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AN APPROACH TO THE TANGENTIAL POISSON COHOMOLOGY BASED ON EXAMPLES IN DUALS OF LIE ALGEBRAS

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In this paper, we are interested in the tangential Poisson cohomology (TP-cohomology) of regular Poisson manifolds, a cohomology which was first defined by Lichnerowicz using contravariant tensor fields. We show that for a regular Poisson manifold M, the TP-cohomology coincides with the leafwise de Rham (or Čech) cohomology of the symplectic foliation of M. Computing the spaces of such a cohomology leads actually to open and quite nontrivial problems. To get a better understanding of these difficulties, we study explicitly many examples coming from nilpotent and 3-dimensional (real) Lie algebras. For the latter, we compare the TP-cohomology and the usual Poisson cohomology (P-cohomology).

1. Introduction and motivation.

This work fits into the study of deformation quantization for the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$, more exactly of star products on $\mathfrak{g}^*$ (or on some natural open subset U of $\mathfrak{g}^*$) which restrict nicely to the coadjoint orbits contained in $\mathfrak{g}^*$ (or U). Such star products are called tangential and for a given Lie algebra $\mathfrak{g}$, they can notably be used to describe the harmonic analysis of the corresponding Lie group.

In general, tangential star products do not exist on the whole dual $\mathfrak{g}^*$ (see [**ACG**] or [**CGR**]), nevertheless we know there always exists such a star product on the dense subset Ω of maximal dimensional coadjoint orbits in $\mathfrak{g}^*$ [**Mas**]. When studying all the possible classes of tangential star products on this set Ω, we became interested in regular Poisson structures and especially in the TP-cohomology of regular Poisson manifolds. (Indeed tangential star products are governed by the TP-cohomology; for instance their classification is described by the second TP-cohomology space.) In this respect, it is worth mentioning Remarks 5 and 8 of the paper, which give an example of how our work applies to the theory of tangential star products.

With this motivation from deformation theory, we present here the result of our attempts to understand and clarify the TP-cohomology. We organize the paper as follows.

In Section 2 below, we prove that, for a regular Poisson manifold, the TP-cohomology is isomorphic to the leafwise de Rham (or Čech) cohomology of the symplectic foliation. Thus it depends only on that foliation and not on the symplectic structure along the leaves. This is a generalization of the fact that the P-cohomology of a symplectic manifold M is just the de Rham cohomology of M. We recall also some classical results of foliation theory computing the TP-cohomology of some particular regular Poisson manifolds. We compare these results with a theorem from [**Va2**] describing the P-cohomology for some specific cases.

The remaining of the paper is devoted to examples arising from Lie algebras. Indeed, each Lie algebra $\mathfrak{g}$ gives rise to a natural regular Poisson manifold, namely the union Ω of all maximal dimensional coadjoint orbits in the dual space $\mathfrak{g}^*$.

In Section 3, we consider nilpotent Lie algebras $\mathfrak{g}$. For such Lie algebras, given a Jordan-Hölder basis B, Ω has a natural layering whose first layer, say V_B, is known as the generic (dense) open subset of $\mathfrak{g}^*$ associated to B ([**ACG**, **Ver**]). It is easy to see that the TP-cohomology of V_B is trivial in degree superior to zero. We prove here that the same is true for the union $\bigcup_B V_B$, which is more canonical than V_B since it does not depend on the choice of the basis B. However, $\bigcup_B V_B$ is sometimes strictly smaller than Ω; this happens for instance in the case of the filiform Lie algebras. We will also see, by studying in details the case of the filiform Lie algebra $\mathfrak{g}_{4,1}$, that the TP-cohomology of Ω can be essentially larger than the TP-cohomology of $\bigcup_B V_B$. Such an example shows that the TP-cohomology of regular Poisson manifolds (and more generally the leafwise de Rham cohomology of foliations) can be huge even if the leaves are cohomologically trivial.

Later, in Section 4, we consider an arbitrary 3-dimensional regular Poisson manifold M and we perform the inductive computations of [**Va2**] to describe the P-cohomology spaces of M. This enables us to observe the influence of the TP-cohomology on the P-cohomology: The TP-cohomology spaces appear naturally in the decomposition of the P-cohomology spaces (see Proposition 6). Then, we examine the TP-cohomology and the P-cohomology of the regular Poisson manifold Ω arising from any 3-dimensional Lie algebra. Some of these Lie algebras can be directly treated with the help of Section 2, the others will require more attention. We conclude with some general remarks.

2. Regular Poisson manifolds and foliation theory.

2.1. Basic definitions. A Poisson manifold is a C^∞ manifold M equipped with a Poisson bracket $\{\,,\,\}$ i.e., a bilinear skew-symmetric operation on

$C^\infty(M)$ with values in $C^\infty(M)$, satisfying the Leibniz rule:

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \quad \forall f, g, h \in C^\infty(M)$$

and the Jacobi identity:

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \quad \forall f, g, h \in C^\infty(M).$$

For any manifold M, we denote by $\mathcal{V}^*(M)$ the graded space of skew-symmetric contravariant tensor fields and by $\Omega^*(M)$ the graded space of forms on M.

Let M be a Poisson manifold. Since the Poisson bracket is skew-symmetric and satisfies the Leibniz rule, there exists a unique tensor field Λ in $\mathcal{V}^2(M)$ such that

$$\{f, g\} = \Lambda(df, dg) \quad \forall f, g \in C^\infty(M).$$

This tensor field is usually called the Poisson bivector of M.

To express the Jacobi identity in terms of Λ, we recall that the commutator bracket of vector fields extends to the Schouten bracket, uniquely defined on $\mathcal{V}^*(M)$ by the relations:

(i) $[P, Q] = -(-1)^{(p-1)(q-1)}[Q, P] \quad \forall P \in \mathcal{V}^p(M), \ \forall Q \in \mathcal{V}^q(M)$.
(ii) For P in $\mathcal{V}^p(M)$, $[P, .]$ is a derivation of degree $p - 1$.

The Schouten bracket satisfies the graded Jacobi identity:

$$[P, [Q, R]] = [[P, Q], R] + (-1)^{(p-1)(q-1)}[Q, [P, R]],$$

for P in $\mathcal{V}^p(M)$, Q in $\mathcal{V}^q(M)$, R in $\mathcal{V}^*(M)$, and thus defines a graded Lie algebra structure on $\mathcal{V}^*(M)$ with the shifted grading: $\deg(S) = s - 1$ if S belongs to $\mathcal{V}^s(M)$. One can then check [**Li1**, **Wei**] that the bracket on $C^\infty(M)$ given by Λ satisfies the Jacobi identity if and only if $[\Lambda, \Lambda] = 0$ holds.

In the sequel, we shall denote by (M, Λ) our Poisson manifold. If f is a C^∞ function on M, we call Hamitonian vector field of f the vector field corresponding to the derivation $\{f, .\}$. With (M, Λ) is associated a bundle map:

$$\begin{array}{rccc} \# : T^*M & \longrightarrow & TM \\ \alpha & \longmapsto & \alpha^\# \end{array}$$

defined by

$$\alpha^\#(\beta) = \Lambda(\alpha, \beta)$$

for any α, β in $T^*_x M$. Finally, the rank of M at a point x is by definition the rank of the linear mapping $\#_x : T^*_x M \to T_x M$. If it is constant, M is said to be regular. In particular, if it is everywhere equal to the dimension of M, $\#$ is an isomorphism and M is a symplectic manifold whose symplectic structure ω is given by $\omega = \#^{-1}(\Lambda)$.

We are now ready to define the Poisson cohomology of the Poisson manifold (M, Λ).

Definition 1. Let $\sigma : \mathcal{V}^*(M) \to \mathcal{V}^{*+1}(M)$ be the operator given by

$$\sigma = [\Lambda, .].$$

Due to the graded Jacobi identity for $[\,,\,]$, σ is a coboundary operator (i.e., $\sigma^2 = 0$). The complex $(\mathcal{V}^*(M), \sigma)$ is called the Poisson complex of M and the corresponding cohomology $H^*_\Lambda(M)$ is the P-cohomology of M.

See [**Hue**] for an algebraic definition of the P-cohomology and [**APP**, **CW**] for some more recent results about the P-cohomology.

The interpretation of the first few P-cohomology spaces is well-known. Indeed, $H^0_\Lambda(M)$ is the space $I(M)$ of Casimir functions over M i.e., those whose Hamiltonian vector fields are trivial; $H^1_\Lambda(M)$ consists of infinitesimal Poisson automorphisms of M (Poisson vector fields) modulo inner automorphisms (Hamiltonian vector fields); $H^2_\Lambda(M)$ classifies (modulo the trivial deformations) the formal deformations of the Poisson structure Λ (with the form $\Lambda + t\alpha_1 + t^2\alpha_2 + \dots$) and finally $H^3_\Lambda(M)$ houses the obstructions to extend a formal deformation from one step (in powers of t) to the next.

Let us just recall that the equivalence classes of star products on (M, Λ) are in one to one correspondence with the equivalence classes of formal deformations of Λ (see [**Kon**] for more details).

Note also that the Poisson bracket gives rise to a bracket $\{\,,\,\}$ on $\Omega^1(M)$, which is the unique extension of the bracket given by $\{df, dg\} = d\{f, g\}$ such that

$$\{\alpha, f\beta\} = f\{\alpha, \beta\} + (\alpha^\# f)\beta \quad \forall f \in C^\infty(M) \quad \forall \alpha, \beta \in \Omega^1(M).$$

This bracket is defined by

$$\{\alpha, \beta\} = L_{\alpha^\#}(\beta) - L_{\beta^\#}(\alpha) - d(\Lambda(\alpha, \beta)) \quad \forall \alpha, \beta \in \Omega^1(M)$$

and one can prove (see [**Va2**, p. 44])

$$\begin{aligned}\sigma Q(\alpha_0, \dots, \alpha_k) = &\sum_{i=0}^{k} (-1)^i \alpha_i^\# (Q(\alpha_0, \dots, \hat{\alpha_i}, \dots, \alpha_k)) \\ &+ \sum_{i<j} (-1)^{i+j} Q(\{\alpha_i, \alpha_j\}, \alpha_0, \dots, \hat{\alpha_i}, \dots, \hat{\alpha_j}, \dots, \alpha_k)\end{aligned}$$

where Q is in $\mathcal{V}^k(M)$ and the α_i are 1-forms on M.

The latter expression can be used ([**Va2**]) to see that the natural extension $\widetilde{\#}$ of $\#$ to forms:

$$\widetilde{\#}\lambda(\alpha_0, \dots, \alpha_{q-1}) = (-1)^q \lambda(\alpha_0^\#, \dots, \alpha_{q-1}^\#)$$

intertwines σ and the de Rham differential d and thus induces a natural homomorphism from $H^*_{DR}(M)$ to $H^*_\Lambda(M)$. This homomorphism is trivially an isomorphism in the symplectic case (see also [**Kos**] or [**Li1**]).

Some preparatory material related to a foliation is now needed. Let $(M,\mathcal{F})$ be an arbitrary foliated manifold and denote by $T\mathcal{F}$ the tangent bundle of $\mathcal{F}$. As in [**DH**] or [**Li2**], one can choose a transversal distribution $\nu\mathcal{F}$ such that

$$TM = T\mathcal{F} \oplus \nu\mathcal{F} \quad \text{and} \quad T^*M = T^*\mathcal{F} \oplus \nu^*\mathcal{F}.$$

These decompositions induce a bigrading of the space $\mathcal{V}^*(M)$ of contravariant tensor fields and of the space $\Omega^*(M)$ of forms on M, namely

$$\mathcal{V}^*(M) = \bigoplus_{p,q} \mathcal{V}_{p,q}(M) \quad \text{and} \quad \Omega^*(M) = \bigoplus_{p,q} \Omega_{p,q}(M)$$

where $\mathcal{V}_{p,q}(M)$ (resp. $\Omega_{p,q}(M)$) denotes the space of sections of the bundle $\wedge^q(T\mathcal{F}) \otimes \wedge^p(\nu\mathcal{F})$ (resp. $\wedge^q(T^*\mathcal{F}) \otimes \wedge^p(\nu^*\mathcal{F})$).

Elements of $\mathcal{V}_{p,q}(M)$ and $\Omega_{p,q}(M)$ are said to be of type (p,q). Moreover, an operator will be homogeneous of type (a,b) if it sends elements of type (p,q) to elements of type $(p+a, q+b)$. We recall that the de Rham differential d can be decomposed into $d = d' + d'' + d_{2,-1}$ where d' is of type $(1,0)$, d'' denotes the leafwise de Rham differential of the foliation $\mathcal{F}$ and is of type $(0,1)$, and $d_{2,-1}$ is of type $(2,-1)$.

Assume now that (M,Λ) is a regular Poisson manifold and denote by $\mathcal{F}$ the symplectic foliation of M. As above, one can choose a transversal distribution $\nu\mathcal{F}$ for M. It was shown in [**Va2**] that with respect to a given choice of $\nu\mathcal{F}$, the coboundary operator σ, introduced in Definition 1, has a well-defined decomposition $\sigma = \sigma' + \sigma''$ where σ' is of type $(-1,2)$ and σ'' is of type $(0,1)$.

On the other hand, Lichnerowicz has shown in [**Li2**] that one gets a consistent theory by restricting the P-cohomology complex $(\mathcal{V}^*(M),\sigma)$ to tangential multivector fields. The resulting cohomology is known as the TP-cohomology of the regular Poisson manifold (M,Λ). In fact, the same cohomology can be defined by using the transversal distribution $\nu\mathcal{F}$ and the types of the tensor fields. Indeed, we have:

Definition 2. The TP-cohomology complex of the regular Poisson manifold (M,Λ) is $\bigoplus_q \mathcal{V}_{0,q}(M)$ with the coboundary operator σ'' and

$$H^q_{\Lambda,\tan}(M) = \frac{\operatorname{Ker}(\sigma'' : \mathcal{V}_{0,q}(M) \to \mathcal{V}_{0,q+1}(M))}{\operatorname{Im}(\sigma'' : \mathcal{V}_{0,q-1}(M) \to \mathcal{V}_{0,q}(M))}$$

is the *qth* TP-cohomology space of (M,Λ).

