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A LIOUVILLE TYPE THEOREM FOR SEMILINEAR ELLIPTIC SYSTEMS

Tomomitsu Teramoto and Hiroyuki Usami

Dedicated to Professor Masayuki Itô on his 60th birthday

This paper treats the second order semilinear elliptic systems of the form

$$\boldsymbol{\Delta u = p(x)v^{\alpha}, \quad \Delta v = q(x)u^{\beta}, \quad x \in \mathbf{R}^N,}$$

where $\boldsymbol{\alpha,\ \beta > 0}$ are constants satisfying $\boldsymbol{\alpha\beta > 1}$, and $\boldsymbol{p, q \in C(\mathbf{R}^N; (0,\infty))}$. We obtain a Liouville type theorem for nonnegative entire solutions of this system.

1. Introduction and statement of the result.

In this paper we consider second order semilinear elliptic systems of the form

$$\left\{ \begin{array}{l} \Delta u = p(x)v^{\alpha} \\ \\ \Delta v = q(x)u^{\beta} \end{array} \right. \quad x \in \mathbf{R}^N, \tag{1}$$

where $N \geq 3$, $\alpha > 0$ and $\beta > 0$ are constants satisfying $\alpha\beta > 1$, and $p,\ q \in C(\mathbf{R}^N; (0,\infty))$. An *entire* solution of system (1) is defined to be a function $(u, v) \in C^2(\mathbf{R}^N) \times C^2(\mathbf{R}^N)$ which satisfies (1) at every point in $\mathbf{R}^N$.

In the previous paper [4] one of the authors has proved the following:

Theorem 0. (i) *Let $\alpha > 1$ and $\beta > 1$. Suppose that*

$$\liminf_{|x|\to\infty} |x|^{\lambda} p(x) > 0 \quad \textit{and} \quad \liminf_{|x|\to\infty} |x|^{\mu} q(x) > 0 \tag{2}$$

hold for some constants λ and μ satisfying

$$\left\{ \begin{array}{l} \lambda \leq \alpha(2-\mu) + 2 \quad \textit{or} \\ \\ \mu \leq \beta(2-\lambda) + 2. \end{array} \right. \tag{3}$$

Then (1) does not possess any positive entire solutions.

(ii) *Let $\alpha\beta > 1$, and p and q have radial symmetry. Suppose moreover that*

$$\limsup_{|x|\to\infty} |x|^{\lambda} p(x) < \infty \quad \textit{and} \quad \limsup_{|x|\to\infty} |x|^{\mu} q(x) < \infty$$

hold for some constants λ *and* μ *satisfying*

$$\begin{cases} \lambda > \alpha(2-\mu)+2 \quad \textit{and} \\ \mu > \beta(2-\lambda)+2. \end{cases}$$

Then (1) *has infinitely many positive radial entire solutions.*

From the above statement a natural question comes to the authors: When $\alpha\beta > 1$ (and one of α and β is less than 1), and (2) and (3) hold for some constants λ and μ, does not (1) possess a positive entire solution? — We will answer this problem partially here. That is, when $\alpha\beta > 1$, we can obtain a Liouville type theorem for nonnegative entire solutions of system (1) to the effect that (1) cannot possess nonnegative entire solutions (u, v) except for the trivial one $(u, v) \equiv (0, 0)$ if it satisfies a kind of growth condition at ∞.

Our result is as follows:

Theorem 1. (i) *Let* $\alpha\beta > 1$, $0 < \alpha < 1$, *and* (2) *hold for some constants* λ *and* μ *satisfying*

$$\lambda \leq \alpha(2-\mu)+2.$$

If (u, v) *is a nonnegative entire solution of* (1) *satisfying*

$$u(x) = O(\exp|x|^{\rho}) \quad \textit{as} \quad |x| \to \infty \quad \textit{for some} \quad \rho > 0, \tag{4}$$

then $(u, v) \equiv (0, 0)$.

(ii) *Let* $\alpha\beta > 1$, $0 < \beta < 1$, *and* (2) *hold for some constants* λ *and* μ *satisfying*

$$\mu \leq \beta(2-\lambda)+2.$$

If (u, v) *is a nonnegative entire solution of* (1) *satisfying*

$$v(x) = O(\exp|x|^{\rho}) \quad \textit{as} \ |x| \to \infty \ \textit{for some} \quad \rho > 0,$$

then $(u, v) \equiv (0, 0)$.

Since the positive radial entire solutions (u, v) constructed in [4] under the assumption of (ii) of Theorem 0 have the asymptotic growth

$$u, \ v = O(|x|^k) \quad \text{as } |x| \to \infty \text{ for some } \ k > 0,$$

the assumption of Theorem 1 is best possible in some sense.

