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# THE HARMONIC FUNCTIONAL CALCULUS AND HYPERREFLEXIVITY

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**A natural $L^\infty$ functional calculus for an absolutely continuous contraction is investigated. It is harmonic in the sense that for such a contraction and any bounded measurable function $\phi$ on the circle, the image can rightly be considered as $\hat{\phi}(T)$, where $\hat{\phi}$ is the solution of the Dirichlet problem for the disk with boundary values $\phi$. The main result shows that if the functional calculus is isometric on $H^\infty$, then it is isometric on all of $L^\infty$. As a consequence we obtain that if the contraction has an isometric $H^\infty$ functional calculus and is in class $C_{00}$, then the range of the harmonic functional calculus is a hyperreflexive subspace of operators. In particular, the space of all Toeplitz operators with a bounded harmonic symbol acting on the Bergman space of the disc is hyperreflexive. Applications of these results to subnormal operators are also presented.**

## 1. Introduction.

Let $\mathcal{H}$ be a complex, separable Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators from $\mathcal{H}$ into itself. Assume $T$ is a contraction in $\mathcal{B}(\mathcal{H})$ that is absolutely continuous. That is, if $T$ has a reducing subspace on which it is unitary, then the spectral measure of this unitary is absolutely continuous. It is a well-known result of Sz.-Nagy [**14**] that $T$ has a unitary dilation. That is, there is a Hilbert space $\mathcal{K}$ that contains $\mathcal{H}$ and a unitary operator $U$ on $\mathcal{K}$ such that $T^n = P_{\mathcal{H}} U^n|\mathcal{H}$ for all $n \geq 0$. (A proof of this can be found in [**10**], p. 200. This book will serve as general background for this paper as will [**7**].) This unitary, moreover, is absolutely continuous. Therefore for any bounded Borel function $\phi$, we can define the operator $\phi(T) \equiv P\phi(U)|\mathcal{H}$. If $L^\infty = L^\infty(\partial\,\mathbb{D})$ is the $L^\infty$-space of Lebesgue measure on the circle, $\partial\,\mathbb{D}$, then this defines a map $\xi : L^\infty \to \mathcal{B}(\mathcal{H})$ given by $\xi(\phi) = \phi(T)$. The properties of this map are summarized below. The weak$^*$ topology on $L^\infty$ referred to in this result is the usual one it has as the Banach space dual of $L^1$; the weak$^*$ topology on $\mathcal{B}(\mathcal{H})$ is the one it has as the dual of the trace class, $\mathcal{B}_1$.

**Theorem 1.1.** *$\xi : L^\infty \to \mathcal{B}(\mathcal{H})$ is a positive linear contraction that is weak* continuous.*

The proof of this theorem is standard and almost immediate from the definition of $\xi$. Note that $\xi$ is not multiplicative unless $T$ is unitary. When this map $\xi$ is restricted to the bounded analytic functions, $H^\infty$, then it is called the Sz.-Nagy-Foias functional calculus for the operator $T$. It is the functional calculus $\xi : L^\infty \to \mathcal{B}(\mathcal{H})$ that is the subject of this paper. Define $\mathcal{T}(T)$ to be the range of $\xi$.

This functional calculus was introduced in a more general setting in [**11**], where many of its properties are deduced and some applications are presented. There is an overlap between [**11**] and the remainder of this section.

The reason that the functional calculus $\xi$ is called the "harmonic functional calculus" is the following. If $p$ and $q$ are analytic polynomials, then $\xi(p + \bar{q}) = p(T) + q(T)^*$. Now $p + \bar{q}$ is the typical trigonometric polynomial. If $\phi \in L^\infty$, then there is a sequence of trigonometric polynomials $\{f_n\}$ such that $\|f_n\|_\infty \leq \|\phi\|_\infty$ and $f_n \to \phi$ weak* in $L^\infty$. Indeed, one can take the $f_n$ to be the Césaro sums of the Fourier series of $\phi$. It follows that $f_n(T) \to \phi(T)$ weak* in $\mathcal{B}(\mathcal{H})$. But on the open disk the trigonometric polynomials $\{f_n\}$ converge uniformly on compact subsets of $\mathbb{D}$ to $\widehat{\phi}$, the harmonic extension of $\phi$. This is more than a slight of hand. Indeed, as will be seen below, in the case of many operators such as the Bergman shift, $\phi(T)$ can be equivalently defined in terms of $\widehat{\phi}$. A further, more explicit connection with harmonic functions can be seen by a consideration of absolutely continuous contractions that are normal operators.