It is clear that $H^0_{\Lambda,\tan}(M) = H^0_\Lambda(M)$. Moreover, as mentioned in the introduction, the TP-cohomology plays an important role in the theory of

tangential star products. Indeed, the derivations of a given tangential star product on M, modulo inner derivations, are parameterized by sequences of elements in $H^1_{\Lambda,\tan}(M)$; similarly equivalences of tangential star products on M are classified at each step by $H^2_{\Lambda,\tan}(M)$ and finally the obstructions to construct such a star product are localized in $H^3_{\Lambda,\tan}(M)$ (this last point could be omitted since a tangential deformation always exists on M [**Mas**]).

2.2. Leafwise de Rham cohomology. In this paragraph, we want to prove that the TP-cohomology of a regular Poisson manifold (M,Λ) is isomorphic to the leafwise de Rham cohomology of the symplectic foliation and therefore does not depend on the symplectic structure along the leaves.

To this end, we consider first the general case of a foliated manifold $(M,\mathcal{F})$. For all p, we denote by $\Phi^p(\mathcal{F})$ the sheaf of (germs of) projectable p-forms (i.e., those induced by forms on the space of leaves). In particular, $\Phi^0(\mathcal{F})$ is the sheaf of the germs of functions on M that are constant along the leaves of $\mathcal{F}$ (see [**Va2**]). Let us fix a transversal distribution $\nu\mathcal{F}$ and consider the sheaf cohomology of $\Phi^p(\mathcal{F})$, that is

$$H^q(M,\Phi^p(\mathcal{F})) = \frac{\operatorname{Ker}\left(d'' : \Omega_{p,q}(M) \to \Omega_{p,q+1}(M)\right)}{\operatorname{Im}\left(d'' : \Omega_{p,q-1}(M) \to \Omega_{p,q}(M)\right)}.$$

A change of $\nu\mathcal{F}$ leads to an isomorphism in the corresponding cohomology spaces. To see this, let us introduce more notations. Let $N\mathcal{F}$ be the normal bundle of $\mathcal{F}$, that is $N\mathcal{F} = TM/T\mathcal{F}$. Denote by $\Omega^q_{[p]}(\mathcal{F})$ the space of sections of the bundle $\wedge^q(T^*\mathcal{F}) \otimes \wedge^p(N^*\mathcal{F})$. The elements of $\Omega^0_{[p]}(\mathcal{F})$ are usually called normal forms and those of $\Omega^q_{[p]}(\mathcal{F})$ are the tangential q-forms with values in the normal p-forms. The Lie algebra of tangential vector fields acts naturally (by Lie derivative) on the normal forms and the leafwise de Rham differential $d_{\mathcal{F}}$ acts on $\Omega^q_{[p]}(\mathcal{F})$ in the usual way:

$$\begin{aligned} &d_{\mathcal{F}}(\omega_x)(X_0,\dots,X_q) \\ &= \sum_{i=0}^{q}(-1)^i L_{X_i}(\omega_x(X_0,\dots,\hat{X}_i,\dots,X_q)) \\ &\quad + \sum_{i<j}(-1)^{i+j}\omega_x([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_q), \end{aligned}$$

if x is in M, ω in $\Omega^q_{[p]}(\mathcal{F})$ and the X_i in $T_x\mathcal{F}$. Since $d^2_{\mathcal{F}} = 0$, we obtain a complex $\left(\oplus_q \Omega^q_{[p]}(\mathcal{F}), d_{\mathcal{F}}\right)$ whose cohomology is denoted hereafter by $H^*_{[p]}(\mathcal{F})$.

With these notations, we can prove:

Lemma 1. *For all p, the complexes $\left(\bigoplus_q \Omega_{p,q}(M), d''\right)$ and $\left(\bigoplus_q \Omega^q_{[p]}(\mathcal{F}), d_{\mathcal{F}}\right)$ are isomorphic. In particular, $H^*(M, \Phi^p(\mathcal{F}))$ and $H^*_{[p]}(\mathcal{F})$ coincide and are independent of the choice of $\nu\mathcal{F}$.*

Proof. For each x in M, let $\psi_x : N_x\mathcal{F} \to \nu_x\mathcal{F}$ be the natural isomorphism of vector spaces. That is, if Y is in T_xM, then $\psi_x(\widetilde{Y}) = \pi(Y)$ where $\widetilde{Y}$ denotes the class of Y in $N_x\mathcal{F}$ and π is the projection $\pi : TM \to \nu\mathcal{F}$.

The mapping $f : \Omega_{p,q}(M) \to \Omega^q_{[p]}(M)$ given by

$$f(\omega_x)(X_1, \dots, X_q)(\widetilde{Y_1}, \dots, \widetilde{Y_p}) = \omega_x(X_1, \dots, X_q, \psi_x(\widetilde{Y_1}), \dots, \psi_x(\widetilde{Y_p})),$$

here ω is in $\Omega_{p,q}(M)$, the X_i in $T_x\mathcal{F}$, and the Y_j in T_xM, is clearly bijective. In addition, we have

$$\begin{aligned}
&d_{\mathcal{F}} f(\omega_x)(X_0, \dots, X_q)(\widetilde{Y_1}, \dots, \widetilde{Y_p}) \\
&= \sum_{i=0}^{q} (-1)^i X_i \Big(f(\omega_x)(X_0, \dots, \hat{X}_i, \dots, X_q)(\widetilde{Y_1}, \dots, \widetilde{Y_p}) \Big) \\
&- \sum_{i=0}^{q} \sum_{l=1}^{p} (-1)^{i+l-1} f(\omega_x)(X_0, \dots, \hat{X}_i, \dots, X_q)(\widetilde{[X_i, Y_l]}, \widetilde{Y_1}, \dots, \hat{Y}_l, \dots, \widetilde{Y_p}) \\
&+ \sum_{0 \le i < j \le q} (-1)^{i+j} f(\omega_x)([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots X_q)(\widetilde{Y_1}, \dots, \widetilde{Y_p}) \\
&= f(d''\omega_x)(X_0, \dots, X_q)(\widetilde{Y_1}, \dots, \widetilde{Y_p}).
\end{aligned}$$

Thus, $d_{\mathcal{F}} \circ f = f \circ d''$. This ends the proof.

Remark that $H^q(M, \Phi^0(\mathcal{F}))$ $\Big($or $H^q_{[0]}(\mathcal{F})\Big)$ is nothing else but the leafwise de Rham cohomology of the foliation $\mathcal{F}$.

Let us now restrict ourselves to the case of a regular Poisson manifold (M, Λ). Recall that $I(M)$ denotes the space of Casimir functions over M, in other words the space of smooth functions on M that are constant along the leaves of the symplectic foliation. Then we have:

Theorem 1. *Let (M, Λ) be a regular Poisson manifold, $\mathcal{F}$ the symplectic foliation of M and $\nu\mathcal{F}$ a transversal distribution for M. Then $\left(\bigoplus_q \mathcal{V}_{0,q}(M), \sigma''\right)$ and $\left(\bigoplus_q \Omega_{0,q}(M), d''\right)$ are isomorphic as complexes of $I(M)$-modules. In particular, for all q, $H^q_{\Lambda,\tan}(M)$ and $H^q(M, \Phi^0(\mathcal{F}))$ are isomorphic $I(M)$-modules.*

Proof. It is not difficult to prove that the natural extension $\widetilde{\#}$ of $\#$ realizes an $I(M)$-modules isomorphism between $\Omega_{0,q}(M)$ and $\mathcal{V}_{0,q}(M)$. This isomorphism satisfies $\sigma'' \circ \widetilde{\#} = -\widetilde{\#} \circ d''$. The result follows.

2.3. Čech cohomology. Let $(M, \mathcal{F})$ be a foliated manifold. We shall say that a locally finite covering $\mathcal{U} = (U_i)$ of M is a good covering if for all $q > 0$, all k and all $i_1, \dots, i_k$,

$$H^q\left(U_{i_1,\dots,i_k}, \Phi^0\left(\mathcal{F}_{|U_{i_1,\dots,i_k}}\right)\right) = \{0\},$$

where $U_{i_1,\dots,i_k} = U_{i_1} \cap \dots \cap U_{i_k}$ and $\mathcal{F}_{|U_{i_1,\dots,i_k}}$ denotes the foliation induced by $\mathcal{F}$ in $U_{i_1,\dots,i_k}$ (i.e., if the leaves of $\mathcal{F}$ are noted $\mathcal{L}_\alpha$, the leaves of $\mathcal{F}_{|U_{i_1,\dots,i_k}}$ are the connected components of the intersections $\mathcal{L}_\alpha \cap U_{i_1,\dots,i_k}$).

We recall that, for each foliated manifold $(M, \mathcal{F})$, there exists affine connections on M, which are torsion free and adapted to $\mathcal{F}$ in the sense of [**Li2**]. Let us now prove the existence of good coverings.

Lemma 2. *Let $(M, \mathcal{F})$ be a foliated manifold and Γ an affine connection on M, which is torsion free and adapted to $\mathcal{F}$. Then, every atlas $\{U_i, \varphi_i\}$ on M of distinguished charts has an open refinement $\{V_l, \psi_l\}$ such that:*

(1) *Each V_l has compact closure.*
(2) *(V_l) is locally finite and is a good covering of M.*

Proof. By taking an open refinement if necessary, we may assume that (U_i) is locally finite and that each U_i has compact closure. Let (U_i') be an open refinement of (U_i) (with the same index set) such that $\overline{U_i'} \subset U_i$ for all i, and note $\varphi_i' = \varphi_{i|_{U_i'}}$. For each x in M, let W_x be a normal neighborhood of x, which is small enough to satisfy the following properties:

- for each point a in W_x, there exists a normal neighborhood N_a of 0 in T_aM such that $\exp : N_a \to W_x$ is a diffeomorphism;
- W_x is geodesically convex;
- W_x is contained in some U_i'.

Note $\varphi_x = \varphi_{i|_{W_x}}'$ and, for each k, set

$$B_k = \{(W_x, \varphi_x) : W_x \cap \overline{U_k'} \neq \emptyset\}.$$

Since $\overline{U_k'}$ is compact, there exists a finite subfamily B_k' of B_k, which covers $\overline{U_k'}$. Then the family $\mathcal{B} = \bigcup_k B_k'$ is an open refinement $\{V_l, \psi_l\}$ of $\{U_i, \varphi_i\}$, which satisfies (1) and (2). In fact, it is clear by construction that each V_l has compact closure and that (V_l) is locally finite. To show that (V_l) is also a good covering of M, we shall now prove that each V_l has geodesically convex plaques. Let y be in some V_l and denote by P_y the plaque of $\mathcal{F}_{|V_l}$ containing y. Take two points a and b in P_y. By assumption, there exists a normal neighborhood N_a of 0 in T_aM such that $\exp : N_a \to V_l$ is a diffeomorphism.

a) First, we shall prove that $\exp(N_a \cap T_a\mathcal{F}) = P_a$. Let Y be in $N_a \cap T_a\mathcal{F}$. Denote by τ the geodesic of V_l with the initial condition (a, Y):

$$\tau(0) = a \quad \text{and} \quad \dot{\tau}(0) = \tau_{*_0}\left(\frac{d}{dt}\right) = Y.$$

This curve τ is at least defined for $0 \leq t \leq 1$. We express it in the distinguished chart (V_l, ψ_l) as follows

$$x(t) := \psi_l(\tau(t)) = (x^i(t), x^u(t))$$

where $i, j \ldots = 1, \ldots, r$ (resp. $u, v, \ldots = 1, \ldots, s$) denote the tangential (resp. transverse) indexes. Since τ is a geodesic, it satisfies

$$\begin{cases} \dfrac{d^2x^i}{dt^2} &= -\displaystyle\sum_{1\leq J,K\leq r+s} \Gamma^i_{JK}(x)\frac{dx^J}{dt}\frac{dx^K}{dt} \qquad \text{for} \quad i = 1, \ldots, r \\ \dfrac{d^2x^u}{dt^2} &= -\displaystyle\sum_{1\leq J,K\leq r+s} \Gamma^u_{JK}(x)\frac{dx^J}{dt}\frac{dx^K}{dt} \qquad \text{for} \quad u = 1, \ldots, s. \end{cases}$$

Moreover, since Γ is torsion free and adapted to $\mathcal{F}$, we have

$$\Gamma^u_{iA} = \Gamma^u_{Ai} = 0 \quad \forall 1 \leq i \leq r,\ 1 \leq u \leq s,\ 1 \leq A \leq r+s.$$

It follows that

$$\frac{d^2x^u}{dt^2} = -\sum_{1\leq v,w\leq s} \Gamma^u_{vw}(x)\frac{dx^v}{dt}\frac{dx^w}{dt} \quad \text{for} \quad u = 1, \ldots, s.$$

Let us introduce the notation

$$f(t) = (x^i(t))_{1\leq i\leq r}, \quad g(t) = (x^u(t))_{1\leq u\leq s}.$$

Then, the above system can be reduced to two ordinary differential equations of the form

$$f''(t) = F(f'(t), f(t), g'(t), g(t)) \tag{1}$$

$$g''(t) = G(g'(t), g(t), f(t)). \tag{2}$$

Since $\dot{\tau}(0) = Y$ is in $T_a\mathcal{F}$, we shall have $g'(0) = 0$. Now, for fixed f and with the initial conditions $g(0)$ and $g'(0) = 0$, (2) has a unique solution namely $g = cst = g(0)$. Let us denote by $f_0(t) = (a^i(t))_{1\leq i\leq r}$ the unique solution of (1) when $g = cst$ and with $f(0)$ and $f'(0)$ as initial conditions. Then, we have $(x^i(t), x^u(t)) = (a^i(t), x^u(0))$ for all $0 \leq t \leq 1$. In particular, $\tau(1) = \exp(Y)$ belongs to P_a. We get thus the inclusion $\exp(N_a \cap T_a\mathcal{F}) \subset P_a$. The equality comes from the fact that $\exp(N_a \cap T_a\mathcal{F})$ is both open and closed in P_a, and that P_a is connected.

b) Now, let γ be the unique minimizing geodesic of V_l, joining a and b. We may write

$$\gamma : [0, 1] \to V_l, \quad \gamma(t) = \exp(tX)$$

where $\gamma(0) = a$, $\gamma(1) = b$ and X is in N_a. Since $\gamma(1) = b = \exp(X)$ is in $P_a (= P_y)$ and using the equality proved in a), we see that X is in fact in $N_a \cap T_a\mathcal{F}$. Thus, γ lies entirely in P_y.

