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Ringel duality exhibits a symmetry for quasi-hereditary algebras, which, in particular, is of interest for blocks of the BGG-category $\mathcal{O}$ and for Schur algebras of classical groups. This symmetry is used to phrase (Kazhdan-)Lusztig conjecture in terms of maps between tilting modules and also in terms of composition factors occuring in certain layers of good or cogood filtrations of tilting modules. The conditions make sense for centralizer subalgebras as well where they can be formulated in terms of non-existence of certain uniserial submodules. Hence, the validity of (Kazhdan-)Lusztig conjecture is equivalent to a 'regularity' condition on the structure of tilting modules.

1. Introduction.

Let $\mathfrak{g}$ be a finite dimensional semisimple complex Lie algebra. Then category $\mathcal{O}$, as defined by Bernstein, Gelfand and Gelfand, decomposes into a direct sum of module categories of certain finite dimensional associative algebras, the blocks of $\mathcal{O}$. Similarly, the category of polynomial representations of the group $GL_n(k)$ (where k is an infinite field of any characteristic) decomposes into a direct sum of module categories of certain finite dimensional associative algebras, the 'classical' Schur algebras. More generally, Schur algebras are defined for any reductive algebraic group G (over an algebraically closed field k) and then describe (in a more complicated way than by a block decomposition) the rational representation theory of G. These two classes of algebras are the main objects of this note. A common language is given by the notion of quasi-hereditary algebras which I am going to use.

Important properties of these algebras are given by Kazhdan-Lusztig theory. More precisely, (Kazhdan-)Lusztig conjecture (a theorem for $\mathcal{O}$, but not yet for Schur algebras) has been phrased as a formula giving certain composition multiplicities, but also via the non-vanishing of certain cohomology spaces over these algebras which in turn is equivalent to the existence or the non-existence of certain modules (see e.g., [5]).

Recently, both for blocks of $\mathcal{O}$ and for Schur algebras a new kind of symmetry has been studied, which is given by so-called Ringel duality. This relates a quasi-hereditary algebra A to its Ringel dual, which is the endomorphism ring of the characteristic tilting module over A. Donkin [**9**, **10**] has shown, that (under certain assumptions) a (q-)Schur algebra of type A is its own Ringel dual. This result has been extended by Adamovich and Rybnikov [**1**] to Ringel dualities between certain Schur algebras of classical groups. Moreover, Soergel [**22**] has shown that category $\mathcal{O}$ is Ringel self-dual.

The aim of this note is to relate Ringel duality with Kazhdan-Lusztig theory, that is, to formulate the validity of (Kazhdan-)Lusztig conjecture for an algebra in terms of its Ringel dual. The main result gives several equivalent formulations of (Kazhdan-)Lusztig conjecture, either in terms of maps between tilting modules or in terms of composition factors or submodules of tilting modules, thus exhibiting several structural consequences of these conjectures.

To state the theorem, some notation is needed.

For a quasi-hereditary algebra A, the poset of weights of A is denoted by $(\Lambda, \leq)$, the A-standard modules by $\Delta(\lambda)$, its costandard modules by $\nabla(\lambda)$, and its indecomposable characteristic tilting modules by $T(\lambda)$ for $\lambda \in \Lambda$. In this paper, a quasi-hereditary algebra always is assumed to have a duality. That is, the category $A - mod$ has an involutory self-equivalence which fixes isomorphism classes of simple modules and interchanges standard and costandard modules.

In a poset, the symbol $\lambda \lessdot \mu$ means that $\lambda < \mu$ and λ and μ are neighbours. For A being a Schur algebra or a block of $\mathcal{O}$, the partial order is given by the Bruhat order on a Coxeter group. In this case, the notion of neighbouring indices has a natural meaning in terms of the Coxeter group; one of these indices is obtained from the other by multiplication with a simple reflection.

An ideal I in a poset has the property that $x < y$ and $y \in I$ implies $x \in I$. A coideal satisfies the dual condition.

We will assume that some given quasi-hereditary algebra satisfies the following condition:

(∗) For all $\lambda \lessdot \mu$, the composition multiplicity $[\Delta(\mu) : L(\lambda)]$ equals one.

This condition is well-known and easy to check for blocks of category $\mathcal{O}$ ([**17**, 4.13]) and for Schur algebras ([**18**, II, 6.24]).

For given $\lambda \lessdot \mu < \eta$ we define certain submodules of the tilting module $T(\eta)$ by short exact sequences:

$0 \to T_{\nabla \leq \mu}(\eta) \to T(\eta) \to X_1 \to 0$ where $T_{\nabla \leq \mu}(\eta)$ is filtered by $\nabla(\nu)$ with $\nu \leq \mu$ and X_1 is filtered by $\nabla(\nu)$ with $\nu \not\leq \mu$,

$0 \to T_{\Delta \geq \lambda}(\eta) \to T(\eta) \to X_2 \to 0$ where $T_{\Delta \geq \lambda}(\eta)$ is filtered by $\Delta(\nu)$ with $\nu \geq \lambda$ and X_2 is filtered by $\Delta(\nu)$ with $\nu \not\geq \lambda$, and

$0 \to T_{\Delta > \lambda}(\eta) \to T(\eta) \to X_3 \to 0$ where $T_{\Delta > \lambda}(\eta)$ is filtered by $\Delta(\nu)$ with $\nu \geq \mu$ and X_3 is filtered by $\Delta(\nu)$ with $\nu \not\geq \mu$.

