Pacific Journal of Mathematics

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Volume 203 No. 2 April 2002

PACIFIC JOURNAL OF MATHEMATICS
Vol. 203, No. 2, 2002

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Let K be a quadratic extension of Q, B a quaternion algebra over Q and $A = B \otimes_{\mathbf{Q}} K$. Let $\mathcal{O}$ be a maximal order in A extending an order in B. The projective norm one group $P\mathcal{O}^1$ is shown to be isomorphic to the spinorial kernel group $O'(L)$, for an explicitly determined quadratic Z-lattice L of rank four, in several general situations. In other cases, only the local structures of $\mathcal{O}$ and L are given at each prime. Both definite and indefinite lattices are covered. Some results for quadratic global field extensions K/F and maximal S-orders are given. There is a description of the F-quaternion subalgebras of A, and also of their norm one groups as stabilizer subgroups and as unitary groups. Conjugacy classes of the Fuchsian subgroups of $P\mathcal{O}^1$ corresponding to stabilizer subgroups are studied.

1. Introduction.

The Bianchi groups were described as the spinorial kernel groups $O'(L)$ of certain specific rank four indefinite lattices L over $\mathbf{Z}$ in [**6**]. This enabled local-global techniques on these orthogonal groups to be used, to classify up to conjugacy, the maximal Fuchsian subgroups of the Bianchi groups. Later, in [**5**], this was generalized to $SL(2, D_S)$ where D_S is the ring of S-integers in a global field K, a quadratic extension of F, and used to classify up to conjugacy the unitary subgroups of $SL(2, D_S)$. This approach utilized a connection between the norm one group in the split quaternion algebra $\mathbf{M}(2, K)$ and a spinor orthogonal group $O'(V)$ over F. These techniques will now be extended to the corresponding norm one groups of S-orders $\mathcal{O}_S$ in quaternion algebras A over global fields K. This work has evolved from questions asked by C. Maclachlan and W. Plesken about the Fuchsian subgroups of arithmetic Kleinian groups.

Let K/F be a quadratic extension of global fields, let B be a quaternion algebra over F and $A = B \otimes_F K$. For a Dedekind set of prime spots S for F (see [**10**]), let R_S be the ring of S-integers in F, and D_S its integral closure in K. For several explicit maximal S-orders $\mathcal{L}_S$ in B we construct an exact sequence

$$1 \to \{\pm 1\} \to \mathcal{O}_S^1 \xrightarrow{\Phi} O'(L) \to 1$$

where $\mathcal{O}_S^1$ is the multiplicative group of elements with norm one in the order $\mathcal{O}_S = \mathcal{L}_S \otimes_{R_S} D_S$, and $O'(L)$ is the spinor kernel group of a S-lattice L in a quadratic space V over F that is explicitly determined by $\mathcal{O}_S$. In particular, the sequence is exact when S contains no dyadic primes so that 2 is a unit in R_S (see Theorem 5.1). Several rational examples with $F = \mathbf{Q}$, for explicit global orders $\mathcal{O}$ and the corresponding Kleinian groups and $\mathbf{Z}$-lattices L, are given in §6. Since the arguments still hold in the split case we get new proofs of results in [**5**] and [**6**] on Hilbert modular and Bianchi groups. Other results where Φ is surjective are given, but $\mathcal{O}$ and L are only described locally. In particular, the dyadic primes give several difficulties. It appears necessary to assume that $\mathcal{O}$ is a maximal order to show Φ surjective (see Theorem 6.3 and other examples in §6). The proofs showing Φ surjective use localization arguments and are independent of whether the underlying quadratic forms are definite or indefinite. The stabilizer subgroups $\mathrm{Stab}(v, O'(V))$, for anisotropic $v \in V$, are isomorphic to the projective norm one groups of the F-quaternion subalgebras of A; these groups are also unitary groups (see §4).

For K an imaginary quadratic field and $R_S = \mathbf{Z}$, the discrete groups $\mathrm{Stab}(v, O'(L))$ give examples of Fuchsian subgroups of the arithmetic Kleinian groups $P\mathcal{O}^1$. The conjugacy classes of these groups are studied in the final section using the local-global method of [**6**] (see also [**9**]).

2. Quaternion algebras.

Let F be a field, with characteristic not two, and $K = F(\alpha)$ where $\alpha \notin F$ and $\alpha^2 \in F$. Then $K = \{a + \alpha b | a, b \in F\}$ has a galois automorphism $\overline{a + \alpha b} = a - \alpha b$. If we let β^J denote the standard conjugate of β in a quaternion algebra B over F, then the F-linear mapping

$$\tau : B \otimes_F K \to B \otimes_F K$$

induced by $\tau(\beta \otimes x) = \beta^J \otimes \bar{x}$ is a conjugate linear map of the K-space $A = B \otimes_F K$ and an anti-homomorphism with respect to multiplication of the quaternion algebra A. Thus, for $\beta, \gamma \in A$ and $a, b \in K$,

$$\tau(a\beta + b\gamma) = \bar{a}\tau(\beta) + \bar{b}\tau(\gamma) \text{ and } \tau(\beta\gamma) = \tau(\gamma)\tau(\beta).$$

The norm form $n : A \to K$ is defined by $n(\beta) = \beta\beta^J$ where now J is the extension of the standard conjugation to A over K. Then $\tau(\beta^J) = \tau(\beta)^J$ so that $n(\tau(\beta)) = \overline{n(\beta)}$.

Let $V = \{v \in A | \tau(v) = v\}$. If $1, i, j, ij = k$ is a standard basis of B, then V is a 4-dimensional F-space with basis $\{1, \alpha i, \alpha j, \alpha k\}$. Moreover, this is an orthogonal basis with respect to the restriction of the norm form, so that V is a quadratic space with symmetric bilinear form

$$f(v,w) = n(v+w) - n(v) - n(w) = vw^J + wv^J$$

for $v, w \in V$. Note that

$$f(v,w) = \tau(f(v,w)) = \tau(vw^J + wv^J) = f(v^J, w^J).$$

If $B = (\frac{a,b}{F})$ so that $i^2 = a, j^2 = b, ji = -ij$, and $\alpha^2 = -d \in F$, then V diagonalizes with f-matrix $\langle 2, 2ad, 2bd, -2abd\rangle$.

Let $A_F^* = \{\beta \in A | n(\beta) \in F^*\}$ and note that the anisotropic vectors of V lie in A_F^*. Define ϕ_β on V by

$$\phi_\beta(v) = n(\beta)^{-1}\beta v \tau(\beta).$$

Then $\phi_\beta \in O(V)$, the orthogonal group of V, and $\Phi : A_F^* \to O(V)$, with $\Phi(\beta) = \phi_\beta$, defines a homomorphism.

Now suppose that $\beta \in \text{Ker } \Phi$. Then $n(\beta)^{-1}\beta v\tau(\beta) = v$ for all $v \in V$. Since $1 \in V$, we have that $\tau(\beta) = \beta^J$ and hence $\beta \in B$. For $v = \alpha i, \alpha j, \alpha k$, the equality $\beta v \beta^{-1} = v$ then implies that $\beta\gamma = \gamma\beta$ for all $\gamma \in B$. Thus $\beta \in Z(B)$. Conversely if $\beta \in Z(B)$, then $\beta \in \text{Ker } \Phi$. Hence $\text{Ker } \Phi = F^*$.

The group $O(V)$ is generated by reflections ρ_y, for y an anisotropic vector in V, where for each $v \in V$,

$$\rho_y(v) = -yv^J(y^J)^{-1} = v - f(y,v)n(y)^{-1}y.$$

Then $\rho_{y_1}\rho_{y_2}(v) = y_1 y_2^{-1} v y_2^J (y_1^J)^{-1} = n(y_1 y_2^J)^{-1}(y_1 y_2^J) v \tau(y_1 y_2^J)$. Thus $\rho_{y_1}\rho_{y_2} = \phi_\beta$ for $\beta = y_1 y_2^J$. Since $SO(V)$ consists of products of an even number of reflections, it follows that $SO(V) \subseteq \Phi(A_F^*)$. If the image of A_F^* properly contained $SO(V)$ then each reflection would lie in the image. In particular, $\rho_{\alpha i} = \phi_\beta$ for some $\beta \in A_F^*$, and

$$n(\beta)^{-1}\beta v\tau(\beta) = -\alpha i v^J((\alpha i)^J)^{-1} = iv^J i^{-1}.$$

As before, taking $v = 1$, we obtain $\beta \in B$. Therefore, $i\beta v^J = vi\beta$ for all $v \in V$, and hence $\beta = 0$. This contradiction shows that $\rho_{\alpha i}$ is not in the image of A_F^*, and we have an exact sequence

$$1 \to F^* \to A_F^* \xrightarrow{\Phi} SO(V) \to 1.$$

Clearly Φ restricted to the norm one group A^1 has kernel $\{\pm 1\}$. Let Θ denote the spinor norm on $SO(V)$. If $\beta = y_1 y_2^J$ as above, then

$$\Theta(\phi_\beta) = \Theta(\rho_{y_1})\Theta(\rho_{y_2}) = n(y_1)n(y_2) = n(\beta)$$

viewed in $F^*/{F^*}^2$. More generally, since $SO(V)$ consists of products of an even number reflections, $\Theta(\phi_\beta) = n(\beta)$ for $\phi_\beta \in SO(V)$. Given $\varphi \in O'(V)$ the spinorial kernel, there exists $\beta \in A_F^*$ with $\Phi(\beta) = \varphi$. Since $\Theta(\varphi) = 1$ we have $n(\beta) \in F^{*2}$ so we may choose $\beta \in A^1$. Thus $\Phi(A^1) = O'(V)$. This establishes the following generalization of results in [**5**, **6**] where only the split case $A = \mathbf{M}(2, K)$ was treated.

