*Pacific Journal of Mathematics*

# ALL LINKS ARE SUBLINKS OF ARITHMETIC LINKS

Mark D. Baker

Volume 203 No. 2 April 2002

PACIFIC JOURNAL OF MATHEMATICS
Vol. 203, No. 2, 2002

# ALL LINKS ARE SUBLINKS OF ARITHMETIC LINKS

Mark D. Baker

**We show that every link in $S^3$ is a sublink of an arithmetic link.**

## 1. Introduction.

In this paper we show that arithmetic links play a central role in the Dehn surgery description of closed 3-manifolds. Let $L \subset S^3$ be a link of (one or more) circles. We prove that $L$ is a sublink of an arithmetic link. Specifically:

**Theorem 1.** *Let $L \subset S^3$ be a link. Then $L$ is a sublink of a link $J$ such that $S^3 \setminus J$ is homeomorphic to $\mathbb{H}^3/\Gamma$, where $\Gamma$ is a torsion-free subgroup of finite index in the Bianchi group $PSL_2(\mathbb{Z}[i])$.*

Since every closed, orientable 3-manifold can be obtained by Dehn surgery on a link in $S^3$ (see [**Li**]), we have:

**Theorem 2.** *Every closed, orientable 3-manifold can be obtained by Dehn surgery on an arithmetic link in $S^3$.*

While it is known that every closed orientable 3-manifold $M$ contains an arithmetic link $L$ (since the figure-eight knot complement is both arithmetic and universal), Theorem 2 asserts that $L$ can be chosen so that $M \setminus L$ is homeomorphic to the complement of a link in $S^3$.

Recall that a link $L$ in $S^3$ (resp. in $M$) is hyperbolic if $S^3 \setminus L$ (resp. $M \setminus L$) is homeomorphic to $\mathbb{H}^3/\Gamma$, where $\mathbb{H}^3$ is hyperbolic 3-space and $\Gamma$ a discrete, torsion-free, finite covolume subgroup of $PSL_2(\mathbb{C})$. We say that $L$ is arithmetic if $\Gamma$ can be chosen commensurable with a Bianchi group $PSL_2(\mathcal{O}_m)$, where $\mathcal{O}_m$ is the integers of the imaginary quadratic number field $\mathbb{Q}(\sqrt{-m})$ (see [**R**] for a more general discussion). Finally, $L$ is a sublink of $J$ if it is a union of components of $J$.

## 2. Proof of Theorem 1.

Denote by $L_1$ the 6-circle link in Figure 1.

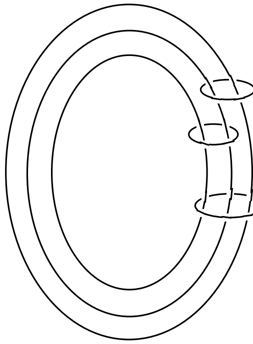

**Figure 1.**

We prove our result by showing that:

i) $L_1$ is an arithmetic link: $S^3 \setminus L_1 \cong \mathbb{H}^3/\Gamma_1$, where $\Gamma_1$ is a torsion-free subgroup of the Bianchi group $PSL_2(\mathbb{Z}[i])$.

ii) Every link $L$ occurs as a sublink of a link $J$ such that $S^3 \setminus J$ is a covering space of $S^3 \setminus L_1$.

Thus $S^3 \setminus J \cong \mathbb{H}^3/\Gamma$, where $\Gamma \subset \Gamma_1 \subset PSL_2(\mathbb{Z}[i])$, and so $J$ is arithmetic and contains $L$ as a sublink. We prove the arithmeticity of $L_1$ in **2.1**. Section **2.2** is devoted to proving property ii).

**2.1. .** The link $L_1$ is arithmetic since $S^3 \setminus L_1$ is a 2-fold cover of $S^3 \setminus L_0$ where $L_0$ is the four component arithmetic link in Figure 2.