Theorem 1. *Let A be a quasi-hereditary algebra with duality and B its Ringel dual. Assume that Condition $(*)$ is satisfied for B.*

Then the following assertions are equivalent:

1. *For all $\lambda \lessdot \mu$, the B-cohomology space $Ext^1_B(L(\mu), L(\lambda))$ is not zero.*
2. *For all $\lambda \lessdot \mu < \eta$ and for all $\varphi \in \operatorname{Hom}_A(T(\mu), T(\eta))$ and $\psi \in \operatorname{Hom}_A(T(\eta), \nabla(\lambda))$, the composition $\varphi\psi$ equals zero.*
3. *For all $\lambda \lessdot \mu < \eta$, the composition multiplicity*
$$[(T_{\nabla_{\leq\mu}}(\eta) \cap T_{\Delta_{\geq\lambda}}(\eta))/(T_{\nabla_{\leq\mu}}(\eta) \cap T_{\Delta_{>\lambda}}(\eta)) : L(\lambda)]$$
equals zero.
4. *For each coideal I in the poset of weights of A the following is true: For all $\lambda \lessdot \mu < \eta$ with λ minimal in I, and for all $\varphi \in \operatorname{Hom}_{eAe}(T(eAe, \mu), T(eAe, \eta))$ and $\psi \in \operatorname{Hom}_{eAe}(T(eAe, \eta), \nabla(eAe, \lambda))$, the composition $\varphi\psi$ equals zero. Here, e is a complete sum of primitive idempotents indexed by elements of I.*
5. *For each coideal I in the poset of weights of A and e its associated idempotent (as in (4)), the following is true: For all $\lambda \lessdot \mu < \eta$ with λ minimal in I, the composition multiplicity*
$$[(T_{\nabla_{\leq\mu}}(eAe, \eta) \cap T_{\Delta_{\geq\lambda}}(eAe, \eta))/(T_{\nabla_{\leq\mu}}(eAe, \eta) \cap T_{\Delta_{>\lambda}}(eAe, \eta)) : L(eAe, \lambda)]$$
equals zero.
6. *For each coideal I in the poset of weights of A and its associated idempotent e, the following is true: For all $\lambda \lessdot \mu < \eta$ with λ minimal in I, let U be a uniserial submodule of $T(eAe, \eta)$ which has simple top $L(eAe, \lambda)$, Loewy length less than or equal to three and only composition factors of type $L(eAe, \nu)$ where ν is either λ or μ. Then U is contained in the radical of $T(eAe, \eta)$.*

For B (or, equivalently by Soergel's result, for A) a block of category $\mathcal{O}$, (1) is equivalent to the Kazhdan-Lusztig conjecture (hence true). For B a Schur algebra with $p = \operatorname{char}(k) \geq h$ (h is the Coxeter number), condition (1) is equivalent to the Lusztig conjecture (hence true at least for p very large).

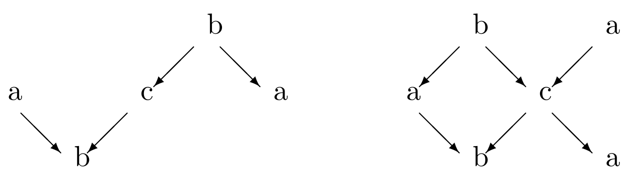
Condition (2) will be rather easily seen to be equivalent to (1) by just applying the mechanism of Ringel duality, which interchanges projective and tilting modules.

Condition (3) might be more interesting, since it really needs properties of tilting modules to be formulated. All known versions of the (Kazhdan-)Lusztig conjecture are based on projective or standard (= Verma or Weyl) or simple modules. For such modules there is no analogue of (3). Condition (3) also should be compared with Irving's results [**15**, **16**] which give (for category $\mathcal{O}$ only) equivalent versions of the Kazhdan-Lusztig conjecture in terms of Loewy series of Verma modules, that is, in terms of 'rigidity'. Usually, tilting modules are much bigger than Verma or Weyl modules, hence Kazhdan-Lusztig conjecture implies less information on the structure of tilting modules. To completely determine the structure of tilting modules seems to be a hard problem (whereas their characters in many cases are known, in particular for category $\mathcal{O}$, see [**22**]).

A more attractive version of this module theoretical condition might be assertion (6) which is formulated over certain centralizer subalgebras. This condition is in terms of the non-existence of a very easy kind of submodules, namely uniserial ones of quite small length.