2. Preliminary lemmas.

Let w be a continuous function in $\mathbf{R}^N$. We denote by $\overline{w}(r)$, $r \geq 0$, the average of $w(x)$ over the sphere $|x| = r$, that is,

$$\overline{w}(r) = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} w(x) dS,$$

where ω_N denotes the surface area of the unit sphere in $\mathbf{R}^N$.

The next lemma is needed in proving Theorem 1:

Lemma 2. *Let $\beta > 1$, (u, v) be a nonnegative entire solution of (1), and $b \in (0, 1)$ a constant. Then its spherical mean $(\bar{u}, \bar{v})$ satisfies the ordinary differential inequalities*

$$\bar{u}'(r) \geq \widetilde{C} r p_*(r) \bar{v}(br)^{\alpha}, \; r > 0, \; \bar{u}'(0) = 0, \tag{5}$$

$$(r^{N-1}\bar{v}'(r))' \geq r^{N-1}\hat{q}(r)\bar{u}(r)^{\beta}, \; r > 0, \; \bar{v}'(0) = 0, \tag{6}$$

where $\widetilde{C} = \widetilde{C}(N, \alpha, b) > 0$ is a constant and

$$p_*(r) = \min_{|x| \leq r} p(x), \quad r \geq 0,$$

and

$$\hat{q}(r) = \left(\frac{1}{\omega_N r^{N-1}} \int_{|x|=r} \frac{dS}{q(x)^{\beta'/\beta}} \right)^{-\beta/\beta'}, \quad r \geq 0$$

with $1/\beta + 1/\beta' = 1$.

To prove Lemma 2, we prepare the following lemma; see [**1**, p. 244] or [**3**, p. 225].

Lemma 3. *Let D be a domain in $\mathbf{R}^N$. Suppose that $\sigma > 0$ is a constant and $x_0 \in D$ and $r > 0$ satisfy $B_{2r}(x_0) \equiv \{x : |x - x_0| \leq 2r\} \subset D$. Then, we can find a constant $C = C(N, \sigma) > 0$ satisfying*

$$\left(\max_{B_r(x_0)} u \right)^{\sigma} \leq \frac{C}{r^N} \int_{B_{2r}(x_0)} u^{\sigma} dx$$

for any function $u \in C^2(D)$ satisfying $u \geq 0$ and $\Delta u \geq 0$ in D.

Proof of Lemma 2. Let (u, v) be a nonnegative entire solution of (1). Since $\beta > 1$, one can prove (6) easily by the same computation as was used in [**2**, p. 508]. We will prove the validity of (5). By taking the mean value of the first equation of (1), we have

$$r^{1-N}(r^{N-1}\bar{u}'(r))' = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} p(x) v(x)^{\alpha} dS, \quad r \geq 0. \tag{7}$$

Since an integration of (6) shows that $\bar{v}$ is nondecreasing on $[0, \infty)$, we may assume $b > 1/2$ in (5). Put $b = 1 - a$, $a \in (0, 1/2)$. Integrating (7) over $[0, r]$, we have

$$\begin{aligned} \bar{u}'(r) &= \frac{1}{\omega_N r^{N-1}} \int_{|x| \leq r} p(x) v(x)^{\alpha} dx \\ &\geq \frac{p_*(r)}{\omega_N r^{N-1}} \int_{|x| \leq r} v(x)^{\alpha} dx. \end{aligned} \tag{8}$$

Let $r > 0$ be fixed. We take $y \in \mathbf{R}^N$ such that

$$v(y) = \max_{|x|=(1-a)r} v(x) \left(= \max_{|x| \leq (1-a)r} v(x) \right),$$

and take $z \in \mathbf{R}^N$ such that $z = My,\ 0 < M < 1$ and $|y - z| = ar$. Then we can see that

$$\int_{|x|\le r} v^\alpha dx \ge \int_{|x-z|\le 2ar} v^\alpha dx.$$

Using Lemma 3, we obtain

$$\begin{aligned}\int_{|x-z|\le 2ar} v^\alpha dx &\ge C_0 r^N \left(\max_{|x-z|\le ar} v(x)\right)^\alpha \\ &= C_0 r^N \left[v(y)\right]^\alpha \\ &= C_0 r^N \left(\max_{|x|=(1-a)r} v(x)\right)^\alpha \\ &\ge C_0 r^N \bar{v}((1-a)r)^\alpha,\end{aligned}$$

where $C_0 = C_0(N, \alpha, a) > 0$ is a constant. From this estimate and (8) we obtain (5). This completes the proof.