Theorem 2.1. *With notation as above, the following sequence is exact*

$$1 \to \{\pm 1\} \to A^1 \xrightarrow{\Phi} O'(V) \to 1.$$

Other forms of this result are given in [**2**, p. 32] and [**3**, §7.3B]. The argument above is a variation of one by Colin Maclachlan, and is derived from that in [**6**] by avoiding a choice of basis.

3. S-orders and integral groups.

Now assume that F is a global field, with characteristic not two, and let S be a Dedekind set of prime spots for F and R_S the ring of S-integers in F (see [**10**]). Denote the integral closure of R_S in $K = F(\alpha)$ by D_S. Let B be a quaternion algebra over F and $A = B \otimes_F K$. Next let $\mathcal{L}$ be an S-order in B so that $\mathcal{L}^J = \mathcal{L}$. Then $\mathcal{O}_S = \mathcal{L} \otimes_{R_S} D_S$ is an S-order in A and $\tau(\mathcal{O}_S) = \mathcal{O}_S$. Put $L = \mathcal{O}_S \cap V$. Since $\mathcal{O}_S$ is a finitely generated D_S-module and D_S is a finitely generated R_S-module, L is a lattice over R_S. Note that $1, \alpha i, \alpha j, \alpha k \in L$ if $\alpha \in D_S$ and $a, b \in R_S$.

Define a subgroup of A_F^* by

$$A_L = \{\beta \in A_F^* | \beta L = L\tau(\beta^J)\}.$$

Then there is a homomorphism

$$\Phi : A_L \to SO(L)$$

given by $\Phi(\beta) = \phi_\beta$, where $\phi_\beta(v) = n(\beta)^{-1}\beta v \tau(\beta) \in L$ for all $v \in L$. This follows since $\phi_\beta(L) = L$ if and only if $\beta v \tau(\beta) \in \beta\beta^J L$, that is, $\beta v \in \tau(\beta^J L) = L\tau(\beta^J)$ for all $v \in L$. The kernel of Φ is

$$\text{Ker } \Phi = \{\beta \in A_L | \beta v = v\tau(\beta^J) \text{ for all } v \in L\} = F^*$$

and so we have an exact sequence

$$1 \to F^* \to A_L \xrightarrow{\Phi} SO(L).$$

Next we show that this mapping Φ is locally surjective (under an assumption at dyadic primes). For non-dyadic primes $p \in S$ the local group $O(L_p)$ is generated by integral symmetries, even without going to the completion in the localization L_p (see [**10**, §92.4]). Hence we can modify the argument for the surjectivity of A_F^* onto $SO(V)$. Let L_p be the local lattice over the local ring $R_p \subseteq F$ (not completed) at a non-dyadic prime $p \in S$. Define, for $v \in L_p$,

$$\rho_y(v) = v - f(v, y)n(y)^{-1}y = -yv^J(y^J)^{-1}$$

where $y \in L_p$ satisfies $f(L_p, y) \subseteq n(y)R_p \neq 0$. Then $\rho_y \in O(L_p)$, and $SO(L_p)$ is generated by pairs of such integral symmetries. As before, for anisotropic $y_1, y_2 \in L_p$, put $\beta = y_1 y_2^J$ so that $\rho_{y_1}\rho_{y_2} = \phi_\beta$. Note that the condition $y_1 y_2^J L_p = L_p \tau(y_2 y_1^J)$ follows from the restrictions on y_1 and y_2 assumed for integral symmetries. Now we have a local exact sequence

$$1 \to F^* \to A_{L_p} \xrightarrow{\Phi} SO(L_p) \to 1$$

where $A_{L_p} = \{\beta \in A_F^* \mid \beta L_p = L_p \tau(\beta^J)\}$. Next restrict to the local exact sequence

$$1 \to \{\pm 1\} \to A^1_{L_p} \xrightarrow{\Phi} O'(L_p) \to 1$$

where $A^1_{L_p} = \{\beta \in A_{L_p} \mid n(\beta) = 1\}$. The map Φ remains surjective. For given $\varphi \in O'(L_p)$, there exists $\beta \in A_{L_p}$ such that $\Phi(\beta) = \varphi$. Since $\Theta(\varphi) = 1$, we have $n(\beta) = {F^*}^2$ and hence we may choose $\beta \in A^1_{L_p}$.

Theorem 3.1. *Let $\mathcal{O}_S = \mathcal{L} \otimes_{R_S} D_S$ be the S-order in A defined above, and assume S contains no dyadic primes. Then, for $L = \mathcal{O}_S \cap V$, the following sequence is exact*

$$1 \to \{\pm 1\} \to A^1_L \xrightarrow{\Phi} O'(L) \to 1.$$

Proof. The local surjectivity established above can be used to show global surjectivity onto $O'(L)$ as follows. For $v \in V$ we have $v \in L$ if and only if $v \in L_p$ for all $p \in S$ (see [**10**]). Let $\beta \in A^1$. Then

$$\begin{aligned} \beta \in A^1_L &\iff \beta L \tau(\beta) = L \\ &\iff \beta L_p \tau(\beta) = L_p \text{ for all } p \in S \\ &\iff \beta \in A^1_{L_p} \text{ for all } p \in S. \end{aligned}$$

Let $\varphi \in O'(L) \subseteq O'(V)$, so there exists $\beta \in A^1$ such that $\Phi(\beta) = \varphi$. Since φ is in $O'(L_p)$, there exists $\beta_p \in A^1_{L_p}$ such that $\Phi(\beta_p) = \varphi$. Then $\beta\beta_p^{-1} \in$ Ker $\Phi = \{\pm 1\}$, so that $\beta \in A^1_{L_p}$ for each prime $p \in S$. Therefore, $\beta \in A^1_L$.

To handle dyadic primes, and study the primes where A is ramified, we go to the completions. Let $\mathcal{P}$, over the prime $p \in S$, be a prime where A is ramified or a dyadic prime, let $K_\mathcal{P}$ be the completion of K at $\mathcal{P}$, and F_p the completion of F at p. The corresponding complete local rings of integers are denoted by $D_\mathcal{P}$ and R_p.

Ramified primes. When A is ramified at $\mathcal{P}$, $A_\mathcal{P} = A \otimes_K K_\mathcal{P}$ is a division ring and, necessarily, $B_p = B \otimes_F F_p$ is also a division ring. Then $\nu(\beta) = \text{ord}_\mathcal{P}\, n(\beta)$ defines a discrete valuation on $A_\mathcal{P}$, and

$$\mathcal{O}_\mathcal{P} = \{\beta \in A_\mathcal{P} | \nu(\beta) \geq 0\}$$

is the unique maximal order of $A_{\mathcal{P}}$, assuming $\mathcal{O}_S$ is locally maximal at $\mathcal{P}$ (see Lemma 1.5 in [**12**, p. 34]). Put $V_p = \{v \in A_{\mathcal{P}} \mid \tau(v) = v\}$, an anisotropic quadratic space over F_p, and

$$L_p = V_p \cap \mathcal{O}_{\mathcal{P}} = \{v \in V_p | n(v) \in R_p\}.$$

Then L_p is a maximal R_p-lattice in the sense of Eichler, and the integral group $O(L_p) = O(V_p)$ (see [**10**, §91A, 91:15]). It follows that $O(L_p)$ is generated by integral symmetries. Moreover, if $\beta \in A^1_{L_p}$ then $n(\beta) = 1$ so that $\beta \in \mathcal{O}_{\mathcal{P}}$, trivially. Hence $A^1_{L_p} \subseteq \mathcal{O}^1_{\mathcal{P}}$.