The main motivation for proving this theorem, in particular for relating conditions on A with conditions on centralizer subalgebras eAe, is the current interest in such centralizer subalgebras for Schur algebras. Decomposition matrices for Schur algebras have a kind of fractal structure which has been explained in the $GL(2)$-case by work of Henke [**13**], who proved isomorphisms $S(2,r) \simeq eS(2,r')e$ between certain Schur algebras and centralizer subalgebras of other Schur algebras. This has subsequently [**14**] been extended to the case of general $GL(n)$. The idempotents e occuring in this context correspond to coideals. Thus Theorem 1 applies. By combining Theorem 1 with these embeddings, computing a single example of a tilting module over a Schur algebra yields information about tilting modules for infinitely many Schur algebras. Much explicit information on small examples can be found in [**11**, **12**].

Let us illustrate the conditions in Theorem 1 by the following examples of tilting modules for a centralizer subalgebra associated to just three indices $\lambda \lessdot \mu < \eta$. In the pictures we denote the corresponding simple modules by $a = L(\lambda), b = L(\mu), c = L(\eta)$.



T T’

The pictures show composition series of two tilting modules T and T'. The module T' satisfies the conditions in Theorem 1, whereas T doesn’t. In fact, the composition factor a ‘on the left hand side of T’ causes condition (5) to be invalid. The same composition factor a generates a uniserial submodule which violates condition (6).

In Section 2, the definitions of quasi-hereditary algebras and tilting modules are recalled. Section three contains the proof of Theorem 1 by a sequence of lemmas.

2. Quasi-hereditary algebras and tilting modules.

An algebra always is supposed to be associative with unit element, and finite dimensional over an algebraically closed field k. Modules are finitely generated left modules.

Definition 2.1 (Cline, Parshall and Scott, [**4**])**.** Let A be a finite dimensional algebra over a field, and Λ the set of isomorphism classes of simple A-modules. Choose representatives $L(\lambda)$ of the elements of Λ. Let $\leq$ be a partial order on I. Then $(A, \leq)$ is called **quasi-hereditary** if and only if the following assertions are true:

(a) For each $\lambda \in \Lambda$, there exists a finite dimensional A-module $\Delta(\lambda)$ with an epimorphism $\Delta(\lambda) \to L(\lambda)$ such that the composition factors $L(\mu)$ of the kernel satisfy $\mu < \lambda$.
(b) For each $\lambda \in \Lambda$, a projective cover $P(\lambda)$ of $L(\lambda)$ maps onto $\Delta(\lambda)$ such that the kernel has a finite filtration with factors $\Delta(\mu)$ satisfying $\mu > \lambda$.

The module $\Delta(i)$ is called **standard module** of index i. Injective A-modules are filtered by modules $\nabla(i)$ (which are dual to the standard modules of the quasi-hereditary algebra $(A^{op}, \leq)$).

An equivalent way of defining quasi-hereditary algebras is via a **heredity chain** $0 \subset J_n \subset J_{n-1} \subset \cdots \subset J_1 \subset J_0 = A$ of twosided ideals in A: For each i, the quotient J_i/J_{i+1} is a **heredity ideal** in A/J_i, that means, it is projective as a left module, it is generated by an idempotent, and it has semisimple endomorphism ring.

Throughout, a quasi-hereditary algebra A will be assumed to have a **duality**; that is, there exists a contravariant self-duality on the category of left A-modules which fixes isomorphism classes of simple modules.

If a quasi-hereditary algebra A is fixed, notations as above always refer to A, whereas reference to other algebras is made explicitly. For example, $L(\lambda)$ is a simple A-module, whereas $L(eAe, \lambda)$ is a simple module over eAe.

Blocks of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ [**3**] of a semisimple complex Lie algebra satisfy these conditions. Schur algebras associated with semisimple algebraic groups (over an algebraically closed field of any characteristic) also are quasi-hereditary [**8**].

Next we are going to define tilting modules in the sense of [**20**].

Theorem 2 (Ringel, [**20**])**.** *Let $(A, \leq)$ be a quasi-hereditary algebra with set Λ of isomorphism classes of simple modules.*

Then, for each $\lambda \in \Lambda$, there is a unique (up to isomorphism) indecomposable module $T(\lambda)$ which has both a filtration with subquotients of the form $\Delta(\mu)$ (for $\mu \leq \lambda$ and $\Delta(\lambda)$ itself occuring with multiplicity one) and another filtration with subquotients of the form $\nabla(\lambda)$ (for $\mu \leq \lambda$ and $\nabla(\lambda)$ itself occuring with multiplicity one).

The module $T(\lambda)$ is characterized by its 'highest weight' λ. That is, $T(\lambda)$ is unique among the $T(\mu)$ with the property that $[T(\lambda) : L(\lambda)] = 1$ and $[T(\lambda) : L(\nu)] = 0$ for $\nu \not\leq \lambda$.

Ringel calls the direct sum $T := \oplus_{\lambda\in\Lambda} T(\lambda)$ the **characteristic tilting module** of $(A, \leq)$. Slightly abusing language, the indecomposable modules $T(\lambda)$ or any sum of those, often are just called 'tilting modules'. A **full tilting module** is one which contains at least one direct summand from each isomorphism class of indecomposable tilting modules.

