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This paper gives a complete characterization of the spectra of composition operators acting on H^∞ in the case that the symbol φ has an interior fixed point. This is done after it proves that the essential norm of a composition operator acting on H^∞ is either 1 or 0.

1. Introduction.

Throughout this paper, $\mathbb{D}$ denotes the unit disk $\{z : |z| < 1\}$, φ denotes an analytic self-map of $\mathbb{D}$, C_φ is the composition operator defined by $C_\varphi(f) = f \circ \varphi$, $\|C_\varphi\|_e$ and ρ_e represent the essential norm and essential spectral radius of C_φ respectively, and $H(\mathbb{D})$ is the space of analytic functions on $\mathbb{D}$.

The essential norm of an operator is the distance from the operator to the space of compact operators. The essential norm of composition operator acting on H^2, the Hardy space of analytic functions f on $\mathbb{D}$ such that $\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta < \infty$, was given by J. H. Shapiro in terms of the Nevanlinna counting function [**Sh1**]. The spectrum of composition operator on H^2 has also been studied extensively. Kamowitz was the first to investigate spectrum of composition operator whose symbol is not an inner function and has a fixed point in the disk. He proved in [**Kam**] that the spectrum of C_φ on H^2 is the set $\{\lambda : |\lambda| \le \rho_e\} \cup \{\varphi'(a)^n : n \in \mathbb{N}\} \cup \{1\}$ if φ is analytic in a neighborhood of $\mathbb{D}$, not an inner function and has an interior fixed point a. Then Cowen and MacCluer proved in [**CM1**] the same conclusion in the case that φ is univalent, not an automorphism and has an interior fixed point. But a complete understanding of the spectrum of C_φ on H^2 is still lacking. While much attention has been devoted to the study of H^2, the behavior of C_φ acting on H^∞, the space of bounded analytic functions on $\mathbb{D}$, has barely been discussed. It is the purpose of this paper to investigate some properties of C_φ acting on H^∞.

There are two main results of this paper. One of the results, which is stated as Theorem 1, is that the essential norm of C_φ acting on H^∞ is either 1 or 0. This leads to the corollary that the essential spectral radius of C_φ on H^∞ is also 1 or 0. Then the two theorems described in the previous paragraph about the spectrum of C_φ on H^2, together with this corollary, suggest that if φ has an interior fixed point a, the spectrum of C_φ on H^∞ is $\overline{\mathbb{D}}$ or the sequence $\{0, 1, \varphi'(a)^n : n = 1, 2, \dots\}$. This is the second main result, stated and proved below as Theorem 4. But unlike the theorems about the spectrum of C_φ acting on H^2 previously referred to, this does not require φ to be univalent or analytic in a neighborhood of $\mathbb{D}$. It only requires φ to have an interior fixed point.

2. Essential norm.

The essential norm of C_φ on H^∞ is defined to be

$$\|C_\varphi\|_e = \inf\{\|C_\varphi - K\| : \ K \text{ is compact operator on } H^\infty\}.$$

Clearly C_φ is compact if and only if its essential norm is zero.

The next result shows that there is only one other possible value for the essential norm of a composition operator on H^∞.

Theorem 1. *If C_φ is not compact on H^∞, then its essential norm is* 1.

In order to prove this theorem, we need the following lemma, which was first proved by Schwartz [**Sch**].

Lemma 2. *C_φ is compact on H^∞ if and only if $\varphi(\mathbb{D})$ is relatively compact in $\mathbb{D}$.*

Proof of Theorem 1. We know that C_φ is compact if and only if its essential norm is 0. The main argument here is that the essential norm must be 1 if C_φ is not compact. Since $\|C_\varphi(f)\|_\infty = \sup_{z\in\mathbb{D}} |f \circ \varphi(z)| \le \sup_{z\in\mathbb{D}} |f(z)| = \|f\|_\infty$, $\|C_\varphi\| \le 1$, and hence $\|C_\varphi\|_e \le 1$. It suffices to prove that $\|C_\varphi\|_e \ge 1$ if C_φ is not compact on H^∞.