By arguments similar to those above for non-dyadic primes, but now with A replaced by $A_{\mathcal{P}}$, and $A^1_{L_p}$ modified accordingly, we get the following exact sequence for the completed groups

$$1 \to \{\pm 1\} \to A^1_{L_p} \xrightarrow{\Phi} O'(L_p) \to 1.$$

Dyadic primes. We still need to consider dyadic primes $\mathcal{P}$ where A is not ramified. Eichler transformations $E(u, x)$ are now needed since there are cases in rank four where $O(L_p)$ is not generated by symmetries (see [**11**]). Let $u, x \in V_p$ satisfy $n(u) = 0$ and $f(u, x) = ux^J + xu^J = 0$, and put $\beta = 1 - xu^J \in A^1_{\mathcal{P}}$. Then, for $v \in V_p$,

$$\phi_\beta(v) = \beta v \tau(\beta) = E(u, x)(v)$$

where

$$E(u, x)(v) = v - f(u, v)x + f(x, v)u - n(x)f(u, v)u$$

since $xu^Jv + vu^Jx = f(u, v)x - f(x, v)u$ and $xu^Jvu^Jx = -n(x)f(u, v)u$. We need the integrality conditions $f(u, L_p)x \subseteq L_p, f(x, L_p)u \subseteq L_p$ and $n(x)f(u, L_p)u \subseteq L_p$ to get $E(u, x) \in O'(L_p)$. Then $\Phi(A^1_{L_p}) = O'(L_p)$ follows whenever $SO(L_p)$ is generated by integral Eichler transformations and double symmetries. In particular, Theorem 4.1 in [**5**] establishes this for the groups $SO(L_p)$ associated with the maximal orders $\mathbf{M}(2, \mathcal{O}_{\mathcal{P}})$ in the dyadic split case where $A_{\mathcal{P}} \cong \mathbf{M}(2, K_{\mathcal{P}})$, but nice generators for the general dyadic case are not known when L_p is not unimodular (see [**11**]).

Theorem 3.2. *Let $\mathcal{O}_S = \mathcal{L} \otimes_{R_S} D_S$ be the S-order in A defined above and put $L = \mathcal{O}_S \cap V$. Assume the complete local group $SO(L_p)$ is generated by integral Eichler transformations and double symmetries at all dyadic $p \in S$. Then the sequence*

$$1 \to \{\pm 1\} \to A^1_L \xrightarrow{\Phi} O'(L) \to 1$$

is exact.

The proof is essentially the same as for Theorem 3.1.

Theorem 3.3. *Let $\mathcal{O}_S$ be an S-order in A with $\tau(\mathcal{O}_S) = \mathcal{O}_S$. Then*

$$\mathcal{O}_S^1 \subseteq A_L^1.$$

Proof. Let $\beta \in \mathcal{O}_S^1$ so that $n(\beta) = \beta\beta^J = 1$. If $v \in L$ we must prove $\beta v \in L\tau(\beta^J)$, that is, $\beta v\tau(\beta) \in L = \mathcal{O}_S \cap V$. Since $\tau(\mathcal{O}_S^1) = \mathcal{O}_S^1$, we have $\beta v\tau(\beta) \in \mathcal{O}_S$ because $v \in \mathcal{O}_S$. Also $\tau(\beta v\tau(\beta)) = \beta v\tau(\beta) \in V$.

We give several examples in §§5,6 where $\mathcal{O}_S^1 = A_L^1$, but this is not true in general, as shown by 6.3 and the other examples in §6.

4. Stabilizer subgroups and quaternion subalgebras.

Let F be any field with $2 \neq 0$ and $K = F(\alpha)$ with $\alpha^2 = -d \in F^*$. As before, assume $A = B \otimes_F K$. Take $v \neq 0$ in V. Then, for $\beta \in A^1$, its image $\Phi(\beta)$ is in $\mathrm{Stab}(v, O'(V)) = \{\phi \in O'(V) | \phi(v) = v\}$ if and only if $\phi_\beta(v) = \beta v\tau(\beta) = v$, or equivalently $\beta v = v\tau(\beta^J)$. Define

$$A(v) = \{\beta \in A \mid \beta v = v\tau(\beta^J)\}.$$

Then $A(v)$ is a F-subalgebra of A with $A(v)^J = A(v)$. Moreover,

$$\Phi(A(v)^1) = \mathrm{Stab}(v, O'(V)).$$

In particular, $A(1) = B$ and $\Phi(B^1) = \mathrm{Stab}(1, O'(V))$.

Theorem 4.1. *Let $v \in V$ with $n(v) \neq 0$. Then $A(v)$ is a quaternion algebra over the field F with conjugation J induced from A. If $V = Fv \perp W$ then $A(v) \cong C^+(W)$, the even Clifford algebra of W.*

Proof. Expand v to an orthogonal basis v, v_1, v_2, v_3 of V and let $\beta_1 = v_2v_3^J, \beta_2 = v_3v_1^J$ and $\beta_3 = v_1v_2^J$. Since $v_iv_j^J + v_jv_i^J = f(v_i, v_j) = 0$ for $i \neq j$, it follows that $\beta_i^J = -\beta_i$ and $\beta_i\beta_j = -\beta_j\beta_i$. Also,

$$\beta_1\beta_2 = v_2v_3^Jv_3v_1^J = -n(v_3)\beta_3,$$

$$\beta_1^2 = v_2v_3^Jv_2v_3^J = -v_2v_3^Jv_3v_2^J = -n(v_2)n(v_3).$$

There are similar results for β_2^2 and β_3^2. Also $\phi_{\beta_1} = \rho_{v_2}\rho_{v_3}$. Hence $\phi_{\beta_i}(v) = v = n(\beta_i)^{-1}\beta_iv\tau(\beta_i)$ and consequently $\beta_i \in A(v)$. If we show that $1, \beta_1, \beta_2, \beta_3$ are linear independent over F and span $A(v)$, it follows that $A(v)$ is the quaternion algebra $(\frac{-n(v_1v_2), -n(v_2v_3)}{F}) \cong C^+(W)$ (see [**2**, p. 29] or [**10**, §54]). Assume that $a_0 + a_1\beta_1 + a_2\beta_2 + a_3\beta_3 = 0$ with $a_i \in F$. Multiply through on the right by v_1 and use $\beta_1v_1 = v_1\tau(\beta_1^J)$ and $\beta_iv_1 = -v_1\tau({\beta_i}^J)$ for $i = 2, 3$. Simplifying, subtracting and repeating variations of this shows that all $a_i = 0$. Finally, note that $\alpha, \alpha\beta_i \notin A(v)$ and A is eight dimensional over F to complete the proof.

For example, $A(\alpha i)$ has F-basis $1, i, \alpha j, \alpha k$ and $A(\alpha i) = \left(\frac{a, -db}{F}\right)$.

Theorem 4.2. *Let $Q \subset A$ be a quaternion F-subalgebra of A with conjugation induced from A. Then $Q = A(v)$ for some $v \in V$ with $n(v) \neq 0$.*

Proof. Let $1, \beta_1, \beta_2, \beta_3$ be a standard basis for Q over F with $\beta_i^J = -\beta_i$, $n(\beta_i) = a_i \in F^*$, and $\beta_1\beta_2 = a\beta_3$. If $\Phi(\beta_1) = \pm I$, then $\Phi(\beta\beta_1) = \Phi(\beta_1\beta)$ for all $\beta \in Q$ with $n(\beta) \in F^*$, and we get contradictions such as $(\beta_1 + \beta_2)\beta_1 = \pm\beta_1(\beta_1 + \beta_2)$. Since $\beta_i\beta_j = -\beta_j\beta_i$ for $i \neq j$, the three maps $\Phi(\beta_i)$ form a set of mutually commuting, extremal, non-central involutions in $SO(V)$. Hence there exists an orthogonal basis v_1, v_2, v_3, v of V with $\Phi(\beta_1) = \rho_{v_2}\rho_{v_3}, \Phi(\beta_2) = \rho_{v_1}\rho_{v_3}$ and $\Phi(\beta_3) = \Phi(\beta_1)\Phi(\beta_2) = \rho_{v_1}\rho_{v_2}$. Since $\Phi(\beta_i)(v) = v = n(\beta_i)^{-1}\beta_i v \tau(\beta_i)$, it follows that $\beta_i \in A(v)$. Hence $Q = A(v)$ since both algebras are four dimensional over F.

The group $A(v)^1 = \{\beta \in A^1 | \beta v \tau(\beta) = v\}$ can be viewed as a subgroup of a unitary group. The special case $a = -b = 1$, where B is the matrix algebra $\mathbf{M}(2, F)$, was considered in [**5**, §5]. We now give a very different approach.

For fixed $v \neq 0$ in V, set $f_v(x, y) = xv\tau(y)$ for $x, y \in A$. Then $f_v(ax, by) = af_v(x, y)b^J$ for $a, b \in B$. Define $h : A \times A \to B$ by

$$h(x, y) = f_v(x, y) + f_v(y, x)^J.$$

Then $h(x, y)^J = h(y, x) = \tau(h(x, y))$ so that $h(x, y) \in B$ and $h(x, x) \in F$. Thus h is an hermitian form on the B-module A (see [**3**, §5.1B]). Note that h is singular when $n(v) = 0$ since then $h(v^J, A) = 0$.

Let $U(A, h)$ be the unitary group of this form. For $\beta \in A(v)^1$, so that $\beta v\tau(\beta) = v$, define a linear map $\psi_\beta : A \to A$ by $\psi_\beta(x) = x\beta$. Then, for $x, y \in A$,

$$h(\psi_\beta(x), \psi_\beta(y)) = h(x\beta, y\beta) = h(x, y)$$

and hence $\psi_\beta \in U(A, h)$. Hence $\Psi(\beta) = \psi_\beta$ defines an anti-monomorphism $\Psi : A(v)^1 \to U(A, h)$.

