ and $z_i = \gamma_i(z)$. Let $\mathcal{C}_i$ be the germ of C_i at z_i and morphism $\gamma_i|_{C_{m-1}} : C_{m-1} \to C_i$ be k_i-sheeted. As γ_i is a composition of basic modifications (which can be viewed locally as monoidal transformations by Remark 2.3) the germ of $\mathcal{T}_i$ at a general point of $\mathcal{C}_i$ consists of smooth branches which meet D_i transversally and the sum of multiplicities of these branches is k_i. Let V_0 be the Euclidean germ of $X = X_0$ at z_0 and let $S^0 \subset V_0$ be an irreducible component of $\mathcal{T}_0$ (i.e., $S_0 = \rho_0(S_i')$ for some i). One can check that $S^0 \setminus D$ is closed in $V_0 \setminus D$ (i.e., $S^0 \setminus D$ is an analytic hypersurface in $V_0 \setminus D$ as the restriction of ρ_0 to $X' \setminus E$ is an embedding) and the closure $\bar{S}^0$ of $S^0 \setminus D$ in V_0 is S_0. Hence $\bar{S}^0 \cap D \subset C$ (in particular, the intersection of $\mathcal{T}_0$ and $D_0 \cap V_0$ is $\mathcal{C}_0$) and by [**BeNa**, Th. 1.2] S^0 is an analytic hypersurface in V_0. As X is locally analytic UFD, in V_0 the divisor $\mathcal{T}_0$ coincides with the divisor of a meromorphic function h_0 on V_0. By the theorem about deleting singularities the restriction of h_0 to $V_0 \cap D_0$ is a holomorphic function whose divisor is $k_0\mathcal{C}_0$. Put $V_1 = \sigma_1^{-1}(V_0)$ and $\mathcal{D}_1 = \sigma_1^{-1}(\mathcal{C}_0)$. As σ_1 is basic $e_1 = h_0 \circ \sigma_1$ is a meromorphic function on V_1 whose divisor is $\mathcal{T}_1 + k_0\mathcal{D}_1$.

By Remark 2.3 the divisor of $g_1 = g \circ \sigma_1$ on V_1 is $\mathcal{D}_1$ whence $\mathcal{T}_1$ is the divisor of the meromorphic function $h_1 := e/g_1^{k_0}$. As for general $x_1 \in \mathcal{C}_1$ each branch of $\mathcal{T}_1$ at x_1 meets $\mathcal{D}_1$ transversally and $\mathcal{T}_1 \cap D_1 = \mathcal{C}_1$, Lemma 3.3 (1) implies that $h_1|_{\mathcal{D}_1}$ is a holomorphic function with divisor $k_1\mathcal{C}_1$. Put $e_2 = h_1 \circ \sigma_2, V_2 = \sigma_2^{-1}(V_1) = \delta_2^{-1}(V_0)$, and $\mathcal{D}_2 = \sigma_2^{-1}(\mathcal{C}_1) = \delta_2^{-1}(\mathcal{C}_0)$ (where δ_i is as in Convention 2.2 (2)). Repeating the procedure we get the germ of a meromorphic function h_2 whose divisor is $\mathcal{T}_2$. Induction yields a meromorphic function h_i on $V_i = \delta_i^{-1}(V_0)$ whose divisor is $\mathcal{T}_i$. Hence $\forall i \leq r$ the restriction of h_i to $\mathcal{D}_i = \delta_i^{-1}(\mathcal{C}_0)$ is a holomorphic function whose divisor is $k_i\mathcal{C}_i$. By Lemma 3.3 the defining ideal of $\mathcal{C}_i$ is principal, and, therefore, the defining ideal of C_i in $\mathbf{C}[D_i]$ is also principal. $\square$

3.3. Decomposition.

Lemma 3.4. *Let Convention 2.2 hold, $A_1 \hookrightarrow A'$ be a basic modification, $D_1 \simeq \mathbf{C} \times C$ where C is a curve, z be an irreducible singular point of C, and C_1 meet $\mathbf{C} \times z \subset D_1$ at $z_1 = 0 \times z$ but $C_1 \neq \mathbf{C} \times z$. Then z_1 is a singular point of C_1.*

Proof. Assume the contrary, i.e., C_1 is smooth at z_1. As the situation is local we can suppose that C is a closed curve in $\mathbf{C}^n$. Consider a normalization $\nu_0 : C^\nu \to C$ and $\nu = (\mathrm{id}, \nu_0) : \mathbf{C} \times C^\nu \to \mathbf{C} \times C \subset \mathbf{C}^{n+1}$. Let $(y, \bar{x}) = (y, x_1, \ldots, x_n)$ be a coordinate system in $\mathbf{C}^{n+1}$, and g, b_1 be an almost complete intersection in A_1 which generates modification $A_1 \hookrightarrow A'$, i.e., b_1 generates the defining ideal of C_1 in $\mathbf{C}[D_1]$. We treat b_1 as a polynomial $b_1(y, \bar{x})$ on $\mathbf{C}^{n+1}$. Let $\beta = b_1|_{D_1} \circ \nu$, C_1^ν be the proper transform of C_1 (i.e., $C_1^\nu = \beta^{-1}(0)$), and let $o = \nu^{-1}(z_1)$. As C_1 is smooth and $\nu|_{C_1^\nu} : C_1^\nu \to C_1$ is a homeomorphism, C_1^ν is biholomorphic to C_1 by [**Pe**, Cor. 1.5] whence C_1^ν is smooth at o. As $A_1 \hookrightarrow A'$ is basic the gradient of β does not vanish at general points of C_1^ν and also at o as C_1^ν is smooth at o. Let (v, t) be a local coordinate system at o where t is a coordinate on the second factor of $\mathbf{C} \times C^\nu$ and v is a coordinate on the first one. The Taylor series of $\beta(v, t) = b_1(v, \bar{x}(t))$ at o does not have a nonzero linear term ct since z is a singular point of C. The linear part of this power series must be nonzero (otherwise the gradient of β at o is zero). Thus the Taylor series of b_1 at z_1 has a nonzero linear term cv. The implicit function theorem implies that the germ of C_1 at z_1 is biholomorphic to the germ of C at z, i.e., C_1 is singular at z_1. Contradiction. $\square$

