Pacific
Journal of
Mathematics

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Volume 203 No. 2 April 2002

PACIFIC JOURNAL OF MATHEMATICS
Vol. 203, No. 2, 2002

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It is shown that a central extension of a Lie groupoid by an Abelian Lie group A has a principal A-bundle structure and the extended Lie groupoid is classified by an Euler es-class. Then we prove that for a symplectic *α-connected, $\alpha\beta$-transversal or α-simply connected* groupoid, there exists at most one central S^1-extension, the Euler es-class of which corresponds to the Poisson cohomology class of the Poisson manifold of units.

Introduction.

Central extensions of a Lie groupoid Γ by an Abelian Lie group A are Lie groupoids and have principal A-bundle structures over Γ (see Lemma 2.1). Using the groupoid cochains consisting of A-valued functions which are smooth in an open neighborhood of the diagonal of the unit space Γ_0 of Γ, Weinstein and Xu [**W-X**, p. 161] defined an identity smooth cohomology $H^*_{\text{es}}(\Gamma; A)$.

Theorem 2.2. *If a Lie groupoid Γ over Γ_0 is generated by arbitrarily small neighborhoods of the identity, then the isomorphism classes of central extensions of Γ by an Abelian Lie group A are mapped isomorphically to the cohomology group $H^2_{\text{es}}(\Gamma; A)$.*

Let S^1 denote the unit circle. Then the central extensions E of Γ by S^1 are principal S^1-bundles over Γ. Suppose that Γ is a symplectic groupoid with a symplectic form ω and let ϖ denote the Poisson tensor on the unit space Γ_0 of Γ. Weinstein and Xu [**W-X**, pp. 162-170] constructed a homomorphism $\Psi : H^*_{\text{es}}(\Gamma; S^1) \to H^*_{\varpi}(\Gamma_0)$ where $H^*_{\varpi}(\Gamma_0)$ is the Poisson cohomology of Γ_0. A Lie groupoid $(\Gamma \rightrightarrows \Gamma_0, \alpha, \beta)$ is called *$\alpha\beta$-transversal* if an α-fiber and a β-fiber are transversal everywhere, providing a trivial vertex bundle. The main result of the present paper is the following:

Theorem 3.2. *Let $((\Gamma, \omega) \rightrightarrows \Gamma_0, \alpha, \beta)$ be a symplectic α-connected, $\alpha\beta$-transversal or α-simply connected groupoid. Then there exists at most one central S^1-extension E of Γ, such that Ψ maps the groupoid Euler es-class of E to the class of Poisson tensor ϖ.*

It is emphasized that Theorem 3.2 includes a non α-simply connected case. As a corollary of the theorem, if $((\Gamma, \omega) \rightrightarrows \Gamma_0, \alpha, \beta)$ is α-connected, α-simply connected and quantizable, or if it is a "covering groupoid" of a pair groupoid of a connected (*not necessarily simply connected*) quantizable symplectic manifold then there exists a unique central S^1-extension E of Γ, the Euler es-class of which corresponds to the class of ϖ. In the argument of Theorem 3.2, a Riemannian metric of Γ plays an essential role. Thus presumably the results only hold if Γ is Hausdorff. Moreover, E is a contact groupoid (cf. [**D**, p. 437]). It is so in stronger senses by P. Libermann [**L**, p. 39] and by Y. Kerbrat and Z. Souici-Benhammadi [**K-SB**, p. 81], too.

In Section 1, we define a central extension of a groupoid by an Abelian group and review its classification by groupoid cohomology. Then we go to the central extension in the Lie groupoid category. In Section 2, we get a principal A-bundle structure on a central A-extension of a Lie groupoid for the Abelian Lie group A. Then, by making use of a technique of V.S. Varadarajan [**Var**, pp. 63-64], we prove that the groupoid Euler es-class classifies the central A-extension of a Lie groupoid, that is Theorem 2.2. In the last section, we examine the Weinstein-Xu homomorphism for $H^2_{\mathrm{es}}(\Gamma; S^1)$ and get its injectivity for symplectic, α-simply connected or $\alpha\beta$-transversal groupoid (Γ, ω) generated by arbitrarily small neighborhoods of Γ_0, which proves Theorem 3.2. Then we show that a central S^1-extension E of a quantizable symplectic groupoid $((\Gamma, \omega) \rightrightarrows \Gamma_0, \alpha, \beta)$ has a contact groupoid structure if the symplectic groupoid $(\Gamma, \omega) \rightrightarrows \Gamma_0$ satisfies the conditions on fibers of $\Gamma$and E corresponds to the Poisson class.

The author expresses his thanks to Professor A. Weinstein for discussions during the preparation of the paper. He also thanks the referee for kind comments on Theorem 2.2, Lemma 3.1 and others.

1. Central extensions of a groupoid.

We begin with algebraic arguments of groupoids without any topology or measures. Let $(\Gamma \rightrightarrows \Gamma_0, \alpha, \beta)$ be a groupoid (cf. [**B-W**, Definition 8.5, p. 115], [**Vai**, p. 138] and [**M**, p. 2]) and A an Abelian group.Let $p : \Gamma_0 \times A \to \Gamma_0$ denote the first factor projection. By taking $\alpha = \beta = p$ and identifying Γ_0 with $\Gamma_0 \times \{e\}$ for the unit element $e \in A$, $\Gamma_0 \times A$ is regarded as a groupoid on Γ_0. A *central extension* $(E \rightrightarrows E_0, \alpha, \beta)$ of the groupoid Γ by the Abelian group A is a sequence

$$\Gamma_0 \times A \overset{\iota}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} \Gamma$$

where ι and π are injective and surjective groupoid morphisms over the identifying map $\Gamma_0 \overset{\cong}{\to} E_0$ and its inverse respectively, satisfying the conditions

$$\mathrm{im}(\iota) = \ker(\pi), \tag{1.1}$$

$$(\iota(\alpha \circ \pi(\xi), u))\xi = \xi(\iota(\beta \circ \pi(\xi), u)) \tag{1.2}$$

for any $\xi \in E$ and $u \in A$. (1.2) is abbreviated by $u\xi = \xi u$. Notice that $\pi|_{E_0}$ is injective.