To determine the image of Ψ first note that $n(\psi_\beta(x)) = n(x)$ and also $f_v(\psi_\beta(x), \psi_\beta(y)) = f_v(x, y)$ for all $x, y \in A$. Therefore, define the special unitary group $SU(A, h)$ to consist of those $\psi \in U(A, h)$ with the two properties $n(\psi(x)) = n(x)$ and $f_v(\psi(x), \psi(y)) = f_v(x, y)$ for all $x, y \in A$.

Theorem 4.3. *Assume h is non-singular. Then the map*

$$\Psi : A(v)^1 \to SU(A, h)$$

is an anti-isomorphism.

Proof. It remains to show that Ψ is surjective. Let $\psi \in SU(A, h)$ and put $\psi(1) = \beta \in A$. Then $n(\beta) = n(\psi(1)) = 1$ so that $\beta \in A^1$. From $f_v(\psi(1), \psi(1)) = f_v(1, 1)$ we get $\beta v\tau(\beta) = v$, so that $\beta \in A(v)^1$. Replacing ψ by $\psi_{\beta^J}\psi$ we may assume that $\beta = 1$. Since ψ is B-linear and $1, \alpha$ is a basis of A over B, it now suffices to show $\psi(\alpha) = \alpha$. Put $\gamma = \psi(\alpha)$. From $f_v(\alpha, \alpha) = f_v(\gamma, \gamma)$ and $n(\gamma) = -d$ it follows that $v\tau(\gamma) = -\gamma^J v$. Then $f_v(\alpha, 1) = f_v(\gamma, 1)$ yields $\alpha v = \gamma v$. Hence $\alpha = \gamma$ provided $n(v) \neq 0$.

5. Norm one groups.

Let F be a global field with $2 \neq 0$, and let S be a Dedekind set of prime spots for F that contains no dyadic primes. For $a, b \in R_S$, let $B = \left(\frac{a,b}{F}\right)$ and take $\mathcal{L}_S = R_S1 + R_Si + R_Sj + R_Sk = R_S[i,j]$, an order in B. Let $K = F(\alpha)$ with $\alpha^2 = -d \in R_S$, and assume that locally $0 \leq \mathrm{ord}_p(abd) \leq 1$ for all $p \in S$. Note that $D_S = R_S[\alpha]$ since 2 is a unit in R_S. Denote by R_p the localization of R_S at $p \in S$, with completion not assumed, and by $D_{\mathcal{P}}$ the localization of D_S at a prime $\mathcal{P}$ over p.

Theorem 5.1. *Let $\mathcal{O}_S = \mathcal{L}_S \otimes_{R_S} D_S$ be an S-order in A, with $\mathcal{L}_S$ as above, and assume S excludes all dyadic primes. Then $\mathcal{O}_S^1 = A_L^1$ where*

$$L = \mathcal{O}_S \cap V = R_S1 \perp R_S\alpha i \perp R_S\alpha j \perp R_S\alpha k \cong \langle 2, 2ad, 2bd, -2abd\rangle$$

and there exists an exact sequence

$$1 \to \{\pm 1\} \to \mathcal{O}_S^1 \xrightarrow{\Phi} O'(L) \to 1.$$

Proof. Since $\mathcal{O}_S^1 \subseteq A_L^1$ by 3.3, it remains to prove $A_L^1 \subseteq \mathcal{O}_S$; then the result follows from 3.1. Let

$$\beta = x + yi + zj + wk \in A_L^1.$$

It suffices to prove that $\beta \in \mathcal{O}_{\mathcal{P}}$, the localization of $\mathcal{O}_S$ at $\mathcal{P}$, for all primes $\mathcal{P}$ over $p \in S$, by using $\beta v\tau(\beta) \in L$ for all $v \in L$, and

$$n(\beta) = \beta\beta^J = x^2 - ay^2 - bz^2 + abw^2 = 1.$$

Let $\mathrm{Tr} : K \to F$ denote the trace. Taking $v = \alpha i$ gives

$$\begin{aligned} \beta\alpha i\tau(\beta) &= \alpha(x + yi + zj + wk)(\bar{x}i - a\bar{y} - \bar{z}k - a\bar{w}j) \\ &= -a\mathrm{Tr}(\alpha x\bar{y} + \alpha bz\bar{w}) + (x\bar{x} - ay\bar{y} + bz\bar{z} - abw\bar{w})\alpha i \\ &\quad -a\mathrm{Tr}(x\bar{w} + y\bar{z})\alpha j - \mathrm{Tr}(x\bar{z} + ay\bar{w})\alpha k \in L. \end{aligned}$$

Similar results, but with different sign patterns, follow for $v = 1, \alpha j, \alpha k$. Thus, $x\bar{x} + ay\bar{y} - bz\bar{z} - abw\bar{w}$, $\mathrm{Tr}(x\bar{y} - bz\bar{w})$, $b\mathrm{Tr}(x\bar{w} - y\bar{z}) \in R_S$ follow from $v = \alpha j$. Hence $x\bar{x}, ay\bar{y}, bz\bar{z}$ and $abw\bar{w}$ are in R_S. Also, $a\mathrm{Tr}(x\bar{y})$, $b\mathrm{Tr}(x\bar{z}), ab\mathrm{Tr}(x\bar{w}), ab\mathrm{Tr}(y\bar{z}), ab\mathrm{Tr}(y\bar{w})$ and $ab\mathrm{Tr}(z\bar{w})$ are in R_S.

First let $p \in S$ be a prime that is either inert or ramified in K with $\mathcal{P}$ the prime ideal in K over p. Then $\mathrm{ord}_{\mathcal{P}}x = \mathrm{ord}_{\mathcal{P}}\bar{x}$. Hence $x \in D_{\mathcal{P}}$ since $x\bar{x} \in R_S$. Similarly y, z, w, are all locally integral at $\mathcal{P}$ since $0 \leq \mathrm{ord}_p ab \leq 1$. Note also, if p is ramified in K, then $\mathrm{ord}_p d = 1$ so that ab is a unit in $R_{\mathcal{P}}$. Thus $\beta \in \mathcal{O}_{\mathcal{P}}$.

Finally let $p \in S$ be a prime that splits in K into two ideals $\mathcal{P}$ and $\overline{\mathcal{P}}$. Consider first $a \in R_p$ a unit. Assume locally $x \notin D_{\mathcal{P}}$, so that $\bar{x} \in \mathcal{P}$ then follows from $x\bar{x} \in R_S$. Since $x\bar{y} + y\bar{x} \in R_S$ it follows that locally $\bar{y} \in \mathcal{P}$ (for if $\bar{y} \notin \mathcal{P}$, then $y\bar{x}$ and y are not locally integral, forcing $\bar{y} \in \mathcal{P}$). If, however, $\mathrm{ord}_{\mathcal{P}} a = 1$, we still get $\bar{y} \in D_{\mathcal{P}}$ from $a(x\bar{y} + y\bar{x}) \in R_S$. Hence $a\bar{y} \in \mathcal{P}$. Similarly, $b\bar{z}, ab\overline{w} \in \mathcal{P}$ which contradicts $1 = \overline{n(\beta)}$. Thus $x \in D_{\mathcal{P}}$. Likewise $y, z, w \in D_{\mathcal{P}}$. Therefore $\beta \in \mathcal{O}_{\mathcal{P}} \cap \mathcal{O}_{\overline{\mathcal{P}}}$, completing the proof.

This result generalizes Theorem 4.2 in [**5**]. The theorem applies to the rational function field $F = \mathbf{F}(X)$ where $\mathbf{F}$ is a finite field with characteristic not two. Let $B = (\frac{a,b}{\mathbf{F}(X)})$ where $a, b \in \mathbf{F}[X] = R_S$. Then $\mathcal{L} = \mathbf{F}[X, i, j]$ is an order in B. For $K = F(\alpha)$ with $\alpha^2 = d \in \mathbf{F}[X]$ and abd square-free, Theorem 5.1 then holds. In particular, one can take $d \in \mathbf{F}$ with $\mathbf{K} = \mathbf{F}(\alpha)$ a quadratic extension of $\mathbf{F}$ so that $D_S = \mathbf{K}[X]$ and $\mathcal{O} = \mathbf{K}[X, i, j]$.

Theorem 5.2. *Let* $B = (\frac{a,b}{\mathbf{Q}})$ *and* $\mathcal{L}_S$ *be as in* 5.1. *Then*

$$P\mathcal{L}_S^1 \cong O'(M)$$

is a subgroup of $P\mathcal{O}_S^1$*, where* M *is the* R_S*-lattice with* f*-form* $\langle a, b, -ab\rangle$*.*

Proof. Let d be a unit in R_S such that $\alpha \notin R_S$. From the previous section, $\Phi(B_L^1) = \mathrm{Stab}(1, O'(L)) = O'(M)$ where $M \cong \langle a, b, -ab\rangle$ after scaling out $2d$. Since $\mathcal{O}_S \cap B = \mathcal{L}_S$, we have $\mathcal{L}_S^1 \subseteq B_L^1 \subseteq A_L^1 = \mathcal{O}_S^1$ and so $\mathcal{L}_S^1 = B_L^1$. Thus $P\mathcal{L}_S^1 \cong O'(M)$.