Now assume C_φ is not compact on H^∞. By Lemma 2, $\sup_{z\in\mathbb{D}} |\varphi(z)| = 1$. There exists a sequence $\{a_k\}_{k=1}^\infty \subset \mathbb{D}$ such that $\varphi(a_k) \to e^{i\beta}$ as $k \to \infty$ for some $\beta \in \mathbb{R}$. Without loss of generality, let's assume $e^{i\beta} = 1$. Let $\{r_n\}_{n=1}^\infty$ be a nonnegative sequence increasing to 1, and

$$\psi_n(z) = \frac{z - r_n}{1 - r_n z}.$$

Then $\|\psi_n\|_\infty = 1$, ψ_n fixes 1 and -1 for all $n \in \mathbb{N}$, and $\psi_n(z) \to -1$ as $n \to \infty$ for all $z \in \mathbb{D}$. Let K be a compact operator on H^∞. We want to show that $\|C_\varphi - K\| \ge 1$. Since K is compact and $\|\psi_n\|_\infty = 1$, there is a subsequence $\{\psi_{n_j}\}_{j=1}^\infty$ and an $f \in H^\infty$ such that $\lim_{j\to\infty} \|K\psi_{n_j} - f\|_\infty = 0$. For $\|C_\varphi - K\| \ge 1$ to be true, it is enough to prove that $\limsup_{j\to\infty} \|(C_\varphi - K)(\psi_{n_j})\|_\infty \ge 1$. But

$$\|(C_\varphi - K)(\psi_{n_j})\|_\infty \ge \|C_\varphi(\psi_{n_j}) - f\|_\infty - \|K\psi_{n_j} - f\|_\infty,$$

which implies

$$\limsup_{j\to\infty} \|(C_\varphi - K)(\psi_{n_j})\|_\infty \geq \limsup_{j\to\infty} \|C_\varphi(\psi_{n_j}) - f\|_\infty.$$

It suffices to prove that $\limsup_{j\to\infty} \|C_\varphi(\psi_{n_j}) - f\|_\infty \geq 1$.

The fact that $\psi_n(z) \to -1$ as $n \to \infty$ for all $z \in \mathbb{D}$ implies that $\psi_{n_j} \circ \varphi(z) \to -1$ as $j \to \infty$, and hence $\lim_{j\to\infty} |\psi_{n_j} \circ \varphi(z) - f(z)| = |-1 - f(z)|$ for all $z \in \mathbb{D}$. If there is $z_0 \in \mathbb{D}$ such that $|-1 - f(z_0)| \geq 1$, we have $\|\psi_{n_j} \circ \varphi - f\|_\infty \geq |\psi_{n_j} \circ \varphi(z_0) - f(z_0)| \to |-1 - f(z_0)| \geq 1$, which implies $\limsup_{j\to\infty} \|\psi_{n_j} \circ \varphi - f\|_\infty \geq 1$ as desired. Otherwise, $|-1-f(z)| < 1$ for all $z \in \mathbb{D}$. Then by the triangle inequality $|1 - f(z)| > 1$ for all $z \in \mathbb{D}$. Consider the sequence $\{a_k\}_{k=1}^\infty \subset \mathbb{D}$ which was obtained at the beginning of the proof. We have $\lim_{k\to\infty} \varphi(a_k) = 1$ and $\{f(a_k)\}_{k=1}^\infty$ is bounded since $f \in H^\infty$. Then there is a subsequence $\{f(a_{k_j})\}_{j=1}^\infty$ converging to some $\omega \in \mathbb{C}$. By re-indexing we may assume, without loss of generality that, $\lim_{k\to\infty} f(a_k) = \omega$. Then by our assumption on f, $|1 - \omega| = \lim_{k\to\infty} |1 - f(a_k)| \geq 1$. Since ψ_n is continuous, $\psi_n(1) = 1$ and $\lim_{k\to\infty} \varphi(a_k) = 1$, it follows that $\lim_{k\to\infty} |\psi_n \circ \varphi(a_k) - f(a_k)| = |1 - \omega| \geq 1$ for all n. Then $\|\psi_{n_j} \circ \varphi - f\|_\infty \geq \lim_{k\to\infty} |\psi_{n_j} \circ \varphi(a_k) - f(a_k)| \geq 1$ for all j. Hence $\limsup_{j\to\infty} \|\psi_{n_j} \circ \varphi - f\|_\infty \geq 1$ as desired.