We choose a section s of π such that $s|_{\Gamma_0}$ coincides with the identifying map $\Gamma_0 \overset{\cong}{\to} E_0$.

Lemma 1.1. *The map s commutes with the maps α and β: For any $x \in \Gamma$, we have $\alpha \circ s(x) = s \circ \alpha(x)$ and $\beta \circ s(x) = s \circ \beta(x)$.*

Proof. Since we have $(\alpha \circ s(x))s(x) = s(x)$, it follows that

$$\begin{aligned} \pi(\alpha \circ s(x))x &= \pi(\alpha \circ s(x))(\pi \circ s(x)) \\ &= \pi((\alpha \circ s(x))s(x)) \\ &= \pi \circ s(x) \\ &= x, \end{aligned}$$

and hence $\pi(\alpha \circ s(x)) = \alpha(x)$. By the bijectivity of $\pi|_{E_0}$, one gets $\alpha \circ s(x) = s \circ \alpha(x)$. A similar computation shows that $\beta \circ s(x) = s \circ \beta(x)$. □

For $x, y \in \Gamma$ with $\beta(x) = \alpha(y)$, the product $s(x)s(y)s(xy)^{-1}$ is well-defined in E by the above lemma. Since we have

$$\begin{aligned} \pi(s(x)s(y)s(xy)^{-1}) &= (\pi \circ s(x))(\pi \circ s(y))(\pi(s(xy)^{-1})) \\ &= xy(\pi \circ s(xy))^{-1} \\ &= xy(xy)^{-1} \\ &= \alpha(x) \in \Gamma_0, \end{aligned}$$

and since E is the central extension of Γ, it follows that

$$s(x)s(y)s(xy)^{-1} \in \iota(\{\alpha(x)\} \times A) \subset E.$$

Therefore it determines an element $\mu(x, y) \in A$. By direct computations, it is shown that μ is a 2-cocycle on the groupoid Γ with coefficients in the Abelian group A and the cohomology class $[\mu] \in H^2(\Gamma; A)$ does not depend on the choice of the section $s : \Gamma \to E$. $[\mu]$ is called the *groupoid Euler class* of the central extension E of Γ by A. One gets the following result on the classification of central extension of a groupoid (see, e.g., [**R**, p. 13]).

Proposition 1.2. *The isomorphism classes of central extensions of a groupoid Γ over Γ_0 by an Abelian group A are mapped isomorphically to the cohomology group $H^2(\Gamma; A)$ by the groupoid Euler classes.*

2. Lie groupoids.

A groupoid $(\Gamma \rightrightarrows \Gamma_0, \alpha, \beta)$ is called a *Lie groupoid* (cf. [**B-W**, pp. 115-116]) if Γ is a smooth manifold and the following properties are satisfied:

1) Γ_0 is a smooth submanifold of Γ;
2) α and β are submersions;

3) the multiplication is a smooth mapping

$$\Gamma_2 = (\alpha \times \beta)^{-1}(\text{diagonal } (\Gamma_0 \times \Gamma_0)) \to \Gamma$$

(notice that Γ_2 is a smooth submanifold since $\alpha \times \beta : \Gamma \times \Gamma \to \Gamma_0 \times \Gamma_0$ is transversal to the diagonal by 2);

4) the inversion $x \mapsto x^{-1} : \Gamma \overset{\cong}{\to} \Gamma$ is a diffeomorphism.

A *central extension* $(E \rightrightarrows E_0, \alpha, \beta)$ of a Lie groupoid Γ by an Abelian Lie group A is a sequence of Lie groupoids and smooth mappings

$$\Gamma_0 \times A \overset{\iota}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} \Gamma$$

where ι and π are imbedding and submersion groupoid morphisms over the diffeomorphism $\Gamma_0 \overset{\cong}{\to} E_0$ and its inverse respectively, satisfying the conditions (1.1) and (1.2).

Lemma 2.1. *The central extension E of a Lie groupoid Γ by an Abelian Lie group A is a principal A-bundle over Γ.*

Proof. We define a smooth action "$\cdot$" of the Abelian Lie group A on the smooth manifold E by Equation (1.2), that is,

$$u \cdot \xi = u\xi = \xi u$$

for any $u \in A$ and $\xi \in E$. Since E is an extension of the Lie groupoid Γ by A, A is mapped diffeomorphically onto each orbit by the action. Therefore one gets a free A-action on E whose orbit space is Γ. By the standard arguments of smooth transformation groups, E is a principal A-bundle over Γ. □

Transitive groupoids by Mackenzie [**M**, Definition 3.1, p. 13] are just extensions of the pair groupoid, which has a principal bundle structure. In this sense any extension of Γ should be thought of as a "transitive groupoid over Γ".

The identity smooth groupoid cohomology $H^*_{\text{es}}(\Gamma; A)$ is defined by the groupoid cochains consisting of A-valued functions which are smooth in a neighborhood of the diagonal of Γ_0 (cf. [**W-X**, p. 161] and [**T-W**, p. 217]). Since $E|_{\Gamma_0}$ is trivial, one can choose a section $s : \Gamma \to E$ which is smooth in a neighborhood of Γ_0 and gets the *groupoid Euler es-class* of the extension E of Γ by A in $H^2_{\text{es}}(\Gamma; A)$. Let

$$\Gamma_n = \{(x_1, \ldots, x_n) \in \Gamma^n \mid \beta(x_i) = \alpha(x_{i+1}), i = 1, \ldots, n-1\}.$$

If for any neighborhood N of Γ_0, we have

$$\bigcup_{n \to \infty} \{m(N_n)\} = \Gamma$$

where $N_n = N^n \cap \Gamma_n$ and m is the groupoid multiplication map, then we say that Γ is *generated by arbitrarily small neighborhoods of the identity.*

We notice that if the Lie groupoid Γ is α-connected, then it is generated by arbitrarily small neighborhoods of the identity.