This generalizes [**3**, §7.3A] where $a = b = 1$ and $\mathcal{L}_S^1 = SL(2, R_S)$.

6. Kleinian groups and Z-lattices.

We now consider the rational case where $F = \mathbf{Q}$, $R_S = \mathbf{Z}$ and $B = \left(\frac{a,b}{\mathbf{Q}}\right)$ with a, b square-free integers. Let $K = \mathbf{Q}(\alpha)$ with $\alpha^2 = -d$ a square-free integer. When $d \equiv 1, 2 \bmod 4$, so that 2 is ramified in K, the integers $\mathbf{Z}_K = \mathbf{Z}[\alpha]$; but for $d \equiv 3 \bmod 4$, so that 2 is inert or split in K, $\mathbf{Z}_K = \mathbf{Z}[\omega]$ with $\omega = (1 + \alpha)/2$. The next result generalizes the isomorphism theorems for Hilbert modular and Bianchi groups in [**5**, **6**], since $\mathcal{O}^1 = SL(2, \mathbf{Z}_K)$ when $a = -b = 1$.

Theorem 6.1. *Let* $B = \left(\frac{a,b}{\mathbf{Q}}\right)$ *with* $a \equiv 1 \bmod 4$ *and* $ab \neq 0$ *square-free. Then* $\mathcal{L} = \mathbf{Z}[1, (1+i)/2, j, (j+k)/2]$ *is a maximal order in* B*. For* $K = \mathbf{Q}(\alpha)$ *with* $\alpha^2 = -d$ *and* $(ab, d) = 1$*, put* $\mathcal{O} = \mathcal{L} \otimes_{\mathbf{Z}} \mathbf{Z}_K$ *and* $L = \mathcal{O} \cap V$*. Then* $\mathcal{O}^1 = A_L^1$*, and the sequence*

$$1 \to \{\pm 1\} \to \mathcal{O}^1 \xrightarrow{\Phi} O'(L) \to 1$$

is exact when $d \equiv 1, 2 \bmod 4, a \equiv 1 \bmod 8$*, b is odd, and*

$$\begin{aligned} L &= \mathbf{Z}1 \perp \mathbf{Z}\alpha i \perp (\mathbf{Z}\alpha j + \mathbf{Z}\alpha(j+k)/2) \\ &\cong \begin{pmatrix} 2 & 0 \\ 0 & 2ad \end{pmatrix} \perp bd \begin{pmatrix} 2 & 1 \\ 1 & (1-a)/2 \end{pmatrix}. \end{aligned}$$

The sequence is also exact when $d \equiv 3 \bmod 4$ *with b odd, or when* $d \equiv 3 \bmod 8$, $a \equiv 1 \bmod 8$ *with b even, but now*

$$\begin{aligned} L &= (\mathbf{Z}1 + \mathbf{Z}(1+\alpha i)/2) \perp (\mathbf{Z}\alpha j + \mathbf{Z}\alpha(j+k)/2) \\ &\cong \begin{pmatrix} 2 & 1 \\ 1 & (1+ad)/2 \end{pmatrix} \perp bd \begin{pmatrix} 2 & 1 \\ 1 & (1-a)/2 \end{pmatrix}. \end{aligned}$$

Proof. We already know $\mathcal{O}^1 \subseteq A_L^1$. It remains to prove $A_L^1 \subseteq \mathcal{O}$, and then the result follows from 3.2 since the complete group $O(L_2)$ is generated by symmetries and Eichler transformations (see [**5**, **10**]). Let $\beta = x + yi + zj + wk \in A_L^1$. It suffices to prove that $\beta \in \mathcal{O}_\mathcal{P}$, the localization of $\mathcal{O}$ at $\mathcal{P}$, for all finite primes $\mathcal{P}$ of K. The odd primes are treated as in 5.1. It remains to show $x \pm y, z \pm w$ are integral at each dyadic prime $\mathcal{P}$, for then

$$\beta = (x-y) + 2y(1+i)/2 + (z-w)j + 2w(j+k)/2 \in \mathcal{O}_\mathcal{P}.$$

As in 5.1, $x\overline{x} - ay\overline{y} + bz\overline{z} - abw\overline{w}$ and traces like $a\mathrm{Tr}(x\overline{w} + y\overline{z})$ are now in $2^{-1}\mathbf{Z}$. Similar results, but with different sign patterns, are obtained by taking $v = 1, \alpha j$ and αk. Hence $8x\overline{x}, 8ay\overline{y}, 8bz\overline{z}$ and $8abw\overline{w}$ are in $\mathbf{Z}$. Also, $4a\mathrm{Tr}(x\overline{y}), 4b\mathrm{Tr}(x\overline{z})$, $4ab\mathrm{Tr}(x\overline{w})$, $4ab\mathrm{Tr}(y\overline{z})$, $4ab\mathrm{Tr}(y\overline{w})$ and $4ab\mathrm{Tr}(z\overline{w})$ are all in $\mathbf{Z}$, as are traces like $4ab\mathrm{Tr}(\alpha x\overline{w})$. From the coefficient of αj in $\beta\alpha k\tau(\beta)$ we also have

$$2a\mathrm{Tr}(x\overline{y} + bz\overline{w}) \in \mathbf{Z}. \tag{1}$$

Adding the coefficients of αj and αk in $\beta\alpha(j+k)\tau(\beta) \in 2L$ gives

$$2(x\overline{x} + ay\overline{y}) + \mathrm{Tr}((a+1)x\overline{y} + b(a-1)z\overline{w}) \in \mathbf{Z} \tag{2}$$

and subtracting these two coefficients gives

$$2b(z\overline{z} + aw\overline{w}) - \mathrm{Tr}((a-1)x\overline{y} + b(a+1)z\overline{w}) \in \mathbf{Z}. \tag{3}$$

From the αk coefficient of $\beta\alpha(j+k)\tau(\beta) \in 2L$, we have

$$x\overline{x} + ay\overline{y} + bz\overline{z} + abw\overline{w} + \mathrm{Tr}(x\overline{y} - bz\overline{w}) \in \mathbf{Z}. \tag{4}$$

First consider 2 inert in K so that $d \equiv 3 \bmod 8$. Then $2x, 2y$ are locally integral at 2, and hence in $\mathbf{Z}[\omega]$ since, for example, $8x\overline{x} \in \mathbf{Z}$ and $\mathrm{ord}_2 x = \mathrm{ord}_2 \overline{x}$. Since $a \equiv 1 \bmod 4$, it follows from (2) that $2(x+y)(\overline{x}+\overline{y})$ is locally integral at 2. Therefore $x \pm y \in \mathbf{Z}[\omega]$. Then $\beta\beta^J = 1$ gives $b(z^2 - aw^2) \in \mathbf{Z}[\omega]$. For b odd we have $z \pm w \in \mathbf{Z}[\omega]$ and hence $\beta \in \mathcal{O}^1$, since $2z, 2w \in \mathbf{Z}[\omega]$. For b even and $a \equiv 1 \bmod 8$, from (4) and since $(x+y)(\overline{x}+\overline{y})$ is integral, it follows that $b(z\overline{z} + w\overline{w} - \mathrm{Tr}(z\overline{w})) = b(z-w)(\overline{z}-\overline{w})$ is integral. Hence $z - w$ is integral, and similarly, from (1) and (4), $z + w$ is integral. Thus $\beta \in \mathcal{O}^1$.

Next consider 2 ramified in K so that $2\mathbf{Z}_K = \mathcal{P}^2$ and $a \equiv 1 \bmod 8$ (so that, in essence, $a = 1$). By combining the coefficients of 1 and αi in $\beta\alpha(j+k)\tau(\beta) \in 2L$, we have $\alpha b(x+y)(\overline{w} - \overline{z})$ is locally integral, since $4b\mathrm{Tr}(\alpha x\overline{w} - \alpha y\overline{z})$ and $8\alpha by\overline{w}$ are locally integral. Since b is odd, it follows that either $x+y$ or $z-w$ is locally integral. A similar calculation, using $\beta\alpha(j-k)\tau(\beta) \in 2L$, gives either $x-y$ or $z+w$ is integral at $\mathcal{P}$. From (4), as with 2 inert, if $x+y$ is locally integral, so is $z-w$, and conversely. Similarly for the pair $x-y$ and $z+w$. Now all four are integral and $\beta \in \mathcal{O}^1$.