3. Proof of Theorem 1.

This section is entirely devoted to proving our Theorem 1. Assume that (2) hold. Then there exist positive constants $K_1,\ K_2$ and r_0 such that

$$p(x) \ge \frac{K_1}{|x|^\lambda},\ q(x) \ge \frac{K_2}{|x|^\mu} \quad \text{for} \ \ |x| \ge r_0. \tag{9}$$

Proof of Theorem 1. We prove only the statement (i); the proof of (ii) is similar. It suffices to treat the case that $\lambda = \alpha(2-\mu)+2$. The proof is done by contradiction. Suppose to the contrary that (1) has a nonnegative nontrivial entire solution (u, v) satisfying (4). Then, by Lemma 2, its spherical mean $(\bar{u}, \bar{v})$ satisfies (5) and (6).

Let $m > 1$ be a number satisfying

$$1 < m < (1+\delta)^{1/\rho},\ \delta = \frac{\alpha\beta - 1}{\beta + 2}, \tag{10}$$

where ρ is the number appearing in (4). We choose the constant b in (5) such that $1/m < b < 1$. The proof is decomposed into three steps.

Step 1. We show that

$$\lim_{r\to\infty} \bar{u}(r) = \infty. \tag{11}$$

Integrating (5) and (6) on $[0, r]$, we see that $\bar{u}$ and $\bar{v}$ are nondecreasing functions on $[0, \infty)$, and

$$\bar{u}(r) \ge \bar{u}(0) + \widetilde{C} \int_0^r s p_*(s) \bar{v}(bs)^\alpha ds, \quad r \ge 0, \tag{12}$$

and

$$\bar{v}(r) \geq \bar{v}(0) + \frac{1}{N-2}\int_0^r s\hat{q}(s)\left[1 - \left(\frac{s}{r}\right)^{N-2}\right]\bar{u}(s)^\beta ds, \quad r \geq 0, \tag{13}$$

respectively. For some point $x^* \in \mathbf{R}^N$ we have $u(x^*) > 0$ or $v(x^*) > 0$; that is $\bar{u}(r^*) > 0$ or $\bar{v}(r^*) > 0$, $r^* = |x^*|$. Therefore we see from (12) and (13) that $\bar{u}(r) > 0$, $\bar{v}(r) > 0$ for $r > r^*$. We may assume that $r_0 > r^*$. From (9) and the monotonicity of $\bar{v}$, we have

$$\bar{u}(r) \geq \bar{u}(0) + \widetilde{C}K_1\bar{v}(r_0)^\alpha \int_{r_0/b}^r s^{1-\lambda} ds, \quad r \geq r_0/b.$$

Accordingly we observe that, if $\lambda \leq 2$, then

$$\bar{u}(r) \geq \begin{cases} Cr^{2-\lambda} & \text{for } \lambda < 2, \\ C\log r & \text{for } \lambda = 2, \end{cases}$$

for some constant $C > 0$ and $r \geq r_1 > 2r_0/b$. Therefore (11) holds if $\lambda \leq 2$.

It remains to consider the case of $\lambda > 2$. Since in this case $\mu < 2$, from (9) and (13), we have

$$\begin{aligned} \bar{v}(r) &\geq \bar{v}(0) + \frac{1}{N-2}\int_{r_0}^r s\hat{q}(s)\left[1 - \left(\frac{s}{r}\right)^{N-2}\right]\bar{u}(s)^\beta ds \\ &\geq \bar{v}(0) + \frac{K_2\bar{u}(r_0)^\beta}{N-2}\int_{r_0}^r s^{1-\mu}\left[1 - \left(\frac{s}{r}\right)^{N-2}\right] ds \\ &\geq \bar{v}(0) + \frac{K_2\bar{u}(r_0)^\beta}{N-2}\left[1 - \left(\frac{1}{2}\right)^{N-2}\right]\int_{r_0}^{r/2} s^{1-\mu} ds \\ &\geq C_1 r^{2-\mu}, \quad r \geq r_2 > 2r_0 \end{aligned} \tag{14}$$

for some constant $C_1 > 0$. Let $r \geq r_3 > r_2/b$. From (14) and (12), we have

$$\begin{aligned} \bar{u}(r) &\geq \bar{u}(0) + \widetilde{C}\int_0^r sp_*(s)\bar{v}(bs)^\alpha ds \\ &\geq \bar{u}(0) + \widetilde{C}K_1C_1^\alpha b^{\alpha(2-\mu)}\int_{r_3}^r s^{1-\lambda+\alpha(2-\mu)} ds \\ &= \bar{u}(0) + \widetilde{C}K_1C_1^\alpha b^{\alpha(2-\mu)}\int_{r_3}^r s^{-1} ds \\ &\geq C_2 \log r, \quad r \geq r_4 > 2r_3 \end{aligned}$$

for some constant $C_2 > 0$. Thus we obtain (11).