If $e^{i\beta} \neq 1$, let $\Psi_n(z) = e^{i\beta}\psi_n(e^{-i\beta}z)$. The same proof holds with ψ_n replaced by Ψ_n, and the boundary points 1 and -1 replaced by $e^{i\beta}$ and $-e^{i\beta}$ respectively. This completes the proof of Theorem 1.

For the rest of this paper, φ_n will denote the n^{th} iterate of φ, i.e., $\varphi_1 = \varphi$ and $\varphi_n = \varphi \circ \varphi_{n-1}$ for $n > 1$.

Definition ([**CM2**, p. 150]). If T is a bounded linear operator on a Hilbert space, then the spectrum of the equivalence class in the Calkin algebra that contains T is called the essential spectrum of T.

Corollary 3. *The essential spectral radius of C_φ on H^∞ is either* 1 *or* 0. *If C_{φ_n} $(= C_\varphi^n)$ is compact for some $n \geq 1$, then $\rho_e(C_\varphi) = 0$. Otherwise $\rho_e(C_\varphi) = 1$.*

Proof. The conclusion follows immediately from Theorem 1 and the formula that $\rho_e(C_\varphi) = \lim_{n\to\infty}(\|C_\varphi^n\|_e)^{1/n} = \lim_{n\to\infty}(\|C_{\varphi_n}\|_e)^{1/n}$.

3. Spectrum.

For C_φ acting on H^∞, the spectrum $\sigma(C_\varphi)$ is contained in $\overline{\mathbb{D}}$. This is because the norm of C_φ acting on H^∞ is always 1, which implies the spectral radius $\rho(C_\varphi) = \lim_{n\to\infty} \|C_{\varphi_n}\|^{1/n} = 1$.

Theorem 4. *If φ is not a constant, not an automorphism, and $\varphi(a) = a$ for some $a \in \mathbb{D}$, then*

$$\sigma(C_\varphi) = \overline{\mathbb{D}}, \text{ if } \|\varphi_n\|_\infty = 1 \text{ for all } n \in \mathbb{N}$$

and

$$\sigma(C_\varphi) = \{\varphi'(a)^k : k = 1, 2, \dots\} \cup \{0, 1\}, \text{ if } \|\varphi_n\|_\infty < 1 \text{ for some } n \in \mathbb{N}.$$

The proof of Theorem 4 will be given after some lemmas. Some ideas and approaches used in the proof are suggested by the work of Kamowitz in [**Kam**], and that of Cowen and MacCluer in [**CM1**].

The following lemma, Lemma 5, follows immediately from Koenigs' Theorem [**Koe**] (see also [**Sh2**, Chapter 6]), if we can show that the Koenigs' function ξ of φ is in H^∞ under the assumption of the Lemma. Since $\xi \circ \varphi = \varphi'(a)\xi$, which implies $\xi \circ \varphi_n = \varphi'(a)^n \xi$, and hence $\xi = \varphi'(a)^{-n}\xi(\varphi_n)$ for all $n \in \mathbb{N}$, we conclude that ξ is in H^∞ under the assumption that $\|\varphi_n\|_\infty < 1$ for some $n \in \mathbb{N}$.