Theorem 2.2. *If a Lie groupoid Γ over Γ_0 is generated by arbitrarily small neighborhoods of the identity, then the isomorphism classes of central extensions of Γ by an Abelian Lie group A are mapped ismorphically to the cohomology group $H^2_{\mathrm{es}}(\Gamma; A)$.*

Proof. Let U be a sufficiently small open neighborhood of Γ_0 generating Γ. For any central A-extension E of Γ, the restriction $E|_U$ has a trivial A-bundle structure. If two central A-extensions $E_{(1)}$ and $E_{(2)}$ correspond to the same class of $H^2_{\mathrm{es}}(\Gamma; A)$, multiplicative structures on $E_{(1)}|_U$ and $E_{(2)}|_U$ are smoothly isomorphic since their Euler es-classes are represented by the same 2-cocycle which is smooth on U. Let $\mathcal{N}$ denote a small neighborhood of Γ_0 such that $\mathcal{N} \subset E_{(1)}|_U \cap E_{(2)}|_U$. An arbitrary element $\bar{\xi} \in E_{(1)}$ splits to a finite product

$$\bar{\xi} = \xi_1 \cdots \xi_k$$

with $\xi_i \in \mathcal{N}, i = 1, \ldots, k$. Let $(\Sigma_i; s_{\alpha,i}, s_{\beta,i})$ denote a local smooth bi-cross section (cf. [**Vai**, 9.10. Definition, p. 146]) such that $\Sigma_i \subset \mathcal{N}$ and

$$s_{\alpha,i} \circ \alpha(\xi_i) = \xi_i, \qquad s_{\beta,i} \circ \beta(\xi_i) = \xi_i$$

for $i = 1, \ldots, k$.

Let $\mathcal{W}$ be a sufficiently small neighborhood of $\beta(\bar{\xi}) = \beta(\xi_k)$ in $\mathcal{N}$. Obviously, one can assume that $\mathcal{W}$ is of the form $W_1 \times W_2$ where W_1 is a smooth coordinate neighborhood around $\beta(\xi_k)$ in Γ_0 and W_2 is that around $\beta(\xi_k)$ in $\mathcal{N} \cap \alpha^{-1}(\beta(\xi_k))$. Then $W_1 \times W_2$ is a smooth coordinate system of a neighborhood $\mathcal{W}_{\bar{\xi}}$ of $\bar{\xi}$ of $\bar{\xi}$ in $E_{(1)}$ by taking a smaller neighborhood $\mathcal{W}$ of $\beta(\xi_k)$ if necessary, since any element $\xi \in \mathcal{W}_{\bar{\xi}}$ has a product form

$$\xi = (s_{\beta,1} \circ \alpha \circ s_{\beta,2} \circ \alpha \circ \because \cdots \circ \alpha \circ s_{\beta,k} \circ \beta(\xi')) \cdots \\ (s_{\beta,k-1} \circ \alpha \circ s_{\beta,k} \circ \beta(\xi'))(s_{\beta,k} \circ \alpha(\xi'))\xi'$$

for $\xi' \in \mathcal{W}$. The algebraic isomorphism $\varphi : E_{(1)} \to E_{(2)}$ of Proposition 1.2 gives us local smooth bi-cross sections $(\varphi(\Sigma_i); \varphi \circ s_{\alpha,i}, \varphi \circ s_{\beta,i})$ around $\varphi(\xi_i) \in \mathcal{N} \subset E_{(2)}|_U$ and hence φ maps smoothly the coordinate meighborhood $\mathcal{W}_{\bar{\xi}}$ of $\bar{\xi}$ in $E_{(1)}$ to that of $\varphi(\bar{\xi})$ in $E_{(2)}$. Therefore the smooth multiplicative structure of $E_{(1)}|_U$ and $E_{(2)}|_U$ is extended to a smooth isomorphism of $E_{(1)}$ to $E_{(2)}$. That is, the Euler es-class maps isomorphism classes of central A-extension of Γ injectively to $H^2_{\mathrm{es}}(\Gamma; A)$.

The surjectivity is shown as follows: Let μ be a representative 2-cocycle of an element of $H^2_{\mathrm{es}}(\Gamma; A)$. Then one can assume that μ is smooth on the neighborhood U of Γ_0, and the groupoid multiplication on $U \times A$ is defined by the formula

$$(x_1, u_1)(x_2, u_2) = (x_1 x_2, u_1 + u_2 - \mu(x_1, x_2))$$

for $(x_1, x_2) \in \Gamma_2 \cap (U \times U)$ and $u_1, u_2 \in A$. Let V be a so small neighborhood of Γ_0 in Γ that $VV^{-1} = m(\Gamma_2 \cap (V \times V^{-1})) \subset U$. For an element $\bar{x} \in \Gamma$, one can find a coordinate neighborhood $U_{\bar{x}}$ around $\bar{x}$ of the form $U_{0,\alpha(\bar{x})} \times B$ where $U_{0,\alpha(\bar{x})}$ is an open ball around $\alpha(\bar{x})$ in Γ_0 and B is an open ball around the origin in $\mathbb{R}^{\dim(\alpha^{-1}(\alpha(\bar{x})))}$ such that $(x_0, 0)^{-1}(\{x_0\} \times B) \subset \alpha^{-1}(\beta((x_0, 0))) \cap V$ for each $x_0 \in U_{0,\alpha(\bar{x})}$. For an element $\bar{\xi} = (\bar{x}, \bar{u})$ of the abstract A-extension E ($= \Gamma \times A$ as a set), we define a coordinate neighborhood around $\bar{\xi}$ in E by $\mathcal{W}_{\bar{\xi}} = U_{\bar{x}} \times \bar{u}D$ where D is an open ball around the identity e in A and $U_{\bar{x}} \subset \Gamma$ is the coordinate neighborhood around $\bar{x}$ stated in the above. The family $\{\mathcal{W}_{\bar{\xi}}\}$ define a fundamental system of neighborhoods in E and E is a topological space. Suppose that $\mathcal{W}_{\bar{\xi},\bar{\xi}'} = \mathcal{W}_{\bar{\xi}} \cap \mathcal{W}_{\bar{\xi}'} \neq \emptyset$ for two elements $\bar{\xi}$ and $\bar{\xi}'$. $\mathcal{W}_{\bar{\xi},\bar{\xi}'}$ is obviously an open submanifold of both $\mathcal{W}_{\bar{\xi}}$ and $\mathcal{W}_{\bar{\xi}'}$. Let $\xi = (x, u)$ be an arbitrary element of $\mathcal{W}_{\bar{\xi},\bar{\xi}'}$ represented in $\mathcal{W}_{\bar{\xi}}$. We denote smooth α-fiber 0-sections in $U_{\bar{x}}$ and $U_{\bar{x}'}$ by ϕ and ϕ' respectively. The coordinate transformation of ξ to $\mathcal{W}_{\bar{\xi}'}$ is

