

*Pacific
Journal of
Mathematics*

THE SECOND COHOMOLOGY OF SMALL IRREDUCIBLE
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Let G be a connected, simply connected, quasisimple algebraic group over an algebraically closed field of characteristic $p > 0$, and let V be a rational G -module such that $\dim V \leq p$. According to a result of Jantzen, V is completely reducible, and $H^1(G, V) = 0$. In this paper we show that $H^2(G, V) = 0$ unless some composition factor of V is a nontrivial Frobenius twist of the adjoint representation of G .

1. Introduction.

Let G be a quasisimple, connected, simply connected algebraic group over the algebraically closed field k of characteristic $p > 0$. By a G -module V , we always understand a rational G -module (one given by a morphism of algebraic groups $G \rightarrow \mathrm{GL}(V)$). In this paper, we study the cohomology of a G -module V such that $\dim V \leq p$. By results of Jantzen [Jan96] one knows that V is semisimple and that $H^1(G, V) = 0$.

Recall that the Lie algebra \mathfrak{g} of G is a G -module via the adjoint action. Our main result is:

Theorem A. *Let V be a G -module with $\dim V \leq p$. Then $H^2(G, V) \neq 0$ if and only if V has a composition factor isomorphic with a Frobenius twist $\mathfrak{g}^{[d]}$ of \mathfrak{g} for some $d \geq 1$.*

Differentiating the representation of G on V gives a representation for the Lie algebra \mathfrak{g} on V . Assume that $V^{\mathfrak{g}} = 0$. Then the theorem says that $H^2(G, V) = 0$. For V of this sort, the vanishing of H^2 is a consequence of the linkage principle for G together with results in Section 2 which give estimates for the dimensions of Weyl modules whose high weights are simultaneously in the low alcove and in the orbit $W_p \bullet 0$. In fact, the same argument shows that $H^i(G, V)$ is 0 for all $i \geq 1$; see Proposition 5.2. It was pointed out to me that an earlier version of this manuscript contained an overly complicated proof of this observation.

The crucial case for Theorem A is when V is simple, nontrivial and $V^{\mathfrak{g}} = V$. There is a unique $d \geq 1$ such that the ‘‘Frobenius untwist’’ $V^{[-d]}$ is a G -module on which \mathfrak{g} acts nontrivially. We have already seen that

$H^i(G, V^{[-d]}) = 0$ for $i = 1, 2$, so Theorem A follows from the following two results (see 5.4). [We denote by h the Coxeter number of the group G .]

Theorem B. *Suppose that $p \geq h$ and that W is a G -module for which $H^i(G, W) = 0$ for $i = 1, 2$. Then $H^2(G, W^{[d]}) \simeq \text{Hom}_G(\mathfrak{g}, W)$ for all $d \geq 1$.*

Theorem C. *If $p > h$, $\dim H^2(G, \mathfrak{g}^{[d]}) = 1$ for all $d \geq 1$. For any p , there is a $d_0 \geq 1$ so that $H^2(G, \mathfrak{g}^{[d]}) \neq 0$ for all $d \geq d_0$.*

Theorem B is proved in 5.3; it immediately implies the first assertion of Theorem C (see 5.5). We give a proof of the second assertion of Theorem C in Section 5.6.

We end the paper by applying the results of Section 2 to calculations of cohomology groups $H^i(G_1, L)$, where G_1 is the Frobenius kernel, and L is a simple G_1 module with $\dim L \leq p$; see Proposition 6.

We make now the following remark concerning our hypothesis on G . Suppose that G is quasisimple, but not necessarily simply connected, and let $\pi : G_{\text{sc}} \rightarrow G$ denote the isogeny from the corresponding simply connected covering group. Then any G representation V is also a G_{sc} representation, and the kernel of π is a diagonalizable group scheme. It follows that π induces an isomorphism $H^i(G, V) \simeq H^i(G_{\text{sc}}, V)$ for each $i \geq 0$; see [CPSvdK77, Remark (2.7)]. I thank W. van der Kallen for pointing this out to me. One may check using Lemma 4.1(A) and Proposition 5.1 that $\text{Lie}(G_{\text{sc}})$ and $\text{Lie}(G)$ are isomorphic simple G_{sc} representations whenever $\dim G \leq p$. Thus the conclusion of Theorem A remains true for G .