Lemma 5. *For φ as in Theorem 4, suppose $\|\varphi_n\|_\infty < 1$ for some $n \in \mathbb{N}$. If $\varphi'(a) \neq 0$, then $\{\varphi'(a)^n, n = 0, 1, 2, \dots\}$ is the set of eigenvalues of C_φ on H^∞. If $\varphi'(a) = 0$, the only eigenvalue of C_φ is 1.*

Lemma 6. *Let φ be the same as in Theorem 4 and suppose $\|\varphi_n\|_\infty < 1$ for some $n \in \mathbb{N}$. Then for C_φ on H^∞,*

$$\sigma(C_\varphi) = \{\varphi'(a)^k : k = 1, 2, \dots\} \cup \{0, 1\}.$$

Proof. Under the hypothesis that $\|\varphi_n\|_\infty < 1$, $C_{\varphi_n} = C_\varphi^n$ is compact, which implies that C_φ is not invertible and hence 0 is in the spectrum. Also, since C_φ^n is compact, C_φ is a Riesz operator. So its nonzero spectrum consists of eigenvalues [**Kön**, p. 19-21], and the result follows from Lemma 5.

Lemma 7. *Suppose $\varphi(0) = 0$. Then $H_m = z^m H^\infty$ is an invariant subspace of C_φ and $\sigma(C_m) \subset \sigma(C_\varphi)$ where $C_m = C_\varphi|_{H_m}$.*

Proof. It's easy to see that H_m is invariant under C_φ. Since $\varphi(0) = 0$, $\varphi(z) = z\phi(z)$ for some $\phi \in H^\infty$. Then if $f \in H^\infty$, $C_\varphi(z^m f) = \varphi^m(f \circ \varphi) \in H_m$.

Suppose λ is in the spectrum of C_m. If λ is an eigenvalue of C_m, it must be an eigenvalue of C_φ and hence in the spectrum of C_φ. If λ is not an eigenvalue of C_m, then $C_m - \lambda I$ is one-one. But it is not invertible, and hence not onto. So there exists $f \in H_m$ with $f \notin (C_m - \lambda I)(H_m)$. If we can show that $C_\varphi - \lambda I$ on H^∞ is not onto, then it will follow that $\lambda \in \sigma(C_\varphi)$ and the conclusion holds. Suppose to the contrary $C_\varphi - \lambda I$ is onto. Then $f \in (C_\varphi - \lambda I)(H^\infty)$ and there is $g \in H^\infty$ with $g \notin H_m$ such that $(C_\varphi - \lambda I)g = f$. Let $g = g_1 + g_2$, where $g_1 \in \operatorname{span}(1, z, z^2, \dots, z^{m-1})$ and $g_2 \in H_m$. We have $g_1 \neq 0$ since $g \notin H_m$. Let $f_1 = (C_\varphi - \lambda I)g_1 = f - (C_\varphi - \lambda I)g_2$. Then $f_1 \in H_m$ since $f, g_2 \in H_m$.