We have proved that V_l has geodesically convex (hence contractible) plaques. It is of course the same for every finite intersection $V_{l_1,\dots,l_k}$. Therefore, each foliation $\mathcal{F}_{|V_{l_1,\dots,l_k}}$ is a product foliation by contractible leaves. Following a result of [**Va2**], we shall mention just in the next paragraph (Theorem 3), this means that $H^q(V_{l_1,\dots,l_k}, \Phi^0(\mathcal{F}_{|V_{l_1,\dots,l_k}})) = \{0\}$ for all $q > 0$. Lemma 2 is proved.

Now, for any sheaf A on M (in particular for $\Phi^0(\mathcal{F})$), we shall denote by $\check{C}^k(\mathcal{U}, A)$ the space of k-Čech cochains of A with respect to a covering $\mathcal{U}$ of M, by $\check{\delta}$ the Čech coboundary and $\check{\mathrm{H}}^*(\mathcal{U}, A)$ the cohomology corresponding to the Čech complex $(\check{C}^*(\mathcal{U}, A), \check{\delta})$.

The purpose of the following proposition is to prove that the leafwise de Rham cohomology of a foliation $\mathcal{F}$ coincides with the Čech cohomology of the sheaf $\Phi^0(\mathcal{F})$. This can be convenient to calculate the TP-cohomology (see later in Section 3).

Proposition 1. *Let $(M, \mathcal{F})$ be a foliated manifold and $\nu\mathcal{F}$ a transversal distribution for M. Let also $\mathcal{U} = (U_i)$ be a good covering of M and (h_i) a partition of unity subordinate to $\mathcal{U}$. For a k-Čech cocycle c, let ω_c be the d''-closed k-form on M defined by*

$$\omega_{c|U_i} = (-1)^{\frac{k(k+1)}{2}} \sum_{i_1,\dots,i_k} c_{i_1,\dots,i_k,i} d''h_{i_1} \wedge \dots d''h_{i_k}.$$

Then, the homomorphism $\varphi : \check{H}^k(\mathcal{U}, \Phi^0(\mathcal{F})) \to H^k(M, \Phi^0(\mathcal{F}))$ mapping the cohomology class $[c]$ to the cohomology class $[\omega_c]$ is an isomorphism for all k.

Proof. The result can be proved in the same way as Brylinski did in [**Bry**] to show that the Čech cohomology of the constant sheaf $\mathbb{R}_M$ on any manifold M coincides with the de Rham cohomology of M. The key point of the proof is to consider the Čech double complex $K^{**} = \check{C}^*(\mathcal{U}, \underline{\Omega_{0,*}(M)})$ where underlining denote sheaves of germs. The total cohomology of K^{**}, also called Čech hypercohomology, is by definition the cohomology corresponding to the complex (K^*, D) where $K^n = \bigoplus_{p+q=n} K^{p,q}$ and $D = \check{\delta} + (-1)^p d''$ in degree (p, q). It is known ([**Bry**, p. 28]) that the natural spectral sequences associated to K^* lead to a canonical isomorphism between $\check{H}^k(\mathcal{U}, \Phi^0(\mathcal{F}))$ and $H^k(M, \Phi^0(\mathcal{F}))$. To prove this isomorphism is just φ, one needs first to see c and ω_c as elements of $\check{C}^k(\mathcal{U}, \underline{\Omega_{0,0}(M)})$ and $\check{C}^0(\mathcal{U}, \underline{\Omega_{0,k}(M)})$ respectively. Then one can show, as in [**Bry**, p. 45], that $[c]$ and $[\underline{\omega_c}]$ correspond to each

other from the point of view of hypercohomology. The proposition follows directly.

2.4. Review of some classical results. We begin this paragraph by mentioning two well-known results of foliation theory. These results, which can be found in [**DH**] and [**Va2**] respectively, provide the computation of the TP-cohomology in some particular cases.

Theorem 2. *Let $(M,\mathcal{F})$ be a foliated manifold and r some integer. Assume that the foliation $\mathcal{F}$ is given by a submersion $\Pi : M \to B$ (B being a Hausdorff manifold), and that any leaf L of $\mathcal{F}$ is connected and satisfies $H^q_{DR}(L) = \{0\}$ for all $0 < q \leq r$. Then,*

$$H^q(M, \Phi^0(\mathcal{F})) = \begin{cases} C^\infty(B) & \text{if } q = 0 \\ \{0\} & \text{if } 0 < q \leq r. \end{cases}$$

Theorem 3. *Let L and R be two smooth manifolds and $\mathcal{F}$ the foliation of $M = L \times R$ by the leaves $L \times \{x\}$ where x is in R. Assume that L has finite Betti numbers. Then,*

$$H^q(M, \Phi^p(\mathcal{F})) = H^q_{DR}(L) \otimes \Omega^p(R).$$

Next, in Theorem 4, we recall an important result from [**Va1**, **Va2**]. This result will be used in Section 4 when we shall compute the P-cohomology spaces associated to 3-dimensional Lie algebras.

Theorem 4. *Let $M = S \times R$ be a regular Poisson manifold whose regular Poisson structure Λ is transversally constant with respect to the transversal distribution $\nu\mathcal{F} = TR$ (i.e., the symplectic foliation $\mathcal{F}$ of $M = S \times R$ is defined by a fixed symplectic structure of S). Suppose that S has finite Betti numbers. Then,*

$$H^q_\Lambda(M) \simeq \bigoplus_{0 \leq k \leq q} H^k_{DR}(S) \otimes \Omega^{q-k}(R).$$

Remark 1. One can use Theorem 4 to show that, with constrast to the TP-cohomology, the P-cohomology not only depends on the symplectic foliation but also on the symplectic structure along the leaves. Indeed, let M be $\mathbb{S}^2 \times \mathbb{R}^*_+$ and denote by ω the standard symplectic structure on the unit sphere $\mathbb{S}^2$. If M is endowed with the regular Poisson structure defined by the same symplectic structure ω on each leaf, then the P-cohomology of M is given by Theorem 4. But, if the same manifold M is viewed as $\mathfrak{su}(2)^* \backslash \{0\}$ with its usual linear Poisson structure, then each leaf $\mathbb{S}^2 \times \{t\}$ ($t \in \mathbb{R}^*_+$) has a different symplectic structure, namely $t\omega$, and Theorem 4 is no more valid for M (we will see the actual computation of $\mathfrak{su}(2)^* \backslash \{0\}$ in §4.7).

Apart from the specific cases of Theorems 2 and 3, the task of computing the TP-cohomology still remains unsolved. To better understand the TP-cohomology of general regular Poisson manifolds and to make some

comparison between the TP-cohomology and the P-cohomology, we devote the next two sections to a large number of explicit computations related to Lie algebras. More precisely, the regular Poisson manifolds we shall consider in the rest of the paper are Poisson submanifolds of the union Ω of all maximal dimensional coadjoint orbits in the dual of a given Lie algebra. That makes sense since the dual $\mathfrak{g}^*$ of any Lie algebra $\mathfrak{g}$ can be endowed with a natural Poisson structure, the well-known Lie Poisson structure [**Wei**]; the leaves of the symplectic foliation of $\mathfrak{g}^*$ being exactly the coadjoint orbits.

3. The nilpotent case.

Suppose that $\mathfrak{g}$ is an m-dimensional nilpotent Lie algebra. Denote by $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$ and by G the connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \ldots \subset \mathfrak{g}_m = \mathfrak{g}$ be a flag of $\mathfrak{g}$ ($\dim \mathfrak{g}_i = i$) such that $[\mathfrak{g}, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i-1}$ for all i in $\{1, \ldots, m\}$. Let also $B = (X_1, \ldots, X_m)$ be a Jordan-Hölder basis adapted to $(\mathfrak{g}_i)_i$ that is $\mathfrak{g}_i = \mathbb{R}X_1 \oplus \ldots \oplus \mathbb{R}X_i$ for all i. Then, $\mathfrak{g}^*$ (or Ω) has a natural layering which can be summarized as follows (see also [**ACG**, **Bon**, **Puk**, **Ver**]).

For μ in $\mathfrak{g}^*$, we define the set of indexes $J_\mu = \{j : X_j \notin \mathfrak{g}_{j-1} + \mathfrak{g}_\mu\}$, where $\mathfrak{g}_\mu = \{X \in \mathfrak{g} : \forall Y \in \mathfrak{g}, \langle \mu, [X, Y]\rangle = 0\}$. If $J_\mu = \{j_1 < \ldots < j_{2r}\}$, we shall have $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathbb{R}X_{j_1} \ldots \oplus \mathbb{R}X_{j_{2r}}$. Let $\Delta = \{J_\mu, \mu \in \mathfrak{g}^*\}$. For e in Δ, we define the following layer

$$\Omega_B^e = \{\mu \in \mathfrak{g}^* : J_\mu = e\}.$$

By construction, each layer is a G-invariant subset of $\mathfrak{g}^*$ and $\mathfrak{g}^*$ (resp. Ω) is a disjoint finite union of layers $\mathfrak{g}^* = \cup_{e\in\Delta} \Omega_B^e$ (resp. $\Omega = \cup_{e\in\Delta}(\Omega_B^e \cap \Omega)$). Note that all the orbits contained in a given layer have the same dimension (card e).

Now, let Ω_B^e be an arbitrary layer of $\mathfrak{g}^*$ and assume that the orbits contained in Ω_B^e are $2r$-dimensional. Then, there exists on $\mathfrak{g}^*$

(i) $m - 2r$ polynomial functions $z_1, \ldots, z_{m-2r}$
(ii) $2r$ rational functions $p_1, \ldots, p_r, q_1, \ldots, q_r$ which are regular on Ω_B^e

such that:

- the polynomial functions $z_1, \ldots, z_{m-2r}$ separate the orbits contained in Ω_B^e;
- for each orbit O contained in Ω_B^e, there is a diffeomorphism (a global Darboux chart) $\varphi : O \to \mathbb{R}^{2r}$ of symplectic manifolds defined by the functions p_i, q_j.

The first layer, noted V_B, has additional properties: It is a Zariski dense open subset of $\mathfrak{g}^*$, it contains only orbits of maximal dimension $2d$ and the polynomial functions $z_1, \ldots, z_{m-2d}$ separating the orbits of V_B are G-invariant. Moreover, if we identify the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$ with the space of polynomial functions on $\mathfrak{g}^*$ and denote by $S(\mathfrak{g})^G$ the subring

of $S(\mathfrak{g})$ of the G-invariant polynomial functions, then the quotient field of $S(\mathfrak{g})^G$ coincides exactly with the field $\mathbb{R}(z_1, \ldots, z_{m-2d})$ of rational functions in the z_k variables. The open set V_B is usually called the generic set associated to the basis B, the orbits contained in V_B are the generic orbits and the corresponding polynomial functions $z_1, \ldots, z_{m-2d}$ are the generic invariants. Since the symplectic foliation of V_B is a product foliation whose leaves are contractible, it follows immediately from Theorem 3 that the TP-cohomology of V_B is trivial in degree superior to zero. The next proposition claims it is even possible to get rid of the choice of the basis B.

Proposition 2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Consider the union $\bigcup_B V_B$ of all possible generic sets associated to Jordan-Hölder bases B of $\mathfrak{g}$ and denote by Λ its regular Poisson structure. Then,*

$$H^0_{\Lambda,\tan}\left(\bigcup_B V_B\right) = I\left(\bigcup_B V_B\right) \quad \text{and} \quad H^q_{\Lambda,\tan}\left(\bigcup_B V_B\right) = \{0\} \quad \forall q > 0.$$

Proof. We observe first that $\Pi : \bigcup_B V_B \to \left(\bigcup_B V_B\right)/G$ is a locally trivial fibration thus a submersion. Now, since the orbits contained in $\bigcup_B V_B$ are connected and cohomologically trivial, the result directly comes from Theorem 2 (Section 2).

Remark 2. It is known that V_B is in general strictly included in the set Ω of all maximal dimensional coadjoint orbits (see [**SG**]). Unfortunately, $\bigcup_B V_B$ can also be strictly smaller than Ω. For instance, in the case of the filiform Lie algebras (defined in [**CG**] or [**GK**]), all the V_B coincide and are distinct from Ω.

Now, the following result is very convenient and quite efficient for many examples.

Proposition 3. *Let $\mathfrak{g}$ be an m-dimensional nilpotent Lie algebra, $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$ and G the connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Denote by Ω the union of all the coadjoint orbits of maximal dimension $(2d)$. Still denote by $B = (X_i)$ a Jordan-Hölder basis of $\mathfrak{g}$, by V_B the generic set associated to the basis B and by $z_1, \ldots, z_{m-2d}$ the generic invariants separating the orbits of V_B. If $\widetilde{\Omega}$ is any open subset of Ω such that the polynomial functions $z_1, \ldots, z_{m-2d}$ separate the orbits contained in $\widetilde{\Omega}$ and that the vectors $dz_1(\mu), \ldots, dz_{m-2d}(\mu)$ are linearly independent for all μ in $\widetilde{\Omega}$, then the TP-cohomology of $\widetilde{\Omega}$ (endowed with its linear Poisson structure Λ) is given by*

$$H^0_{\Lambda,\tan}(\widetilde{\Omega}) = I(\widetilde{\Omega}) \quad \text{and} \quad H^q_{\Lambda,\tan}(\widetilde{\Omega}) = \{0\} \quad \forall q > 0.$$

Proof. Let us consider the smooth mapping

$$\begin{aligned} f: \ \widetilde{\Omega}\times\widetilde{\Omega} &\longrightarrow \mathbb{R}^{m-2d} \\ (\mu,\eta) &\longmapsto (z_1(\mu)-z_1(\eta),\dots,z_{m-2d}(\mu)-z_{m-2d}(\eta)). \end{aligned}$$

Note that $\Sigma = f^{-1}(0)$ is not empty (it contains the diagonal set $\Delta = \{(\mu,\mu) : \mu \in \widetilde{\Omega}\}$). Moreover, for all (μ,η) in Σ, the rank of the linear mapping $f_{*(\mu,\eta)}$ is, by assumption, equal to $m-2d$. As a result, Σ is a closed submanifold of $\widetilde{\Omega}\times\widetilde{\Omega}$. This exactly means (see [**Die**, p. 58]) that the space of leaves $\widetilde{\Omega}/G$ is Hausdorff and that the canonical projection $\Pi : \widetilde{\Omega} \to \widetilde{\Omega}/G$ is a submersion. The result is thus again a straightforward consequence of Theorem 2.