Step 2. We show that

$$\bar{u}(mr) \geq M\bar{u}(r)^{\delta+1} \quad \text{near} \quad +\infty \tag{15}$$

for some constant $M > 0$, where m is the number appearing in (10).

Let us fix $R > r_5 > \max\{r_1, r_4\}$ arbitrarily for a moment. Integrating (5) and (6) over $[R, r]$, we have

$$\overline{u}(r) \geq \overline{u}(R) + \widetilde{C} \int_R^r s p_*(s) \overline{v}(bs)^\alpha ds, \quad r \geq R,$$

and

$$\overline{v}(r) \geq \overline{v}(R) + \frac{1}{N-2} \int_R^r s \left[1 - \left(\frac{s}{r}\right)^{N-2}\right] \hat{q}(s) \overline{u}(s)^\beta ds, \quad r \geq R,$$

respectively. Using (9) and the inequality

$$s \left[1 - \left(\frac{s}{r}\right)^{N-2}\right] \geq \frac{N-2}{m^{N-2}}(r-s), \quad R \leq s \leq r \leq mR,$$

we have

$$\overline{u}(r) \geq C_3 R^{1-\lambda} \int_R^r \overline{v}(bs)^\alpha ds, \quad R \leq r \leq mR, \tag{16}$$

and

$$\overline{v}(r) \geq C_4 R^{-\mu} \int_R^r (r-s) \overline{u}(s)^\beta ds, \quad R \leq r \leq mR, \tag{17}$$

where C_3 and C_4 are positive constants independent of r and R. Now let us define the functions $f(r; R)$ and $g(r; R)$ for $R \leq r \leq mR$, by the right hand sides of (16) and (17), respectively. Then f and g satisfy $f(R; R) = g(R; R) = 0$. We denote simply $f(r; R) = f(r)$ and $g(r; R) = g(r)$, when there is no ambiguity. We then have

$$\begin{aligned} f'(r) &= C_3 R^{1-\lambda} \overline{v}(br)^\alpha \\ &\geq C_3 R^{1-\lambda} g(br)^\alpha, \quad b^{-1}R \leq r \leq mR; \end{aligned} \tag{18}$$

$$g'(r) = C_4 R^{-\mu} \int_R^r \overline{u}(s)^\beta ds \geq 0, \quad R \leq r \leq mR; \ g'(R) = 0,$$

and

$$g''(r) = C_4 R^{-\mu} \overline{u}(r)^\beta \geq C_4 R^{-\mu} f(r)^\beta, \quad R \leq r \leq mR. \tag{19}$$

Multiplying (18) by $g'(r) \geq 0$ and integrating by parts the resulting inequality on $[b^{-1}R, r]$, we have

$$\begin{aligned} & f(r)g'(r) - f(b^{-1}R)g'(b^{-1}R) - \int_{b^{-1}R}^{r} f(s)g''(s)ds \\ & \geq C_3 R^{1-\lambda} \int_{b^{-1}R}^{r} g(bs)^{\alpha} g'(s)ds \\ & \geq C_3 R^{1-\lambda} \int_{b^{-1}R}^{r} g(bs)^{\alpha} g'(bs)ds \\ & = \frac{C_3}{b(\alpha+1)} R^{1-\lambda} g(br)^{\alpha+1}, \quad b^{-1}R \leq r \leq mR. \end{aligned}$$

Hence

$$f(r)^{\beta} g'(r)^{\beta} \geq C_5 R^{(1-\lambda)\beta} g(br)^{(\alpha+1)\beta}, \quad b^{-1}R \leq r \leq mR,$$

where $C_5 = \left(\frac{C_3}{b(\alpha+1)}\right)^{\beta}$. From now on, we use C to denote various positive constants independent of r and R, as we will have no confusion. Combining this inequality with (19), we obtain

$$g''(r)g'(r)^{\beta} \geq C R^{(1-\lambda)\beta-\mu} g(br)^{(\alpha+1)\beta}, \quad b^{-1}R \leq r \leq mR.$$

Multiplying this inequality by $g'(r) \geq 0$ and integrating over $[b^{-1}R, r]$, we have

$$g'(r) \geq C R^{\frac{(1-\lambda)\beta-\mu}{\beta+2}} g(br)^{\delta+1}, \quad b^{-1}R \leq r \leq mR.$$