Let $N$ be a normal operator on $\mathcal{H}$ that is an absolutely continuous contraction and let $N = \int z\, dE(z)$ be its spectral decomposition. For vectors $x$ and $y$ in $\mathcal{H}$, let $\mu_{x,y}$ be the measure defined on $\operatorname{cl}\mathbb{D}$ by $\mu_{x,y}(\Delta) = \langle E(\Delta)x, y\rangle$. Denote by $\widetilde{\mu}_{x,y}$ the sweep of $\mu_{x,y}$ to $\partial\mathbb{D}$. (See p. 311 of [**9**].) It follows that $\mu_{x,y}$ is absolutely continuous on the circle and for every $\phi$ in $L^\infty$,

$$\int_{\partial\mathbb{D}} \phi\, d\widetilde{\mu}_{x,y} = \int_{\operatorname{cl}\mathbb{D}} \widehat{\phi}\, d\mu_{x,y} = \langle \widehat{\phi}(N)x, y\rangle.$$

In particular, $P(\Delta) = \widehat{\chi}_\Delta(N)$ defines a positive operator-valued measure on the circle with $P(\partial\mathbb{D}) = 1$. Call $P$ the *sweep of the spectral measure $E$*. By the Naimark Dilation Theorem (see p. 197 of [**10**]), there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a spectral measure $F$ on the circle with values in $\mathcal{B}(\mathcal{K})$ such that $P(\Delta) = P_\mathcal{H} F(\Delta)|\mathcal{H}$ for every Borel subset $\Delta$ of $\partial\mathbb{D}$.

**Proposition 1.2.** *Let $N$ be a normal operator that is an absolutely continuous contraction and let $N = \int z\, dE(z)$ be its spectral decomposition. If $P$ is the sweep of the spectral measure $E$, $F$ is the minimal spectral measure with values on $\mathcal{B}(\mathcal{K})$ that dilates $P$, $\mathcal{H} \subseteq \mathcal{K}$, then $U = \int z\, dF(z)$ is the minimal unitary dilation of $N$.*

*Proof.* Using the notation that preceded the statement of the proposition, if $x, y \in \mathcal{H}$ and $n \geq 0$, then $\langle U^n x, y\rangle = \int z^n \, d\langle F(z)x, y\rangle = \int z^n \, d\widetilde{\mu}_{x,y} = \langle N^n x, y\rangle$, since $z^n$ is harmonic. Thus $U$ is a unitary dilation. Now to show that it is minimal.

Let $\mathcal{L}$ be the closed linear span of $\{U^n\mathcal{H} : n \in \mathbb{Z}\}$. To show the minimality of $U$ it must be shown that $\mathcal{L} = \mathcal{K}$. But $\mathcal{L}$ is clearly a reducing subspace for the unitary operator $U$, so $F(\Delta)\mathcal{L} \subseteq \mathcal{L}$ for every Borel subset $\Delta$ of $\partial\,\mathbb{D}$. Now the fact that $F$ is the minimal spectral measure that dilates $P$ implies that $\mathcal{L} = \mathcal{K}$ . $\square$

The next result is a corollary of the preceding proposition, but it can also be proved directly.

**Corollary 1.3.** *If $N$ is a normal operator that is an absolutely continuous contraction, then the functional calculus $\xi : L^\infty \to \mathcal{B}(\mathcal{H})$ is given by*

$$\xi(\phi) = \widehat{\phi}(N) = \int \widehat{\phi}\, dE,$$

*where $\widehat{\phi}$ is the solution of the Dirichlet problem with boundary values $\phi$.*

**Corollary 1.4.** *If $S$ is a subnormal, absolutely continuous contraction on $\mathcal{H}$ with minimal normal extension $N$ on $\mathcal{K}$, then $\xi(\phi) = P_{\mathcal{H}}\widehat{\phi}(N)|\mathcal{H}$ for every $\phi$ in $L^\infty$.*

In the case of a subnormal operator $S$ as described in the preceding corollary, there is a richer linear functional calculus. If $\mu$ is a scalar-valued spectral measure for $N$, then we can define $\phi(S) = P_{\mathcal{H}}\phi(N)$ for every $\phi$ in $L^\infty(\mu)$. These operators $\{\phi(S) : \phi \in L^\infty(\mu)\}$ are sometimes called the Toeplitz operators associated with the subnormal operator $S$. Indeed, if $S$ is the unilateral shift, then these are the classical Toeplitz operators. Extending results from the classical case to this more general setting seems almost hopeless. There is evidence, however, that some results will extend from the classical case to the Toeplitz operators with a harmonic symbol.

## 2. The functional calculus.

The key to the proof of the main result of this paper is a result of Tomiyama and Yabuta [**17**], which we state here for reference. Recall the definition of a uniform algebra and its Shilov boundary.