Remark 3. If some $\widetilde{\Omega}$ satisfies the conditions of Proposition 3 for a particular Jordan-Hölder basis, it satisfies these conditions for any Jordan-Hölder basis. Indeed, since the generic invariants $z_1,\dots,z_{m-2d}$ associated to any basis B generate the quotient field of $S(\mathfrak{g})^G$, the fact that the coadjoint orbits contained in $\widetilde{\Omega}$ are separated or not by the polynomial functions $z_1,\dots,z_{m-2d}$, and therefore Proposition 3, do not depend on the choice of the basis B.

As one can see by studying the examples of Pedersen [**Pe1**, **Pe2**], there are some nilpotent Lie algebras for which Proposition 3, and more generally Theorem 2, cannot be applied to compute the TP-cohomology of the union Ω of all maximal dimensional coadjoint orbits. To deal with these cases which are actually the most fascinating, we propose first to examine the example of $\mathfrak{g} = \mathfrak{g}_{4,1}$. The brackets of this filiform Lie algebra are

$$[X_4,X_3] = X_2, \quad [X_4,X_2] = X_1.$$

Let us identify $\mathfrak{g}^*$ with $\mathbb{R}^4$ by means of the coordinates system (x_i) of $\mathfrak{g}^*$ associated to the basis (X_i). The 2-dimensional orbits in $\mathfrak{g}^*$ are of two kinds. There are first the orbits of the points $\mu = (\mu_1,\mu_2,\mu_3,\mu_4)$ with $\mu_1 \neq 0$, which are parabolic cylinders of the form

$$O_\mu = \left\{ \left(\mu_1, s, \frac{2\mu_1\mu_3 - \mu_2^2 + s^2}{2\mu_1}, \mu_1 t \right) : (s,t) \in \mathbb{R}^2 \right\}.$$

Moreover, for the limiting case ($\mu_1 = 0$), there are the orbits of the points $\mu = (0,\mu_2,\mu_3,\mu_4)$ with $\mu_2 \neq 0$, which are affine varieties of the form

$$O_\mu = \{(0,\mu_2,s,\mu_2 t) : (s,t) \in \mathbb{R}^2\}.$$

In this example, the regular Poisson manifold (Ω,Λ) is thus the set

$$\Omega = \{(x_1,x_2,x_3,x_4) \in \mathfrak{g}^* : x_1^2 + x_2^2 \neq 0\},$$

endowed with the regular Poisson structure Λ coming from the Lie bracket. It is clear that the generic invariants associated to the basis (X_i), namely

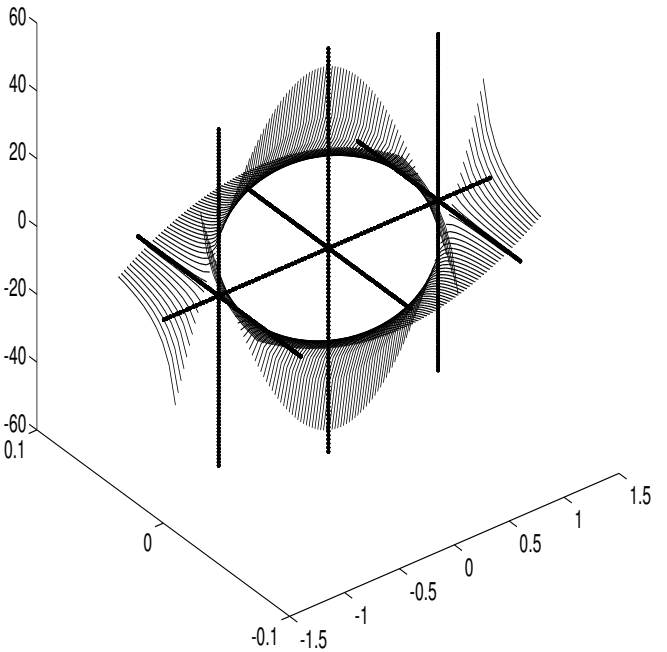

Figure 1. Orbits of the points $(0,1,0,0)$, $(0,-1,0,0)$ and $(\alpha,1,0,0)$ where $0<|\alpha|<0,1$.

$z_1 = x_1$ and $z_2 = x_2^2 - 2x_1x_3$, do not separate the 2-dimensional orbits. In fact, the space of leaves Ω/G (where G stands for the Lie group corresponding to $\mathfrak{g}$) equipped with the quotient topology is not Hausdorff. To see this, consider the orbits O^+ and O^- of the points $(0,1,0,0)$ and $(0,-1,0,0)$. Denote by Π the projection $\Pi : \Omega \to \Omega/G$. Let U (resp. V) be any neighborhood of O^+ (resp. O^-). Then $\Pi^{-1}(U)$ (resp. $\Pi^{-1}(V)$) is an open subset of Ω cointaining $(0,1,0,0)$ (resp. $(0,-1,0,0)$). As illustrated by Figure 1, $\Pi^{-1}(U)\cap\Pi^{-1}(V)$ intersects the orbits of the points $(\alpha,1,0,0)$ for sufficiently small α. It follows that $U\cap V = \Pi(\Pi^{-1}(U)\cap\Pi^{-1}(V))$ is not empty.

Let us now study the TP-cohomology of (Ω,Λ). As always,

$$H^0_{\Lambda,\tan}(\Omega) = I(\Omega).$$

To describe $H^1_{\Lambda,\tan}(\Omega)$, we shall observe that any tangential vector field X can be written in the form $X = a\partial_4 + bH_{x_4}$ where a, b are in $C^\infty(\Omega)$ and that, for such a X, $\sigma(X) = 0$ if and only if

$$(*) \qquad H_{x_4}(b) + \partial_4(a) = x_2\partial_3(b) + x_1\partial_2(b) + \partial_4(a) = 0.$$

Let us reduce the study of $H^1_{\Lambda,\tan}(\Omega)$ to the resolution of the partial differential equation

$$H_{x_4}(g) = a_0$$

when a_0 and g depend only on the variables x_1, x_2, x_3.

Suppose that $X = a\partial_4 + bH_{x_4}$ satisfies $\sigma(X) = 0$, that is $(*)$ holds. Then, there exists f in $C^\infty(\Omega)$ such that $X = \sigma f$ if and only if

$$X(dx_i) = \sigma f(dx_i) = -\{f, x_i\} \quad \forall i,$$

or equivalently, if and only if

$$b = -\partial_4(f) \quad \text{and} \quad a = H_{x_4}(f).$$

Thus, the existence of f is equivalent to the existence of $g(x) = g(x_1, x_2, x_3)$ such that

$$f(x) = -\int_0^{x_4} b(x_1, x_2, x_3, t)dt + g(x_1, x_2, x_3)$$

and

$$H_{x_4}(f)(x) = -\int_0^{x_4} H_{x_4}b(x_1, x_2, x_3, t)dt + H_{x_4}(g)(x) = a(x).$$

Because of $(*)$, it exactly means that

$$\begin{aligned} H_{x_4}(f)(x) &= \int_0^{x_4} \partial_4(a)(x_1, x_2, x_3, t)dt + H_{x_4}(g)(x) \\ &= a(x_1, x_2, x_3, x_4) - a(x_1, x_2, x_3, 0) + H_{x_4}(g)(x) \\ &= a(x_1, x_2, x_3, x_4), \end{aligned}$$

and thus, as announced, that $H_{x_4}(g)(x) = a(x_1, x_2, x_3, 0)$. We want now to prove:

Lemma 3. *Let a be a function in $C^\infty(\Omega)$ depending only on the variables x_1, x_2, x_3. Assume there exists a function g in $C^\infty(\Omega)$, depending also on the variables x_1, x_2, x_3, such that*

$$H_{x_4}(g) = x_2\partial_3(g) + x_1\partial_2(g) = a.$$

Then,

$$\lim_{x_1 \to 0} \int_{-1}^{1} a\left(x_1, s, -\frac{1}{2x_1} + \frac{s^2}{2x_1}\right)\frac{ds}{x_1}$$

exists.

Proof. Using the change of variables on the open set $U = \{x \in \Omega : x_1 \neq 0\}$,

$$u = x_3 - \frac{x_2^2}{2x_1}, v = \frac{x_2}{x_1}, w = x_1,$$

we can see that g is necessarily of the form

$$\begin{aligned} g(x_1,x_2,x_3) &= \int_0^{\frac{x_2}{x_1}} a\left(x_1, x_1t, x_3 - \frac{x_2^2}{2x_1} + \frac{t^2x_1}{2}\right) dt + c\left(x_1, x_3 - \frac{x_2^2}{2x_1}\right) \\ &= \int_0^{x_2} a\left(x_1, s, x_3 - \frac{x_2^2}{2x_1} + \frac{s^2}{2x_1}\right) \frac{ds}{x_1} + c\left(x_1, x_3 - \frac{x_2^2}{2x_1}\right). \end{aligned}$$

Thus,

$$g(x_1,1,0) = \int_0^1 a\left(x_1, s, -\frac{1}{2x_1} + \frac{s^2}{2x_1}\right) \frac{ds}{x_1} + c\left(x_1, -\frac{1}{2x_1}\right)$$

$$\text{and} \quad g(x_1,-1,0) = \int_0^{-1} a\left(x_1, s, -\frac{1}{2x_1} + \frac{s^2}{2x_1}\right) \frac{ds}{x_1} + c\left(x_1, -\frac{1}{2x_1}\right).$$

Therefore, $g(x_1,1,0) - g(x_1,-1,0) = \int_{-1}^1 a\left(x_1, s, -\frac{1}{2x_1} + \frac{s^2}{2x_1}\right) \frac{ds}{x_1}$. We get the result from the fact that g is continuous at the points $(0,1,0)$ and $(0,-1,0)$ (see also Figure 1).

While the leaves of Ω are cohomologically trivial, the first TP-cohomology space of Ω is very large. Indeed, we have:

Proposition 4. *Let Ω be the set of maximal dimensional orbits associated to $\mathfrak{g} = \mathfrak{g}_{4,1}$ and Λ its regular Poisson structure. Let G be the connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Denote by $S(\mathfrak{g})^G$ (resp. $\mathcal{A}(\mathfrak{g})^G$) the ring of G-invariant polynomial (resp. analytic) functions over $\mathfrak{g}^*$. For each $\alpha > \frac{1}{2}$, denote by T_α and K_α the vector fields defined by $T_\alpha = t_\alpha \partial_4$ and $K_\alpha = k_\alpha \partial_4$ where*

$$t_\alpha = \frac{1}{(x_1^2+x_2^2)^\alpha} \quad \text{and} \quad k_\alpha = \frac{x_1 \exp\left(\frac{\alpha}{x_1^2+x_2^2}\right)}{(x_1^2+x_2^2)^2}.$$

Then, the following assertions hold.

(i) *The classes $[T_\alpha]$ generate an infinite dimensional space over $\mathbb{R}$.*

(ii) *The classes $[K_\alpha]$ are linearly independent not only over $\mathbb{R}$ but also over $S(\mathfrak{g})^G$ or over $\mathcal{A}(\mathfrak{g})^G$, i.e., $H^1_{\Lambda,\tan}(\Omega)$, as a $S(\mathfrak{g})^G$-module or as a $\mathcal{A}(\mathfrak{g})^G$-module, is not finitely generated.*

Remark 4 (and convention)**.** We use in (ii) the integral domains $S(\mathfrak{g})^G$ and $\mathcal{A}(\mathfrak{g})^G$, but, from a differential geometry point of view, it would be more interesting to consider the whole ring $I(\Omega)$ of G-invariant smooth functions over Ω. In fact, due to the complexity of the non-integral domain $I(\Omega)$,

we do not know the $I(\Omega)$-module structure of $H^1_{\Lambda,\tan}(\Omega)$. Nevertheless we conjecture that it is still not finitely generated.

In the sequel, when we say for some Lie algebra $\mathfrak{g}$ that a TP-cohomology (or a P-cohomology) space of Ω is infinite dimensional, we will mean both infinite dimensional as a vector space and not finitely generated as a module over $S(\mathfrak{g})^G$ or $\mathcal{A}(\mathfrak{g})^G$.