Theorem 3.1. *Let Convention 2.2 (1) hold for an affine modification $A \hookrightarrow A'$, X' be a threefold, and $H_3(X') = H_2(X' \setminus E) = H_3(X' \setminus E) = 0$. Let*

(i) *D be isomorphic to $\mathbf{C}^2$;*
(ii) *X be a locally analytic UFD, and*
(iii) *C be not contained in the singularities of X.*

Let m, A_i, C_i, and $\bar{C}_i$ be the same as in Proposition 2.12 [3] *and Convention 2.2 (3).*

(1) *Then the algebras A_i's can be chosen so that $A_m = A'$, $C_i = \bar{C}_i$ for every i, and if C_i is a curve its defining ideal in $\mathbf{C}[D_i]$ is principal.*

(2) *Each C_i is either a point or an irreducible contractible curve, and in the case when E has at most isolated singularities these curves are smooth contractible.*

Proof. We use induction on m. Suppose first that C_0 is a point. Assumption (i) allows us to choose $b_1, b_2 \in A$ such that g, b_1, b_2 generate the defining ideal I_1 of C_0 in A. Hence the exceptional divisor $E_1 = D_1$ of modification $A \hookrightarrow A_1 = A[I_1/g]$ with locus (I_1, g) is isomorphic to $\mathbf{C}^2$. By Proposition 2.3 and Convention 2.2 (1) $A_1 \subset A'$. If $m = 1$ in Proposition 2.12 then Lemma 2.2 implies that ρ_1 is an isomorphism, i.e., $A_1 = A'$. Let $m \geq 2$. Note that $E_1 \simeq \mathbf{C}^2$ and it is the divisor of the modification $A_1 \hookrightarrow A'$ from Convention 2.2. By (iii) C_0 is a smooth point of X whence $\sigma_1^{-1}(C_0)$ is contained in the smooth part of X_1 by Remark 2.3, i.e., X_1 is a locally analytic UFD and normal Cohen-Macaulay. Thus the assumptions of this Theorem hold also for the modification $A_1 \hookrightarrow A'$. The decomposition of $A_1 \hookrightarrow A'$ contains $m-1$ factors and induction implies the desired conclusion in this case. Let C_0 be a curve. As there is a surjective morphism $C_{m-1} \to C_0$ the normalization of C_0 is $\mathbf{C}$ by Lemma 3.1. This implies that C_0 is closed in X, i.e., $C = C_0$. The defining ideal of $C_0 \subset D \simeq \mathbf{C}^2$ is generated by $\bar{b} \in \mathbf{C}[D]$ whence the defining ideal I_1 of C in A is generated by g and b where b is an extension of $\bar{b}$ to X. Thus the exceptional divisor $E_1 \simeq \mathbf{C} \times C_0$ of $A \hookrightarrow A_1 = A[I_1/g]$ is irreducible and by Proposition 2.11 A_1 is normal Cohen-Macaulay. By Proposition 3.1 the defining ideal of C_1 in $\mathbf{C}[D_1]$ (recall $D_1 = E_1$ by Convention 2.2) is principal. Let $\bar{b}_1$ be its generator and b_1 be an extension of $\bar{b}_1$ to X_1. The defining ideal I_2 of C_1 in A_1 is generated by g and b_1. Thus the exceptional divisor E_2 of $A \hookrightarrow A_2 = A[I_2/g]$ is again irreducible. Repeating this procedure we can construct basic modifications σ_i from Proposition 2.12 so that E_m is irreducible. As ρ_m is pseudo-basic relative to σ_m, Lemma 2.2 implies that $X' \simeq X_m$ and ρ_m coincides with σ_m under this isomorphism. If C_i has a double point then one can check that $C_{i+1} \subset D_{i+1} \simeq \mathbf{C} \times C_i$ has also a double point as the defining ideal of C_{i+1} in $\mathbf{C}[D_{i+1}]$ is principal. Thus C_{m-1} has a double point. For every $z \in C_{m-1}$ the number of components in $\sigma_m^{-1}(z)$ is one, since $\sigma_m^{-1}(z) \simeq \mathbf{C}$. By Lemma 3.1 the number of irreducible components of the germ of C_{m-1} at z is one, i.e., C_{m-1} and, therefore, each C_i have no double points. By the same Lemma the normalization of C_{m-1} is $\mathbf{C}$ whence C_{m-1} and similarly C_i's are contractible. If C_i has an irreducible singularity then, by Lemma 3.4,

[3] If we allow σ_1 in Proposition 2.12 to be only locally basic then instead of (i) one can suppose that D is only smooth, or it is a cylinder over a curve.

C_{i+1} and, therefore, C_{m-1} have singularities. But if C_{m-1} is singular then $E \simeq \mathbf{C} \times C_{m-1}$ has non-isolated singularities which yields the last statement of (2). $\square$

4. Applications of the decomposition.

4.1. The proof of Lemma I. We shall reduce first Lemma I to a problem about affine modifications.

Lemma 4.1. *Let X' be a UFD of dimension 3 which contains a $\mathbf{C}^2$-cylinder Z over a smooth affine curve U.*

(i) *Then U is rational and the natural projection $p_0 : Z \to U$ can be extended to a function $p \in \mathbf{C}[X']$ whose general fibers are still isomorphic to $\mathbf{C}^2$.*
(ii) *Furthermore, let x, y, z be coordinates on $X = \mathbf{C}^3$. Then there exists an affine modification $\sigma : X' \to X$ such that its coordinate form is $\sigma = (p, p_1, p_2)$ and the divisor of this modification coincides with the zeros of some polynomial $f(x)$ on $\mathbf{C}^3$.*