We conclude this introduction by remarking that the result of Jantzen [Jan96] cited above is one of several recent results studying the semisimplicity of low dimensional representations of groups in characteristic p . See [Ser94], [McN98], [McN99], [Gur99] and [McN00] for related work.

The author would like to acknowledge the hospitality of Bob Guralnick and the University of Southern California during a visit in June 1999; in particular, questions of Guralnick encouraged the author to consider the problems addressed in this paper, and several conversations inspired some useful ideas.

2. Root systems.

2.1. We denote by R an indecomposable root system in its weight lattice X with simple roots $S \subset R^+$. For each $\alpha \in S$, there is a fundamental dominant weight $\varpi_\alpha \in X$; the ϖ_α form a \mathbb{Z} basis of X .

We write α_0 for the dominant short root, and $\tilde{\alpha}$ for the dominant long root in R (these coincide in case there is only one root length).

The Coxeter number of R is given by

$$h - 1 = \sup_{\alpha \in R^+} \{ \langle \rho, \alpha^\vee \rangle \} = \langle \rho, \alpha_0^\vee \rangle.$$

For $m \in \mathbb{Z}$ and $\alpha \in R$, let $s_{\alpha,m}$ denote the affine reflection of $X_{\mathbb{R}} = X \otimes_{\mathbb{Z}} \mathbb{R}$ in the hyperplane $H_{\alpha,m} = \{x \in X_{\mathbb{R}} : \langle x, \alpha^{\vee} \rangle = m\}$.

Let $l > h$ be an integer. The affine Weyl group W_l is the group of affine transformations of $X_{\mathbb{R}}$ generated by all $s_{\alpha,ln}$ for $n \in \mathbb{Z}$. According to [Bou72, Ch. VI, §2.1, Prop. 1] W_l is isomorphic to the semidirect product of W (the finite Weyl group) with $l\mathbb{Z}R$. The normalizer of W_l in the full affine transformation group of $X_{\mathbb{R}}$ contains all translations by lX , hence W_l is a normal subgroup of \widehat{W}_l , the semidirect product of W and lX . Moreover, $\widehat{W}_l/W_l \simeq lX/l\mathbb{Z}R \simeq X/\mathbb{Z}R$ is the fundamental group of R , which we will denote by π .

Let $\rho = \frac{1}{2} \sum_{\alpha \in S} \alpha$. We always consider the dot action of \widehat{W}_l (also of W and W_l) on X : For $w \in \widehat{W}_l$ and $\lambda \in X$, this is given by $w \bullet \lambda = w(\lambda + \rho) - \rho$.

The closure of the subset C_l of $X_{\mathbb{R}}$ given by

$$C_l = \{\lambda \in X_{\mathbb{R}} \mid 0 < \langle \lambda + \rho, \alpha^{\vee} \rangle < l \text{ for each } \alpha \in R^+\}$$

is a fundamental domain for the dot action of W_l on X ; the conjugates of C_l under W_l are known as alcoves, and C_l is the lowest alcove. Since \widehat{W}_l normalizes W_l , [Bou72, Ch. VI, §2.1] shows that \widehat{W}_l permutes the alcoves.

Let Ω be the stabilizer in \widehat{W}_l of C . Since W_l permutes the alcoves simply transitively, one deduces that \widehat{W}_l is the semidirect product of Ω and W_l . Thus $\Omega \simeq \widehat{W}_l/W_l \simeq \pi$.

Since $l > h$, the intersection $C_l \cap X^+$ is nonempty. [Note that if $l \leq h$ had been allowed, we would have $C_l \cap X^+ = \{0\}$ in case $l = h$, and $C_l \cap X^+ = \emptyset$ if $l < h$.] It is then clear that $\widehat{W}_l \bullet 0 \cap C_l = \{\omega \bullet 0 \mid \omega \in \Omega\}$.

2.2. . Let I index the simple roots $S = \{\alpha_i\}$, write $\alpha_0^{\vee} = \sum_{i \in I} n_i \alpha_i^{\vee}$, and put $J = \{i \in I \mid n_i = 1\}$. A dominant weight $0 \neq \varpi \in X$ is *minuscule* if whenever $\lambda \leq \varpi$ and λ is a dominant weight, then $\varpi = \lambda$. According to [Bou72, Ch. VI, Exerc. 23, 24], ϖ is minuscule just in case $\varpi = \varpi_i$ for some $i \in J$.