Proof of Proposition 4. For all $0 < x_1 < 1$, we have

$$\int_0^1 \frac{ds}{(x_1^2+s^2)^\alpha x_1} \geq \int_0^{x_1} \frac{ds}{(x_1^2+s^2)^\alpha x_1} \geq \frac{1}{2^\alpha x_1^{2\alpha}} \quad (\forall \alpha)$$

and

$$\begin{aligned}\int_0^1 \frac{ds}{(x_1^2+s^2)^\alpha x_1} &\leq \int_0^{x_1} \frac{ds}{(x_1^2+s^2)^\alpha x_1} + \int_{x_1}^1 \frac{ds}{s^{2\alpha} x_1} \\ &\leq c_\alpha \frac{1}{x_1^{2\alpha}} \quad \left(\forall \alpha > \frac{1}{2}\right)\end{aligned}$$

where $c_\alpha = \frac{2\alpha}{2\alpha-1}$. Now, assume that for some p, $\sum_{i=1}^{p} \lambda_{\alpha_i}[T_{\alpha_i}] = 0$, the λ_{α_i} being in $\mathbb{R}$ and $\alpha_1 < \ldots < \alpha_p$. Using Lemma 3, we directly see that

$$E = \int_0^1 \sum_{i=1}^{p} \lambda_{\alpha_i} \frac{ds}{(x_1^2+s^2)^{\alpha_i} x_1}$$

must have a limit when x_1 tends to zero. But, when $0 < x_1 < 1$,

$$\begin{aligned}|E| \quad &\geq \left| \int_0^1 \frac{\lambda_{\alpha_p} ds}{x_1 (x_1^2+s^2)^{\alpha_p}} \right| - \left| \int_0^1 \frac{\lambda_{\alpha_{p-1}} ds}{x_1 (x_1^2+s^2)^{\alpha_{p-1}}} \right| \\ &\quad \ldots - \left| \int_0^1 \frac{\lambda_{\alpha_1} ds}{x_1 (x_1^2+s^2)^{\alpha_1}} \right| \\ &\geq \frac{|\lambda_{\alpha_p}|}{2^{\alpha_p} x_1^{2\alpha_p}} - \frac{|\lambda_{\alpha_{p-1}}| c_{\alpha_{p-1}}}{x_1^{2\alpha_{p-1}}} - \cdots - \frac{|\lambda_{\alpha_1}| c_{\alpha_1}}{x_1^{2\alpha_1}}.\end{aligned}$$

Thus, λ_{α_p} must be 0 and a step-by-step application of the same argument shows that $\lambda_{\alpha_i} = 0$ for all i. It implies that the classes $[T_\alpha]$ generate an infinite dimensional vector space over $\mathbb{R}$. That ends the proof of (i).

To prove (ii), it is enough to see that

$$\begin{aligned}\int_0^1 \frac{2\exp\left(\frac{\alpha}{x_1^2+s^2}\right)}{(x_1^2+s^2)^2} ds \quad &\geq \quad \int_0^1 \frac{2s\exp\left(\frac{\alpha}{x_1^2+s^2}\right)}{(x_1^2+s^2)^2} ds \\ &\geq \quad \frac{1}{\alpha}\left(\exp\left(\frac{\alpha}{x_1^2}\right) - \exp\left(\frac{\alpha}{x_1^2+1}\right)\right),\end{aligned}$$

and that

$$\int_0^1 \frac{2\exp\left(\frac{\alpha}{x_1^2+s^2}\right)}{(x_1^2+s^2)^2}ds \leq \frac{2\exp\left(\frac{\alpha}{x_1^2}\right)}{x_1^4}.$$

Thus we can use the same argument as in (i).

To finish the discussion about this example, we shall prove the vanishing of $H^2_{\Lambda,\tan}(\Omega)$. Let A be a tangential 2-tensor field. Necessarily,

$$A = \varphi\Lambda = \varphi(x_2\partial_4 \wedge \partial_3 + x_1\partial_4 \wedge \partial_2)$$

for some function φ in $C^\infty(\Omega)$ and $\sigma(A) = 0$. We have thus to find a tangential vector field B such that $A = \sigma(B)$, or equivalently, to find a, b in $C^\infty(\Omega)$ such that $H_{x_4}(b) + \partial_4(a) = \varphi$. We immediately check that

$$a = \int_0^{x_4} \varphi(x_1, x_2, x_3, t)dt \quad \text{and} \quad b = 0$$

are convenient. Therefore, $H^2_{\Lambda,\tan}(\Omega) = \{0\}$.

This fact can also be deduced from Proposition 1 (Section 2), which identifies the TP-cogomology with a Čech cohomology. Indeed, it is easy to see that Ω admits a good covering $\mathcal{U} = (U_i)$ without any nontrivial intersections of three open sets U_i.

Now, since the rank of Ω is 2, we have $H^k_{\Lambda,\tan}(\Omega) = \{0\}$ for all $k > 2$.

Remark 5. An immediate consequence of the above analysis concerning the nilpotent case is that the tangential star products on $\underset{B}{\cup} V_B$ (and on any regular Poisson manifold $\widetilde{\Omega}$ satisfying the conditions of Proposition 3) are all equivalent because of the vanishing of the second TP-cohomology space (Propositions 2 and 3). The same is true for the union Ω of all maximal dimensional coadjoint orbits in $\mathfrak{g}^*_{4,1}$.

The computation of the TP-cohomology spaces for $\mathfrak{g}_{5,4}$, $\mathfrak{g}_{6,18}$ or for all the filiform Lie algebras, studied for instance in [**Pe1**, **Pe2**, **CG**, **GK**, **BLM**], leads to the same results as in the $\mathfrak{g}_{4,1}$-case. Due to these examples, we believe that the TP-cohomology spaces of a regular Poisson manifold M are huge and rather complicated to compute whenever the quotient space of M by the foliation is not Hausdorff. To confirm this observation, we shall now study more varied examples.

4. Further examples.

Let (M, Λ) be a 3-dimensional regular Poisson manifold. If we exclude the trivial case where $\Lambda = 0$, we can suppose M to be of rank 2. To describe the P-cohomology of M, we are going to use Vaisman's notations and computations ([**Va2**, p. 69]). Of course,

$$H^0_\Lambda(M) = H^0_{\Lambda,\tan}(M) = I(M).$$

Now, we have by definition

$$H^1_\Lambda(M) = \frac{\{Q \in \mathcal{V}^1(M) : \sigma(Q) = 0\}}{\{\sigma f : f \in C^\infty(M)\}}.$$

Let us then choose a transversal distribution $\nu\mathcal{F}$. Let us also use the decompositions $\bigoplus_q \mathcal{V}_{0,q}(M)$, $\bigoplus_q \Omega_{0,q}(M)$, $\sigma = \sigma' + \sigma''$ and $d = d' + d'' + d_{2,-1}$. Each element Q of $\mathcal{V}^1(M)$ can thus be written in the form

$$Q = Q_{0,1} + Q_{1,0}$$

where $Q_{0,1}$ and $Q_{1,0}$ are of type $(0,1)$ and $(1,0)$ respectively. Furthermore, $\sigma(Q) = 0$ if and only if $\sigma(Q_{0,1} + Q_{1,0}) = 0$, or else, if and only if

$$\sigma'(Q_{1,0}) + \sigma''(Q_{1,0}) + \sigma''(Q_{0,1}) = 0,$$

that is, if and only if

$$\sigma''(Q_{1,0}) = 0 \quad \text{and} \quad \sigma'(Q_{1,0}) + \sigma''(Q_{0,1}) = 0.$$

Consider now the linear mapping p defined by

$$\begin{array}{rl} p : H^1_\Lambda(M) & \longrightarrow \widetilde{\mathcal{V}}_{1,0}(M) \\ [A] & \longmapsto A_{1,0} \end{array}$$

where

$$\widetilde{\mathcal{V}}_{1,0}(M) = \{A \in \mathcal{V}_{1,0}(M) : \sigma''(A) = 0\}.$$

It follows that

$$H^1_\Lambda(M) \cong \mathrm{Ker}(p) \oplus \mathrm{Im}(p).$$

Now

$$\begin{aligned} \mathrm{Ker}\,(p) &= \{[A] \in H^1_\Lambda(M) : A_{1,0} = 0\} \\ &= \frac{\{A \in \mathcal{V}_{0,1}(M) : \sigma(A) = 0\}}{\{\sigma f : f \in C^\infty(M)\}} \\ &= \frac{\{A \in \mathcal{V}_{0,1}(M) : \sigma''(A) = 0\}}{\{\sigma'' f : f \in C^\infty(M)\}} \\ &= H^1_{\Lambda,\tan}(M). \end{aligned}$$

Moreover,

$$\mathrm{Im}\,(p) = \big\{A \in \mathcal{V}_{1,0}(M) : \sigma''(A) = 0 \qquad \text{and} \qquad \exists B \in \mathcal{V}_{0,1}(M)/\sigma(A) + \sigma''(B) = 0\big\}.$$

Let us compute the second order space:

$$\begin{aligned} H^2_\Lambda(M) &= \frac{\{Q \in \mathcal{V}^2(M) : \sigma(Q) = 0\}}{\{\sigma(V) : V \in \mathcal{V}^1(M)\}} \\ &= \frac{\{Q = Q_{0,2} + Q_{1,1} \in \mathcal{V}^2(M) : \sigma''(Q_{1,1}) = 0\}}{\{\sigma(V) : V = V_{0,1} + V_{1,0} \in \mathcal{V}^1(M)\}} \\ &= \frac{\{Q_{0,2} \in \mathcal{V}_{0,2}(M)\}}{\{\sigma''(V_{0,1}) + \sigma'(V_{1,0})\}} \oplus \frac{\{Q_{1,1} \in \mathcal{V}_{1,1}(M) : \sigma''(Q_{1,1}) = 0\}}{\{\sigma''(V_{1,0})\}} \\ &= \frac{\dfrac{\{Q_{0,2}\}}{\{\sigma''(V_{0,1})\}}}{\sigma(\mathcal{V}_{1,0}(M))} \oplus \frac{\{Q_{1,1} : \sigma''(Q_{1,1}) = 0\}}{\{\sigma''(V_{1,0})\}} \\ &= \frac{H^2_{\Lambda,\tan}(M)}{\sigma(\mathcal{V}_{1,0}(M))} \oplus \frac{\operatorname{Ker}\left(\sigma''_{|\mathcal{V}_{1,1}(M)}\right)}{\sigma''(\mathcal{V}_{1,0}(M))}. \end{aligned}$$

In the same way, we get the third order space:

$$H^3_\Lambda(M) = \frac{\{Q_{1,2}\}}{\{\sigma(V_{0,2} + V_{1,1})\}} = \frac{\{Q_{1,2}\}}{\{\sigma''(V_{1,1})\}} = \frac{\mathcal{V}_{1,2}(M)}{\sigma''(\mathcal{V}_{1,1}(M))}.$$

As seen in Section 2, for all regular Poisson manifold (M, Λ), $\left(\bigoplus_q \Omega_{0,q}(M), d''\right)$ and $\left(\bigoplus_q \mathcal{V}_{0,q}(M), \sigma''\right)$ are isomorphic as complexes of $I(M)$-modules. In fact, it is always possible to define an isomorphism between $\Omega_{p,q}(M)$ and $\mathcal{V}_{p,q}(M)$ for all p and q. For this, one can consider, as Vaisman did in [**Va2**], an Euclidean metric on $\nu^*\mathcal{F}$. That leads to an isomorphism between $\nu^*\mathcal{F} \oplus T^*\mathcal{F}$ and $\nu\mathcal{F} \oplus T\mathcal{F}$, which can naturally be extended to the required isomorphism between $\Omega_{p,q}(M)$ and $\mathcal{V}_{p,q}(M)$. In general, however, this isomorphism is not an isomorphism of complexes. Via the next proposition, we give a simple situation where $\left(\bigoplus_q \Omega_{p,q}(M), d''\right)$ and $\left(\bigoplus_q \mathcal{V}_{p,q}(M), \sigma''\right)$ are effectively isomorphic complexes for all p.

Proposition 5. *Let (M, Λ) be a regular Poisson manifold. Denote by $\mathcal{F}$ the symplectic foliation of M. Choose a transversal distribution $\nu\mathcal{F}$ and assume that $\Omega_{1,0}(M)$ and $\mathcal{V}_{1,0}(M)$ are isomorphic free $I(M)$-modules with bases (β_i) and (X_i) respectively (β_i and X_i being globally defined). Suppose also that $d''\beta_i = 0$ and $\sigma''(X_i) = 0$. Then, for all p, $\left(\bigoplus_q \Omega_{p,q}(M), d''\right)$ and $\left(\bigoplus_q \mathcal{V}_{p,q}(M), \sigma''\right)$ are isomorphic complexes of $I(M)$-modules. In particular, $H^q(M, \Phi^p(\mathcal{F}))$ and $H^q\left(\bigoplus_k \mathcal{V}_{p,k}(M), \sigma''\right)$ are isomorphic as $I(M)$-modules.*

Proof. Recall that $\# : T^*M \to TM$ can be extended to an $I(M)$-modules isomorphism $\widetilde{\#} : \Omega_{0,q}(M) \to \mathcal{V}_{0,q}(M)$ satisfying $\sigma'' \circ \widetilde{\#} = -\widetilde{\#} \circ d''$. Consider now the mapping $\hat{\#} : \Omega_{p,q}(M) \to \mathcal{V}_{p,q}(M)$ defined by

$$\hat{\#}\left(\sum_{i_1,\dots,i_p} \alpha_{i_1,\dots,i_p} \wedge \beta_{i_1} \wedge \ldots \wedge \beta_{i_p}\right) = \sum_{i_1,\dots,i_p} (\widetilde{\#}(\alpha_{i_1,\dots,i_p}) \wedge X_{i_1} \wedge \ldots \wedge X_{i_p})$$

where the $\alpha_{i_1,\dots,i_p}$ are in $\Omega_{0,q}(M)$. It is not difficult to check that $\hat{\#}$ is an $I(M)$-modules isomorphism and that $\sigma'' \circ \hat{\#} = -\hat{\#} \circ d''$. This ends the proof.

The following definitions are quite standard, we recall them for completeness. Let $V \to B$ be a vector bundle whose fibers are q-dimensional. We shall say that $V \to B$ (or simply V) is orientable if the bundle $\Lambda^q V \to B$ admits a global nonsingular (i.e., nowhere vanishing) section. If $V \to B$ is orientable, so is its dual $V^* \to B$. Recall also that a manifold M is said to be orientable if TM (or T^*M) is orientable.

Moreover, we shall say that a foliation $\mathcal{F}$ on M is (co)orientable if its normal bundle $N\mathcal{F}$ is orientable. In the important case of a 1-codimensional foliation $\mathcal{F}$, this foliation $\mathcal{F}$ is orientable if and only if there exists a nonsingular 1-form β vanishing exactly on vectors tangent to the leaves of $\mathcal{F}$. In this case, we say that β defines the foliation.