Finally consider $d \equiv 7 \bmod 8$ so that 2 splits in K and $2\mathbf{Z}_K = \mathcal{P}\overline{\mathcal{P}}$. From (2), $2(x+y)(\overline{x}+\overline{y})$ is locally integral, and hence either $\mathrm{ord}_{\mathcal{P}}(x+y) \geq 0$ or $\mathrm{ord}_{\mathcal{P}}(\overline{x}+\overline{y}) \geq 0$. Since $4a\mathrm{Tr}(x\overline{y}) \in \mathbf{Z}$, also $\mathrm{ord}_{\mathcal{P}}(x-y) \geq 0$ or $\mathrm{ord}_{\mathcal{P}}(\overline{x}-\overline{y}) \geq 0$. A similar argument, using the coefficients of 1 and αi in $\beta(1+\alpha i)\tau(\beta) \in 2L$, and $\alpha^2 \equiv 1 \bmod 8$, shows that $2(x-y)(\overline{x}+\overline{y})$ is integral at $\mathcal{P}$; hence $\mathrm{ord}_{\mathcal{P}}(x-y) \geq 0$ or $\mathrm{ord}_{\mathcal{P}}(\overline{x}+\overline{y}) \geq 0$. Now either $x \pm y$ are both locally integral at $\mathcal{P}$, or $\overline{x} \pm \overline{y}$ are both integral at $\mathcal{P}$ so that $x \pm y$ are locally integral at $\overline{\mathcal{P}}$. Since b is odd, from (3) either $z \pm w$ are both integral at $\mathcal{P}$, or both are integral at $\overline{\mathcal{P}}$. If $x \pm y, z \pm w$ are all locally integral at $\mathcal{P}$, then $\beta \in \mathcal{O}^1_{\mathcal{P}}$. Assume, therefore, $x \pm y, \overline{z} \pm \overline{w}$ are locally integral at $\mathcal{P}$. Since $4(x\overline{x} + bz\overline{z}), 4b\mathrm{Tr}(x\overline{z}) \in \mathbf{Z}$, it follows that $4(x+z)(\overline{x}+\overline{z})$ is locally integral at $\mathcal{P}$ and hence $2(x+z)$ is integral at $\mathcal{P}$ or $\overline{\mathcal{P}}$. In the first case, $2z$ is now integral at $\mathcal{P}$; from $n(\beta) = 1$ we then have $z \pm w$ integral at $\mathcal{P}$, and again $\beta \in \mathcal{O}^1_{\mathcal{P}}$. In the second case, $2x$ is integral at $\overline{\mathcal{P}}$ so that $\beta \in \mathcal{O}^1_{\overline{\mathcal{P}}}$. By symmetry, we may now assume $\beta \in \mathcal{O}^1_{\mathcal{P}}$. But $\beta\tau(\beta) \in L \subseteq \mathcal{O}$ so that $\tau(\beta) \in \beta^J\mathcal{O}^1 \subseteq \mathcal{O}^1_{\mathcal{P}}$. Hence $\beta \in \tau(\mathcal{O}^1_{\mathcal{P}}) = \mathcal{O}^1_{\overline{\mathcal{P}}}$.

Remarks. Let B be a quaternion algebra over a number field F with $\mathcal{L}_S$ a maximal S-order in B. Let $A = B \otimes_F K$ for a quadratic extension K/F. Assume the order $\mathcal{O}_S = \mathcal{L}_S \otimes_{R_S} D_S$ is maximal and put $L = \mathcal{O}_S \cap V$. Then is $A^1_L = \mathcal{O}^1_S$ so that $\Phi : \mathcal{O}^1_S \to O'(L)$ is surjective? The main difficulty is with the dyadic primes since 5.1 essentially covers non-dyadic primes. As observed in §3, for primes $\mathcal{P}$ where A is ramified, $A^1_{L_p} \subseteq \mathcal{O}_{\mathcal{P}}$ since $\mathcal{O}_{\mathcal{P}}$ is now maximal. In general, the order $\mathcal{O}$ and the lattice L will have to be given locally. In particular, 5.1 can be easily generalized by giving $\mathcal{O}_S$ and L locally, but then the explicitness of the global data is lost. Also, what is the index $[O'(L) : \Phi(\mathcal{O}^1_S)]$ when $\mathcal{O}_S$ is not maximal? The orders in 5.1 and 6.1 are maximal although the proofs only use this indirectly. Some restrictions on the orders $\mathcal{L}_S$ and $\mathcal{O}_S$ are necessary as the following examples show. Similar examples could be given with the values of a, b, d changed modulo 8 since this has little effect dyadically, and the odd primes are well behaved when abd is square-free.

Example 1. Let $a = 1 = -b$ and $\mathcal{L}' = \mathbf{Z}[1, i, j, k] \subset \mathcal{L}$, as in 6.1, so that $\mathcal{L}'$ is not maximal. Take $d = 3$ and $\beta = x + yi + \overline{x}j + \overline{y}k$ in A with $2x = 1+\omega, 2y = 1-\omega$ and $\omega = (1+\alpha)/2$. Then $n(\beta) = 1$ and $\beta \notin \mathcal{O}' = \mathcal{L}' \otimes_{\mathbf{Z}} \mathbf{Z}[\omega]$.

However, from $x\bar{x} = 3/4, y\bar{y} = 1/4$, $\mathrm{Tr}(x\bar{y}) = 0$ and $2\mathrm{Tr}(\alpha y\bar{x}) = 3$ it can be checked that $\beta \in A_{L'}$ where $L' = \mathbf{Z} + \mathbf{Z}\alpha i + \mathbf{Z}\alpha j + \mathbf{Z}\alpha k = \mathcal{O}' \cap V$. Hence $\mathcal{O}'^1 \neq A^1_{L'}$.

Example 2. Let $d = a = 1, b = -2$ and $\beta = (j + \alpha k)/2$ with $\mathcal{O}, L$ as in 6.1. Then $n(\beta) = 1, \beta\tau(\beta) = \alpha i$, $\beta\alpha i\tau(\beta) = 1, \beta\alpha j\tau(\beta) = \alpha j$ and $\beta\alpha k\tau(\beta) = -\alpha k$. Hence $\beta \in A^1_L$. Put $\pi = \alpha - 1$ so that $\pi\bar{\pi} = 2$. Then $\beta = 2^{-1}(j+k) + \bar{\pi}^{-1}k \notin \mathcal{O}$ and $A^1_L \neq \mathcal{O}^1$. Since $\mathcal{O}[\beta] = \mathbf{Z}_K[(1+i)/2, \beta, k/\pi]$ is an order, $\mathcal{O}$ is not maximal in A.

The next three theorems extend our approach to other explicit situations.

Theorem 6.2. *Let* $B = \left(\frac{a,b}{\mathbf{Q}}\right)$ *with* $a \equiv 3 \bmod 4$, *b even, and ab square-free. Then* $\mathcal{L} = \mathbf{Z}[1, i, (1 + i + j)/2, (j + k)/2]$ *is a maximal order in B. Let* $K = \mathbf{Q}(\alpha)$ *with* $\alpha^2 = -d \equiv 5 \bmod 8$ *and* $(ab, d) = 1$. *Put* $\mathcal{O} = \mathcal{L} \otimes_{\mathbf{Z}} \mathbf{Z}_K$ *and*

$$\begin{aligned} L = \mathcal{O} \cap V &= \mathbf{Z}1 + \mathbf{Z}\alpha i + \mathbf{Z}\alpha(j + k)/2 + \mathbf{Z}(1 + \alpha i + \alpha j)/2 \\ &\cong \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2ad & 0 & ad \\ 0 & 0 & (1-a)bd/2 & bd/2 \\ 1 & ad & bd/2 & (1 + ad + bd)/2 \end{pmatrix}. \end{aligned}$$

Then $\mathcal{O}^1 = A^1_L$, *and the following sequence is exact*

$$1 \to \{\pm 1\} \to \mathcal{O}^1 \xrightarrow{\Phi} O'(L) \to 1.$$

Proof. Locally, for odd primes the proof is essentially the same as in 5.1, but again 2 needs careful treatment. Let $\beta \in A^1_L$ be as in 6.1. Then (1)–(4) still hold since they are derived from $\alpha(j + k) \in 2L$. Also 2 is inert in $\mathbf{Z}_K = \mathbf{Z}[\omega]$, and hence $2x, 2y, 4z, 4w \in \mathbf{Z}_K$ as in 6.1. Again from (1) and (2), since $a \equiv 3 \bmod 4$, $x \pm y$ are integral at 2, and $x^2 - y^2 \in \mathbf{Z}_K$. It follows from $n(\beta) = 1$ that $(a - 1)y^2 + bz^2 - abw^2$ is integral. Therefore, $\mathrm{ord}_2 z = -2$ if and only if $\mathrm{ord}_2 w = -2$. Moreover, if $2z$ and $2w$ are integral, then $2(y^2 + z^2 + w^2) \in \mathbf{Z}_2$ so that $y + z + w$ is integral. Therefore,

$$\beta = x - y + 2y\frac{1 + i + j}{2} + (z - y - w)j + 2w\frac{j + k}{2} \in \mathcal{O}^1.$$

Finally $\mathrm{ord}_2 z = -2$ is not possible. For let $4z \equiv z_0 + 2z_1 \bmod 4$ where $z_i \in \{0, 1, \omega, \bar{\omega}\}$ (the residue class field is $\mathbf{F}_4$), with a similar 2-adic expression for $4w$. Then $(4z)^2 \equiv z_0^2 \bmod 4$. Since $8bz^2 \equiv 8abw^2 \bmod 4$, it follows that $z_0^2 \equiv -w_0^2 \bmod 4$, and then $z_0 = w_0 = 0$, completing the proof.