The tilting module $T(\lambda)$ comes with two important exact sequences which relate it to the theory of left and right approximations ([**2**]) which was the starting point of the theory of tilting modules:

$$0 \to \Delta(\lambda) \to T(\lambda) \to C \to 0$$

where C has a filtration by standard modules, and

$$0 \to K \to T(\lambda) \to \nabla(\lambda) \to 0$$

where K has a filtration by costandard modules. Each sequence can be used as a starting point for constructing $T(\lambda)$ by iterated universal extensions (see [**20**]). Constructing universal extensions in an inductive way, one produces in particular all the submodules of $T(\lambda)$ which occur in Theorem 1.

Ringel has shown that the endomorphism ring $B = End_A(T)$ again is a quasi-hereditary algebra, with the same poset Λ but reversed partial order. In order to avoid confusion, I will stick to the symbol $\leq$ denoting the partial order which defines the quasi-hereditary structure on A.

The algebra B usually is called the 'Ringel dual' of A. The functor $\mathrm{Hom}_A(T, -)$ sends costandard A-modules to standard B-modules and tilting A-modules to projective B-modules.

3. Proof of Theorem 1.

If $(A, \leq)$ is a quasi-hereditary algebra, then certain quotients and also certain centralizer subalgebras of A are quasi-hereditary as well. In fact, let $(\Lambda, \leq)$ be the poset of simple A-modules and fix a coideal I and an ideal J in Λ. This gives us two decompositions $1 = e + (1 - e)$ and $1 = f + (1 - f)$ where e or f is a complete sum of idempotents corresponding to elements in I or J respectively, and $1-e$ or $1-f$ are sums of idempotents lying 'outside' I or J (that is, in the respective complement, which is an ideal in the case of I and a coideal in the case of J). Then both the subalgebra eAe and the quotient algebra $A/A(1-f)A$ are quasi-hereditary (with respect to the restriction of $\leq$). Such idempotents e or f in the following will be called 'idempotents defined by ideals or coideals'. For the Ringel dual algebra B one gets a dual picture. In fact, if T is a full tilting module for A, then cancelling all direct summands indexed by elements in J one gets a full tilting module T' for $A/A(1-f)A$, and the endomorphism ring of T' is of the form eBe (where I is the complement of J).

The first lemma is well-known (and contained in a more complicated to formulate result on highest weight categories in [4]).

Lemma 3.1. *Let $(A,\leq)$ be quasi-hereditary, e an idempotent defined by a coideal I, $\lambda \in I$ an index, $\Delta(\lambda)$ the corresponding standard module, $\nabla(\lambda)$ the costandard module, and $T(\lambda)$ the indecomposable tilting module.*

Then $e\Delta(\lambda)$ is the standard eAe-module $\Delta(eAe,\lambda)$, and $e\nabla(\lambda)$ is the costandard eAe-module $\nabla(eAe,\lambda)$. Moreover, $eT(\lambda)$ is an eAe-tilting module, and it decomposes as $eT(\lambda) = T(eAe,\lambda) \oplus \sum_{\mu \not\geq \lambda} T(eAe,\mu)^{n_\mu}$ (for some n_μ).

Proof. The projective module Ae by assumption is filtered by standard A-modules $\Delta(\lambda)$. Applying the exact functor $\mathrm{Hom}_A(Ae,-)$ to Ae produces a filtration of the regular eAe-module with factors of the form $e\Delta(\lambda)$. For $\lambda \notin I$, the module $eL(\lambda)$ equals zero, whereas $eL(\lambda) \neq 0$ for $\lambda \in I$. Then (since I is a coideal) all $L(\mu)$ with index $\mu < \lambda$ are killed as well and $e\Delta(\lambda)$ also equals zero. Hence eAe is filtered by modules $e\Delta(\lambda)$ with $\lambda \in I$. These modules satisfy the ordering condition on standard modules. Moreover, the indecomposable projective module $eP(\lambda)$ maps onto $e\Delta(\lambda)$. Hence $e\Delta(\lambda)$ has simple top $L(\lambda)$ as well. Thus the modules $e\Delta(\lambda)$ form the standard modules of the quasi-hereditary algebra eAe.

Of course, a similar statement is valid for costandard modules.

A tilting module T is characterized by having both a filtration by standard modules and one by costandard modules. Therefore eT also is a tilting module. The biggest index of a composition factor of $T(\lambda)$ is λ which has multiplicity one. We arrived at the desired decomposition. □

Homomorphisms between tilting modules can be defined 'piecewise' according to the following lemma.

Lemma 3.2. *Let $(A,\leq)$ be quasi-hereditary, X an A-module with a Δ-filtration and Y an A-module with a ∇-filtration. Then as a k-space, $\mathrm{Hom}_A(X,Y)$ is a direct sum of $\mathrm{Hom}(\Delta(\lambda),\nabla(\lambda)) \simeq k$ where $\Delta(\lambda)$ runs through all the Δ-factors in a filtration of X and, for fixed λ, $\nabla(\lambda)$ runs through all ∇-factors of this type in a filtration of Y.*

Proof. This follows immediately from the long exact cohomology sequences attached to the two filtrations, combined with the well-known fact that

Hom $(\Delta(\lambda), \nabla(\mu))$ equals zero if $\lambda \neq \mu$. Thus the isomorphism is induced from compositions of injections and projections which relate T to these subquotients. □

In the next lemma, the general assumption is needed that A has a duality.