Note that neither the leaves nor the total manifold M of an orientable foliation need to be orientable. However, if (M, Λ) is a regular Poisson manifold (of rank $2n$) and $\mathcal{F}$ is the symplectic foliation of M, then the situation is somewhat simpler. Since the tensor Λ^n defines a global nonsingular section of $\Lambda^{2n} T\mathcal{F}$, the tangent bundle $T\mathcal{F}$ of $\mathcal{F}$ is orientable. In other words, the symplectic foliation of a regular Poisson manifold M is orientable if and only if M is orientable.

We turn back now to the case where (M, Λ) is 3-dimensional and give a result which will be useful later.

Proposition 6. *Let (M, Λ) be a regular Poisson manifold of dimension 3 and rank 2. Denote by $\mathcal{F}$ the symplectic foliation of M. Suppose that M is orientable and that one can choose the 1-form β defining $\mathcal{F}$ such that $d\beta = 0$. Then:*

(i) *The second and third P-cohomology spaces of M are*

$$\begin{aligned} H^2_\Lambda(M) &\simeq \frac{H^2_{\Lambda,\tan}(M)}{\sigma(\mathcal{V}_{1,0}(M))} \oplus H^1(M, \Phi^1(\mathcal{F})) \\ H^3_\Lambda(M) &\simeq H^2(M, \Phi^1(\mathcal{F})). \end{aligned}$$

(ii) *For all q, $H^q_{\Lambda,\tan}(M)$ and $H^q(M, \Phi^1(\mathcal{F}))$ are isomorphic as $I(M)$-modules.*

Proof. Let us identify the normal bundle $N\mathcal{F}$ of $\mathcal{F}$ with a sub-bundle $\nu\mathcal{F}$ of TM. Since M (or $\mathcal{F}$) is orientable, there exists a nonsingular vector field X (globally defined) such that, for each x, $\nu_x\mathcal{F}$ (resp. $\nu_x^*\mathcal{F}$) is spanned by X_x (resp. β_x). By construction, $\sigma''X = 0$ and $d\beta = d''\beta = 0$. Moreover, $\Omega_{1,0}(M)$ and $\mathcal{V}_{1,0}(M)$ are isomorphic free and with basis (β) and (X) respectively. The point (i) is thus a direct corollary of Proposition 5.

To prove (ii), denote by $\Phi : H^q(M, \Phi^0(\mathcal{F})) \to H^q(M, \Phi^1(\mathcal{F}))$ the mapping defined by $\Phi([\alpha]) = [\alpha \wedge \beta]$ for all α in $\Omega_{0,q}(M)$ such that $d''\alpha = 0$. Clearly, Φ is both well-defined and bijective. Therefore, $H^q(M, \Phi^0(\mathcal{F}))$ (which is isomorphic to $H^q_{\Lambda,\tan}(M)$ by Theorem 1) coincides with $H^q(M, \Phi^1(\mathcal{F}))$. This ends the proof.

As we already said, each Lie algebra provides a natural regular Poisson manifold: The union Ω of all maximal dimensional coadjoint orbits. Let us now study the TP-cohomology and the P-cohomology of Ω for any 3-dimensional Lie algebra.

First recall that every nonabelian 3-dimensional Lie algebra is isomorphic to exactly one in the following list (see [**Br**] for instance):

- a nilpotent Lie algebra, namely the 3-dimensional Heisenberg algebra, given by: $[X_1, X_2] = X_3$;
- a solvable non-exponential Lie algebra, namely $\mathfrak{e}(2)$, defined by: $[X_1, X_2] = -X_3$, $[X_1, X_3] = X_2$;
- several (solvable) exponential Lie algebras:
  - the algebra spanned by X_1, X_2, X_3 with $[X_1, X_2] = X_2$;
  - the "book algebra": $[X_1, X_2] = X_2$, $[X_1, X_3] = X_3$;
  - "Grélaud's Lie algebras": $[X_1, X_2] = X_2 - \sigma X_3$, $[X_1, X_3] = X_3 + \sigma X_2$ where $\sigma > 0$;
  - $[X_1, X_2] = X_2$, $[X_1, X_3] = \frac{1}{\tau}X_3$ where $\tau > 1$;
  - $[X_1, X_2] = X_2 + X_3$, $[X_1, X_3] = X_3$;
  - $[X_1, X_2] = X_2$, $[X_1, X_3] = -X_3$;
  - $[X_1, X_2] = X_2$, $[X_1, X_3] = \frac{1}{\tau}X_3$ where $\tau < -1$;
- two simple Lie algebras: $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$.

Among them, there are some easy examples. Let us pass them in review.

4.1. The 3-dimensional Heisenberg algebra. This nilpotent Lie algebra is defined by the bracket $[X_1, X_2] = X_3$.

For this example, the nontrivial orbits are planes and the regular Poisson manifold (Ω, Λ) is the set

$$\Omega = \{(x_1, x_2, x_3) : x_3 \neq 0\},$$

endowed with its regular Poisson structure Λ. We identify Ω with $\mathbb{R}^2\times\mathbb{R}^*$ by means of the Weinstein chart [SG, Wei] mapping an element $x=(x_1,x_2,x_3)$ of Ω to $(p=x_1, q=\frac{x_2}{x_3}, z=x_3)$.

Using Theorem 3 (or Theorem 2), we see that the TP-cohomology of Ω is trivial in degree superior to zero.

Furthermore, Λ is transversally constant with respect to the transversal distribution $\nu\mathcal{F}=T\mathbb{R}^*$. Using Theorem 4 or by direct computation, we get the P-cohomology of Ω:

$$\begin{aligned} H^0_\Lambda(\Omega) &= H^0_{\Lambda,\tan}(\Omega)=I(\Omega)=C^\infty(\mathbb{R}^*)\\ H^1_\Lambda(\Omega) &\simeq \{u\,dz : u\in I(\Omega)\}\simeq I(\Omega)\\ H^k_\Lambda(\Omega) &= \{0\}\quad \forall k>1. \end{aligned}$$

4.2. The Lie-algebra $\mathfrak{e}(2)$ of the Euclidean 2-dimensional group. The brackets are:

$$[X_1,X_2]=-X_3,\quad [X_1,X_3]=X_2.$$

The nontrivial orbits are cylinders C_r with radius $r>0$, and the regular Poisson manifold (Ω,Λ) associated to this Lie algebra is the set

$$\Omega=\{(x_1,x_2,x_3)\in\mathfrak{e}(2)^* : x_2^2+x_3^2\neq 0\},$$

endowed with its regular Poisson structure Λ. We identify Ω with $\mathbb{R}\times\mathbb{T}\times\mathbb{R}^*_+$ by means of the Weinstein chart mapping an element $x=(x_1,x_2,x_3)$ of Ω to the point (p,q,r) defined by

$$p=x_1,\quad e^{\imath q}=\frac{x_2+\imath x_3}{\sqrt{x_2^2+x_3^2}},\quad r=\sqrt{x_2^2+x_3^2}.$$

Using Theorem 3, we obtain the TP-cohomology of Ω:

$$\begin{aligned} H^0_{\Lambda,\tan}(\Omega) &= I(\Omega)=C^\infty(\mathbb{R}^*_+)\\ H^1_{\Lambda,\tan}(\Omega) &\simeq H^1_{DR}(\mathbb{R}\times\mathbb{T})\otimes C^\infty(\mathbb{R}^*_+)\\ &\simeq \{[\rho(r)dq] : \rho(r)\in I(\Omega)\}\simeq I(\Omega)\\ H^k_{\Lambda,\tan}(\Omega) &= \{0\}\quad \forall k>1. \end{aligned}$$

Now, Λ is transversally constant with respect to the transversal distribution $\nu\mathcal{F}=T\mathbb{R}^*_+$. By Theorem 4, the P-cohomology of Ω is:

$$\begin{aligned} H^0_\Lambda(\Omega) &= I(\Omega)\\ H^1_\Lambda(\Omega) &\simeq H^1_{\Lambda,\tan}(\Omega)\oplus\{u\,dr : u\in I(\Omega)\}\\ &\simeq I(\Omega)\oplus I(\Omega)\\ H^2_\Lambda(\Omega) &\simeq H^1_{DR}(\mathbb{R}\times\mathbb{T})\otimes\Omega^1(\mathbb{R}^*_+)\\ &\simeq \{[\rho(r)dq\wedge dr] : \rho(r)\in I(\Omega)\}\simeq I(\Omega)\\ H^k_\Lambda(\Omega) &= \{0\}\quad \forall k>2. \end{aligned}$$

4.3. The Lie algebra $\mathfrak{a}$ of the affine group. The only non-vanishing bracket is $[X_1, X_2] = X_2$.

The nontrivial orbits are half planes (x_3 and $\operatorname{sign}(x_2)$ fixed) and the regular Poisson manifold (Ω, Λ) associated to this Lie algebra is the set

$$\Omega = \{(x_1, x_2, x_3) \in \mathfrak{a}^* : x_2 \neq 0\},$$

endowed with its regular Poisson structure Λ. We identify Ω with $\mathbb{R}^* \times \mathbb{R} \times \mathbb{R}$ by means of the Weinstein chart mapping an element $x = (x_1, x_2, x_3)$ of Ω to the point (p, q, z) defined by

$$p = x_2, \quad q = \frac{-x_1}{x_2}, \quad z = x_3.$$

By Theorem 3 (or Theorem 2), the TP-cohomology of Ω is trivial in degree superior to zero.

Again, Λ is transversally constant with respect to the transversal distribution $\nu\mathcal{F} = T\mathbb{R}$. By Theorem 4, the P-cohomology of Ω is:

$$\begin{aligned} H^0_\Lambda(\Omega) &= H^0_{\Lambda,\tan}(\Omega) = I(\Omega) \\ &\simeq C^\infty((\{1\} \times \mathbb{R}) \cup (\{-1\} \times \mathbb{R})) \\ &\simeq C^\infty(\mathbb{R}) \oplus C^\infty(\mathbb{R}) \\ H^1_\Lambda(\Omega) &\simeq \{u\, dz : u \in I(\Omega)\} \simeq I(\Omega) \\ H^k_\Lambda(\Omega) &= \{0\} \quad \forall k > 1. \end{aligned}$$

4.4. The book algebra. This Lie algebra is given by the following brackets:

$$[X_1, X_2] = X_2, \quad [X_1, X_3] = X_3.$$

The 2-dimensional orbits are characterized by an invariant θ and are of the form

$$O_\theta = \{(s, e^t \cos(\theta), e^t \sin(\theta)) : (s, t) \in \mathbb{R}^2\}.$$

The corresponding regular Poisson manifold (Ω, Λ) is thus the set

$$\Omega = \{(x_1, x_2, x_3) : x_2^2 + x_3^2 \neq 0\},$$

with its regular Poisson structure Λ. We identify Ω with $\mathbb{R} \times \mathbb{R} \times \mathbb{T}$ by means of the Weinstein chart mapping an element $x = (x_1, x_2, x_3)$ of Ω to the point (p, q, θ) defined by

$$p = x_1, \quad q = \frac{1}{2}\ln(x_2^2 + x_3^2), \quad e^{\imath\theta} = \frac{x_2 + \imath x_3}{\sqrt{x_2^2 + x_3^2}}.$$

By Theorem 3 (or Theorem 2), the TP-cohomology of Ω is trivial in degree superior to zero.

Moreover, Λ is transversally constant with respect to $\nu\mathcal{F} = T\mathbb{T}$. By Theorem 4, the P-cohomology of Ω is:

$$\begin{aligned} H^0_\Lambda(\Omega) &= H^0_{\Lambda,\tan}(\Omega) = I(\Omega) = C^\infty(\mathbb{T}) \\ H^1_\Lambda(\Omega) &\simeq \{u\, d\theta : u \in I(\Omega)\} \simeq I(\Omega) \\ H^k_\Lambda(\Omega) &= \{0\} \quad \forall k > 1. \end{aligned}$$

4.5. Similar to the book algebra: Grélaud's Lie algebras. The brackets of these Lie algebras, studied by Grélaud in [**Gré**], are

$$[X_1, X_2] = X_2 - \sigma X_3, \quad [X_1, X_3] = X_3 + \sigma X_2 \quad \sigma > 0.$$

The 2-dimensional orbits are a "spiral" version of those of the book algebra, they are of the form

$$O_\theta = \{(s, e^t \cos(\theta + \sigma t), e^t \sin(\theta + \sigma t)) : (s,t) \in \mathbb{R}^2\}$$

where θ is defined by

$$e^{\imath\theta} = \frac{x_2 + \imath x_3}{\sqrt{x_2^2 + x_3^2}} e^{-\frac{\imath\sigma}{2}\ln(x_2^2+x_3^2)}.$$

The regular Poisson manifold (Ω, Λ) is thus the set

$$\Omega = \{(x_1, x_2, x_3) : x_2^2 + x_3^2 \neq 0\},$$

with its regular Poisson structure Λ.

The situation here is identical to that of the book algebra. First, Ω can be identified with $\mathbb{R} \times \mathbb{R} \times \mathbb{T}$ and the TP-cohomology of Ω is trivial in degree superior to zero. Moreover, Λ is transversally constant with respect to $\nu\mathcal{F} = T\mathbb{T}$ so that the P-cohomology of Ω is:

$$\begin{aligned} H^0_\Lambda(\Omega) &= H^0_{\Lambda,\tan}(\Omega) = I(\Omega) = C^\infty(\mathbb{T}) \\ H^1_\Lambda(\Omega) &= \{u\, d\theta : u \in I(\Omega)\} \simeq I(\Omega) \\ H^k_\Lambda(\Omega) &= \{0\} \quad \forall k > 1. \end{aligned}$$

4.6. Other examples very close to the book algebra. Consider the family of Lie algebras defined by the brackets:

$$[X_1, X_2] = X_2, \quad [X_1, X_3] = \frac{X_3}{\tau} \quad \tau > 1.$$

The 2-dimensional orbits can be parameterized with obvious notations by

$$O_\mu = \left\{(s, e^{-t}\mu_2, e^{-\frac{t}{\tau}}\mu_3) : (s,t) \in \mathbb{R}^2\right\},$$

they are all cohomologically trivial and the regular Poisson manifold (Ω, Λ) is the set

$$\Omega = \{(x_1, x_2, x_3) : x_2^2 + x_3^2 \neq 0\}$$

with its regular Poisson structure Λ.