Proof. Let $\overline{F}_c$ be the closure of the fiber $F_c = \{p_0 = c\} \subset Z$ in X' (where $c \in U$). Assume that $\overline{F}_c \cap \overline{F}_{c'} \neq \emptyset$ for some $c \neq c' \in U$. Since X' is a UFD there exists $g \in \mathbf{C}[X']$ whose zero fiber is $\overline{F}_{c'}$. Thus the zero locus of $g|_{\overline{F}_c}$ is $\overline{F}_c \cap \overline{F}_{c'}$. But $g|_{\overline{F}_c}$ is nowhere zero on $\overline{F}_c \setminus (\overline{F}_c \cap \overline{F}_{c'}) \supset F_c \simeq \mathbf{C}^2$ whence this function must be a nonzero constant on F_c and, therefore, $\overline{F}_c$. Contradiction. Thus $\overline{F}_c \cap \overline{F}_{c'} = \emptyset$ for every $c' \neq c \in U$. Assume $F_c \simeq \mathbf{C}^2$ if different from $\overline{F}_c$. Let one of the irreducible components of $\overline{F}_c \setminus F_c$ be a point. Then a normalization G of $\overline{F}_c$ contains $\mathbf{C}^2$ and one of the irreducible components of $G \setminus \mathbf{C}^2$ is also a point o. By [**Rem**, Ch. 13] every holomorphic function on $\mathbf{C}^2$ can be extended to o whence $\mathbf{C}^2$ is not Stein. Contradiction. Thus $\overline{F}_c \setminus F_c$ is a curve. As $\overline{F}_c \cap \overline{F}_{c'} = \emptyset$ the closure of $\bigcup_{c \in U} (\overline{F}_c \setminus F_c)$ is a divisor in X'. As X' is a UFD there exists $h \in \mathbf{C}[X']$ whose zero fiber is this divisor. Thus the zero locus of $h|_{\overline{F}_c}$ is $\overline{F}_c \setminus \overline{F}_c$ and we get a contradiction in the same way we did for function g. Hence $F_c = \overline{F}_c$. This implies that $Z \to U$ can be extended to continuous map p from X' to the completion $\overline{U}$ of U, and $p^{-1}(U) = Z$. In particular, general fibers of p are isomorphic to $\mathbf{C}^2$. As X' is a UFD p must be holomorphic [**Rem**, Ch. 13] and, therefore, regular (e.g., see [**Ka1**]). Since X' is a UFD Z is also a UFD whence U is a UFD. This implies that U is rational, i.e., $\overline{U} = \mathbf{P}^1$. Assume that $p : X' \to \mathbf{P}^1$ is surjective. Let $X_0 = p^{-1}(\mathbf{C})$ and $q = p|_{X_0}$. We can suppose that $Z \subset X_0$, i.e., $U \subset \mathbf{C}$. Extend the isomorphism $Z \simeq U \times \mathbf{C}^2 \subset \mathbf{C} \times \mathbf{C}^2$ to a rational map from X_0 to $\mathbf{C}^3$ (with coordinate x, y, z) and then multiply the two last coordinates by polynomials in q to make this mapping regular. We obtain a birational morphism $\sigma : X_0 \to \mathbf{C}^3$ which is an affine modification by

[**KaZa1**, Th. 1.1]. Clearly, $q = x \circ \sigma$ and the divisor of this modification is the zero fiber of $f \in \mathbf{C}[x]$. As $q : X_0 \to \mathbf{C}$ is surjective, every invertible function on X_0 is constant by Proposition 2.5. But $p^{-1}(\infty)$ is the zero divisor of $g \in \mathbf{C}[X']$ as X' is a UFD. Hence $g|_{X_0}$ is invertible and nonconstant. Contradiction. Thus one can suppose that $p = q$ and $X' = X_0$. □

Lemma 4.2. *Let $A \simeq \mathbf{C}[x, y, z]$, $f = x^n$, and $A \hookrightarrow A'$ be an affine modification. Let A' be a UFD, $E \neq \emptyset$, X' be smooth, and $H_3(X') = 0$. If E have at most isolated singularities then A' is also a polynomial ring which contains x as variable.*

Proof. By Proposition 2.4 $x \circ \sigma$ is irreducible in A' whence E is irreducible as A' is a UFD. Thus Convention 2.2 (1) holds which makes Theorem 3.1 applicable to modification $\sigma : X' \to X$. Let the notation from Theorem 3.1 and Proposition 2.12 hold. Then the reduced center C_i of each element σ_{i+1} of the decomposition is either a point or a smooth contractible irreducible curve. Let $C_0 = C$ be a point (say, the origin $o = \{x = y = z = 0\}$) and M be the maximal ideal in A that vanishes at o. By Theorem 3.1 $A_1 = A[M/x]$ whence $A_1 \simeq \mathbf{C}[x, y/x, z/x]$. Suppose that j is the first number for which C_j and, therefore, by Proposition 2.12 (2) every C_k with $k > j$ are curves. By induction we can suppose that $A_j \simeq \mathbf{C}[x, \xi, \zeta]$ (in particular, the divisor D_{j+1} of σ_{j+1} is the $\xi\zeta$-plane). By Theorem 3.1 C_j is isomorphic to $\mathbf{C}$ and by the Abhyankar-Moh-Suzuki theorem one can assume that it is given by $x = \xi = 0$. Let I_j be the ideal generated by x and ξ. By Theorem 3.1 $A_{j+1} = A_j[I_j/x]$ whence $A_{j+1} \simeq \mathbf{C}[x, \xi/x, \zeta]$. Induction concludes the proof. □

Lemma 4.3. *Let $A = \mathbf{C}[x, y, z]$, $f \in \mathbf{C}[x]$, and $A \hookrightarrow A'$ be an affine modification. Let A' be a UFD and $f - c$ be a non-unit in A' for every root c of f. If X' is smooth, $H_3(X') = 0$, and every irreducible component of E has at most isolated singularities then $X' \simeq \mathbf{C}^3$ and $x \circ \sigma$ is a variable on this sample of $\mathbf{C}^3$.*