**Proposition 2.1.** *Let $\eta : C(X) \to \mathcal{B}(\mathcal{H})$ be a contractive linear representation such that $\eta(1) = 1$, and let $A$ be a uniform algebra on $X$. If $\eta$ is isometric on $A$ and $X$ is a Shilov boundary of $A$, then $\eta$ is an isometry on $C(X)$.*

The main result of the paper is the following:

**Theorem 2.2.** *If $T$ is an absolutely continuous contraction and if the functional calculus $\xi : L^\infty \to \mathcal{B}(\mathcal{H})$ is isometric on $H^\infty$, then $\xi$ is isometric on $L^\infty$, $\mathcal{T}(T)$ is weak* closed, and $\xi$ is a weak* homeomorphism from $L^\infty$ onto $\mathcal{T}(T)$.*

*Proof.* Let $X$ be the maximal ideal space of $L^\infty$ and let $\gamma_L : L^\infty \to C(X)$ be the Gelfand map; so $\gamma_L$ is an isometric isomorphism. We will consider $\xi \circ \gamma_L^{-1} : C(X) \to \mathcal{T}(T)$ and show that it is an isometry, which will complete the proof. This is done by applying Proposition 2.1 to $A = \gamma_L(H^\infty)$. Before showing that the assumptions of Proposition 2.1 are fulfilled, we need some notation.

Let $\mathfrak{M}$ denote the maximal ideal space of $H^\infty$ and let $\gamma_H : H^\infty \to C(\mathfrak{M})$ be the Gelfand map. Define $\rho : X \to \mathfrak{M}$ to be the restriction map, $\rho(\alpha) = \alpha | H^\infty$. By [**12**], p. 174, $\rho$ is a homeomorphism of $X$ onto $\rho(X)$ and $\rho(X)$ is the Shilov boundary of $H^\infty$.

Note that the diagram

$$\begin{array}{ccc} L^\infty & \xrightarrow{\gamma_L} & C(X) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \rho_*} \\ H^\infty & \xrightarrow{\gamma_H} & C(\mathfrak{M}) \end{array}$$

is commutative, where $i$ is the inclusion map and $\rho_*(g) = g \circ \rho$ for $g$ in $C(\mathfrak{M})$. Indeed, for $\phi$ in $H^\infty$ and $\alpha$ in $X$, $(\rho_* \circ \gamma_H(\phi))(\alpha) = \gamma_H(\phi)(\rho(\alpha)) = \rho(\alpha)(\phi) = \alpha(\phi) = \gamma_L(\phi)(\alpha)$.

It is clear that $A$ is norm closed and $1 \in A$. To show that $A$ is a uniform algebra in $C(X)$, it remains to show that $A$ separates the points of $X$. Let $\alpha_1, \alpha_2 \in X$, and assume that $\gamma_L(\phi)(\alpha_1) = \gamma_L(\phi)(\alpha_2)$ for all $\phi$ in $H^\infty$. Since the diagram commutes, this implies that $\rho(\alpha_1)(\phi) = \gamma_H(\phi) \circ \rho(\alpha_1) = \gamma_H(\phi) \circ \rho(\alpha_2) = \rho(\alpha_2)(\phi)$. Thus, by the definition of a homomorphism, $\rho(\alpha_1) = \rho(\alpha_2)$. But $\rho$ is injective, so $\alpha_1 = \alpha_2$. Hence $A$ is a uniform algebra.

Note that $X$ is a boundary for $A \subset C(X)$. If $F$ is a proper closed subset of $X$, then $\rho(F)$ is a proper closed subset of the Shilov boundary of $H^\infty$, $\rho(X)$. Hence there is $\phi$ in $H^\infty$ such that $1 = \|\gamma_H(\phi)\| = \|\gamma_H(\phi)\|_{\rho(X)}$ and $\|\gamma_H(\phi)\|_{\rho(F)} < 1$. If $f = \gamma_L(\phi) \in A$, then, for $\alpha$ in $X$,

$$|f(\alpha)| = |\gamma_L(\phi)(\alpha)| = |(\rho_* \circ \gamma_H(\phi))(\alpha)| = |\gamma_H(\phi)(\rho(\alpha))|.$$

Hence $\|f\| = 1$, but $|f(\alpha)| < 1$ for $\alpha \in F$. Thus $F$ cannot be a boundary for $A$ and so $X$ is the Shilov boundary for $A$. By Proposition 2.1, $\xi$ is an isometry.