$$
\begin{aligned}
(x', u') = (x'(x), u + \mu((\phi \circ \alpha(x))^{-1}x, x^{-1}(\phi' \circ \alpha(x))) \\
- \mu((\phi' \circ \alpha(x))^{-1}x, x^{-1}(\phi' \circ \alpha(x))))
\end{aligned}
$$

which is smooth in (x, u) since $x'(x)$ is a smooth coordinate transformation in the smooth manifold Γ and the second coordinate in the right side is smooth too as we have $(\phi \circ \alpha(x))^{-1}x \in \alpha^{-1}(\beta \circ \phi \circ \alpha(x)) \cap V$ and $(\phi' \circ \alpha(x))^{-1}x \in \alpha^{-1}(\beta \circ \phi' \circ \alpha(x)) \cap V$ so that $(\phi \circ \alpha(x))^{-1}\phi' \circ \alpha(x) \in \alpha^{-1}(\beta \circ \phi \circ \alpha(x)) \cap U$. Therefore the family $\{\mathcal{W}_{\bar{\xi}}\}_{\bar{\xi} \in E}$ of coordinate neighborhoods defines a smooth structure on E and $\pi : E \to \Gamma$ is smooth.

We use a Lie groupoid version of the proof of [**Var**, Lemma 2.6.1, pp. 63-64] to prove the smoothness of groupoid operations of E. Obviously Γ is a topological groupoid and A is a topological Abelian group. Since the central A-extension E of Γ is a principal A-bundle over Γ by Lemma 2.1, the algebraic groupoid multiplication formula shows that $E|_U$ is a topological groupoid with respect to the manifold topology of $E|_U$. By the definition of the smooth structure on E, the left action of a local smooth bi-cross section is smooth. Let $\mathcal{U}$ be a sufficiently small neighborhood of Γ_0 in E such that $\mathcal{U} \subset \pi^{-1}(U)$ and let $\mathcal{V}$ be a sufficiently small neighborhood of Γ_0 in E that

$$\mathcal{V}\mathcal{V}^{-1} = m(E_2 \cap (\mathcal{V} \times \mathcal{V}^{-1})) \subset \mathcal{U}.$$

We take a neighborhood $\mathcal{N}$ of Γ_0 in E with $\mathcal{N} = \mathcal{N}^{-1}$ and

$$\mathcal{N}\mathcal{N}\mathcal{N} = m(E_3 \cap (\mathcal{N} \times \mathcal{N} \times \mathcal{N})) \subset \mathcal{V}.$$

For an element $\bar{\zeta} \in \mathcal{N}$, we take a bi-cross section $(\Sigma_{\bar{\zeta}}; s_{\alpha,\bar{\zeta}}, s_{\beta,\bar{\zeta}})$ such that $\Sigma_{\bar{\zeta}} \subset \mathcal{N}$ and

$$s_{\alpha,\bar{\zeta}} \circ \alpha(\bar{\zeta}) = \bar{\zeta}, \qquad s_{\beta,\bar{\zeta}} \circ \beta(\bar{\zeta}) = \bar{\zeta}.$$

Let $\mathcal{N}_{\beta(\bar{\zeta})}$ be a sufficiently small smooth coordinate neighborhood of $\beta(\bar{\zeta})$ with $\mathcal{N}_{\beta(\bar{\zeta})} \subset \mathcal{N}$ and,

$$\beta(\mathcal{N}_{\beta(\bar{\zeta})}) \cap \alpha(\mathcal{N}_{\beta(\bar{\zeta})}) \subset \beta(\Sigma_{\bar{\zeta}}).$$

Then for any element $\zeta \in \mathcal{N}_{\beta(\bar{\zeta})}$, the product

$$(s_{\beta,\bar{\zeta}} \circ \alpha(\zeta))\zeta(s_{\beta,\bar{\zeta}} \circ \beta(\zeta))^{-1} \in \mathcal{N}$$

is well-defined and is smooth with respect to ζ.

Suppose that $\bar{\zeta} = \zeta_1\zeta_2$ with $\zeta_i \in \mathcal{N}, i = 1, 2$. We take local smooth bi-cross sections $(\Sigma_i; s_{\alpha,i}, s_{\beta,i})$ such that $\Sigma_i \subset \mathcal{N}$ and

$$s_{\alpha,i} \circ \alpha(\zeta_i) = \zeta_i, \qquad s_{\beta,i} \circ \beta(\zeta_i) = \zeta_i.$$

Now we take a sufficiently small smooth coordinate neighborhood $\mathcal{N}_{\beta(\bar{\zeta})}$ of $\beta(\bar{\zeta}) = \beta(\zeta_2)$ with $\mathcal{N}_{\beta(\zeta_2)} \subset \mathcal{N}$, and

$$\beta(\mathcal{N}_{\beta(\zeta_2)}) \cap \alpha(\mathcal{N}_{\beta(\zeta_2)}) \subset \beta(\Sigma_{\zeta_2}),$$

such that for $\zeta \in \mathcal{N}_{\beta(\zeta_2)}$,

$$(s_{\beta,1} \circ \alpha \circ s_{\beta,2} \circ \alpha(\zeta))(s_{\beta,2} \circ \alpha(\zeta))\zeta(s_{\beta,2} \circ \beta(\zeta))^{-1} \cdot (s_{\beta,1} \circ \alpha \circ s_{\beta,2} \circ \beta(\zeta))^{-1} \in \mathcal{N}.$$