Note that $dL = -a^2b^2d^3$. Locally at odd p, $L_p \cong \langle 1, ad, bd, -abd\rangle$. At the prime 2, the vectors 1 and $(1 + \alpha i + \alpha j)/2$ span a binary even unimodular lattice J_0 with discriminant $dJ_0 = (a + b)d$ which splits $L_2 = J_0 \perp J_1$ where J_1 is the 2-modular even lattice spanned by αk and $\alpha(bi - aj - ak)/2$, with discriminant $dJ_1 = -4(a + b)$. Since $a + b \equiv 1, 5 \bmod 8$, either J_0 or J_1 is isotropic in the completion when $d \equiv 3 \bmod 8$. Thus A is not dyadically ramified when $d \equiv 3 \bmod 8$ (see [**10**, §58.7]). Again the dyadic condition for 3.2 follows as in [**5**].

Example 3. The analogue of 6.2 fails when $d = 1$. Take $b = 2a = -2$ and $\beta = 1 + (j + \alpha k)/2$. Then $n(\beta) = 1$ and $\beta \in A_L^1$ where now $L = \mathcal{O} \cap V$ is as in 5.1. But $\beta \notin \mathcal{O}^1$, and again $\mathcal{O}$ is not maximal.

In Example 3, and also in the next result, B is ramified at the dyadic prime, but A is not dyadically ramified, and Φ is not surjective. The algebra B is ramified at the prime 2 whenever the Hilbert symbol $(a, b)_2 = -1$; for example when $a \equiv b \equiv 3 \bmod 4$.

Theorem 6.3. *Let $B = \left(\frac{a,b}{\mathbf{Q}}\right)$ with $a \equiv b \equiv 3 \bmod 4$, ab square-free, and with a, b not both negative. Then $\mathcal{L} = \mathbf{Z}[i, j, (1 + i + j + k)/2]$ is a maximal order in B. Let $K = \mathbf{Q}(\alpha)$ with $\alpha^2 = -d \equiv 2, 3 \bmod 4$ and abd square-free. Then for $\mathcal{O} = \mathcal{L} \otimes_{\mathbf{Z}} \mathbf{Z}_K$ and*

$$L = \mathcal{O} \cap V = \mathbf{Z}1 \perp \mathbf{Z}\alpha i \perp \mathbf{Z}\alpha j \perp \mathbf{Z}\alpha k \cong \langle 2, 2ad, 2bd, -2abd\rangle$$

the index

$$[A_L^1 : \mathcal{O}^1] = [O'(L) : \Phi(\mathcal{O}^1)] = 2.$$

Proof. Let $\beta \in A_L^1$ be as in 6.1. Locally, for odd primes the proof is essentially the same as in 5.1 and $\beta \in \mathcal{O}_{\mathcal{P}}$ at all non-dyadic primes. The condition in 3.2 on $SO(L_2)$ follows from [**5**, **11**]. The prime 2 is ramified in $\mathbf{Z}_K$ with $2\mathbf{Z}_K = \mathcal{P}^2$. Since L has an orthogonal basis, $x\overline{x} \pm ay\overline{y} \pm bz\overline{z} \pm abw\overline{w} \in \mathbf{Z}$ for all choices of an even number of negative signs. Hence, for example, $4x\overline{x} \in \mathbf{Z}$ and therefore $2x, 2y, 2z, 2w \in \mathbf{Z}_K$. Also, all traces such as $2a\mathrm{Tr}(x\overline{y})$ and $2b\mathrm{Tr}(x\overline{z})$ are in $\mathbf{Z}$. Since $a \equiv 3 \bmod 4$,

$$2(x - y)(\overline{x} + \overline{y}) = 2(x\overline{x} - y\overline{y}) - 4y\overline{x} + 2\mathrm{Tr}(x\overline{y}) \in \mathbf{Z}$$

and thus $\pi(x - y) \in \mathbf{Z}_{\mathcal{P}}$ where $\mathcal{P} = \pi\mathbf{Z}_{\mathcal{P}}$. Similarly, $\pi(x - z)$ and $\pi(x - w)$ are in $\mathbf{Z}_{\mathcal{P}}$. Let $2x \equiv x_0 + x_1\pi + 2x_2 \bmod 2\mathcal{P}$ where $x_0, x_1, x_2 \in \{0, 1\}$. Then

$$(2x)^2 \equiv x_0^2 + x_1^2\pi^2 + 2x_0x_1\pi + 4x_2^2 + 4x_0x_2 \bmod 4\mathcal{P}$$

with similar expressions for $2y, 2z$ and $2w$. Then $x_0 = y_0 = z_0 = w_0$ follows from $\pi(x - y) \in \mathbf{Z}_{\mathcal{P}}$ and similar facts. Put

$$s_i = s_i(\beta) = x_i + y_i + z_i + w_i.$$

Substituting into $4n(\beta) \equiv 4 \bmod 4\mathcal{P}$ gives $s_1 \equiv 0 \bmod 2$. If $x_0 = 1$, since $(1 - a)(1 - b) \equiv 4 \bmod 8$, we get the stronger result $4|s_1$, so that $x_1 = y_1 = z_1 = w_1$. Thus $x - w, y - w, z - w \in \mathbf{Z}_K$ and

$$\beta = (x - w) + (y - w)i + (z - w)k + 2w(1 + i + j + k)/2 \in \mathcal{O}.$$

It remains to consider β with $x_0 = 0$ and $s_1(\beta) = 2$. Use the surjectivity argument in [**6**, 4.1B] to construct various $\gamma \in A_L^1$ with $s_1(\gamma) = 2$ so that each $\gamma \notin \mathcal{O}^1$. The assumptions ensure that the norm form is indefinite so that the strong approximation theorem can be applied. Thus, if $x_1 = y_1 = 1$ for β, we can find $\gamma \in A_L^1$ with $x_1(\gamma) = z_1(\gamma) = 1$. Then $s_0(\beta\gamma) = 4$ and as above $\beta\gamma \in \mathcal{O}^1$. It easily follows if $\beta, \beta' \in A_L^1$ with $\beta, \beta' \notin \mathcal{O}^1$, then $\beta' \in \beta\mathcal{O}^1$. Hence $[A_L^1 : \mathcal{O}^1] = 2$.

The order $\mathcal{O} = \mathcal{L} \otimes_{\mathbf{Z}} \mathbf{Z}_K$ in 6.3 is not maximal. By enlarging to a maximal order we can get Φ surjective, but then both $\mathcal{O}$ and L have to be prescribed locally.

Theorem 6.4. *Let $K = \mathbf{Q}(\alpha)$ with $\alpha^2 = -d \equiv 2 \bmod 4$, and let A be the quaternion algebra $\left(\frac{a,b}{K}\right)$ where $a, b \in \mathbf{Z}$ with $ab(a+b)d$ square-free, $a \equiv b \equiv 3 \bmod 4$ and $a + b \equiv d \bmod 8$. Let $\mathcal{O}$ be the maximal order in A with localizations*

$$\mathcal{O}_{\mathcal{P}} = \mathbf{Z}_{\mathcal{P}}[1, (i+j)/\alpha, (j+k)/\alpha, (1+i+j+k)/2]$$

at the dyadic prime $\mathcal{P} = \alpha\mathbf{Z}_K + 2\mathbf{Z}_K$, and $\mathcal{O}_{\mathcal{Q}} = \mathbf{Z}_{\mathcal{Q}}[i, j]$ for each odd prime $\mathcal{Q}$. Put $L = \mathcal{O} \cap V$. Then $\mathcal{O}^1 = A_L^1$ and the following sequence is exact

$$1 \to \{\pm 1\} \to \mathcal{O}^1 \xrightarrow{\Phi} O'(L) \to 1.$$

Proof. It remains to check the result locally at the dyadic prime where

$$L_{\mathcal{P}} = \mathcal{O}_{\mathcal{P}} \cap V = \mathbf{Z}_{\mathcal{P}}1 \perp \mathbf{Z}_{\mathcal{P}}\alpha^{-1}(i+j) \perp \mathbf{Z}_{\mathcal{P}}\alpha^{-1}(bi - aj) \perp \mathbf{Z}_{\mathcal{P}}\alpha k.$$

Let $a + b = cd$ where $c \in \mathbf{Z}_{\mathcal{P}}$. Put $i' = (i+j)/\alpha, j' = (bi - aj)/\alpha \in \mathcal{O}_{\mathcal{P}}$ so that $i'^2 = -c \equiv 3 \bmod 4, j'^2 = -abc \equiv 3 \bmod 4$ and $i'j' = ck = k' = -j'i'$. Let $\beta = x + yi' + zj' + wk' \in A_L^1$ and repeat the line of argument in 6.3 to show $\beta \in \mathcal{O}_{\mathcal{P}}$. As before $2x, 2y, 2z, 2w, \alpha(x-y), \alpha(x-z), \alpha(x-w)$ are all in $\mathbf{Z}_{\mathcal{P}}$. From $n(\beta) = 1$ we again conclude that $s_0(\beta) = 4$ when $x_0 = 1$, and hence $\beta \in \mathcal{O}^1$. Finally consider $x_0 = y_0 = z_0 = w_0 = 0$ and $s_1(\beta) = 2$. But now $(i' + j')/\alpha = ((1+b)i + (1-a)j)/\alpha^2 \in \mathcal{O}_{\mathcal{P}}$ since a, b are odd. Then $(1 + k')/\alpha \in \mathcal{O}_{\mathcal{P}}$. Also, if $\gamma = (1 + i')/\alpha$, then $\gamma(i' + j')/\alpha = (-c + i' + j' + k')/\alpha^2 \in \mathcal{O}_{\mathcal{P}}$ (as already shown in the $x_0 = 1$ case). Since $n((i' + j')/\alpha) = -c(1 + ab)/d$ is invertible in $\mathbf{Z}_{\mathcal{P}}$, it follows that $\gamma \in \mathcal{O}_{\mathcal{P}}$. Thus $(1 + j')/\alpha \in \mathcal{O}_{\mathcal{P}}$ and hence $\beta \in \mathcal{O}^1$, completing the proof.