Let us now prove that the symplectic foliation $\mathcal{F}$ of Ω is given by a submersion Π from Ω to the circle $\mathbb{S}^1$. Let

$$F : \mathbb{R} \times \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$$

be the mapping defined by

$$F(t, x_2, x_3) = e^{-2t} x_2^2 + e^{-\frac{2t}{\tau}} x_3^2 - 1.$$

Using the standard Implicit Function Theorem, we see there is a unique smooth function $\varphi : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ of the variables x_2, x_3 such that

$$F(\varphi(x_2, x_3), x_2, x_3) = 0 \quad \forall (x_2, x_3) \in \mathbb{R}^2 \backslash \{(0,0)\}.$$

Moreover, the partial derivatives of φ are

$$\begin{aligned} \frac{\partial \varphi}{\partial x_2} &= \frac{e^{-2\varphi} x_2}{e^{-2\varphi} x_2^2 + e^{\frac{-2\varphi}{\tau}} \frac{x_3^2}{\tau}} \\ \frac{\partial \varphi}{\partial x_3} &= \frac{e^{\frac{-2\varphi}{\tau}} x_3}{e^{-2\varphi} x_2^2 + e^{\frac{-2\varphi}{\tau}} \frac{x_3^2}{\tau}}. \end{aligned}$$

It is easy to check that the map $\Pi : \Omega \to \mathbb{S}^1$ defined by

$$(x_1, x_2, x_3) \longmapsto \Pi(x_1, x_2, x_3) = \left(e^{-\varphi(x_2, x_3)} x_2, e^{\frac{-\varphi(x_2, x_3)}{\tau}} x_3 \right)$$

is a submersion. Thus, by Theorem 2, the TP-cohomology of Ω is trivial in degree superior to zero just like in the book algebra example.

Furthermore, observe that Ω is orientable and that there is a 1-form β defining the foliation $\mathcal{F}$ such that $d\beta = 0$, namely $\beta = d\theta$ where θ is given by

$$e^{\imath\theta} = e^{-\varphi} x_2 + \imath e^{\frac{-\varphi}{\tau}} x_3.$$

Therefore, by Proposition 6, the P-cohomology of Ω is the same as in the case of the book algebra:

$$\begin{aligned} H^0_\Lambda(\Omega) &= H^0_{\Lambda,\tan}(\Omega) = I(\Omega) \\ H^1_\Lambda(\Omega) &\simeq \{u\, d\theta : u \in I(\Omega)\} \simeq I(\Omega) \\ H^k_\Lambda(\Omega) &\simeq \{0\} \quad \forall k > 1. \end{aligned}$$

An analogous example is the Lie algebra defined by the brackets:

$$[X_1, X_2] = X_2 + X_3, \quad [X_1, X_3] = X_3.$$

In this case, the 2-dimensional orbits can be parameterized by

$$O_\mu = \{(s, e^{-t}(\mu_2 - \mu_3 t), e^{-t}\mu_3) : (s,t) \in \mathbb{R}^2\},$$

they are all cohomologically trivial and the regular Poisson manifold (Ω, Λ) is the set

$$\Omega = \{(x_1, x_2, x_3) : x_2^2 + x_3^2 \neq 0\}$$

with its regular Poisson structure Λ.

Now, let G denote the following map

$$G(t, x_2, x_3) = e^{-2t}((x_2 - x_3 t)^2 + x_3^2) - 1.$$

By the Implicit Function Theorem, the equation $G(t, x_2, x_3) = 0$ determines t implicitely as a smooth function $\psi : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ of the variables x_2, x_3. The partial derivatives of ψ are

$$\begin{aligned} \frac{\partial \psi}{\partial x_2} &= \frac{x_2 - \psi x_3}{(x_2 - (\psi - \frac{1}{2})x_3)^2 + \frac{3}{4}x_3^2} \\ \frac{\partial \psi}{\partial x_3} &= \frac{x_3 + \psi^2 x_3 - \psi x_2}{(x_2 - (\psi - \frac{1}{2})x_3)^2 + \frac{3}{4}x_3^2}. \end{aligned}$$

As before, the symplectic foliation $\mathcal{F}$ of Ω is given by a submersion, namely the map $\Pi : \Omega \to \mathbb{S}^1$ defined by

$$\Pi(x_1, x_2, x_3) = (e^{-\psi(x_2,x_3)}(x_2 - x_3\psi(x_2, x_3)), e^{-\psi(x_2,x_3)}x_3).$$

Thus, by Theorem 2, the TP-cohomology of Ω is trivial in degree superior to zero.

Moreover, Ω is orientable and there exists a 1-form β defining $\mathcal{F}$ such that $d\beta = 0$, namely $\beta = d\theta$ where θ is now defined by

$$e^{\imath\theta} = e^{-\psi}((x_2 - x_3\psi) + \imath x_3).$$

The P-cohomology of Ω is thus again:

$$\begin{aligned} H^0_\Lambda(\Omega) &= H^0_{\Lambda,\tan}(\Omega) = I(\Omega) \\ H^1_\Lambda(\Omega) &\simeq \{u\, d\theta : u \in I(\Omega)\} \\ &\simeq I(\Omega) \\ H^k_\Lambda(\Omega) &\simeq \{0\} \quad \forall k > 1. \end{aligned}$$

4.7. The simple Lie algebra $\mathfrak{su}(2)$. This Lie algebra is defined by the following brackets:

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2.$$

The nontrivial orbits are 2-spheres and the regular Poisson manifold (Ω, Λ) is $\Omega = \mathfrak{su}(2)^* \backslash \{0\}$ endowed with its regular Poisson structure Λ. We naturally identify Ω with $\mathbb{S}^2 \times \mathbb{R}^*_+$. Using Theorem 3, we get the TP-cohomology of Ω:

$$\begin{aligned} H^0_{\Lambda,\tan}(\Omega) &= I(\Omega) = C^\infty(\mathbb{R}^*_+) \\ H^1_{\Lambda,\tan}(\Omega) &= \{0\} \\ H^2_{\Lambda,\tan}(\Omega) &= \{[u\Lambda] : u \in I(\Omega)\} \\ &\simeq \{[u\omega_\Lambda] : u \in I(\Omega)\} \simeq I(\Omega) \end{aligned}$$

where ω_Λ denotes the foliated 2-form associated to Λ i.e.,

$$\omega_\Lambda = \frac{x_3 dx_1 \wedge dx_2 + x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1}{r^2} \quad \left(r = \sqrt{x_1^2 + x_2^2 + x_3^2}\right).$$

As for the other TP-cohomology spaces,

$$H^k_{\Lambda,\tan}(\Omega) = \{0\} \quad \forall k > 2.$$

It is important to note that, in this example, Ω is a product $S \times R$ like in Theorem 4. However, Λ is not transversally constant so that Theorem 4 cannot be used.

We shall here compute the P-cohomology of (Ω, Λ). First, we have

$$H^0_\Lambda(\Omega) = I(\Omega) = C^\infty(\mathbb{R}^*_+).$$

Now, note $\beta = dr$ and let X be the Euler vector field ($X = \sum x_i \partial_i$). Consider the tranversal distribution $\nu\mathcal{F}$ where, for each x, $\nu^*_x\mathcal{F}$ (resp. $\nu_x\mathcal{F}$) is spanned by β_x (resp. X_x). Recall also that $H^1_\Lambda(\Omega)$ is reduced to the set:

$$\{A \in \mathcal{V}_{1,0}(\Omega) : \sigma''(A) = 0 \text{ and } \exists B \in \mathcal{V}_{0,1}(\Omega) \text{ such that } \sigma(A) + \sigma''(B) = 0\}.$$

It is easy to see that the set $\{A \in \mathcal{V}_{1,0}(\Omega) : \sigma''(A) = 0\}$ coincides with the set $\{uX : u \in I(\Omega)\}$. Now, since $\sigma(uX) = u\sigma(X) = u\Lambda$ for all u in $I(\Omega)$ and that the class $[u\Lambda]$ in $H^2_{\Lambda,\tan}(\Omega)$ does not vanish unless $u = 0$, we obtain

$$H^1_\Lambda(\Omega) = \{0\}.$$

Using Proposition 6 (i), we get

$$H^2_\Lambda(\Omega) \simeq \frac{H^2_{\Lambda,\tan}(\Omega)}{\sigma(\mathcal{V}_{1,0}(\Omega))} \oplus H^1(\Omega, \Phi^1(\mathcal{F})).$$

We saw that $\mathcal{V}_{1,0}(\Omega) \supset \{uX : u \in I(\Omega)\}$ and that $\sigma(uX) = u\Lambda$ for all u in $I(\Omega)$. Moreover, by Proposition 6 (ii), $H^1(\Omega, \Phi^1(\mathcal{F}))$ and $H^1_{\Lambda,\tan}(\Omega)$ are isomorphic. It follows that

$$H^2_\Lambda(\Omega) = \{0\}.$$

Finally, again by Proposition 6, we have

$$H^3_\Lambda(\Omega) \simeq H^2(\Omega, \Phi^1(\mathcal{F})) \quad \text{and} \quad H^2(\Omega, \Phi^1(\mathcal{F})) \simeq H^2_{\Lambda,\tan}(\Omega).$$

More explicitly,

$$\begin{aligned} H^3_\Lambda(\Omega) &= \{u[\Lambda \wedge X] : u \in I(\Omega)\} \\ &\simeq \{u[w_\Lambda \wedge dr] : u \in I(\Omega)\} \\ &\simeq I(\Omega). \end{aligned}$$

Lastly,

$$H^k_\Lambda(\Omega) = \{0\} \quad \forall k > 3.$$

Remark 6. The Lie algebra $\mathfrak{su}(2)$ provides an example of a regular Poisson manifold, namely $\Omega = \mathfrak{su}(2)^* \backslash \{0\}$, which is exact ($\Lambda = \sigma(X) = \sigma(\sum x_i \partial_i)$) without being tangentially exact (Λ cannot be written in the form $\Lambda = \sigma(T)$ for any tangential vector field T).

In fact, since for $\Omega = \mathfrak{su}(2)^* \backslash \{0\}$, $H^2_\Lambda(\Omega) = \{0\}$ and $H^2_{\Lambda,\tan}(\Omega) \neq \{0\}$, the case of $\mathfrak{su}(2)$ illustrates the fact that the TP-cohomology spaces are generally not imbedded in the corresponding P-cohomology spaces, except in degree 1.

Note also that there is in [**Xu**] a different and beautiful method to calculate the P-cohomology of $\mathfrak{su}(2)^* \backslash \{0\}$ by means of symplectic groupoids. It consists of converting the P-cohomology to the de Rham cohomology of certain manifolds.

We propose now to discuss the remaining 3-dimensional Lie algebras. As we shall see in the sequel, all of them are pathological cases.

4.8. An interesting pathological example. Consider the Lie algebra $\mathfrak{h}$ given by the following brackets:

$$[X_1, X_2] = X_2, \quad [X_1, X_3] = -X_3$$

and denote by H the connected and simply connected Lie group with Lie algebra $\mathfrak{h}$. For this Lie algebra, the 2-dimensional coadjoint orbits are the connected components of the hyperbolic cylinders $x_2 x_3 =$const. and the half planes $x_2 = 0$, sign (x_3) fixed and $x_3 = 0$, sign (x_2) fixed. Each of them is cohomologically trivial and the regular Poisson manifold (Ω, Λ) associated to $\mathfrak{h}$ is the set

$$\Omega = \{(x_1, x_2, x_3) \in \mathfrak{h}^* : x_2^2 + x_3^2 \neq 0\},$$

endowed with its regular Poisson structure Λ. However, Theorem 2 cannot be applied because, as it was the case for $\mathfrak{g}_{4,1}$, the space of leaves Ω/H is not Hausdorff.

What is the TP-cohomology of Ω?

As always, $H^0_{\Lambda,\tan}(\Omega) = I(\Omega)$. To describe $H^1_{\Lambda,\tan}(\Omega)$, we proceed as we did for $\mathfrak{g}_{4,1}$. We first observe that every tangential vector field X is of the form $X = a\partial_1 + bH_{x_1}$ with a, b in $C^\infty(\Omega)$ and satisfies the equality $\sigma(X) = 0$ if and only if $H_{x_1}(b) + \partial_1(a) = 0$. Then, we fix a tangential vector field $X = a\partial_1 + bH_{x_1}$ such that $\sigma(X) = 0$. X can be written in the form $X = \sigma f$ ($f \in C^\infty(\Omega)$) if and only if there exists $g(x) = g(x_2, x_3)$ in $C^\infty(\Omega)$ such that $H_{x_1}(g)(x) = a(0, x_2, x_3)$.

Consider now the change of variables: $u = x_2 x_3$, $v = \frac{x_2^2 - x_3^2}{2}$. It gives rise to a diffeomorphism φ from $\{(u, v) : u > 0\}$ to $\{(x_2, x_3) : x_2 > 0, x_3 > 0\}$

defined by

$$\varphi(u,v) = \left(\sqrt{v+\sqrt{u^2+v^2}}, \frac{u}{\sqrt{v+\sqrt{u^2+v^2}}}\right) \quad \text{if} \quad v>0$$
$$\varphi(u,v) = \left(\frac{u}{\sqrt{-v+\sqrt{u^2+v^2}}}, \sqrt{-v+\sqrt{u^2+v^2}}\right) \quad \text{if} \quad v<0$$
$$\varphi(u,0) = \left(\sqrt{u}, \sqrt{u}\right).$$

φ can naturally be extended to $\{(u,v) : u>0\} \cup \{(0,v) : v \neq 0\}$. We still denote by φ this natural extension.

The following result is, for $\mathfrak{h}$, the analog of Lemma 3 given in Section 3 for $\mathfrak{g}_{4,1}$.

Lemma 4. *Let $a(x) = a(x_2, x_3)$ be in $C^\infty(\Omega)$. Suppose there exists some function $g(x) = g(x_2, x_3)$ in $C^\infty(\Omega)$ such that $H_{x_1}(g) = a$ and note $A = a \circ \varphi$, where φ is as above. Then, the limit*

$$\lim_{\substack{u\to 0\\ u>0}} \int_{-1}^{1} \frac{A(u,t)}{2\sqrt{u^2+t^2}} dt,$$

exists.