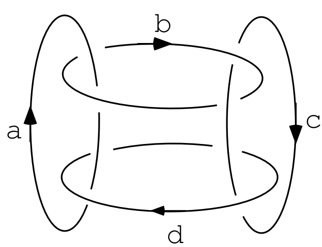


**Figure 2.**

Indeed, $S^3 \setminus L_0 \cong \mathbb{H}^3/\Gamma_0$ where

$$\Gamma_0 = \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2i \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1-i & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1+i & 1 \end{bmatrix} \right\rangle \subset PSL_2(\mathbb{Z}[i])$$

(see [**Wi**] Example 3 for a detailed treatment), and $S^3 \setminus L_1$ is the 2-fold cover corresponding to the kernel of the map $\theta : \pi_1(S^3 \setminus L_0) \to \mathbb{Z}/2\mathbb{Z}$ given by $\theta(a) = \theta(c) = 1$ and $\theta(b) = \theta(d) = 0$.

A second proof of the arithmeticity of $L_1$ goes as follows (see [**R**]): $\Gamma_1$ is a subgroup of $PSL_2(\mathbb{Z}[i])$ if and only if $tr(\Gamma_1) = \{\mathrm{tr}\,(\gamma) \mid \gamma \in \Gamma_1\} \subset \mathbb{Z}[i]$, which is true if and only if, for a set of generators $\gamma_1, \dots, \gamma_n$ of $\Gamma_1$, the following traces are in $\mathbb{Z}[i]$: $\mathrm{tr}\,(\gamma_i)$ and $\mathrm{tr}\,(\gamma_i\gamma_j)$, $i < j$. We used SnapPea ([**W**]) to compute a matrix representation for $\pi_1(S^3 \setminus L_1)$ and verify that the above traces are indeed in $\mathbb{Z}[i]$.

**2.2. .** By the Alexander braiding theorem (see [**B-Z**]) any link can be realized as the closure of an $n$-braid (Figure 3).

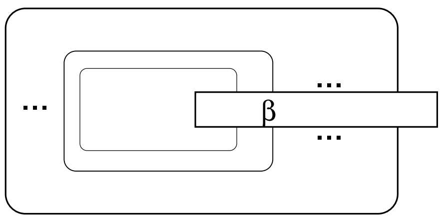


**Figure 3.**

Here $\beta = \alpha_{i_k}^{s_k} \cdots \alpha_{i_1}^{s_1}$ is a product of powers of the standard generators of the braid group $B_n$ (Figure 4).

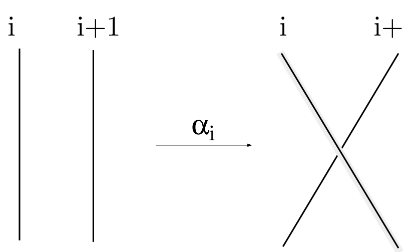


**Figure 4.**

We show that $L$ is a sublink of a $J$ such that $S^3 \setminus J$ is a cover of $S^3 \setminus L_1$ hence arithmetic. Before giving the construction of $J$ in the general case, we first illustrate the process by treating the case when $L$ is the trefoil knot.

**2.2.1. .** The trefoil knot is the closure of the 2-strand braid $\beta = \alpha^3$ (Figure 5).

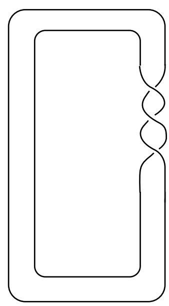

**Figure 5.**

Let $X^1$ be the 2-fold cover of $X \cong S^3 \setminus L_1$ branched over the circle $c$ (see Figure 6. We draw only the braid part of the vertical components in order to save space). This cover is again a link complement since we are branching over an unknotted component of $L_1$.