We set

$$s_{\beta,\bar{\zeta}}(z) = (s_{\beta,1} \circ \alpha \circ s_{\beta,2}(z))(s_{\beta,2}(z)),$$

for $z \in \Gamma_0$ sufficiently close to $\beta(\bar{\zeta}) = \beta(\zeta_2)$. It is a local smooth bi-cross section around $\bar{\zeta}$, and

$$(s_{\beta,\bar{\zeta}} \circ \alpha(\zeta))\zeta(s_{\beta,\bar{\zeta}} \circ \beta(\zeta))^{-1}$$

is smooth since $\mathcal{N}_{\beta(\bar{\zeta})}$ is sufficiently small. Now let $\bar{\zeta}$ be an arbitrary element of E. Since E is generated by $\mathcal{N}$, $\bar{\zeta}$ splits to a finite product

$$\bar{\zeta} = \zeta_1 \cdots \zeta_k$$

with $\zeta_i \in \mathcal{N}, i = 1, \dots, k$. By a back way induction on i and using the argument in the above, one can find a small smooth coordinate neighborhood $\mathcal{N}_{\beta(\bar{\zeta})}$ of $\beta(\bar{\zeta}) = \beta(\zeta_k)$ and a local bi-cross section $s_{\beta,\bar{\zeta}}$, around $\bar{\zeta}$ such that

$$(s_{\beta,\bar{\zeta}} \circ \alpha(\zeta))\zeta(s_{\beta,\bar{\zeta}} \circ \beta(\zeta))^{-1} \in \mathcal{V}$$

for $\zeta \in \mathcal{N}_{\beta(\bar{\zeta})}$ and it is smooth.

Let $\bar{\xi}, \bar{\eta} \in E$ with $\beta(\bar{\xi}) = \beta(\bar{\eta})$, then any elements in a smooth coordinate neighborhoods around $\bar{\xi}$ and $\bar{\eta}$ are of the forms

$$\xi = (s_{\beta,\bar{\xi}} \circ \alpha(\xi'))\xi', \qquad \eta = (s_{\beta,\bar{\eta}} \circ (\eta'))\eta'$$

where ξ', η' are in a sufficiently small smooth coordinate neighborhood $\mathcal{N}_{\beta(\bar{\xi})}$ of $\beta(\bar{\xi}) = \beta(\bar{\eta}) \in \Gamma_0$ in E, and $s_{\beta,\bar{\xi}}, s_{\beta,\bar{\eta}}$ are local smooth bi-cross sections around $\bar{\xi}, \bar{\eta}$ respectively. Suppose that $\beta(\xi) = \beta(\eta)$, then we have

$$\begin{aligned}\xi\eta^{-1} &= (s_{\beta,\bar{\xi}} \circ \alpha(\xi'))\xi'\eta'^{-1}(s_{\beta,\bar{\eta}} \circ \alpha(\eta'))^{-1} \\ &= (s_{\beta,\bar{\xi}} \circ \alpha(\xi'))(s_{\beta,\bar{\eta}} \circ \alpha(\xi'))^{-1}[(s_{\beta,\bar{\eta}} \circ \alpha(\xi'))\xi'\eta'^{-1}(s_{\beta,\bar{\eta}} \circ \alpha(\eta'))^{-1}]\end{aligned}$$

where the term in [ ] is smooth with respect to ξ' and η'. Therefore its left translation by the bi-cross section $(s_{\beta,\bar{\xi}} \circ \alpha(\xi'))(s_{\beta,\bar{\eta}} \circ \alpha(\xi'))^{-1}$ is smooth and hence $\xi\eta^{-1}$ is smooth with respect to ξ and η. In particular, the inversion $\eta \mapsto \eta^{-1}$ is smooth by taking $\xi = \beta(\eta)$, and any groupoid multiplication $\xi\eta^{-1}$ is smooth in E too. □

3. Weinstein-Xu homomorphism for $H^2_{\text{es}}(\Gamma; S^1)$.

The *Lie algebroid* $\mathcal{A}$ of a Lie groupoid $(\Gamma \rightrightarrows \Gamma_0, \alpha, \beta)$ is a vector bundle over Γ_0 whose sections consist of all left invariant fields on Γ, with the anchor map $\rho : \mathcal{A} \to T\Gamma_0$, given by

$$(\rho(X)f)(z) = X(\beta^* f)(z)$$

for $z \in \Gamma_0$ and for all $X \in \Gamma^\infty(\mathcal{A})$ and $f \in C^\infty(\Gamma_0)$. The bracket on sections of $\mathcal{A}$ satisfies the axiom $[\phi X, Y] = \phi[X, Y] - (\rho(Y)\phi)X$ for each scalar function ϕ. Weinstein and Xu [**W-X**, pp. 162-166] introduced a cohomology algebra homomorohism

$$\Psi : H^n_{\text{es}}(\Gamma; \mathbb{R}) \to H^n(\mathcal{A}; \mathbb{R})$$

which is defined in cochain levels by

$$(\Psi\sigma)(X_1, \dots, X_n)(z) = \sum (-1)^{\tau(k_1,\dots,k_n)}(X_{k_1} \cdots X_{k_n}\sigma)(z)$$

for $z \in \Gamma_0$ and for any $\sigma \in C^n_{\text{es}}(\Gamma; \mathbb{R})$, $X_i \in \Gamma^\infty(\mathcal{A})$ $(i = 1, \dots, n)$, where the sum is over all the permutations $(k_1, \dots, k_n)$ of $(1, \dots, n)$ and $\tau(k_1, \dots, k_n)$ is the sign of the permutation $(k_1, \dots, k_n)$. The function $(X_{k_1} \cdots X_{k_n}\sigma)(z)$ on Γ_0 is defined inductively on n: By fixing variables $(x_1, \dots, x_{n-1}) \in \Gamma_{n-1}$ (that is, $x_i \in \Gamma$ with $\beta(x_i) = \alpha(x_{i+1}), i = 1, \dots, n-2$), we regard $\sigma(x_1, \dots, x_{n-1}, x_n)$ as a function of x_n alone defined on an α-fiber, then by applying X_{k_n} on it and evaluating at $x_n = \beta(x_{n-1})$ we obtain a function of $n-1$ arguments defined on Γ_{n-1}.