Also by the assumption that $C_\varphi - \lambda I$ is onto, for each function z^i, $i = 1, 2, \dots, m-1$, there exists $h_i \in H^\infty$ such that $(C_\varphi - \lambda I)h_i(z) = z^i$. Let $h_i = k_i + l_i$ where $k_i \in \operatorname{span}(1, z, z^2, \dots, z^{m-1})$ and $l_i \in H_m$. The next step is to show that $g_1, k_0, k_1, \ldots, k_{m-1}$ are linearly independent. Suppose $\beta g_1 + \sum_{i=0}^{m-1} \alpha_i k_i = 0$ for some β and $\alpha_i, i = 0, 1, \ldots, m-1$. Then $\beta g_1 + \sum_{i=0}^{m-1} \alpha_i h_i = (\beta g_1 + \sum_{i=0}^{m-1} \alpha_i k_i) + \sum_{i=0}^{m-1} \alpha_i l_i = \sum_{i=0}^{m-1} \alpha_i l_i \in H_m$. So $(C_\varphi - \lambda I)(\beta g_1(z) + \sum_{i=0}^{m-1} \alpha_i h_i(z)) = \beta f_1(z) + \sum_{i=0}^{m-1} \alpha_i z^i \in H_m$. Since $f_1 \in H_m$, we have $\alpha_i = 0$ for $i = 0, 1, ..., m-1$. Then it follows that $\beta = 0$. So $g_1, k_0, k_1, \ldots, k_{m-1}$ are linearly independent. But this is impossible since $\{g_1, k_0, k_1, \ldots, k_{m-1}\} \subset \mathrm{span}(1, z, \ldots, z^{m-1})$, which is only m dimensional. So $C_\varphi - \lambda I$ is not onto, and hence λ is in the spectrum of C_φ.

Definition. We say the sequence of points $\{z_k\}_{k=K}^{M}$ is an iteration sequence for φ if $\varphi(z_k) = z_{k+1}$ for $K \le k < M$ where $K \ge -\infty$ and $M \le \infty$.

Lemma 8 ([**CM2**, p. 292, Lemma 7.34]). *If φ is not an automorphism and $\varphi(0) = 0$, then given $0 < r < 1$, there exists M_r with $1 \le M_r < \infty$ such that if $\{z_k\}_{-K}^{\infty}$ is an iteration sequence with $|z_l| \ge r$ for some $l \ge 0$ and if $\{w_k\}_{-K}^{l}$ is arbitrary, there is $h \in H^\infty$ such that $h(z_k) = w_k$ for $-K \le k \le l$ and $\|h\|_\infty \le M_r \sup\{|w_k| : -K \le k \le l\}$.*

Lemma 9 ([**CM2**, p. 293, Lemma 7.35]). *For φ in Lemma 8 and $\{z_k\}$ any iteration sequence, there exists $c < 1$ such that $|z_{k+1}|/|z_k| \le c$ whenever $|z_k| \le 0.5$.*

Proof of Theorem 4. The statement in the case that $\|\varphi_n\|_\infty < 1$ for some $n \in \mathbb{N}$ is the result of Lemma 6. Now suppose $\|\varphi_n\|_\infty = 1$ for all $n \in \mathbb{N}$. We want to prove $\sigma(C_\varphi) = \overline{\mathbb{D}}$. If the interior fixed point $a \ne 0$, let $\tau(z) = (a - z)/(1 - \bar{a}z)$ and $\psi = \tau \circ \varphi \circ \tau$. Then $\tau^{-1} = \tau$, $\psi(0) = 0$ and $C_\psi = C_\tau \circ C_\varphi \circ C_\tau = C_\tau \circ C_\varphi \circ C_\tau^{-1}$. C_φ and C_ψ are similar and hence have the same spectrum. So without loss of generality, we can assume $\varphi(0) = 0$.

Since $\sigma(C_\varphi) \subset \overline{\mathbb{D}}$ and $\sigma(C_\varphi)$ is closed, it suffices to prove $\mathbb{D} - \{0\} \subset \sigma(C_\varphi)$. Let $0 \ne \lambda \in \mathbb{D}$, $H_m = z^m H^\infty$, and $C_m = C_\varphi|_{H_m}$. By Lemma 7, it suffices to prove that λ is in the spectrum of C_m for some positive integer m. Since $C_m - \lambda I$ is not onto if and only if $(C_m - \lambda I)^*$ is not bounded from below, it is enough to find a positive integer m with $(C_m - \lambda I)^*$ not bounded from below.