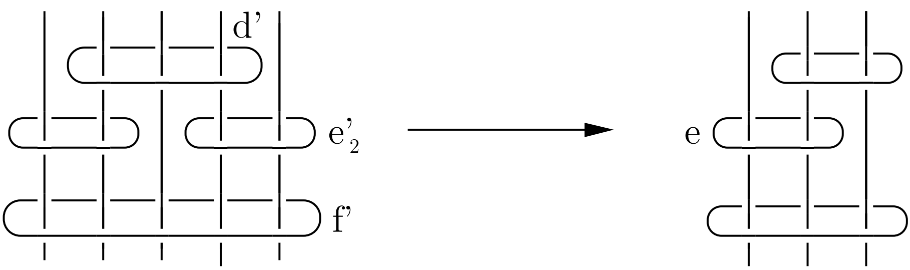


**Figure 6.**

Now we transform $X^1$ into $S^3 \setminus J$ by performing a 3/2 twist about the circle $d'$ i.e., cutting along the disk $D'$ bounded by $d'$, twisting through $3\pi$ and regluing. This has the effect of $\alpha^3$ on the circles $b'_1$ ,$b'_2$, changing them into the desired trefoil knot (Figure 7). The key point here is that $S^3 \setminus J$ is also a 2-fold cover of $S^3 \setminus L_1$. We now examine this point in greater detail.

**2.2.2. .** We show (with notation as above) that any $n/2$ twist about $d'$ transforms $X^1$ into another 2-fold cover of $X$. Since integer twists about $d'$ are homeomorphisms, it suffices to consider the case of a 1/2 twist. Note that $D'$ 2-fold covers $D$, a disk bounded by circle $d$ in $X$ (Figure 8). Since the complement of an unknotted circle in $S^3$ is a solid torus, cutting along $D'$ and $D$ transforms $X^1$ and $X$ into solid cylinders (minus circles and arcs): $Y^1$ 2-fold covering $Y$. Now glue the two copies of $D$ by the identity to get $X$, and note that there are two gluings of the $D'$'s that give a cover of $X$: The identity which gives back $X^1$ and the order 2 automorphism of $D'$ which yields the cover corresponding to the 1/2 twist mentioned above.

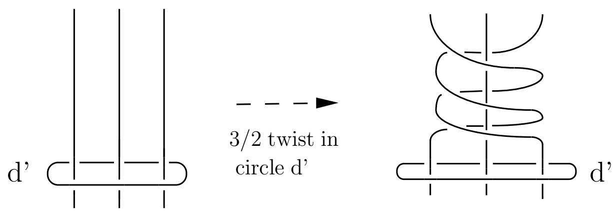


**Figure 7.**

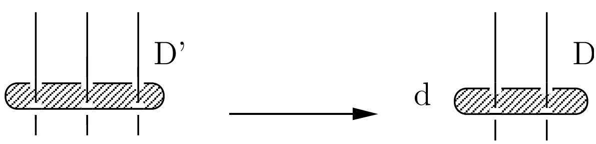


**Figure 8.**

**2.2.3. .** Given $L$ the closed $n$-braid corresponding to $\beta = \alpha_{i_k}^{s_k} \cdots \alpha_{i_1}^{s_1}$, we obtain $J$ by the construction in steps 1–4 below.

1) Let $X^1$ be the 2-fold cover of $X \cong S^3 \setminus L_1$ branched over the circle $c$.
2) For $r > 1$, let $X^r$ be the 2-fold cover of $X^{r-1}$ branched over the right-most preimage of the circle $a$.

The cover $X^r$ is a link complement (since branched over an unknotted circle in $X^{r-1}$) containing $2^r$ unknotted, unlinked preimages of the circle $b$. Choose $r$ so that $2^r > n$.

3) Let $\widetilde{X}$ be the $k$-fold cyclic cover of $X^r$ branched over the preimage of circle $f$ (see Figure 9).

4) Transform $\widetilde{X}$ into $S^3 \setminus J$ by twists corresponding to $\beta = \alpha_{i_k}^{s_k} \cdots \alpha_{i_1}^{s_1}$ in the appropriate preimages of circles $d$ and $e$.