Lemma 3.3. *Let $(A, \leq)$ be quasi-hereditary and B its Ringel dual (which is quasi-hereditary with respect to $\geq$).*

Then there is an equality

$$[\Delta(B,\lambda) : L(B,\mu)] = [T(\mu) : \Delta(\lambda)]$$

of composition and filtration multiplicities.

The 'multiplicity' $[T(\mu) : \Delta(\lambda)]$ is well-defined, since it equals, by Lemma 3.2, the k-dimension of $\mathrm{Hom}_A(T(\mu), \nabla(\lambda))$.

Proof. By assumption, the algebra A has a duality which sends a standard to a costandard module and vice versa. Thus the dual of a tilting module again is a tilting module. Since an indecomposable tilting module $T(\mu)$ is characterized by its highest weight μ, it must be self-dual.

In particular, there is an equality

$$[T(\mu) : \Delta(\lambda)] = [T(\mu) : \nabla(\lambda)]$$

(which need not be true if A does not have a duality). Ringel duality sends an A-tilting module to a projective B-module with the same index, and an A-costandard to a B-standard module with the same index. Hence it gives

$$[T(\mu) : \nabla(\lambda)] = [P(B,\mu) : \Delta(B,\lambda)].$$

Finally, BGG-reciprocity [4] for the quasi-hereditary algebra B states that

$$[P(B,\mu) : \Delta(B,\lambda)] = [\Delta(B,\lambda) : L(B,\mu)].$$

Putting everything together yields the assertion. □

Standard methods now yield the first part of Theorem 1.

Lemma 3.4. *Let $(A, \leq)$ be quasi-hereditary and B its Ringel dual.*

Suppose the algebra B satisfies the condition:

- ($*$) *For all $\lambda \lessdot \mu$, the composition multiplicity $[\Delta(B,\lambda) : L(B,\mu)]$ equals one.*

Then the following conditions are equivalent:

- (1) *For all $\lambda \lessdot \mu$, the B-cohomology space $\mathrm{Ext}^1_B(L(\mu), L(\lambda))$ is not zero.*
- (1′) *For all pairs of neighbouring indices $\lambda \lessdot \mu$, the standard module $\Delta(B,\lambda)$ has a quotient of length two with top composition factor isomorphic to $L(B,\lambda)$ and socle isomorphic to $L(B,\mu)$.*
- (2) *The algebra A satisfies: For all $\lambda \lessdot \mu < \eta$ and for all $\varphi \in \mathrm{Hom}_A(T(\mu), T(\eta))$ and $\psi \in \mathrm{Hom}_A(T(\eta), \nabla(\lambda))$, the composition $\varphi\psi$ equals zero.*

Moreover, if A is either a block of $\mathcal{O}$ or a Schur algebra (and then $p \geq h$), then (1) is equivalent to the validity of Kazhdan-Lusztig or Lusztig conjecture, respectively.

Proof. By the universal property of standard modules, condition (1) implies (1′). The converse implication is obvious.

Let us now prove the equivalence between (1′) and (2). Fix a full tilting module T over A such that $B = End_A(T)$. In Ringel's [**20**] proof that B is quasi-hereditary, the standard modules for B are defined as follows:

$$\Delta(B,\lambda) := \mathrm{Hom}_A(T, \nabla(\lambda)).$$

(This works because of the Ext-orthogonality between the Δ-good and the ∇-good modules: $Ext^l(\Delta(\lambda), \nabla(\mu)) = 0$, unless $l = 0$ and $\lambda = \mu$.)

Decompose T into $T = \bigoplus_\lambda T(\lambda)$. The B-module $\Delta(B,\lambda) = \mathrm{Hom}_A(T, \nabla(\lambda))$ has a unique simple quotient, which is $L(B,\lambda)$ and which is generated by the A-projection $T(\lambda) \to \nabla(\lambda)$.

Combining Lemma 3.3 with Lemma 3.2 and with (∗) implies the existence of a (unique up to scalar) nonzero homomorphism $\alpha : T(\mu) \to \nabla(\lambda)$. Ext-orthogonality and the Δ-filtration of $T(\mu)$ imply that φ is the lift of the unique nonzero morphism $\Delta(\lambda) \to \nabla(\lambda)$. All the other composition factors of $\Delta(B,\lambda)$ arise in a similar way via morphisms starting at other indecomposable tilting modules.

Define the B-module M to be the unique quotient of $\Delta(B,\lambda)$ which has simple socle $L(\mu)$ (which occurs with multiplicity one by (∗)). This socle is generated by the homomorphism α.

The module M has at least one other composition factor, say $L(\eta)$ (with $\eta \neq \lambda, \mu$), if and only if the morphism $\alpha : T(\mu) \to \nabla(\lambda)$ factors through a direct sum of $T(\eta)$ (again with indices satisfying $\eta \neq \lambda, \mu$). Such a factorization yields a nonzero composition $\varphi\psi$ which contradicts (2).