Let M be the constant M_r in Lemma 8 corresponding to r=0.25 and suppose we have an iteration sequence $\{z_k\}_{-K}^{\infty}$ with $K \ge 0$ and $|z_0| > 0.5$ (this sequence will be determined later on). Let $n = \max\{k : |z_k| \ge 0.25\}$. Then $n \ge 0$ and $|z_k| < 0.25 < 0.5$ for all $k \ge n+1$. By Lemma 9, there is a number $c_1 < 1$ so that $|z_{k+1}| \le c_1 |z_k|$ whenever $k \ge n+1$. If $z_n \le 0.5$, the inequality also holds for $k = n$. If $|z_n| > 0.5$, since $|z_{n+1}| < 0.25$, we have $|z_{n+1}| < 0.5|z_n|$. Let $c = \max\{c_1, 0.5\}$. Then $|z_{k+1}| \le c|z_k|$ for all $k \ge n$. It follows that $|z_k| \le c^{k-n}|z_n|$ whenever $k \ge n$. Since $c < 1$, there exists a positive integer m such that $c^m/|\lambda| < 1/(2M+1) < 1$. For this m, $(C_m - \lambda I)^*$ is not bounded from below. To see that, let $\varepsilon > 0$ and we will construct a bounded linear functional L_λ on H_m with $\|(C_m - \lambda I)^* L_\lambda\|/\|L_\lambda\| < \varepsilon$.

Let's define L_λ by $L_\lambda(f) = \sum_{k=-K}^{\infty} \lambda^{-k} f(z_k)$ for $f \in H_m$. We will see that L_λ is well-defined and indeed it is bounded.

For $f \in H_m$, $z^{-m}f(z)$ is analytic and $\|f\|_\infty = \|z^{-m}f(z)\|_\infty$. So

$$|f(z_k)| = |z_k|^m |z_k^{-m} f(z_k)| \le |z_k|^m \|z^{-m} f(z)\|_\infty = |z_k|^m \|f\|_\infty.$$

Then

$$\sum_{k=-K}^{\infty} |\lambda|^{-k} |f(z_k)| \le \|f\|_\infty \sum_{k=-K}^{\infty} |\lambda|^{-k} |z_k|^m = \|f\|_\infty \left(\sum_{k=-K}^{n} |\lambda|^{-k}|z_k|^m + \sum_{k=n+1}^{\infty} |\lambda|^{-k}|z_k|^m \right). \tag{1}$$

For $k > n$, $|z_k| \le c^{k-n}|z_n|$ and so $|z_k|^m \le (c^m)^{k-n}|z_n|^m$. We see that

$$\sum_{k=n+1}^{\infty} |\lambda|^{-k} |z_k|^m \le \sum_{k=n+1}^{\infty} \frac{(c^m)^{k-n}|z_n|^m}{|\lambda|^{k-n}|\lambda|^n} = \frac{|z_n|^m}{|\lambda|^n} \sum_{k=n+1}^{\infty} \left(\frac{c^m}{|\lambda|}\right)^{k-n} < \infty. \tag{2}$$

It follows that $\sum_{k=-K}^{\infty} \lambda^{-k} f(z_k)$ converges. Hence L_λ is well-defined and by (1) it is bounded.

Now let's estimate $\frac{\|(C_m - \lambda I)^* L_\lambda\|}{\|L_\lambda\|}$. For $f \in H_m$,

$$\begin{aligned} \langle f, (C_m - \lambda I)^* L_\lambda \rangle &= \langle (C_m - \lambda I) f, L_\lambda \rangle \\ &= \langle f \circ \varphi - \lambda f, L_\lambda \rangle \\ &= \sum_{k=-K}^{\infty} \lambda^{-k} (f \circ \varphi(z_k) - \lambda f(z_k)) \\ &= \sum_{k=-K}^{\infty} (\lambda^{-k} f(z_{k+1}) - \lambda^{-(k-1)} f(z_k)) \\ &= -\lambda^{K+1} f(z_{-K}). \end{aligned}$$