Let $\varepsilon > 0$ be sufficiently small fixed number. Integrating this relation over $[\frac{1+\varepsilon}{b}R, mR]$, we see that

$$g(mR; R) \geq C R^{\frac{(2-\lambda)\beta+2-\mu}{\beta+2}} g((1+\varepsilon)R; R)^{\delta+1}. \tag{20}$$

On the other hand, from the definition of g and the monotonicity of $\bar{u}$, we have

$$\begin{aligned} g(mR; R) &= \frac{C_4}{R^{\mu}} \int_{R}^{mR} (mR - s)\bar{u}(s)^{\beta} ds \\ &\leq \frac{C_4}{R^{\mu}} \bar{u}(mR)^{\beta} \int_{R}^{mR} (mR - s) ds \\ &= \frac{C_4 (m-1)^2}{2} R^{2-\mu} \bar{u}(mR)^{\beta}, \quad R > r_5, \end{aligned} \tag{21}$$

and

$$
\begin{aligned}
(22) \qquad g((1+\varepsilon)R;R) &= \frac{C_4}{R^\mu}\int_R^{(1+\varepsilon)R}((1+\varepsilon)R-s)\overline{u}(s)^\beta ds \\
&\geq \frac{C_4\overline{u}(R)^\beta}{R^\mu}\int_R^{(1+\varepsilon)R}((1+\varepsilon)R-s)ds \\
&= \frac{C_4\varepsilon^2}{2}R^{2-\mu}\overline{u}(R)^\beta, \quad R > r_5.
\end{aligned}
$$

From (20), (21) and (22) we observe that

$$
\begin{aligned}
\overline{u}(mR) &\geq CR^{\frac{(2-\mu)\alpha+2-\lambda}{\beta+2}}\overline{u}(R)^{\delta+1} \\
&= C\overline{u}(R)^{\delta+1}, \quad R > r_5.
\end{aligned}
$$

This implies that (15) holds.

Step 3. This is the final step. Let $\widetilde{r}$ be so large that

$$
(23) \qquad M^{1/\delta}\overline{u}(\widetilde{r}) \geq e,
$$

and

$$
(24) \qquad \overline{u}(mr) \geq M\overline{u}(r)^{1+\delta}, \quad r \geq \widetilde{r}
$$

hold, where $M > 0$ is the constant appearing in (15). This choice of $\widetilde{r}$ is possible by Steps 1 and 2. For $l \in \mathbf{N}$ we obtain from (24)

$$
\begin{aligned}
\overline{u}(m^l\widetilde{r}) &\geq M\overline{u}(m^{l-1}\widetilde{r})^{1+\delta} \\
&\geq M^{1+(1+\delta)}\overline{u}(m^{l-2}\widetilde{r})^{(1+\delta)^2} \\
&\geq \cdots\cdots \\
&\geq M^{1+(1+\delta)+\cdots+(1+\delta)^{l-1}}\overline{u}(\widetilde{r})^{(1+\delta)^l} \\
&= M^{-1/\delta}\left[M^{1/\delta}\overline{u}(\widetilde{r})\right]^{(1+\delta)^l}.
\end{aligned}
$$

Thus (23) yields

$$
(25) \qquad \overline{u}(m^l\widetilde{r}) \geq M^{-1/\delta}\exp\{(1+\delta)^l\}.
$$

Let $r \geq m\widetilde{r}$. Then we can find a unique positive integer $l = l(r)$ satisfying $m^l\widetilde{r} \leq r < m^{l+1}\widetilde{r}$, that is,

$$
l > \frac{\log r - \log\widetilde{r}}{\log m} - 1.
$$

It follows therefore from (25) that

$$\begin{aligned}\bar{u}(r) &\geq \bar{u}(m^l \widetilde{r}) \geq M^{-1/\delta} \exp\left\{(1+\delta)^l\right\} \\ &\geq M^{-1/\delta} \exp\left\{(1+\delta)^{-\frac{\log \widetilde{r}}{\log m}-1} \cdot (1+\delta)^{\frac{\log r}{\log m}}\right\} \\ &= M^{-1/\delta} \exp\left\{(1+\delta)^{-\frac{\log \widetilde{r}}{\log m}-1} r^{\frac{\log(1+\delta)}{\log m}}\right\}.\end{aligned} \tag{26}$$

On the other hand, because $u(x) = O(\exp|x|^\rho)$ as $|x| \to \infty$, we obviously have

$$\bar{u}(r) = O(\exp r^\rho) \quad \text{as} \quad r \to \infty.$$

Since $\log(1+\delta)/\log m > \rho$ from our choice of m, (26) gives a contradiction. Therefore $u \equiv v \equiv 0$ in $\mathbf{R}^N$. The proof is finished.

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