Equivalence of (1) with (Kazhdan-)Lusztig conjecture is well-known, see [**5**] and the references therein. □

Lemma 3.5. *Let $(A, \leq)$ be quasi-hereditary and suppose that it satisfies condition (∗).*

Fix a coideal I and denote by e the associated idempotent. Then the quasi-hereditary algebra eAe also satisfies condition (∗).

Fix an ideal J and denote by f the associated idempotent. Then the quasi-hereditary algebra $A/A(1-f)A$ also satisfies condition (∗).

Proof. Fix $\lambda \lessdot \mu$ both in I. By Lemma 3.1, the standard module $\Delta(eAe, \mu)$ equals $e\Delta(\mu)$. Denote by f a primitive idempotent of class λ which is contained in e, that is, it satisfies $f = ef = fe$. Then the assumption that A satisfies (∗) means $\dim_k(f\Delta(\mu)) = 1$. This implies $\dim_k(ef\Delta(\mu)) = 1$, which we had to prove in the first case.

The other case is trivial since standard modules do not change when passing to a quasi-hereditary quotient. □

Lemma 3.6. *Let* $(A, \leq)$ *and* B *be as above* (*in particular,* B *satisfies condition* $(*)$). *Fix* $\lambda \lessdot \mu$ *inside a coideal* I *which defines an idempotent* e *and such that* λ *is minimal in* I. *If some composition* $T(\mu) \to T(\eta) \to \nabla(\lambda)$ (*with* $\eta > \mu$ *varying*) *over* A *is not zero, then there exists some nonzero composition* $T(eAe, \mu) \to T(eAe, \eta) \to \nabla(eAe, \lambda)$ *over* eAe *as well.*

Proof. If there is such a nonzero composition over A, then its image in $\nabla(\lambda)$ must contain a composition factor $L(\lambda)$. Such composition factors are left unchanged when passing to eAe. Hence multiplication by e leaves the composition nonzero. By Lemma 3.1, the involved A-tilting modules decompose over eAe into direct sums of tilting modules. Thus we only have to show that the nonzero map has a nonzero summand which starts at $T(eAe, \mu)$.

The original composition on the level of A comes from lifting the map $\Delta(\lambda) \to \nabla(\lambda)$ to $T(\mu)$. Here, the standard module $\Delta(\lambda)$ is unique in the Δ-good filtration of $T(\mu)$. We are going to prove the following claim:

Claim. The standard filtration of $T(eAe, \mu)$ contains a factor $\Delta(eAe, \lambda)$.

Once the claim has been shown, it follows that all the other summands of $eT(\mu)$ do not contribute to the map $eT(\mu) \to e\nabla(\lambda)$, since their Δ-filtration does not contain a factor $\Delta(eAe, \lambda)$. Thus the summand starting at $T(eAe, \mu)$ must be nonzero.

Proof of the claim. Lemma 3.5 shows that the multiplicity $[\Delta(B, \lambda) : L(B, \mu)]$ does not change when passing to the index set I. Thus Lemma 3.3 implies that $(*)$ remains valid. □

Lemma 3.7. *Let* A *be a quasi-hereditary algebra and* B *its Ringel dual. Assume that the condition* $(*)$ *is satisfied for* B.

Then the following assertions are equivalent:

(I) *For all* $\lambda \lessdot \mu < \eta$ *and for all* $\varphi \in \mathrm{Hom}_A(T(\mu), T(\eta))$ *and* $\psi \in \mathrm{Hom}_A(T(\eta), \nabla(\lambda))$, *the composition* $\varphi\psi$ *equals zero.*

(II) *For all* $\lambda \lessdot \mu < \eta$, *the composition multiplicity*

$$[(T_{\nabla_{\leq\mu}}(\eta) \cap T_{\Delta_{\geq\lambda}}(\eta))/(T_{\nabla_{\leq\mu}}(\eta) \cap T_{\Delta_{>\lambda}}(\eta)) : L(\lambda)]$$

equals zero.

Proof. Fix $\lambda \lessdot \mu < \eta$. We also fix several exact sequences.

(a) $$0 \to T_{\Delta_{\geq\lambda}}(\mu) \to T(\mu) \to T_{\Delta_{<\lambda}}(\mu) \to 0;$$

(b) $$0 \to T_{\nabla_{\leq\mu}}(\eta) \to T(\eta) \to T_{\nabla_{>\mu}}(\eta) \to 0;$$

(c) $$0 \to T_{\Delta_{\geq\lambda}}(\eta) \to T(\eta) \to T_{\Delta_{<\lambda}}(\eta) \to 0.$$

We decompose $\mathrm{Hom}_A(T(\mu), T(\eta))$ and $\mathrm{Hom}_A(T(\eta), \nabla(\lambda))$ according to Lemma 3.2. Recall that $\mathrm{Hom}_A(\Delta(\nu), \Delta(\nu'))$ is zero if $\nu \neq \nu'$. Hence, the sequence (a) provides us with an isomorphism $\mathrm{Hom}_A(T(\mu), \nabla(\lambda)) \simeq \mathrm{Hom}_A(T_{\Delta\geq\lambda}(\mu), \nabla(\lambda))$ which is induced by the inclusion. Thus we can restrict our attention to homomorphisms starting in $T_{\Delta\geq\lambda}(\mu)$. This module is filtered by copies of $\Delta(\lambda)$ and of $\Delta(\mu)$. Hence (again by Lemma 3.2) a morphism starting there and ending in $T(\eta)$ has its image inside $T_{\nabla\leq\mu}(\eta)$.