Let $\mathbb{Z}$ denote the subgroup of integers in the group $\mathbb{R}$ of real numbers. The circle S^1 is identified with the quotient group $\mathbb{R}/\mathbb{Z}$. A cochain $\bar{\sigma} \in C^n_{\text{es}}(\Gamma; S^1) \cong C^n_{\text{es}}(\Gamma; \mathbb{R}/\mathbb{Z})$ is represented by a cochain $\sigma \in C^n_{\text{es}}(\Gamma; \mathbb{R})$ and $\Psi\sigma \in C^n(\mathcal{A}; \mathbb{R})$ does not depend on the choice of the representative σ of $\bar{\sigma}$. Therefore a cochain map $\Phi : C^n_{\text{es}}(\Gamma; S^1) \to C^n(\mathcal{A}; \mathbb{R})$ is induced and we get a homomorphism $\Psi : H^n_{\text{es}}(\Gamma; S^1) \to H^n(\mathcal{A}; \mathbb{R})$. If an α-fiber of a central S^1-extension E of Γ is an orientable S^1-bundle, then E is called

α-*orientable*. We take a Riemannian metric g on Γ and let g_α denote the restriction of g on each α-fiber. The vector bundle $\ker(\alpha_*)|_{\Gamma_0}$ has an open neighborhood U_{α_*} of Γ_0 which is mapped diffeomorphically onto an open neighborhood U of Γ_0 in Γ by the exponential map in each fiber. We call U an α-*vector bundle neighborhood* of Γ_0. A Lie groupoid $(\Gamma \rightrightarrows \Gamma_0, \alpha, \beta)$ is called a *symplectic groupoid* if Γ is a symplectic manifold with a symplectic 2-form ω such that the graph of the multiplication of Γ is a Lagrangian submanifold of $\Gamma \times \Gamma \times (-\Gamma)$ where $-\Gamma$ is Γ endowed with $-\omega$. It is denoted by $((\Gamma, \omega) \rightrightarrows \Gamma_0, \alpha, \beta)$ or simply by (Γ, ω). Γ_0 is a Lagrangian submanifold of Γ (see, e.g., [**Vai**, 9.8. Proposition, p. 144]).

Lemma 3.1. *Let $((\Gamma, \omega) \rightrightarrows \Gamma_0, \alpha, \beta)$ be a symplectic α-connected $\alpha\beta$-transversal groupoid. The homomorphism $\Psi : H^2_{\mathrm{es}}(\Gamma; S^1) \to H^2(\mathcal{A}; \mathbb{R})$ maps the groupoid Euler* es*-classes of central S^1-extensions of Γ, in an injective way.*

Proof. An element of $H^2_{\mathrm{es}}(\Gamma; S^1)$ defines a central S^1-extension E of Γ, by Theorem 2.2 and E is a principal S^1-bundle by Lemma 2.1. Moreover, since an α-fiber of a symplectic groupoid $(\Gamma, \omega) \rightrightarrows \Gamma_0$ gives us a local smooth β-fiber section and since left actions of E preserve S^1-bundle structures of α-fibers, E is α-orientable. We take a connection 1-form θ_α on an α-fiber of E, which is invariant by a left groupoid action. Since a β-fiber is diffeomorphic to an α-fiber by the groupoid inversion, the 1-form θ_α on the α-fiber induces a connection 1-form θ_β on a β-fiber, which is invariant by a right groupoid action. The 1-forms θ_α and θ_β together define a family of horizontal subspaces on E, that is a left (right) invariant connection 1-form θ_E on E. Let ω_E denote its curvature 2-form. We notice that $\theta_E|_{\pi^{-1}(\Gamma_0)} = 0$. Let $U \subset \Gamma$ be an α-vector bundle neighborhood of Γ_0. One can assume that each α-fiber in U is an open ball in $(\dim \Gamma_0)$-vector spaces. For $x \in \Gamma$ we define a piecewise smooth curve γ_x from x to $\alpha(x)$ in the α-fiber $\alpha^{-1}(\alpha(x))$ such that γ_x is the line segment from x to $\alpha(x)$ for $x \in U$. We take the horizontal lift $\widetilde{\gamma}_x$ of γ_x starting from the point $\pi^{-1}(\alpha(x)) \in E_0 \cong \Gamma_0$. Then the end point of $\widetilde{\gamma}_x$ defines a section $s : \Gamma \to E$ which is smooth on the open neighborhood U of Γ_0. We get an extension 2-cocycle $\sigma : \Gamma_2 \to S^1$ defined by $\sigma(x, y) = s(x)s(y)s(xy)^{-1}$. σ is smooth on $\Gamma_2 \cap (U \times U)$ and vanishes on $(\Gamma_0 \times \Gamma) \cup (\Gamma \times \Gamma_0)$, that is, σ is an identity smooth 2-cocycle in the sense of Weinstein and Xu [**W-X**, p. 161]. Since E is α-orientable, $\sigma(x, y)$ is the total holonomy along the closed curve $\gamma_x(x\gamma_y)\gamma_{xy}^{-1}$ in $\alpha^{-1}(\alpha(x))$, and hence we have

$$\sigma(x, y) = \int_{D(\alpha(x), x, y)} \omega_E \qquad (\mathrm{mod}\ \mathbb{Z}),$$

for $x, y \in U$, where $D(\alpha(x), x, y)$ is a surface surrounded by the closed curve in the *open ball* $\alpha^{-1}(\alpha(x)) \cap U$. For any point $z \in \Gamma_0$ and any vectors X_1, X_2 over z in the Lie algebroid $\mathcal{A} \to \Gamma_0$, we extend them locally to vector fields of the tangent bundle $T_\alpha U$ arising from the α-fibration on U, by the parallel displacement in the α-fiber $\alpha^{-1}(z) \cap U$ of z. By the groupoid left action, these define local smooth sections on $\mathcal{A}$, which are denoted by the same symbols X_1, X_2.