Then

$$\begin{aligned} \|(C_m - \lambda I)^* L_\lambda\| &= \sup_{0 \ne f \in H_m} \frac{|\langle f, (C_m - \lambda I)^* L_\lambda \rangle|}{\|f\|_\infty} \\ &= \sup_{0 \ne f \in H_m} \frac{|\lambda|^{K+1}|f(z_{-K})|}{\|f\|_\infty} \\ &\le |\lambda|^{K+1}. \end{aligned}$$

We also need a lower bound for $\|L_\lambda\|$. If we apply Lemma 8 to the iteration sequence $\{z_k\}_{-K}^{\infty}$ with r=0.25, we can find a function $h \in H^\infty$ with $\|h\|_\infty \le M$, $|h(z_k)| = 1$ and $\lambda^{-k} z_k^m h(z_k) > 0$ for $-K \le k \le n$. Let $g(z) = z^m h(z) \in$

H_m. Then $\|g\|_\infty \le M$ and

$$\begin{aligned} L_\lambda(g) &= \sum_{k=-K}^{\infty} \lambda^{-k} z_k^m h(z_k) \\ &= \sum_{k=-K}^{n-1} |\lambda|^{-k}|z_k|^m + |\lambda|^{-n}|z_n|^m + \sum_{k=n+1}^{\infty} \lambda^{-k} z_k^m h(z_k). \end{aligned}$$

By the estimate in (2) and because $M \ge 1$ and $c^m/|\lambda| < 1/(2M+1)$ from the choice of m, we have

$$\begin{aligned} \left|\sum_{k=n+1}^{\infty} \lambda^{-k} z_k^m h(z_k)\right| &\le \sum_{k=n+1}^{\infty} |\lambda|^{-k}|z_k|^m |h(z_k)| \\ &\le M|\lambda|^{-n}|z_n|^m \sum_{k=n+1}^{\infty} \left(\frac{c^m}{|\lambda|}\right)^{k-n} \\ &= M|\lambda|^{-n}|z_n|^m \frac{\frac{c^m}{|\lambda|}}{1-\frac{c^m}{|\lambda|}} \\ &\le M|\lambda|^{-n}|z_n|^m \frac{\frac{1}{2M+1}}{1-\frac{1}{2M+1}} \\ &= \frac{1}{2}|\lambda|^{-n}|z_n|^m. \end{aligned}$$

This shows that

$$\begin{aligned} |L_\lambda(g)| &\ge \sum_{k=-K}^{n-1} |\lambda|^{-k}|z_k|^m + |\lambda|^{-n}|z_n|^m - \left|\sum_{k=n+1}^{\infty} \lambda^{-k} z_k^m h(z_k)\right| \\ &\ge \sum_{k=-K}^{n-1} |\lambda|^{-k}|z_k|^m + \frac{1}{2}|\lambda|^{-n}|z_n|^m \ge \frac{1}{2}|z_0|^m. \end{aligned}$$

Then

$$\|L_\lambda\| \ge \frac{|L_\lambda(g)|}{\|g\|_\infty} \ge \frac{|z_0|^m}{2M} \ge \frac{(0.5)^m}{2M}.$$

It follows that

$$\frac{\|(C_m - \lambda I)^* L_\lambda\|}{\|L_\lambda\|} \le \frac{2M|\lambda|^{K+1}}{0.5^m}.$$

Since $|\lambda| < 1$, this is less than ε if we choose K sufficiently large. For the chosen K, we can determine the iteration sequence $\{z_k\}_{-K}^{\infty}$. Since $\|\varphi_K\|_\infty = 1$ by assumption, there exists $w \in \mathbb{D}$ with $|\varphi_K(w)| > 0.5$. Let $z_{-K} = w$ and $z_{k+1} = \varphi(z_k)$ for $k > -K$. Then $|z_0| = |\varphi_K(z_{-K})| > 0.5$. The above calculation follows, thus $(C_m - \lambda I)^*$ is not bounded from below as desired.

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