Similarly, $\mathrm{Hom}_A(T(\eta), \nabla(\lambda))$ is isomorphic to $\mathrm{Hom}_A(T_{\Delta\geq\lambda}(\eta), \nabla(\lambda))$, again via the inclusion. Moreover, $\mathrm{Hom}_A(T_{\Delta>\lambda}(\eta), \nabla(\lambda))$ equals zero.

Suppose condition (I) not to be satisfied. Then there are morphisms φ and ψ with a nonzero composition. We can restrict $\varphi\psi$ to $T_{\Delta\geq\lambda}(\mu)$, and it still will be nonzero. The image of $\varphi\psi$ in $\nabla(\lambda)$ must intersect nontrivially with $S := \mathrm{soc}\,(\nabla(\lambda)) \simeq L(\lambda)$. Let f be a primitive idempotent in the equivalence class λ. Then S must have a nonzero preimage in $fT_{\Delta\geq\lambda}(\mu)$. That is, there is an element $x \in fT_{\Delta\geq\lambda}(\mu)$ which is mapped to an element $y = \varphi(x)$ in $(fT_{\Delta\geq\lambda}(\eta)) \cap T_{\nabla\leq\mu}(\eta)$ by φ and to a generator of S by $\varphi\psi$. In particular, $y = fy$ cannot be an element of $T_{\Delta>\lambda}(\eta)$, since it would be sent to zero otherwise. Consequently, $y = fy$ is a nonzero element in $(T_{\nabla\leq\mu}(\eta) \cap T_{\Delta\geq\lambda}(\eta))/(T_{\nabla\leq\mu}(\eta) \cap T_{\Delta>\lambda}(\eta))$. This gives the existence of a composition factor of type $L(\lambda)$ as required to contradict (II). The situation is described in the following diagram:

$$\begin{array}{ccccc}
T(\mu) & \xrightarrow{\varphi} & T(\eta) & \xrightarrow{\psi} & \nabla(\lambda) \\
\subset\uparrow & & \subset\uparrow & & =\uparrow \\
T_{\Delta\geq\lambda}(\mu) & \xrightarrow{\varphi} & T_{\nabla\leq\mu}(\eta) & \xrightarrow{\psi} & \nabla(\lambda) \\
\subset\uparrow & & \subset\uparrow & & =\uparrow \\
T_{\Delta\geq\lambda}(\mu) \cap \varphi^{-1}(T_{\Delta\geq\lambda}(\eta)) & \xrightarrow{\varphi} & T_{\nabla\leq\mu}(\eta) \cap T_{\Delta\geq\lambda}(\eta) & \xrightarrow{\psi} & \nabla(\lambda)\,.
\end{array}$$

Conversely, suppose condition (II) is not satisfied. That is, there is a composition factor $L(\lambda)$ generated by an element $y = fy$ as before. This element generates a factor $\Delta(\lambda)$ in the Δ-filtration of $T(\eta)$. Lemma 3.2 implies the existence of a map ψ starting in $T(\eta)$ and sending y to a generator of $S := \mathrm{soc}\,(\nabla(\lambda)) \simeq L(\lambda)$. Thus it remains to define the morphism φ in such a way that y is in the image, which can be done using sequence (a) as follows. It is enough to define the restriction of φ to $T_{\Delta\geq\lambda}(\mu)$. By $(*)$ and Lemma 3.3, this module has a Δ-filtration consisting of one copy of $\Delta(\mu)$ and one copy of $\Delta(\lambda)$. Hence it is generated by the top composition factor of $\Delta(\lambda)$. We want the image of φ (restricted to $T_{\Delta\geq\lambda}(\mu)$) to be inside $T_{\nabla\leq\mu}(\eta)$. Consider the composition factor $L(\lambda)$ defined above. It must be a composition factor either of a subquotient $\nabla(\mu)$ or of a subquotient $\nabla(\lambda)$, since no other subquotient in the ∇-filtration of $T_{\nabla\leq\mu}(\eta)$ has such a composition factor. If it is in a factor $\nabla(\lambda)$ then we define φ by lifting the nontrivial map $\Delta(\lambda) \to \nabla(\lambda)$. In the other case we define φ by lifting the map $\Delta(\mu) \to \nabla(\mu)$. The top composition factor $L(\lambda)$ which generates $T_{\Delta \geq \lambda}(\mu)$ is sent to a composition factor of the same isomorphism class which must generate the image. By construction, this image contains the socle of $\nabla(\mu)$. Hence it also contains the composition factor $L(\lambda)$ of $\nabla(\mu)$. Thus, in both cases the image of φ contains the composition factor $L(\lambda)$ which is a top composition factor of $T_{\Delta \geq \lambda}(\eta)$ and not in the kernel of ψ. □

Lemma 3.8. *Let A be a quasi-hereditary algebra and B its Ringel dual. Assume that the condition $(*)$ is satisfied for B.*

Fix $\lambda \lessdot \mu < \eta$ in a coideal I whose associated idempotent is called e.