Since $\xi$ is weak* continuous (1.1), the fact that $\mathcal{T}(T) = \xi(L^\infty)$ is weak* closed and that $\xi$ is a weak* homeomorphism from $L^\infty$ onto $\mathcal{T}(T)$ is a standard consequence of the Krein-Smulian Theorem. (For example, see [**8**], Proposition I.2.7.) □

Some special cases of this result have appeared in the literature. In [**13**] this theorem is shown for a class of weighted Bergman operators on the disk (Theorem 10), though the proof there is not correct.

The standard terminology is that $\mathbb{A}$ is the set of all absolutely continuous contractions for which the $H^\infty$ functional calculus is isometric. See [**5**]. So the preceding theorem says that every contraction in class $\mathbb{A}$ has an isometric harmonic functional calculus.

This section concludes with an application of this result to all subnormal operators. Let $S$ be a subnormal operator on $\mathcal{H}$ and let $N$ be its minimal normal extension acting on $\mathcal{K}$. If $\mu$ is a scalar-valued spectral measure for $N$, then a natural multiplicative, functional calculus for $S$ is $\phi \to \phi(S) = \phi(N)|\mathcal{H}$ for $\phi$ in $P^\infty(\mu)$, the weak$^*$ closure of the polynomials in $L^\infty(\mu)$. By a result of Sarason [**16**], there is a "special" open set $G$ and a decomposition $\mu = \mu_\infty + \mu_0$, $\mu_0 \perp \mu_\infty$, such that $P^\infty(\mu) = L^\infty(\mu_0) \oplus H^\infty(G, \mu_\infty)$, where $H^\infty(G, \mu_\infty)$ is an isometric, weak$^*$ homeomorphic embedding of $H^\infty(G)$ onto $P^\infty(\mu_\infty) \subseteq L^\infty(\mu_\infty)$. (Also see [**8**], p. 301.) The open set $G$ is called the *Sarason hull* of $\mu$.

The next result gathers information about this. The reader can consult [**8**] for details.

**Theorem 2.3.** *Let $S$ be a subnormal operator on $\mathcal{H}$ with minimal normal extension $N$ acting on $\mathcal{K}$ and scalar-valued spectral measure $\mu$. If $G$ is the Sarason hull of $\mu$ and $G_1, G_2, \dots$ are its components, then there is a decomposition of $\mu$, $\mu = \mu_0 + \mu_1 + \cdots$, where $\mu_n \perp \mu_m$ for $n \neq m$, such that the following hold:*

(a) $P^\infty(\mu) = L^\infty(\mu_0) \oplus P^\infty(\mu_1) \oplus \cdots$ ;
(b) *the polynomials are weak$^*$ dense in $L^\infty(\mu_0)$;*
(c) *for $n \geq 1$, the identity map on polynomials extends to an isometric, weak$^*$ homeomorphism of $P^\infty(\mu_n)$ onto $H^\infty(G_n)$;*
(d) *there is a corresponding decomposition of the Hilbert space $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots$, where each $\mathcal{H}_n$ reduces $S$ and, for $n \geq 0$, the weak$^*$ closed algebra generated by $S_n \equiv S|\mathcal{H}_n$ and the identity is precisely $\{f(S)|\mathcal{H}_n : f \in P^\infty(\mu_n)\} = \{f(S_n) : f \in P^\infty(\mu_n)\}$;*
(e) *for $n \geq 1$, the set $G_n$ is simply connected and if $\tau : G_n \to \mathbb{D}$ is a Riemann map, then $\tau$ is a weak$^*$ generator of $H^\infty(G_n)$ as an algebra and $\tau^{-1}$ is a weak$^*$ generator of $H^\infty(\mathbb{D})$ as an algebra;*
(f) *for $n \geq 1$, $\mu_n$ is supported on $\operatorname{cl} G_n$ and $\mu_n|\partial G_n$ is absolutely continuous with respect to harmonic measure for $G_n$;*
(g) *the Riemann map $\tau : G_n \to \mathbb{D}$ extends to a subset of $\partial G_n$ having full harmonic measure and on this set is a one-to-one, measurable map onto a subset of $\partial \mathbb{D}$ having full measure such that its inverse is also measurable.*

**Remarks.**

1. Condition (b) of this theorem says that $S_0$ is a reductive normal operator.
2. The statement in (e) means that the polynomials in $\tau$ are weak* dense in $H^\infty(G_n)$ and polynomials in $\tau^{-1}$ are weak* dense in $H^\infty(\mathbb{D})$.
3. In light of (c) no distinction between $P^\infty(\mu_n)$ and $H^\infty(G_n)$ will be made. That is, when a function $\phi$ in $P^\infty(\mu_n)$ is considered, it will be assumed that $\phi$ is a bounded analytic function on $G_n$.
4. Using the results from the statement of Theorem 2.3 it can be shown that $G$ is the set of all $\lambda$ in $\mathbb{C}$ such that the map $p \to p(\lambda)$ defined on polynomials extends to a weak* continuous homomorphism from $P^\infty(\mu)$ into $\mathbb{C}$.
5. If $L^\infty(\partial G_n)$ denotes the $L^\infty$ space of harmonic measure for $G_n$, then (g) says that the restriction of $\tau$ to $\partial G_n$ induces an isometric isomorphism and a weak* homeomorphism of $L^\infty(\partial\,\mathbb{D})$ onto $L^\infty(\partial G_n)$.