In order to compute $\Psi\sigma(X_1, X_2)(z) = (X_1 X_2 \sigma - X_2 X_1 \sigma)(z)$, one can assume that X_1, X_2 are linearly independent, or the right hand side of the equation vanishes. Via the transformation of variables $(x, y) \mapsto (x, xy)$, we restrict σ to the plane in $\alpha^{-1}(z) \cap U$ determined by the two vectors X_1, X_2, and take the plane coordinate system (t_1, t_2) with coordinate vectors X_1 and X_2. Let $\Delta(z, x, y)$ denote the triangle with vertices $z = (0, 0), x = (t_1, 0)$ and $xy = (t_1, t_2)$. Then ω_E takes the form $f(t_1, t_2) dt_1 \wedge dt_2$ and we have

$$\sigma(x, y) = \int_{\Delta(z,x,y)} f(t_1, t_2) dt_1 \wedge dt_2 + o(t_2) \pmod{\mathbb{Z}},$$

since the difference area is $E(D(z, x, y) - \Delta(z, x, y)) = o(t_2) \pmod{\mathbb{Z}}$. Similarly we have $E(D(z, x, y) - \Delta(z, x, y)) = o(t_1) \pmod{\mathbb{Z}}$ for $x = (0, t_2)$ and $xy = (t_1, t_2)$. From the estimation of f by Taylor expansion, it follows that

$$\Psi\sigma(X_1, X_2)(z) - \omega_E(X_1, X_2)(z) = o(t_1) + o(t_2).$$

Since t_1, t_2 are arbitrary, one obtains $\Psi\sigma(X_1, X_2) = \omega_E(X_1, X_2)$. If X_1, X_2 are linearly dependent, then both sides vanish. Therefore we conclude that $\Psi\sigma = \omega_E$. Suppose that $\Psi[\sigma] = 0$. Then there exists a global left invariant 1-form ϕ along the α-fibers on Γ such that $d\phi = \omega_E$ by [**W-X**, Theorem 1.2, p. 167]. Define a 1-cochain $c_\phi \in C^1(\Gamma; S^1)$ by $c_\phi(x) = \int_{\gamma_x} \phi \pmod{\mathbb{Z}}$ for $x \in \Gamma$. Since ϕ is left invariant along α-fibers, we have

$$\begin{aligned} c_\phi(x) + c_\phi(y) - c_\phi(xy) &= \int_{D(\alpha(x),x,y)} \omega_E \pmod{\mathbb{Z}} \\ &= \sigma(x, y) \end{aligned}$$

for $x, y \in \Gamma_2$, by Stokes theorem, and hence $\sigma = \delta c_\phi$. □

If $((\Gamma, \omega) \rightrightarrows \Gamma_0, \alpha, \beta)$ is a symplectic groupoid, then the manifold Γ_0 carries a unique Poisson structure for which α is a Poisson map. The Lie algebroid $\mathcal{A}$ of the symplectic groupoid (Γ, ω) is the cotangent bundle $T^*\Gamma_0 \to \Gamma_0$ with the anchor map $\rho : T^*\Gamma_0 \to T\Gamma_0$ naturally induced from the Poisson tensor ϖ. For each n, $C^n(\mathcal{A}; \mathbb{R})$ is naturally isomorphic to $\Gamma^\infty(\wedge^n T\Gamma_0)$, and the Lie algebroid differential d turns out to the Poisson differential d_ϖ for the multi-vector fields over Γ_0 by [**H**, Proposition 3.12.4, p. 86]. Hence the Lie algebroid cohomology of $\mathcal{A}$ with trivial coefficients in $\mathbb{R}$ is isomorphic to the Poisson cohomology of Γ_0:$H^*(\mathcal{A}; \mathbb{R}) \cong H^*_\varpi(\Gamma_0)$ by [**W-X**, Lemma 2.1, p. 169]. We examine central S^1-extensions of symplectic groupoids which correspond to the Poisson cohomology class of the unit space.

Theorem 3.2. *Let $((\Gamma, \omega) \rightrightarrows \Gamma_0, \alpha, \beta)$ be a symplectic α-connected, $\alpha\beta$-transversal or α-simply connected groupoid. Then there exists at most one* *central S^1-extension E of Γ, such that Ψ maps the groupoid Euler es-class of E to the class of Poisson tensor ϖ.*

Proof. By Theorem 2.2 the Euler es-class maps isomorphism classes of central S^1-extension of Γ ismorphically to the group $H^2_{\mathrm{es}}(\Gamma; S^1)$, since an α-connected Lie groupoid is generated by arbitrarily small neighborhoods of the identity. Suppose that E and $\overline{E}$ are central S^1-extensions of the theorem. Their Euler es-classes correspond to the class of the Poisson tensor ϖ on Γ_0 under the homomorphism Ψ. If (Γ, ω) is $\alpha\beta$-transversal by Lemma 3.1 the central S^1-extension E of Γ with the Poisson condition on Ψ is isomorphic to the central S^1-extension $\overline{E}$ that is, E is the unique central S^1-extension of Γ with the Poisson condition on Ψ if it exists. If (Γ, ω) is α-simply connected, a central S^1-extension E of Γ is obviously α-orientable. Then by the proof of Lemma 3.1, we get the same conclusion. □