Proof. By assumption, $H_{x_1}(g) = (x_2\partial_2 - x_3\partial_3)(g) = a$. Changing variables, we get:

$$2\sqrt{u^2+v^2}\partial_v(G) = A$$

where $G = g \circ \varphi$. Thus, on the set $\{(u,v) : u>0\}$, G is necessarily of the form

$$G(u,v) = \int_{-1}^{v} \frac{A(u,t)}{2\sqrt{u^2+t^2}} dt + \psi(u).$$

Moreover, since $\varphi(0,-1) = (0,\sqrt{2})$ and g is continuous at $(0,\sqrt{2})$, $\lim\limits_{\substack{u\to 0\\ u>0}} G(u,-1)$ exists. Thus, $\lim\limits_{\substack{u\to 0\\ u>0}} \psi(u)$ exists too. In the same way, $\lim\limits_{\substack{u\to 0\\ u>0}} G(u,1)$ exists and Lemma 3 is proved.

Now, using the vector fields $\widetilde{T_\alpha} = \widetilde{t_\alpha}\partial_1$ and $\widetilde{K_\alpha} = \widetilde{k_\alpha}\partial_1$ where

$$\widetilde{t_\alpha} = \frac{1}{(x_2^2+x_3^2)^\alpha} \quad \text{and} \quad \widetilde{k_\alpha} = \frac{\exp\left(\frac{\alpha}{(x_2^2+x_3^2)^2}\right)}{(x_2^2+x_3^2)^3},$$

one can see that the space $H^1_{\Lambda,\tan}(\Omega)$ is infinite dimensional.

As for the other TP-cohomology spaces,

$$H^k_{\Lambda,\tan}(\Omega) = \{0\} \quad \forall k>1.$$

To compute the P-cohomology of Ω, let us say that the symplectic foliation $\mathcal{F}$ of Ω is orientable and is defined by the nonsingular 1-form $\beta = x_2dx_3 + x_3dx_2$ ($d\beta = 0$). By Proposition 6 and if we note $X = \dfrac{x_2\partial_3 + x_3\partial_2}{x_2^2 + x_3^2}$, we have

$$\begin{aligned} H^0_\Lambda(\Omega) &= I(\Omega) \\ H^1_\Lambda(\Omega) &= H^1_{\Lambda,\tan}(\Omega) \oplus \{uX : u \in I(\Omega)\} \\ H^2_\Lambda(\Omega) &\simeq H^1(\Omega, \Phi^1(\mathcal{F})) \simeq H^1_{\Lambda,\tan}(\Omega) \\ H^3_\Lambda(\Omega) &\simeq H^2(\Omega, \Phi^1(\mathcal{F})) \simeq H^2_{\Lambda,\tan}(\Omega) = \{0\} \\ H^k_\Lambda(\Omega) &= \{0\} \quad \forall k > 3. \end{aligned}$$

Thus, the large space $H^1_{\Lambda,\tan}(\Omega)$ happens in the P-cohomology. In other words, the spaces $H^1_\Lambda(\Omega)$ and $H^2_\Lambda(\Omega)$ are infinite dimensional.

Remark 7. Consider the family of 3-dimensional Lie algebras given by:

$$[X_1, X_2] = X_2, \quad [X_1, X_3] = \frac{1}{\tau}X_3 \quad \tau < -1.$$

One can prove that the TP-cohomology of Ω in these examples is essentially the same as in the case of $\mathfrak{h}$ because of a similar leaf structure. The situation seems more complicated for the P-cohomology since Proposition 6 cannot be used.

4.9. The case of $\mathfrak{sl}(2)$. We want now to show how the TP-cohomology computation for $\mathfrak{sl}(2)$ leads to the same conclusions as in the case of $\mathfrak{g}_{4,1}$ or $\mathfrak{h}$.

The Lie algebra $\mathfrak{sl}(2)$ is given by the brackets:

$$[X_1, X_2] = -X_3, \quad [X_3, X_1] = X_2, \quad [X_3, X_2] = -X_1.$$

Let us identify $\mathfrak{sl}(2)$ with its dual $\mathfrak{sl}(2)^*$ by means of the Killing form. Then the orbit decomposition is as follows. There are three orbits in the light cone:

$$\begin{aligned} W_0 &= \{0\}; \\ W_+ &= \{(\mu_1, \mu_2, \mu_3) : \mu_1^2 + \mu_2^2 - \mu_3^2 = 0, \mu_3 > 0\}; \\ W_- &= \{(\mu_1, \mu_2, \mu_3) : \mu_1^2 + \mu_2^2 - \mu_3^2 = 0, \mu_3 < 0\}. \end{aligned}$$

Moreover, the hyperbolic orbits are single sheeted hyperboloids outside the cone:

$$W_k = \{(\mu_1, \mu_2, \mu_3) : \mu_1^2 + \mu_2^2 - \mu_3^2 = k^2\} \quad (k > 0)$$

and double sheeted hyperboloids inside the cone (each of the sheets being a different orbit):

$$W_h = \{(\mu_1, \mu_2, \mu_3) : \mu_1^2 + \mu_2^2 - \mu_3^2 = -h^2, \mu_3 h > 0\} \quad (h \neq 0).$$

The regular Poisson manifold (Ω, Λ) associated to $\mathfrak{sl}(2)$ is thus the set

$$\Omega = \{(x_1, x_2, x_3) \in \mathfrak{sl}(2)^* : x_1^2 + x_2^2 + x_3^2 \neq 0\},$$

with its regular Poisson structure Λ and the space of leaves is clearly not Hausdorff.

As always, $H^0_{\Lambda,\tan}(\Omega) = I(\Omega)$.

How about $H^1_{\Lambda,\tan}(\Omega)$? Assume first that X is a tangential vector field of the form

$$X = aH_{x_3}$$

where a is in $C^\infty(\Omega)$. One can easily check that $X = \sigma f$ for a function f in $C^\infty(\Omega)$ if and only if f satisfies the following properties:

$$\begin{aligned} H_{x_1}(f) &= x_2 a \\ H_{x_2}(f) &= -x_1 a \\ H_{x_3}(f) &= 0. \end{aligned}$$

Consider the open set $U = \{(x_1, x_2, x_3) \in \Omega : x_1^2 + x_2^2 - x_3^2 > 0\}$ which can be parameterized by:

$$\begin{aligned} x_1 &= -p\sin(q) + z\cos(q) \\ x_2 &= -p\cos(q) - z\sin(q) \\ x_3 &= p. \end{aligned}$$

Using this parameterization, we see that, if $X = \sigma f$, then $\partial_p(f) = -a$ and $\partial_q(f) = 0$. In other words, if $X = \sigma f$, then $f_{|U}$ must depend only on (p, z) and be of the form

$$f = f(p, z) = -\int_{-1}^{p} a(s, q, z)ds + \psi(z).$$

Consider the vector fields $\hat{T_\alpha}$ ans $\hat{K_\alpha}$, $\alpha > \frac{1}{2}$, defined by

$$\hat{T_\alpha} = \hat{t_\alpha} H_{x_3} \quad \text{and} \quad \hat{K_\alpha} = \hat{k_\alpha} H_{x_3}$$

where

$$\hat{t_\alpha} = \frac{1}{(x_1^2 + x_2^2 + x_3^2)^\alpha} \quad \text{and} \quad \hat{k_\alpha} = \frac{\exp\left(\frac{\alpha}{x_1^2+x_2^2+x_3^2}\right)}{(x_1^2 + x_2^2 + x_3^2)^2}.$$

Then, with the same arguments as in the examples of $\mathfrak{g}_{4,1}$ and $\mathfrak{h}$, one can prove that the space $H^1_{\Lambda,\tan}(\Omega)$ is infinite dimensional.

The same is true for $H^2_{\Lambda,\tan}(\Omega)$. Indeed, take a tangential 2-tensor field A. Necessarily, there is a function φ in $C^\infty(\Omega)$ such that

$$A = \varphi\Lambda = \varphi(x_3\partial_2 \wedge \partial_1 + x_2\partial_3 \wedge \partial_1 + x_1\partial_2 \wedge \partial_3)$$

and A satisfies $\sigma(A) = 0$. Now, suppose there exists a tangential vector field B such that $A = \sigma(B)$. If we note $B = aH_{x_1} + bH_{x_2} + cH_{x_3}$ where a, b, c are in $C^\infty(\Omega)$, then we obtain

$$(\#) \quad H_{x_1}(a) + H_{x_2}(b) + H_{x_3}(c) = \varphi.$$

Translating $(\#)$ on the open set $U = \{(x_1, x_2, x_3) \in \Omega : x_1^2 + x_2^2 - x_3^2 > 0\}$, we get

$$(-x_2\partial_p - \sin(q)\partial_q)(a) + (x_1\partial_p - \cos(q)\partial_q)(b) + \partial_q(c) = \varphi.$$

Let us integrate the above expression between 0 and 2π as follows

$$\begin{aligned}
&p\partial_p\left(\int_0^{2\pi}(\cos(q)a - \sin(q)b)dq\right) + z\partial_p\left(\int_0^{2\pi}(\sin(q)a + \cos(q)b)dq\right)\\
&\quad + \quad \int_0^{2\pi}(-\sin(q)\partial_q(a) - \cos(q)\partial_q(b))dq\\
&= p\partial_p\left(\int_0^{2\pi}(\cos(q)a - \sin(q)b)dq\right) + z\partial_p\Big(\int_0^{2\pi}(\sin(q)a + \cos(q)b)dq\Big)\\
&\quad + \quad \int_0^{2\pi}(\cos(q)a - \sin(q)b)dq\\
&= \int_0^{2\pi}\varphi(p, q, z)dq.
\end{aligned}$$

Assume that φ depends only on the variables p, z. Moreover, define the functions H and g by

$$H(p,z) = \int_0^{2\pi}(\cos(q)a - \sin(q)b)dq, \quad g(p,z) = \int_0^{2\pi}(\sin(q)a + \cos(q)b)dq.$$

Then we have

$$p\partial_p H + H = 2\pi\varphi - z\partial_p(g).$$

Thus, on the open set $U \cap \{p > 0\}$, H is necessarily of the form

$$H = \frac{1}{p}\Big(\int_1^p 2\pi\varphi(s,z)ds - zg(p,z) + zg(1,z) + \psi_1(z)\Big).$$

Similarly, on the open set $U \cap \{p < 0\}$, H is necessarily of the form

$$H = \frac{1}{p}\Big(\int_{-1}^p 2\pi\varphi(s,z)ds - zg(p,z) + zg(-1,z) + \psi_2(z)\Big).$$

The function $H(p, z)$ can be extended at the points $(p, 0)$ with $p \neq 0$, therefore $\psi_1(z)$ et $\psi_2(z)$ have a limit when $z(> 0)$ tends to zero. Moreover, $H(p, z)$ can also be extended at the points $(0, z)$ with $z \neq 0$. Thus, for all $z \neq 0$, we get

$$\int_1^0 (2\pi\varphi(s,z))ds - zg(0,z) + zg(1,z) + \psi_1(z) = 0$$

and

$$\int_{-1}^{0}(2\pi\varphi(s,z))ds - zg(0,z) + zg(-1,z) + \psi_2(z) = 0.$$

It follows that

$$\int_{-1}^{1}(2\pi\varphi(s,z))ds - zg(1,z) + zg(-1,z) - \psi_1(z) + \psi_2(z) = 0.$$

In other words, if $A = \varphi\Lambda$ where $\varphi_{|U} = \varphi(p,z)$ and if there exists a tangential vector field B such that $A = \sigma(B)$, then the limit

$$\lim_{\substack{z\to 0\\ z>0}} \int_{-1}^{1} 2\pi\varphi(s,z)ds$$

should exist. Now to see that the space $H^2_{\Lambda,\tan}(\Omega)$ is infinite dimensional, one can take the vector fields $\hat{t_\alpha}\Lambda$ where $\hat{t_\alpha} = \dfrac{1}{(x_1^2+x_2^2+x_3^2)^\alpha}$ and $\hat{k_\alpha}\Lambda$ where $\hat{k_\alpha} = \dfrac{\exp\left(\frac{\alpha}{x_1^2+x_2^2+x_3^2}\right)}{(x_1^2+x_2^2+x_3^2)^2}$.

Lastly,

$$H^k_{\Lambda,\tan}(\Omega) = 0 \quad \forall k > 2.$$

Let us also say that the symplectic foliation $\mathcal{F}$ of Ω is orientable and is defined by the nonsingular 1-form $\beta = x_1dx_1 + x_2dx_2 - x_3dx_3$, which satisfies $d\beta = 0$. Thus, using Proposition 6, one can see that all the P-cohomology spaces $H^k_\Lambda(\Omega)$ $(1 \le k \le 3)$ are infinite dimensional.

We finish with an application of our cohomology calculations for the classification of tangential star products:

Remark 8. As we just proved in this section, the second TP-cohomology space of Ω is zero for any 3-dimensional Lie algebra except for $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$. This implies the uniqueness (up to equivalence) of the tangential star products on Ω for any non-simple 3-dimensional Lie algebra.

5. Concluding remarks.

As shown in Section 2, for a regular Poisson manifold, the TP-cohomology coincides with the leafwise de Rham cohomology of the symplectic foliation, thus unlike the P-cohomology, does not depend on the symplectic structure along the leaves. That relates the task of computing the TP-cohomology of regular Poisson manifolds to nontrivial questions of foliation theory. Roughly speaking, the TP-cohomology not only contains the de Rham cohomology of the leaves, but also translates the foliation complexity which includes essentially the relative position of the leaves. As seen from the computations of Sections 3 and 4, when the space of leaves is not Hausdorff, this TP-cohomology is very large and hardly describable. To finish,

let us say that the TP-cohomology spaces are involved in the P-cohomology, for instance we have the inclusion $H^1_{\Lambda,\tan}(M) \subset H^1_\Lambda(M)$ for any regular Poisson manifold M. Thus, our calculations and comments can be of some help to understand why the P-cohomology itself is, as often said, so difficult to compute.

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