As mentioned after Corollary 1.4, for a subnormal operator $S$ there is an additional linear functional calculus. Using the preceding notation, for every $\phi$ in $L^\infty(\mu)$, define $\phi(S) \equiv P_{\mathcal{H}}\phi(N)$. In the proposition below the notation of Theorem 2.3 is used. If $\phi \in L^\infty(\partial G_n)$ and $\widehat{\phi}$ is the solution of the Dirichlet problem for $G_n$ with boundary values $\phi$, then $\widehat{\phi}$ can be considered as an element of $L^\infty(\mu_n)$ by letting $\widehat{\phi}$ be itself on $G_n$ while being $\phi$ on $\partial G_n$. Note that this makes sense in light of the fact that $\mu_n|\partial G_n$ is absolutely continuous with respect to harmonic measure. Also note that $\|\widehat{\phi}\|_{L^\infty(\mu_n)} = \|\widehat{\phi}\|_{G_n}$.

**Proposition 2.4.** *Let $S$ be a subnormal operator and adopt the notation in Theorem 2.3. If $\phi \in L^\infty(\partial G_n)$, let $\widehat{\phi}$ denote the solution of the Dirichlet problem for $G_n$ with boundary values $\phi$. If $\rho : L^\infty(\mu_0) \oplus L^\infty(\partial G_1) \oplus L^\infty(\partial G_2) \oplus \cdots \to \mathcal{B}(\mathcal{H})$ is defined by*

$$\rho(\phi_0 \oplus \phi_1 \oplus \phi_2 \oplus \cdots) = \phi_0(S_0) \oplus \widehat{\phi}_1(S_1) \oplus \widehat{\phi}_2(S_2) \oplus \cdots ,$$

*then $\rho$ is a linear isometry, its range is weak* closed, and $\rho$ is a weak* homeomorphism onto its image.*

*Proof.* Since $S_0$ is normal, it suffices to show that for $n \geq 1$ and $\phi$ in $L^\infty(\partial G_n)$, $\|\widehat{\phi}(S_n)\| = \|\widehat{\phi}\|_{G_n} \equiv \sup\{|\widehat{\phi}(z)| : z \in G_n\}$. If $\tau : G_n \to \mathbb{D}$ is a Riemann map, then $\tau(S_n)$ is an absolutely continuous contraction for which the $H^\infty$ functional calculus is isometric. Indeed, for $f$ in $H^\infty$, $f \circ \tau \in H^\infty(G_n)$. Therefore by Theorem 2.3 we have that $\|f(\tau(S_n))\| = \|f \circ \tau(S_n)\| = \|f \circ \tau\|_{G_n} = \|f\|_{\mathbb{D}}$. By Theorem 2.2, for every bounded harmonic function $u$ on the disk, $\|u(\tau(S_n))\| = \|u\|_{\mathbb{D}}$. Taking $u = \widehat{\phi} \circ \tau^{-1}$ implies that $\|\widehat{\phi}(S_n)\| = \|(\widehat{\phi} \circ \tau^{-1})(\tau(S))\| = \|\widehat{\phi} \circ \tau^{-1}\|_{\mathbb{D}} = \|\widehat{\phi}\|_{G_n}$.

Again, the rest of the proposition follows from Proposition 2.7 of [8]. $\square$

## 3. Hyperreflexivity.

Recall that a subspace $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ is said to be *reflexive* if every operator $B$ in $\mathcal{B}(\mathcal{H})$ satisfying $Bx \in \mathrm{cl}\,[\mathcal{M}x]$ for all $x$ in $\mathcal{H}$ necessarily belongs to $\mathcal{M}$. For any $B$ in $\mathcal{B}(\mathcal{H})$, let $\mathrm{dist}\,(B, \mathcal{M})$ denote the usual distance from $B$ to $\mathcal{M}$ in $\mathcal{B}(\mathcal{H})$, and let

$$\alpha(B, \mathcal{M}) = \sup\{\|Q^{\perp}BP\| : \ P, Q \text{ projections with } Q^{\perp}\mathcal{M}P = (0)\}.$$

The linear space $\mathcal{M}$ is called *hyperreflexive* if there is a constant $C > 0$ such that $\mathrm{dist}\,(B.\mathcal{M}) \le C\,\alpha(B, \mathcal{M})$ for every $B$ in $\mathcal{B}(\mathcal{H})$. The smallest constant $C$ is called the *hyperreflexive constant* and is denoted by $\kappa(\mathcal{M})$. (See [**10**], Chapter 8, for the elementary properties of reflexive and hyperreflexive subspaces.) It is straightforward that hyperreflexive subspaces are reflexive, and reflexive spaces are weakly (WOT) closed.