Let (M, ω) be a connected symplectic manifold and let $\hat{\pi}$ be a normal subgroup of the fundamental group $\pi_1(M)$, with an Abelian quotient group. Let $\hat{\Pi}(M)$ denote the space of homotopy classes of paths of M modulo $\hat{\pi}$, relative to end points and let $\hat{\Pi}(M)_0$ denote the subspace of homotopy classes of constant paths modulo $\hat{\pi}$. Path compositions together with the starting and the terminal point mappings define a Lie groupoid $\hat{\Pi}(M) \rightrightarrows \hat{\Pi}(M)_0 \cong M$, which is called a *reduced fundamental groupoid* of M. We get a covering map $p : \hat{\Pi}(M) \to M \times M$ by taking a pair of the starting and the terminal points of a path class. Obviously, the 2-form $\omega^{\hat{\Pi}} = p^*(\omega, -\omega)$ is a symplectic structure on $\hat{\Pi}(M)$ and $\hat{\Pi}(M) \rightrightarrows M$ is a symplectic groupoid with respect to the 2-form $\omega^{\hat{\Pi}}$, which is $\alpha\beta$-transversal. For a symplectic groupoid (Γ, ω), we denote the 2-cycle group of Γ (as a topological space) by $Z_2(\Gamma)$ and ω is called an *integral symplectic structure* if $\mathrm{Per}(\omega) = \mathrm{im}\,(\omega|_{Z_2(\Gamma)})$ is contained in the integral subgroup $\mathbb{Z} \subset \mathbb{R}$. Then the symplectic manifold (Γ, ω) is called *(pre)quantizable*. Let E be a principal S^1-bundle over the symplectic manifold Γ. If the first Chern class $c_1(E)$ of E for the standard unitary representation of S^1 is represented by the symplectic form ω, ω is an integral symplectic structure since $c_1(E)$ is an integral class. A *prequantization* of a symplectic manifold (Γ, ω) is a principal S^1-bundle $\pi : E \to \Gamma$ equipped with a connection θ having curvature ω. (See, e.g., [**B-W**, Definition 7.2, p. 95], [**T-W**, pp. 239-240] and [**K-N**, pp. 305-310].) If the symplectic groupoid (Γ, ω) in Theorem 3.2 is α-simply connected and quantizable, or if it is a reduced fundamental groupoid of a connected quantizable symplectic manifold, then the prequantization bundle E has the connection without holonomy over Γ_0. Therefore it carries a structure of a central S^1-extension of Γ with the Euler es-class corresponding to the Poisson class by [**W-X**, Theorem 3.1, pp. 174-180, Theorem 3.3, pp. 182-184]. From Theorem 3.2 we get immediately:

Corollary 3.3. *If the symplectic groupoid* $((\Gamma,\omega)\rightrightarrows\Gamma_0,\alpha,\beta)$ *is* α*-connected,* α*-simply connected and quantizable, or if it is a reduced fundamental groupoid of a connected* (*not necessarily simply connected*) *quantizable symplectic manifold then there exists a unique central* S^1*-extension* E*, the Euler es-class of which corresponds to the Poisson class of the unit space* Γ_0*.*

If we take $\pi_1(M)$ itself as the subgroup $\hat{\pi}$, the groupoid $\hat{\Pi}(M)\rightrightarrows M$ coincides with the pair groupoid $M\times M\rightrightarrows M$ and if we take $\hat{\pi}=\{1\}\subset\pi_1(M)$, the groupoid $\hat{\Pi}(M)\rightrightarrows M$ coincides with the fundamental groupoid $\Pi(M)\rightrightarrows M$ which is α-simply connected.

Let $\pi:E\to\Gamma$ be a prequantization of a symplectic manifold (Γ,ω) equipped with a connection θ having curvature ω. Since ω is the curvature of the connection θ, we have $d\theta=\pi^*\omega$ and obviously $\theta\wedge(d\theta)^n=\theta\wedge(\pi^*\omega^n)\neq 0$ everywhere for $\dim\Gamma=2n$, that is, θ is a contact form. A *contact structure* on a smooth manifold E is a hyperplane field $\mathcal{H}$ defined by the kernel of local contact form. The manifold E equipped with the contact structure $\mathcal{H}$ is called a *contact manifold* and is denoted by $(E,\mathcal{H})$. If $(E\rightrightarrows E_0,\alpha,\beta)$ is a Lie groupoid, the tangent groupoid of E, $(TE\rightrightarrows TE_0,T\alpha,T\beta)$ is the Lie groupoid with the inverse law $X\mapsto Tj(X)$ for the inversion mapping $j:E\overset{\cong}{\to}E$ and the product law $Tm:(TE)_2=(T\alpha\times T\beta)^{-1}(\text{diagonal}\,(TE_0\times TE_0))\to TE,(X,Y)\mapsto X\oplus Y=Tm(X,Y)$. $(E,\mathcal{H})$ is a *contact groupoid* (see [**D**, p. 437]) if and only if (i) for $X\in\mathcal{H}$, we have $Tj(X)\in\mathcal{H}$, (ii) for $(X,Y)\in(\mathcal{H}\times\mathcal{H})\cap(TE)_2$, we have $X\oplus Y\in\mathcal{H}$. If $\pi:E\to\Gamma$ is a prequantization of the symplectic groupoid $(\Gamma,\omega)\rightrightarrows\Gamma_0$ with a connection 1-form θ without holonomy over Γ_0 such that $d\theta=\pi^*\omega$, then E carries a contact groupoid structure $(E,\mathcal{H})$ with $\mathcal{H}=\ker(\theta)$ by [**W-X**, Theorem 3.1, pp. 174-180]. Therefore we have:

Corollary 3.4. *The central* S^1*-extension* E *of the symplectic groupoid* $((\Gamma,\omega)\rightrightarrows\Gamma_0,\alpha,\beta)$ *in Corollary* 3.3 *is a contact groupoid if* E *corresponds to the Poisson class.*

A contact groupoid structure $(\overline{E},\mathcal{H})$ is obtained on a central S^1-extension from the Poisson manifold Γ_0, by [**D**, Théorème 6.1 (ii), pp. 454-457]. By Theorem 3.2 we get $\overline{E}=E$.

Remark 3.5. We have more strict notions of a contact groupoid by P. Libermann [**L**, p. 39], and Y. Kerbrat and Z. Souici-Benhammadi [**K-SB**, p. 81]. Our contact groupoid E is not only Dazord's but also Libermann's and Kerbrat-Souici-Benhammadi's. The central S^1-extension in Corollary 3.3 is obtained from a prequantization $\pi:E\to\Gamma$ with a connection θ such that E is without holonomy over Γ_0 and it satisfies Libermann's conditions (1), (2) by the proof of [**W-X**, Lemma 3.2, pp. 176-178]. Moreover, it is a contact groupoid $((\Gamma,\omega)\rightrightarrows\Gamma_0,\theta,f)$ of Kerbrat-Souici-Benhammadi with $f=1$, again by [**W-X**, Lemma 3.2, p. 176].

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Received June 13, 2000 and revised April 25, 2001.

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