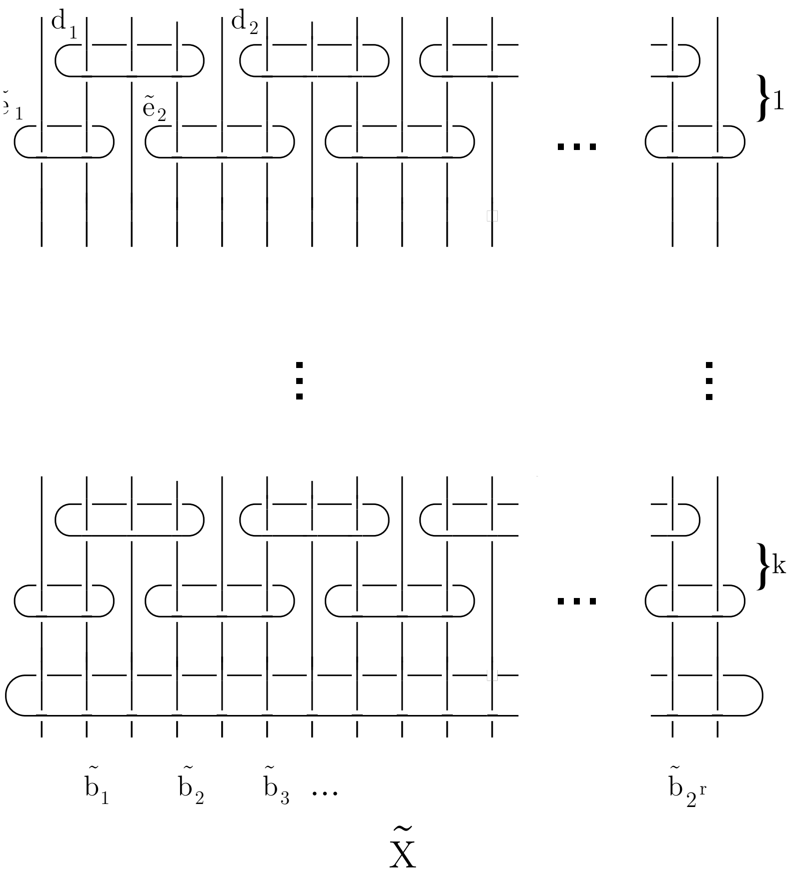


**Figure 9.**

Consider the $n$ left-most preimages $\widetilde{b}_1, \ldots, \widetilde{b}_n$ in $\widetilde{X}$ of circle $b$. Adjacent circles are linked by preimages of $d$ and $e$: $\widetilde{b}_1$,$\widetilde{b}_2$ by $\widetilde{d}_1$; $\widetilde{b}_2$,$\widetilde{b}_3$ by $\widetilde{e}_2$ and so forth. This pattern is repeated in $k$ blocks from top to bottom in Figure 9. Now, for each of the $k$ factors $\alpha_{i_j}^{s_j}$ in $\beta$ perform a $s_j/2$ twist in the $\widetilde{d}$ or $\widetilde{e}$ linking $\widetilde{b}_{i_j}$,$\widetilde{b}_{i_j+1}$ in the $j$-th block. This changes the circles $\widetilde{b}_1, \ldots, \widetilde{b}_n$ into $L$. Finally, $S^3 \setminus J$ is a $(2^r k)$-fold cover of $S^3 \setminus L_1$ as explained in **2.2.2**.

**Acknowledgements.** This paper grew out of questions put to me by D. Long. I also thank J. Hubbard for help in getting started with TEX.

## References

[B-Z] G. Burde and H. Zieschang, *Knots*, deGruyter Studies in Mathematics, **5**, W. deGruyter, Berlin, New York, 1985, MR 87b:57004, Zbl 0568.57001.

[Li] W.B.R. Lickorish, *A representation of orientable combinatorial* 3*-manifolds*, Annals of Math., **76** (1962), 531-538, MR 27 #1929, Zbl 0106.37102.

[R] A. Reid, *Arithmeticity of knot complements*, J. Lond. Math. Soc.(2), **43** (1991), 171-184, MR 92a:57011, Zbl 0847.57013.

[W] J. Weeks, *SnapPea* 2.4 PPC, Available from http://www.northnet.org/weeks.

[Wi] N. Wielenberg, *The structure of certain subgroups of the Picard group*, Math. Proc. Camb. Phil. Soc., **84** (1978), 427-436, MR 80b:57010, Zbl 0399.57005.

Received June 27, 2000 and revised February 14, 2001.

IRMAR
Université de Rennes 1
35042 Rennes cedex
France
*E-mail address*: baker@univ-rennes1.fr