If $S$ is the unilateral shift on the Hardy space $H^2$, it was shown in [**1**] (also see [**10**], Prop. 56.8) that the space $\mathcal{T}(S)$ is far from being reflexive, though it is WOT closed. Indeed, it is transitive. However, $\mathcal{T}(S)$ contains many reflexive subspaces, for example $\mathcal{A}(S)$, the weakly closed algebra generated by $S$. All the reflexive subspaces of $\mathcal{T}(S)$ were characterized in [**1**], where it is also shown that every weak* closed subspace of $\mathcal{T}(S)$ is either reflexive or transitive. Below we will show that $\mathcal{T}(T)$ is hyperreflexive for all $C_{00}$ contractions $T$ in the class $\mathbb{A}$.

Recall that $\mathcal{B}(\mathcal{H})$ is the Banach space dual of the trace class $\mathcal{B}_1$. If $\mathcal{M}$ is a linear manifold in $\mathcal{B}(\mathcal{H})$, then the preannihilator of $\mathcal{M}$ is the space of weak* continuous linear functionals ${}_{\perp}\mathcal{M} \equiv \{L \in \mathcal{B}_1 : L(\mathcal{M}) = (0)\}$. Let $\mathcal{Q}_{\mathcal{M}} = \mathcal{B}_1/{}_{\perp}\mathcal{M}$. So the Banach space dual of $\mathcal{Q}_{\mathcal{M}}$ is the weak* closure of $\mathcal{M}$. For any $L$ in $\mathcal{B}_1$, $[L] = [L]_{\mathcal{M}}$ denotes the coset in $\mathcal{Q}_{\mathcal{M}}$.

Now consider an absolutely continuous contraction $T$ and the subspace $\mathcal{T}(T)$. Since $\xi : L^{\infty} \to \mathcal{T}(T)$ is weak* continuous, there is a bounded linear map $\theta : \mathcal{Q}_{\mathcal{T}(T)} \to L^1$such that $\xi = \theta^*$. Thus for every $L$ in $\mathcal{B}_1$ and $\phi$ in $L^{\infty}$, $\int \phi\theta([L])\,dm = \langle [L], \xi(\phi)\rangle = \mathrm{tr}\,(L\xi(\phi))$. If $x$ and $y$ are vectors in $\mathcal{H}$, $x \otimes y$ is the rank one operator on $\mathcal{H}$ given by $(x \otimes y)(h) = \langle h, y\rangle x$. Using the notation of [**6**], let $x \overset{T}{\cdot} y \equiv \theta(x \otimes y)$. So for every $\phi$ in $L^{\infty}$,

$$\mathrm{tr}\,[\xi(\phi)(x \otimes y)] = \int_{\partial\mathbb{D}} \phi\left(x \overset{T}{\cdot} y\right)\,dm.$$

Denote by $\mathcal{X}_0(\mathcal{M})$ the set of all $[L]$ in $\mathcal{Q}_{\mathcal{M}}$ such that there exist sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ with $\|x_n\| \le 1$ and $\|y_n\| \le 1$ for all $n$ such that

$$\textbf{3.1}. \qquad \begin{cases} \text{(a)} \lim_{n\to\infty} \|[x_n \otimes y_n] - [L]\| = 0, & \\ \text{(b)} \lim_{n\to\infty} \|[x_n \otimes w]\| = 0 & \text{for all} \ \ w \in \mathcal{H}, \\ \text{(c)} \lim_{n\to\infty} \|[w \otimes y_n]\| = 0 & \text{for all} \ \ w \in \mathcal{H}. \end{cases}$$

Say that $\mathcal{M}$ has *property* $X_{0,1}$ if the unit ball of $\mathcal{Q}_{\mathcal{M}}$ is contained in the closed convex hull of $\mathcal{X}_0(\mathcal{M})$. (It was shown in [**5**] that $\mathcal{X}_0(\mathcal{M})$ is in fact absolutely convex and closed.)