Proof. Let $f(x) = x^n q(x)$ where $q(0) \neq 0$, $J = I[1/q]$ and $B = A[1/q]$. By Proposition 2.1 $B' = B[J/x^n]$ coincides with $A'[1/q]$. Hence the exceptional divisor E^0 of modification $B \hookrightarrow B'$ is not empty by the assumption on $f \circ \sigma$. Let L be the ideal in A generated by I and x^n, i.e., $I[1/q] = L[1/q] = J$. Put $A^1 = A[L/x^n]$. Note that $B' = A^1[1/q]$ whence the exceptional divisor of $A \hookrightarrow A^1$ is not empty. By Lemma 4.2 A^1 is a polynomial ring in three variables. Let K be the ideal in A^1 generated by I/x^n. By [**KaZa1**, Prop. 1.2] $A' = A^1[K/q]$. Now the induction by the degree of f implies the desired conclusion. □

Lemmas 4.1 and 4.3 yield Lemma I.

Remark 4.1. Miyanishi's theorem can be proven by this technique as follows. Assumption (3′) implies that E^0 (from the proof of Lemma 4.3) is a

UFD, and it is enough show that $e(E_0) = 1$ (see Remark 3.1). By Proposition 2.12 we can present $B \hookrightarrow B'$ as a composition of basic modifications. Hence either E^0 is isomorphic to $C_{m-1} \times \mathbf{C}^2$ where C_{m-1} is a point or it is as in Remark 3.1, i.e., $e(E_0) \leq 1$. Let D_0 be the plane $x = 0$. Then by the additivity of Euler characteristics [**Du**] $e(X')$ differs from $e(X) = e(\mathbf{C}^3) = 1$ by the sum of terms of form $e(E_0) - e(D_0)$ (which should be considered for each root of f). As $e(X') = 1$ we have $e(E_0) = e(D_0) = 1$ which makes Lemma 4.2 applicable.

4.2. How to present X' as a closed algebraic subvariety of $\mathbf{C}^N$.

Proposition 4.1. *Let $A = \mathbf{C}[x, y, z]$, $f \in \mathbf{C}[x]$ have roots $c_0 = 0, c_1, \ldots,$ $A \hookrightarrow A'$ be an affine modification, and X' satisfy assumptions (0) and (1) of the Main Theorem. Then X' is contractible and either $X' \simeq \mathbf{C}^3$ or there exists a root of f (say c_0) so that X' can be viewed as the subvariety of $\mathbf{C}^N$ given by polynomial equations*

$$\begin{aligned}
&xv_1 - q_0(y, z) = 0\\
&xv_2 - v_1^{n_1} + q_1(y, z, v_1) = 0\\
&\qquad \ldots\\
&xv_m - v_{m-1}^{n_{m-1}} + q_{m-1}(y, z, v_1, \ldots, v_{m-1}) = 0\\
&(x - c_1)u_{1,1} - r_{1,0}(y, z) = 0\\
&(x - c_1)u_{1,2} - u_{1,1}^{n_{1,1}} + r_{1,1}(y, z, u_{1,1}) = 0\\
&\qquad \ldots\\
&(x - c_1)u_{1,m_1} - u_{1,m_1-1}^{n_{1,m_1-1}} + r_{1,m_1-1}(y, z, u_{1,1}, \ldots, u_{1,m_1-1}) = 0\\
&(x - c_2)u_{2,1} - r_{2,0}(y, z) = 0\\
&\qquad \ldots
\end{aligned}$$

where $q_0(y, z) = y^k - z^l$, $(k, l) = 1, k > l \geq 2$, $m > 1$, the standard degree of q_j with respect to v_i is less than n_i $\forall i = 1, \ldots, j$, and the standard degree of $r_{s,j}$ with respect to $u_{s,i}$ is less than $n_{s,i}$ $\forall i = 1, \ldots, j$. Furthermore, the defining ideal I' of X' in $\mathbf{C}^{[N]}$ is prime and generated by the left-hand sides of the equations above.

Proof. We can suppose that $f(x) = x^n$ since Remark 2.1 reduces the problem to the case when f has one root only. As $X' \setminus E \simeq \mathbf{C}^3 \setminus \{x = 0\}$ we have $H_2(X' \setminus E) = H_3(X' \setminus E) = 0$. By Theorem 3.1 $\sigma : X' \to X$ is a composition of basic modifications $X' = X_m \stackrel{\sigma_m}{\to} X_{m-1} \to \ldots \to X_1 \stackrel{\sigma_1}{\to} X$. Let $A_j = \mathbf{C}[X_j]$ and C_j be as in Convention 2.2 (3). By Theorem 3.1 each C_j is either a point or an irreducible contractible curve. If C_0 is either a point or a smooth curve then $X_1 \simeq \mathbf{C}^3$ and one can use induction on m (see the proof of Lemma 4.2). When C_0 is not a smooth curve, by the Lin-Zaidenberg theorem [**LiZa**] one can assume C_0 is given in $\mathbf{C}^3$ by $x = y^k - z^l = 0$ where $(k, l) = 1$ and $k > l \geq 2$. Let I_1 be the ideal in A

generated by x and $y^k - z^l$. By Theorem 3.1 $A_1 = A[I_1/x]$. By Theorem 2.1 that $A_1 = \mathbf{C}[x, y, z, (y^k - z^l)/x]$ and X_1 is the irreducible hypersurface in $\mathbf{C}^4$ with coordinates (x, y, z, v_1) given by $xv_1 = q_0(y, z) := y^k - z^l$. By Theorem 3.1 and by Lemma 3.3 C_1 is the zero fiber of a regular function on the exceptional divisor $E_1 = X_1 \cap \{x = 0\}$ which is of form $v_1^{n_1} + q_1(y, z, v_1)$ where the standard degree of q_1 with respect to v_1 is at most $n_1 - 1$. Let I_2 be the ideal in A_1 generated by x and $v_1^{n_1} + q_1(y, z, v_1)$. By Theorem 3.1 $A_2 = A_1[I_2/x]$. Therefore, by Theorem 2.1 X_2 may be viewed as the irreducible complete intersection in $\mathbf{C}^5$ (with coordinates (x, y, z, v_1, v_2)) given by the equations $xv_1 - q_0(y, z) = xv_2 - v_1^{n_1} + q_1(y, z, v_1) = 0$. Repeating the above argument we see that X' can be viewed as the desired irreducible complete intersection in $\mathbf{C}^{3+m}$. Contractibility of X' follows from Proposition 2.10. In order to check that $m > 1$ when X' is smooth it is enough to note that X_1 is singular at the origin. □