When $d = \pm 2$ with $a + b = cd$, the order and lattice in 6.4 can be given globally, since now $\mathcal{O} = \mathbf{Z}_K[1, (i+j)/\alpha, (j+k)/\alpha, (1+i+j+k)/2]$ and

$$\begin{aligned} L = \mathcal{O} \cap V &= \mathbf{Z}1 \perp \mathbf{Z}\alpha^{-1}(i+j) \perp \mathbf{Z}\alpha^{-1}(bi - aj) \perp \mathbf{Z}\alpha k \\ &\cong \langle 2, 2c, 2abc, -2abd \rangle. \end{aligned}$$

In general, the global lattice L in 6.4 need not have an orthogonal basis. For the special case where $a = b = -1, c = 1$ and $\alpha^2 = 2$, the definite $\mathbf{Z}$-lattice L diagonalizes uniquely as $\langle 2, 2, 2, 4 \rangle$. It follows that $|O'(L)| = 24$ and $|\mathcal{O}^1| = 48$, as in [**12**, p. 141].

Theorem 6.5. *Let* $B = \left(\frac{a,b}{\mathbf{Q}}\right)$ *and* $\mathcal{L}$ *be as in* 6.1 *with* $a \equiv 1 \bmod 4$, *and* ab *square-free. Then the sequence*

$$1 \to \{\pm 1\} \to \mathcal{L}^1 \xrightarrow{\Phi} O'(M) \to 1$$

is exact, where

$$M \cong (2a) \perp b \begin{pmatrix} 2 & 1 \\ 1 & (1-a)/2 \end{pmatrix}.$$

The proof is the same as for Theorem 5.2 (take $d = 1$ if $a \equiv 1 \bmod 8$ with b odd, otherwise take $d = 3$ and scale M). There is a similar result for the order $\mathcal{L}$ in Theorem 6.2. Note, for a and b both negative, the group $O'(M)$ is finite since the underlying quadratic form is then definite. For example, for $a = -3$ and $b = -1$, both $P\mathcal{L}^1$ and $O'(M)$ can be shown directly to be cyclic groups of order 6.

7. Fuchsian subgroups.

Again assume that F is a global field and $L = \mathcal{O}_S \cap V$. Define

$$A_L(v) = A_L \cap A(v)$$

where $v \in L$ is primitive. Then, assuming the dyadic conditions in 3.2,

$$\Phi(A_L(v)^1) = \mathrm{Stab}(v, O'(L)).$$

Therefore $\Phi(B_L^1) = \mathrm{Stab}(1, O'(L))$ where $B_L^1 = \{\beta \in B^1 \mid \beta L = L\beta\}$.

Let $K = \mathbf{Q}(\sqrt{-d})$ with $d > 0$ so that K is an imaginary quadratic number field. Assume $a > 0$ in $B = \left(\frac{a,b}{\mathbf{Q}}\right)$ so that the space V has signature (3,1). Take $v \in V$ with $n(v) = D > 0$. Then $V = Fv \perp W$ with W an indefinite space. Now $A(v) \cong C^+(W)$ is a quaternion algebra over $\mathbf{Q}$ with an indefinite norm, and $A_L(v)^1$ is an infinite Fuchsian subgroup of the arithmetic Kleinian group A_L^1, since $\mathrm{Stab}(v, O'(L))$ is infinite when $D > 0$. The conjugacy classes of these non-elementary Fuchsian subgroups correspond to the orbits of primitive $v \in L$ under the action of $O'(L)$, with the length $n(v) = D$ an invariant of an orbit. The number of orbits is finite for fixed $D > 0$, and can be determined via a product formula by using the strong approximation theorem to relate the global orbits under $O'(L)$ to local orbits under $O'(L_p)$, provided the local structure of L_p is known, as in [**6**]. We now give another example of this. See [**8**, **9**] for more connections between quaternion algebras, arithmetic Kleinian groups and Fuchsian groups.

Let $N(L_p, D)$ denote the number of spinor equivalence classes of primitive representations of D, and $N(L, D)$ the corresponding global number.

Theorem 7.1. *Let* $L_p = J_0 \perp J_1$ *where* J_0 *is unimodular of rank two and* J_1 *is* p*-modular of rank two. Assume either* p *is odd, or* $p = 2$ *with* J_0, J_1 *both even lattices. Then:*

1. *$N(L_p, D) = 0$ when $\mathrm{ord}_p D \geq 2$ and J_0, J_1 are both anisotropic.*
2. *$N(L_p, D) = 2$ when J_0 is hyperbolic with $p|D$, and either J_1 is hyperbolic or $\mathrm{ord}_p D = 1$.*
3. *$N(L_p, D) = 1$ otherwise, including $(p, D) = 1$.*

Proof. Let $v \in L_p$ be primitive with $n(v) = D$. When $(p, D) = 1$, we may assume $v \in J_0$. The group $O(L_p)$ acts transitively on such v with the same norm, and since rank two even modular components have isometries with spinor norms of all possible values (see [**10**, §92:5]), it follows that $N(L_p, D) = 1$. When $p|D$, v can be embedded in either J_0 or J_1, and these two possibilies are not equivalent under the action of $O(L)$ (see [**4**], [**7**]). Therefore $N(L_p, D) \leq 2$, since not all these primitive representations of D need exist.

If J_0 is hyperbolic, then J_0 primitively represents all D. Otherwise, J_0 only primitively represent units. Likewise, if J_1 is hyperbolic, then J_1 primitively represents all D with $\mathrm{ord}_p D \geq 1$; otherwise only the values D with $\mathrm{ord}_p D = 1$ are represented primitively. This then converts into the values given for $N(L_p, D)$.

For the lattices in 6.1 and 6.2 with p odd and $p|b$ so that $(p, d) = 1$, the local discriminants $dJ_0 = ad$ and $dJ_1 = -p^2 a$. Hence J_0 is hyperbolic when $(\frac{-ad}{p}) = 1$, and J_1 is hyperbolic when $(\frac{a}{p}) = 1$. For $p = 2$, the even lattice J_0 in 6.1 or 6.2 is isotropic only when the discriminant $dJ_0 = ad \equiv -1 \bmod 8$, and J_1 is isotropic only when $2^{-2} dJ_1 = -a \equiv -1 \bmod 8$. When the two even Jordan components of L_p in 6.1 are anisotropic, the lattice L_p is maximal and anisotropic. Then, for odd p, $(\frac{-d}{p}) = 1$ so that p splits in the extension $K = \mathbf{Q}(\alpha)$ into $\mathcal{P}$ and $\overline{\mathcal{P}}$. The space V_p is now anisotropic over $\mathbf{Q}_p$ if and only if the norm form of $A_{\mathcal{P}}$ is anisotropic (see [**10**, §58:7]), so that A is then ramified at $\mathcal{P}$. We already observed in §3 that $N(L_p, D) \leq 1$ for these p since L_p is a maximal lattice. The algebra A can not ramify at any other odd prime since the norm form is isotropic.

Some other values for $N(L_p, D)$ are given in [**6**, §5]. Note, however, in [**6**] we consider $n(v) = dD$ and a slightly different form of primitivity. The general problem for $p = 2$ splits into many cases. For an analogue of 7.1 with J_0 or J_1 odd, use Proposition 10 in [**4**] together with Theorem 3.14 in [**1**] to get at spinor equivalence.

Theorem 7.2. *Let L be the lattice in Theorem* 6.1, 6.2 *or* 6.4. *Assume $d, D > 0$ and either a or b is positive. Then almost all $N(L_p, D) = 1$ and*

$$N(L, D) = \prod_p N(L_p, D).$$

The proof is the same as for Theorem 4.1 in [**6**], since the sign assumptions ensure that the strong approximation theorem can be applied.

The number of conjugacy classes of the subgroups $\mathrm{Stab}(v, O'(L))$ with $n(v) = D$, under the action of $O'(L)$, is also $N(L, D)$. To determine the number of conjugacy classes of the maximal Fuchsian subgroups corresponding to $\mathrm{Stab}(\pm v, O'(L))$ it is necessary to also take into account the action of $-I$ on the local $O'(L_p)$ orbits, as in [**6**].

Acknowledgement. The author is grateful to the Marsden Fund for the opportunity to attend the Tolaga Bay, N.Z. Conference on *Discrete groups and hyperbolic manifolds.* The present work is a consequence of this meeting.

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Received June 19, 2000.

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