**Theorem 3.2.** *If $T$ is a $C_{00}$ contraction in the class $\mathbb{A}$, then $\mathcal{T}(T)$ is hyperreflexive with constant at most* 3. *Moreover, each weak* closed subspace of $\mathcal{T}(T)$ is hyperreflexive.*

*Proof.* We will show that $\mathcal{T}(T)$ has property $X_{0,1}$. By Theorem 3.1 of [**4**], this proves the hyperreflexivity of $\mathcal{T}(T)$ with the constant 3. The last statement of the theorem follows from hereditary behavior of the property $X_{0,1}$. Note also that the weak* and weak operator topologies coincide. (See [**2**], Theorem 2.)

By [**6**], Lemma 4.2, the fact that $T$ is a $C_{00}$ operator implies that the following is satisfied.

**Condition 3.3.** For any $f$ in $L^1$ with $\|f\|_1 \leq 1$ there are sequences $\{x_n\}$, $\{y_n\}$ in $\mathcal{H}$ with $\|x_n\| \leq 1, \|y_n\| \leq 1$ such that for every $w$ in $\mathcal{H}$,

$$\left\| f - x_n \overset{T}{\cdot} y_n \right\|_1 \to 0$$

$$\left\| x_n \overset{T}{\cdot} w \right\|_1 + \left\| w \overset{T}{\cdot} y_n \right\|_1 \to 0.$$

Let $[L]$ be any weak* continuous linear functional on $\mathcal{T}(T)$ with $\|[L]\| \leq 1$. Applying (3.3) to $\theta([L])$ and using the hypothesis that $\xi$, and hence $\theta$, is an isometry, we get that

$$\|[L] - [x_n \otimes y_n]\| \;= \left\| \theta([L]) - x_n \overset{T}{\cdot} y_n \right\|_1 \to 0,$$

Similarly

$$\|[x_n \otimes w]\| + \|[w \otimes y_n]\| \to 0 \;\text{ for every } w \text{ in } \mathcal{H}.$$

This implies that ball $\mathcal{Q}_{\mathcal{T}(T)} \subseteq \mathcal{X}_{0,1}$, so that $\mathcal{T}(T)$ has property $X_{0,1}$. □

**Remark.** Theorem 3.2 remains true if we assume that $T$ is an absolutely continuous contraction in the class $\mathbb{A}$ and Condition (3.3) is fulfilled.

This theorem has application to a large collection of subnormal operators. Once again the notation introduced in connection with Theorem 2.3 is used. Note that if $S$ is a pure subnormal operator, then the reductive normal summand in (2.3), $S_0$, is not present. For an open set $\Omega$, $h^\infty(\Omega)$ denotes the bounded harmonic functions on $\Omega$.

**Theorem 3.4.** *Let $S$ be a pure subnormal operator and adopt the notation of Theorem* 2.3 *and Proposition* 2.4. *If $\mu_n(\partial G_n) = 0$ for all $n \geq 1$, then*

$$\mathcal{T} = \{\widehat{\phi}_1(S_1) \oplus \widehat{\phi}_2(S_2) \oplus \cdots : \phi_n \in L^\infty(\partial G_n) \;\; \textit{for} \;\; n \geq 1\}$$

*is hyperreflexive with constant at most* 3*. Moreover each weak* closed subspace of* $\mathcal{T}$ *is hyperreflexive with constant at most* 3*.*

*Proof.* Let $\tau_n : G_n \to \mathbb{D}$ be a Riemann map. Recall that $\tau_n(S_n)$ is a contraction. Put $\mathcal{T}_n = \{u(S_n) : u \in h^\infty(G_n)\}$ and $\mathcal{M}_n = \mathcal{T}(\tau_n(S_n))$.

**Claim.** $\mathcal{T}_n = \mathcal{M}_n$ for all $n \geq 1$.

Indeed, $\mathcal{M}_n = \{v(\tau_n(S_n)) : v \in h^\infty(\mathbb{D})\}$. Since $h^\infty(G_n) = \{v \circ \tau_n : v \in h^\infty(\mathbb{D})\}$, the claim is clearly true.

Theorem 2.3 (g) implies that the scalar-valued spectral measure for the subnormal operator $\tau_n(S_n)$ is $\mu_n \circ \tau_n^{-1}$. Thus the assumption that $\mu_n(\partial G_n) = 0$ implies that $\tau_n(S_n)$ is a $\mathrm{C}_{00}$ contraction. Hence Theorem 3.2 implies that $\mathcal{T}_n$ is hyperreflexive. More importantly for this proof, Condition (3.3) is satisfied.