In combination with Lemma 4.1 and [**ChDi**] we get:

Corollary 4.1. *Suppose that X' satisfies Assumptions (0) and (1) of the Main Theorem and Assumption (2′) of Lemma I. Then either $X' \simeq \mathbf{C}^3$ or X' is diffeomorphic to $\mathbf{R}^6$ and given by the system of equations from Proposition 4.1.*

5. The Makar-Limanov invariant.

5.1. General facts about locally nilpotent derivations. Recall that a derivation ∂ on A is called locally nilpotent if for each $a \in A$ there exists an $k = k(a)$ such that $\partial^k(a) = 0$. For $t \in \mathbf{C}$ the mapping $\exp(t\partial) : A \to A$ is an automorphism whence ∂ generates a $\mathbf{C}_+$-action on X [**Ren**]. Every locally nilpotent derivation defines a degree function $\deg_\partial$ on the domain A with natural values (e.g., see [**FLN**]) given by the formula $\deg_\partial(a) = \max\{k \,|\, \partial^k(a) \neq 0\}$ for every nonzero $a \in A$.

Proposition 5.1 (cf. [**Za**], proof of Lemma 9.3)**.** *Let ∂ be a nonzero locally nilpotent derivation of $A = \mathbf{C}[X]$ and let $F = (f_1, \dots, f_s) : X \to Y \subset \mathbf{C}^s$ and $G : Y \to Z \subset \mathbf{C}^j$ be dominant morphisms of reduced affine algebraic varieties. Put $H = G \circ F = (h_1, \dots, h_j) : X \to Z$. Suppose that for general point $\xi \in Z$ there exists a (Zariski) dense subset T_ξ of $G^{-1}(\xi)$ such that the image of any nonconstant morphism from $\mathbf{C}$ to $G^{-1}(\xi)$ does not meet T_ξ. If $h_1, \dots, h_j \in A^\partial$ then $f_1, \dots, f_s \in A^\partial$.*

Proof. Consider the $\mathbf{C}_+$-action on X generated by ∂. Choose a general point $\xi \in Z$. Let O_ζ be the orbit of $\zeta \in H^{-1}(\xi)$. As $h_1, \dots, h_j \in A^\partial$ the fiber $H^{-1}(\xi)$ is invariant under the action and $O_\zeta \subset H^{-1}(\xi)$. Note that $F(O_\zeta)$ is a point $\forall \zeta \in F^{-1}(T_\xi)$. As $F^{-1}(T_\xi)$ is dense in $H^{-1}(\xi)$ this is also true $\forall \zeta \in H^{-1}(\xi)$ whence each orbit is contained in a fiber of F which yields the desired conclusion. □

Definition 5.1. The Makar-Limanov invariant of A is $\mathrm{ML}(A) = \bigcap_{\partial \in \mathrm{LND}(A)} \operatorname{Ker} \partial$ where $\mathrm{LND}(A)$ is the set of all locally nilpotent derivations on A. Equivalently, $\mathrm{ML}(A)$ is the subset of A which consists of those regular functions on X that are invariant under any regular $\mathbf{C}_+$-action.

5.2. The associated algebra. Let $A' = \mathbf{C}^{[N]}/I'$ where I' is a prime ideal in $\mathbf{C}^{[N]}$. For every $a \in A'$ put $[a] = \{p \in \mathbf{C}^{[N]} | \, p|_{X'} = a\}$ and for every $p \in \mathbf{C}^{[N]} \setminus \{0\}$ we denote by $M(p)$ the set of monomials that are summands of p.

Definition 5.2. A weight degree function on $\mathbf{C}^{[N]}$ is a degree function d such that $d(p) = \max\{d(\mu) \,|\, \mu \in M(p)\}$, where $p \in \mathbf{C}^{[N]} \setminus \{0\}$. Let $\bar{p} = \sum_{\mu \in M(p), d(\mu)=d(p)} \mu$ be the leading d-homogeneous part of p. Consider the ideal $\widehat{I}'_d$ generated by such $\bar{p}$ when p runs over $I' \setminus \{0\}$ and the variety $\widehat{X}'_d \subset \mathbf{C}^N$ defined by $\widehat{I}'_d$. Then we call $\widehat{A}'_d = \mathbf{C}[\widehat{X}'_d]$ the associated graded algebra.

Proposition 5.2 ([**KaM-L2**]). *Let $X' \subset \mathbf{C}^N$ contain the origin of $\mathbf{C}^N$. Then:*

(1) $\forall a \in A' \setminus \{0\}$ *there exists* $p \in [a]$ *so that* $\bar{p} \notin \widehat{I}'_d$. *Furthermore, the map* $\operatorname{gr}_d : A' \setminus \{0\} \to \widehat{A}'_d \setminus \{0\}$ *given by* $\operatorname{gr}_d(a) = \bar{p}|_{\widehat{X}'_d}$ *is well-defined.*

(2) *Every nonzero* $\partial \in \mathrm{LND}(A')$ *generates a nonzero* (*associated*) $\widehat{\partial} \in \mathrm{LND}(\widehat{A}'_d)$ *such that for every* $a \in A' \setminus \{0\}$ *we have* $\deg_\partial(a) \geq \deg_{\widehat{\partial}_d}(\operatorname{gr}_d(a))$.