Suppose that the composition multiplicity

$$[(T_{\nabla \leq \mu}(\eta) \cap T_{\Delta \geq \lambda}(\eta))/(T_{\nabla \leq \mu}(\eta) \cap T_{\Delta > \lambda}(\eta)) : L(\lambda)]$$

equals zero.

Then the composition multiplicity

$$[(T_{\nabla \leq \mu}(eAe, \eta) \cap T_{\Delta \geq \lambda}(eAe, \eta))/(T_{\nabla \leq \mu}(eAe, \eta) \cap T_{\Delta > \lambda}(eAe, \eta)) : L(\lambda)]$$

also equals zero.

Proof. Multiplication by e sends a Δ- (resp. ∇-) filtration of $T(\eta)$ to a corresponding filtration of $eT(\eta)$. Hence the corresponding assertion on composition multiplicities of $eT(\eta)$ is clear. But, by Lemma 3.1, $T(eAe, \eta)$ is a direct summand of $eT(\eta)$. □

Lemma 3.9. *Let $(A, \leq)$ be a quasi-hereditary algebra and B its Ringel dual. Suppose that B satisfies $(*)$. Let $\lambda \lessdot \mu$ be two indices with λ minimal in Λ.*

Let η be any other index.

Then the following two assertions are equivalent:

(a) *Any composition of morphisms $T(\mu) \to T(\eta) \to \nabla(\lambda)$ is zero.*
(b) *Let U be a submodule of $T(\eta)$, which is uniserial with simple top $L(\lambda)$, of Loewy length less than or equal to three, and with all composition factors of type $L(\nu)$ where ν is λ or μ. Then U is contained in the radical of $T(\eta)$.*

Note that such a module U of Loewy length three a priori need not exist. It will be shown that $(*)$ implies $T(\mu)$ to be such a module (which, in general, need not be a submodule of $T(\eta)$).

Proof. Let us first describe the structure of such a module U (if it exists) in more detail: Loewy length less than or equal to three means $\mathrm{rad}^3(U) = 0$. By assumption, $U/\mathrm{rad}(U)$ is isomorphic to $L(\lambda)$. Since A is quasi-hereditary, there are no self-extensions of simple modules. Hence, the composition factor $\mathrm{rad}(U)/\mathrm{rad}^2(U)$ must be isomorphic to $L(\mu)$. For the same reason, the composition factor $\mathrm{rad}^2(U)/\mathrm{rad}^3(U)$ (if it exists) must be $L(\lambda)$ again.

Now suppose that there is a nonzero composition $T(\mu) \xrightarrow{\alpha} T(\eta) \xrightarrow{\beta} \nabla(\lambda)$. Since λ is minimal and μ is a direct neighbour, we can precisely determine some of the modules we are dealing with: The standard module $\Delta(\lambda)$ is simple. The costandard module $\nabla(\lambda)$ is simple as well. The standard module $\Delta(\mu)$ (by $(*)$) has length two, with top composition factor $L(\mu)$ and simple socle $L(\lambda)$. Again by $(*)$, the tilting module $T(\mu)$ is filtered by one copy of $\Delta(\mu)$ (at the bottom) and one copy of $\Delta(\lambda)$ (at the top), hence it is uniserial and satisfies the conditions imposed on U.

Thus the image, say U, of $T(\mu)$ under α again is uniserial, with simple top $L(\lambda)$. The composition $\alpha\beta$ is assumed to be nonzero. Hence, β restricted to U must be the projection onto the top factor $L(\lambda) \simeq \nabla(\lambda)$. Thus U cannot be contained in the radical of $T(\eta)$.

Conversely, suppose that $T(\eta)$ contains a submodule U as before. As U is not contained in $\mathrm{rad}(T(\eta))$, we can define a projection $T(\eta) \to \nabla(\lambda) \simeq L(\lambda)$ whose restriction to U is not zero. By the structure of $T(\mu)$, we can write U as a quotient of $T(\mu)$, say with quotient map α. Therefore, the composition $T(\mu) \xrightarrow{\alpha} U \subset T(\eta) \xrightarrow{\beta} \nabla(\lambda)$ is not zero. □

Proof of Theorem 1. By Lemma 3.5, passing from B to eBe or B/BfB preserves condition $(*)$. The equivalence of (1) and (2) is Lemma 3.4. Lemma 3.7, applied to either A or eAe, proves the equivalence between (2) and (3) and between (4) and (5), respectively. The implication from (4) to (2) is in Lemma 3.6. Lemma 3.8 shows that (3) implies (5). Finally, Lemma 3.9 yields the equivalence between (5) and (6). □

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Department of Mathematics and Computer Science
University of Leicester
Leicester LE1 7RH
United Kingdom
E-mail address: sck5@mcs.le.ac.uk