Let $\mathcal{Q}_n$ be the predual of $\mathcal{T}_n$. Since $\mathcal{T} = \bigoplus_n \mathcal{T}_n$, the predual of $\mathcal{T}$ is $\mathcal{Q} = \bigoplus_n \mathcal{Q}_n$, where this direct sum is an $\ell^1$ direct sum. That is, $\|[L]\| = \sum_n \|[L_n]\|$ for all $[L] = \oplus_n [L_n]$ in $\bigoplus_n \mathcal{Q}_n$. Let $\xi_n : L^\infty \to \mathcal{T}_n$ be the harmonic functional calculus for the contraction $T_n = \tau_n(S_n)$, $\xi_n(\phi) = \widehat{\phi}(T_n)$, and let $\theta_n : \mathcal{Q}_n \to L^1$ be its predual.

Let $[L] \in \mathcal{Q}$, $\|[L]\| \leq 1$, let $X$ be a finite subset of $\mathcal{H}$, and let $\epsilon > 0$; let $X_n$ be the set of $n$-th coordinates of the vectors in $X$. Using the fact that each $\theta_n$ is an isometry, (3.3), when applied to $\mathcal{T}_n$, implies there are vectors $x_n, y_n$ in $\mathcal{H}_n$ with $\|x_n\|, \|y_n\| < \|[L_n]\|^{1/2}$ satisfying

$$\|[L_n] - [x_n \otimes y_n]\| < 2^{-n}\epsilon$$
$$\|[x_n \otimes w_n]\| + \|[w_n \otimes y_n]\| < 2^{-n}\epsilon$$

for all $w_n$ in $X_n$. Put $x = \oplus_n x_n$ and $y = \oplus_n y_n$. So $\|x\|, \|y\| \leq 1$. The reader can verify that as elements of $\mathcal{Q} = \bigoplus_n \mathcal{Q}_n$, $[x \otimes y]_{\mathcal{T}} = \oplus_n [x_n \otimes y_n]_{\mathcal{T}_n}$. Thus

$$\|[L] - [x \otimes y]\| = \sum_n \|[L_n] - [x_n \otimes y_n]\| < \epsilon.$$

Similarly, for $w = \oplus w_n$ in $X$, $\|[x \otimes w]\| = \sum_n \|[x_n \otimes w_n]\| < \epsilon$ and $\|[w \otimes y]\| < \epsilon$.

Thus $\mathcal{T}$ has property $X_{0,1}$. By Theorem 3.1 of [4], $\mathcal{T}$ is hyperreflexive with constant at most 3. As mentioned in the proof of Theorem 3.2, the last statement of the theorem follows from hereditary behavior of the property $X_{0,1}$. □

It is worth singling out the Bergman operators. For a bounded open set $\Omega$ in the plane, $L^2_a(\Omega)$ is the Bergman space of all analytic functions that are square integrable with respect to area measure on $\Omega$. The Bergman operator for $\Omega$ is the operator $S$ defined on $L^2_a(\Omega)$ by $(Sf)(z) = zf(z)$. This is a subnormal operator, and, as for the general subnormal operator, we can define $\phi(S)f \equiv P(\phi f)$ for $f$ in $L^2_a(\Omega)$ and $\phi$ in $L^\infty(G)$. The next corollary is a direct consequence of the preceding theorem.

**Corollary 3.5.** *If $S$ is the Bergman operator for the bounded open set $\Omega$ and $\mathcal{T}$ is the weak* closure in $\mathcal{B}(L^2_a(\Omega))$ of*

$$\{p(S) + q(S)^* : p \;\; \text{and} \;\; q \;\; \text{are analytic polynomials}\},$$

*then every weak* closed subspace of $\mathcal{T}$ is hyperreflexive with constant at most* 3.

The next result follows from the preceding corollary or directly from Theorem 3.2. Indeed, it was the attempt to investigate this example that led the authors to the results that are contained in this paper.

**Corollary 3.6.** *If $S$ is the Bergman operator for the unit disk $\mathbb{D}$, then every weak* closed subspace of*

$$\mathcal{T} = \{u(S) : u \in h^\infty(\mathbb{D})\}$$

*is hyperreflexive with constant at most* 3.

These results raise additional questions. The problem of whether the general subnormal operator is hyperreflexive remains open. But if this is settled, especially if it is settled affirmatively, there will be the question of which subnormal operators $S$ have the property that $\mathcal{T}(S)$, the weak* closure of $\{p(S) + q(S)^* : p, q$ are analytic polynomials$\}$, is reflexive or hyperreflexive. A rich collection of subnormal operators having these properties is shown to exist in this paper. On the other hand, the unilateral shift has neither. An interesting example to explore would be the operator defined as multiplication by the independent variable on $P^2(\mu)$, where $\mu$ equals area measure on the bottom half of the unit disk and arc length measure on the top half of the unit circle.

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