Convention 5.1. Let $q_0(y,z) = y^k - z^l, m_i, n_{j,i}$, and coordinates $(x, y, z, v_1, \ldots, v_m, \; u_{1,1}, \ldots, u_{j,i}, \; \ldots)$ in $\mathbf{C}^N$ be as in Proposition 4.1. Put $d_x = d(x), d_y = d(y), d_z = d(z), d_i = d(v_i)$ and $d_{j,i} = d(u_{j,i})$ where d is a weight degree function. From now on we suppose that

(1) $kd_y = ld_z$ (in particular, $\bar{q}_0 = q_0 = y^k - z^l$);
(2) $d_1 + d_x = kd_y$, and d_1, d_x are $\mathbf{Q}$-independent;
(3) $d_x < 0$ and $d_1 >> d_y > 0$;
(4) $d_x + d_{i+1} = m_i d_i$ for $i \geq 1$;
(5) $d_x + d_{j,i+1} = n_{j,i} d_{j,i} > 0$ for every $j, i \geq 1$.

This Convention implies the following non-difficult Proposition:

Proposition 5.3. *Let X' be the zero set of the system of polynomial equations from Proposition 4.1 and $A' = \mathbf{C}[X']$. Then under Convention 5.1 the associated graded algebra $\widehat{A}'_d = \mathbf{C}[\widehat{X}'_d]$ where $\widehat{X}'_d$ is isomorphic to the zero* *set of the following system*

$$\begin{array}{l} xv_1 - q_0(y,z) = 0, xv_2 - v_1^{n_1} = 0, \dots, xv_m - v_{m-1}^{n_{m-1}} = 0 \\ -c_1u_{1,1} = 0, -c_1u_{1,2} - u_{1,1}^{n_{1,1}} = 0, \dots, -c_1u_{1,m_1} - u_{1,m_1-1}^{n_{1,m_1-1}} = 0 \\ -c_2u_{2,1} = 0, -c_2u_{2,2} - u_{2,1}^{n_{2,1}} = 0, \dots, -c_2u_{2,m_2} - u_{2,m_2-1}^{n_{2,m_2-1}} = 0 \\ \qquad \dots \end{array}$$

Furthermore, the defining ideal $\widehat{I}'_d$ of $\widehat{X}'_d$ is prime and generated by the left-hand sides of the equations above.

Remark 5.1. The variety $\widehat{X}'_d$ is independent on the choice of d satisfying Convention 5.1 and it is isomorphic to the zero set of the following polynomial equations in $\mathbf{C}^{3+m}$ with coordinates $(x, y, z, v_1, \dots, v_m)$

$$\begin{array}{l} P_1(x,y,z,v_1) = xv_1 - q_0(y,z) = 0 \\ P_2(x,v_1,v_2) = xv_2 - v_1^{n_1} = 0 \\ \qquad \dots \\ P_m(x,v_{m-1},v_m) = xv_m - v_{m-1}^{n_{m-1}} = 0. \end{array}$$

Therefore, we shall write further $\widehat{I}'$, $\widehat{A}'$, and $\widehat{X}'$ instead of $\widehat{I}'_d$, $\widehat{A}'_d$, and $\widehat{X}'_d$ provided it does not cause misunderstanding.

5.3. Locally nilpotent derivation of Jacobian type. We say that $a \in \widehat{A}'$ is d-homogeneous if a is the restriction to $\widehat{X}' \subset \mathbf{C}^N$ of a d-homogeneous polynomial. In the rest of the paper we denote $q|_{\widehat{X}'}$ by $\tilde{q}$ for every $q \in \mathbf{C}^{[m+3]}$.

Lemma 5.1. *Let $a \in \widehat{A}'$ be an irreducible d-homogeneous element. Then up to a constant factor a is of one of the following elements $\tilde{v}_i, \tilde{x}, \tilde{y}, \tilde{z}$, or $\tilde{y}^k + c\tilde{z}^l$ where $c \in \mathbf{C}^*$ and k, l are the same as in Proposition 4.1.*

Proof. Let q be d-homogeneous and $a = \tilde{q}$ (in particular, q is irreducible). By Remark 5.1 we can suppose that $\forall \mu \in M(q)$ is non-divisible by xv_i $\forall i = 1, \dots, m$. Each $\tilde{v}_i$ coincides with a rational function on $\widehat{X}'$ of form q_0^s/x^j where $s, j > 0$. If we extend d naturally to the field of rational functions then $d(v_i) = d(q_0^s/x^j)$ by Convention 5.1. Assume that $\mu_1, \mu_2 \in M(q)$ are such that μ_1 is divisible by x but μ_2 is not. Then μ_1 and μ_2 coincides with the restriction to $\widehat{X}'$ of functions $x^{j_1}y^{\alpha_1}z^{\beta_1}$ and $y^{\alpha_2}z^{\beta_2}q_0^s/x^{j_2}$ where $j_1 > 0, j_2 \geq 0$. As $d(\mu_1) = d(\mu_2)$ we have $(j_1 + j_2)d_x = d(y^{\alpha_2-\alpha_1}z^{\beta_2-\beta_1}q_0^s)$. As $d_y = (l/k)d_z$ and $d(q_0) = kd_y$ we get $\mathbf{Q}$-dependence of d_x and d_y which contradicts Convention 5.1. Thus if $q \neq cx$, $c \in \mathbf{C}^*$, none of $\mu \in M(q)$ is divisible by x. Let $\mu_1, \mu_2 \in M(q)$ and $\mu_i = y^{\alpha_i}z^{\beta_i}\nu_i$ where ν_i is a monomial which depends on $v_1, \dots, v_m$ only. The restriction of μ_i to $\widehat{X}'$ coincides with $y^{\alpha_i}z^{\beta_i}q_0^{s_i}/x^{j_i}$. The same argument as above shows that $j_1 = j_2$ since otherwise d_x and d_y are $\mathbf{Q}$-dependent. Hence $d(y^{\alpha_1}z^{\beta_1}q_0^{s_1}) = d(y^{\alpha_2}z^{\beta_2}q_0^{s_2})$. As $d(q_0) = kd_y = ld_z$ and $(k, l) = 1$ we have $\alpha_i = \alpha_0 + t_ik$ and $\beta_i = \beta_0 + \tau_il$

where $0 \leq \alpha_0 \leq k-1$, $0 \leq \beta_0 \leq l-1$, and $t_1 - t_2 + \tau_1 - \tau_2 = s_2 - s_1$. Therefore, the restriction of q to $\widehat{X}'$ coincides with $y^{\alpha_0} z^{\beta_0} \varphi(y^k, z^l) q_0^s / x^j$ where $\varphi(y^k, z^l)$ is d-homogeneous and the restriction of q_0^s/x^j to $\widehat{X}'$ coincides with the restriction of a monomial ν which depends on $v_1, \ldots, v_m$ only. Now the Lemma follows from the fact that that $\varphi(y^k, z^l)$ is the product of factors of type $c_1 y^k + c_2 z^l$ where $c_1, c_2 \in \mathbf{C}$. $\square$

Corollary 5.1. *Let $a = \tilde{q}$ where $q \notin \mathbf{C}[y,z]$ is a d-homogeneous polynomial which does not depend on x. Then q is divisible by some v_i.*

Note that $(\tilde{x}, \tilde{y}, \tilde{z})$ is a local (holomorphic) coordinate system at each point of $\widehat{X}_0 = \widehat{X}' \setminus \{x = 0\}$. For $a_1, a_2, a_3 \in \widehat{A}'$ we denote by $\mathrm{Jac}\,(a_1, a_2, a_3)$ the Jacobian of these regular functions on $\widehat{X}_0$ with respect to $\tilde{x}, \tilde{y}$, and $\tilde{z}$. This is a rational function on $\widehat{X}'$ but $\tilde{x}^m \mathrm{Jac}\,(a_1, a_2, a_3)$ is already regular on $\widehat{X}'$ since x^m is the determinant of the matrix $\{\partial P_i / \partial v_j \,|\, i, j = 1, \ldots, m\}$ where P_i are as in Remark 5.1. Fix $a_1, a_2 \in \widehat{A}'$ and let $a \in \widehat{A}'$. Then one can see that $\partial(a) = \tilde{x}^m \mathrm{Jac}\,(a_1, a_2, a)$ is a derivation on $\widehat{A}'$.

Proposition 5.4. *Let $m \geq 2$, a_1 and a_2 be d-homogeneous, and $\partial(a) = \tilde{x}^m \mathrm{Jac}\,(a_1, a_2, a)$ be nontrivial locally nilpotent. Then:*

(1) *If a_1 and a_2 are irreducible then (a_1, a_2) coincides (up to the order) with one of the pairs $(\tilde{x}, \tilde{y})$ or $(\tilde{x}, \tilde{z})$.*
(2) *$\tilde{x} \in \mathrm{Ker}\,\partial$ and $\deg_\partial(\tilde{v}_i) \geq 2$ for every $i = 1, \ldots, m$.*

Proof. If (a_1, a_2) is one of the pairs in (1) it is easy to check that ∂ is nontrivial and locally nilpotent, and (2) holds also. Show that if we use other possible irreducible d-homogeneous elements from Lemma 5.1 as a_1, a_2 then ∂ cannot be a nontrivial locally nilpotent derivation. Note that a_1 and a_2 are algebraically independent in $\widehat{A}'$ as otherwise ∂ is trivial.

Case 1. Let $(a_1, a_2) = (\tilde{y}, \tilde{z})$. The direct computation shows that $\partial(\tilde{x}) = \tilde{x}^m$ whence ∂ cannot be locally nilpotent. Indeed, one can see that $\deg_\partial(\partial(\tilde{x})) = \deg_\partial(\tilde{x}) - 1$. But $\deg_\partial(\tilde{x}^m) = m \deg_\partial(\tilde{x})$ which yields a contradiction.

Case 2. Either a_1 or a_2 is of form $\tilde{y}^k + c\tilde{z}^l$ where $c \in \mathbf{C}^*$ and k and l are as in Proposition 4.1. By [**M-L2**] $\tilde{y}, \tilde{z} \in \mathrm{Ker}\,\partial$ as $\tilde{y}^k + c\tilde{z}^l \in \mathrm{Ker}\,\partial$. By [**KaM-L1**, Lemma 5.3] the derivation $\tilde{x}^m \mathrm{Jac}\,(\tilde{y}, \tilde{z}, a)$ must be also nonzero locally nilpotent whence this case does not hold.

Case 3. Let $(a_1, a_2) = (\tilde{v}_{i_1}, \tilde{v}_{i_2})$ where $i_1 < i_2$. Consider the identical morphism $F : \widehat{X}' \to \widehat{X}' \subset \mathbf{C}^{m+3}$ and morphism $G : \widehat{X}' \to \mathbf{C}^2$ given by $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{v}_1, \ldots, \tilde{v}_m) \to (\tilde{v}_{i_1}, \tilde{v}_{i_2})$. Recall that $\tilde{v}_{i_k} = \tilde{q}_0^{s_k} / \tilde{x}^{j_k}$. It is easy to check that $\tilde{v}_{i_1}$ and $\tilde{v}_{i_2}$ are algebraically independent in $\widehat{A}'$ which means that the pairs (s_1, j_1) and (s_2, j_2) are not proportional. Consider a general point $\xi \in \mathbf{C}^2$. Each component of the fiber $G^{-1}(\xi)$ is a curve in $\mathbf{C}^{m+3}$ given by equations $v_i = c_i, x = c'$, and $q_0(y,z) = y^k - z^l = c$ where $c_i, c' \in \mathbf{C}$

and $c \in \mathbf{C}^*$. This curve is hyperbolic and thus it does not admit nonconstant morphisms from $\mathbf{C}$. By Proposition 5.1 if ∂ is locally nilpotent then $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{v}_i \in \operatorname{Ker} \partial$ whence ∂ is trivial. Thus this case does not hold.

Case 4. Let $(a_1, a_2) = (\tilde{x}, \tilde{v}_i)$. The same argument is in Case 3 works.

Case 5. Let $(a_1, a_2) = (\tilde{y}, \tilde{v}_i)$ (or, similarly $(\tilde{z}, \tilde{v}_i)$). Consider the identical morphism $F : \widehat{X}' \to \widehat{X}'$ and $G : \widehat{X}' \to \mathbf{C}^2$ given by $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{v}_1, \ldots, \tilde{v}_m) \to (\tilde{y}, \tilde{v}_i)$. As $\tilde{v}_i = \tilde{q}_0^s/\tilde{x}^j$ (where $j \geq 2$ if $i > 1$) the fiber $G^{-1}(\xi)$ where $\xi = (c_1, c_2) \in \mathbf{C}^2$ is isomorphic to the curve $(c_1^k - z^l)^s - c_2 x^j = 0$. When $j \geq 2$ and s is not divisible by j the last curve have no contractible components for general ξ. Proposition 5.1 implies that ∂ must be trivial. If $j \geq 2$ and s is divisible by j then each irreducible component of $G^{-1}(\xi)$ is contractible and contains double points of $G^{-1}(\xi)$. As $G^{-1}(\xi) \subset \widehat{X}'$ is invariant under the associated $\mathbf{C}_+$-action and has singular points this action is trivial on $G^{-1}(\xi)$ and, thus, on $\widehat{X}'$. Hence ∂ is trivial. Let $j = 1$, i.e., $(a_1, a_2) = (\tilde{y}, \tilde{v}_1) = (\tilde{y}, \tilde{q}_0/\tilde{x})$. The direct computation shows that $\partial(\tilde{x}) = c\tilde{x}^{m-1}\tilde{z}^{l-1}$, $c \in \mathbf{C}$. As $m \geq 2$, ∂ cannot be nontrivial locally nilpotent (indeed, compare $\deg_\partial(\tilde{x})$ and $\deg_\partial(\partial(\tilde{x}))$) and we have to disregard this case. In order to see statement (2) in the case when a_1 and a_2 are not irreducible, we note that one can replace a_1 and a_2 with their irreducible factors in the definition of ∂ and obtain a locally nilpotent derivation equivalent to ∂ [**KaM-L2**]. □

5.4. The computation of $\mathrm{ML}(A')$**.** A locally nilpotent derivation ∂ on A' is called perfect if its associated derivation $\widehat{\partial}_d$ is of form $\widehat{\partial}_d(a) = \tilde{x}^m \operatorname{Jac}(a_1, a_2, a)$ where $a_1, a_2 \in \widehat{A}'_d$ are d-homogeneous and algebraically independent. The set of all perfect derivations will be denoted by $\operatorname{Per}(A')$.

Proposition 5.5. *Let A' be as in Proposition 4.1 and let d satisfy Convention 5.1. For every $\partial \in \operatorname{Per}(A')$ we have $x \in \operatorname{Ker} \partial$.*

Proof. Let $a \in A'$ with $\deg_\partial(a) \leq 1$. Show that there exist a polynomial q with $q|_{X'} = a$ such that none of $\mu \in M(q)$ is divisible by v_i or $u_{s,j}$ for all i, s, j, i.e., $q \in \mathbf{C}[x, y, z]$. By Proposition 4.1 we can suppose that none of $\mu \in M(q)$ is divisible by xv_i or $xu_{s,j}$. Thus $M(q) = M_1(q) \cup M_2(q)$ where $M_1(q) \subset \mathbf{C}[x, y, z]$ and $M_2(q)$ consists of monomials which do not depend on x and do not belong to $\mathbf{C}[y, z]$. Assume that $\mu \in M_2(q)$. Under Convention 5.1 one can keep d_y, d_z fixed, decrease d_x, and increase $d_i, d_{s,j}$ so that $d(\mu) > d(\nu)\, \forall \nu \in M_1(q)$. Hence if $\bar{q}_d$ is the leading d-homogeneous part of q then $M(\bar{q}_d) \subset M_2(q)$. By Proposition 5.2 $\deg_{\widehat{\partial}_d}(\operatorname{gr}_d(a)) \leq 1$. The element $\operatorname{gr}_d(a) = \bar{q}_d|_{\widehat{X}'}$ is the product of irreducible d-homogeneous elements of $\widehat{A}'$. By Corollary 5.1 one of them is $\tilde{v}_i$ whence $\deg_{\widehat{\partial}_d}(\tilde{v}_i) \leq \deg_{\widehat{\partial}_d}(\operatorname{gr}_d(a)) \leq 1$ which contradicts Proposition 5.4. Thus $M_2(q)$ is empty. Let $b \in A'$

with $\deg_\partial(b) = 1$. By [**M-L1**] there exist $a', a_0, \dots, a_s \in \operatorname{Ker}\partial$ such that $a'\tilde{v}_1 = \sum_{j=0}^{s} a_j b^j$ where $s = \deg_\partial \tilde{v}_1$. Hence $\tilde{v}_1 = (q(x,y,z)/r(x,y,z))|_{X'}$ where $a' = r(x,y,z)|_{X'}$. But $A' \subset \mathbf{C}[x,y,z,1/f(x)]$ where f is as in Proposition 4.1. Since $v_1 \notin \mathbf{C}[x,y,z]$ we have $r(x,y,z)$ divisible by some $x - c$ where c is a root of f. Therefore, $x - c \in \operatorname{Ker}\partial$ as a divisor of an element from $\operatorname{Ker}\partial$. □

Proof of Lemma II and the Main Theorem. By Proposition 5.3 $\widehat{A'}$ is a domain whence $\mathrm{ML}(A') = \bigcap_{\partial \in \mathrm{Per}\,(A')} \operatorname{Ker}\partial$ [**KaM-L2**]. By Propositions 5.5 $\mathrm{ML}[A'] \supset \mathbf{C}[x]|_{X'}$ which implies Lemma II [4] and, therefore, the Main Theorem. □

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