For $i \in I \cup \{0\}$, let $S_i = S \setminus \{\alpha_i\}$ (so $S_0 = S$). Write $R_i \subset R$ for the root subsystem determined by S_i , and W_i for the parabolic subgroup of W associated with R_i . Let $w_i \in W_i$ be the unique element which makes all positive roots in R_i negative.

For $x \in X$, let $t(x)$ denote the affine translation by x ; for $i \in J$, let $\gamma_i = t(l\varpi_i)w_0w_i \in \widehat{W}_l$. Note that γ_i represents $\varpi_i \in X/\mathbb{Z}R \simeq lX/l\mathbb{Z}R \simeq \widehat{W}_l/W_l$.

Applying [Bou72, Ch. VI, §2.2 Prop. 6 and Cor.] one obtains:

Proposition.

- (a) *Each non-0 coset of $\mathbb{Z}R$ in X is uniquely represented by a minuscule weight. In particular, $|\pi| = |J| + 1$.*

(b) *The nonidentity elements of Ω are precisely the γ_i for $i \in J$. We have*

$$\widehat{W}_l \bullet 0 \cap C_l = \{0\} \cup \{\gamma_i \bullet 0 = (l - h)\varpi_i \mid i \in J\}.$$

2.3. . For a dominant weight λ , let

$$(1) \quad d(\lambda) = \prod_{\alpha > 0} \frac{\langle \lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}$$

be the value of Weyl’s degree formula at λ .

Proposition. *Let $\lambda = (l - h)\varpi_i$ for some $i \in J$.*

- (a) $d(\lambda) \geq \binom{l-1}{l-h}$, with equality if and only if $h - 1 = \ell(w_0 w_i)$.
- (b) If $l - h \geq 2$ and $h \geq 3$, then $d(\lambda) > l$.

Proof. For $1 \leq k \leq h - 1$, let $e(k)$ be the number of $\alpha \in R^+ \setminus R_i^+$ with $\langle \rho, \alpha^\vee \rangle = k$. The argument in the remark on p. 520-521 of [Ser94] (following Prop. 6) shows that $e(k) \geq 1$ for each $1 \leq k \leq h - 1$. Thus, we have

$$d(\lambda) = \prod_{k=1}^{h-1} \left(\frac{l - h + k}{k} \right)^{e(k)} \geq \prod_{k=1}^{h-1} \frac{l - h + k}{k} = \binom{l - 1}{l - h}.$$

If $\ell(w_0 w_i) = |R^+| - |R_i^+| = h - 1$, then $e(k) = 1$ for each $1 \leq k \leq h - 1$ and equality holds. This proves (a).

For (b), note that under the given hypothesis we have $l \geq 5$. Since $\binom{l-1}{l-h} \geq \binom{l-1}{2} > l$ for all such l , (b) follows immediately. \square

Remark. Using the table in the proof of Proposition 2.4 below, it is straightforward to verify that equality holds in (a) if and only if either $R = A_r$ and $i \in \{1, r\}$ or $R = C_r$ and $i = 1$. (Since $B_2 = C_2$, the latter case includes B_2 and $i = 2$.)

2.4. . In the following, let me emphasize the standing assumption $l > h$.

Proposition. *If $0 \neq \lambda \in \widehat{W}_l \bullet 0 \cap C$ and $d(\lambda) < l$ then $d(\lambda) = \ell - 1$ and (R, λ) is listed in the following table. If the rank of R is ≥ 2 , then $l = h + 1$.*

R	l	λ
A_1	any	$(l - 2)\varpi_1$
A_{l-2}		ϖ_1, ϖ_{l-2}
B_2	$l = 5$	ϖ_2
$C_{(l-1)/2}$	l odd	ϖ_1

Proof. The rank 1 situation leads to the item listed in the table. When the rank is at least 2, one applies Proposition 2.3 to obtain $l = h + 1$, whence $\lambda = \varpi_i$ for some $i \in J$; i.e., λ is minuscule.

We handle the minuscule cases by classification. For each indecomposable root system R for which $J \neq \emptyset$, we list in the following table the Coxeter number, the set J , and the value $d(\varpi_i)$ for each $i \in J$. The simple roots are indexed as in the tables in [Bou72, Planche I-X]; the data recorded here, with the exception of the values $d(\varpi_i)$, may be verified by inspecting those tables as well. The values $d(\varpi_i)$ are well-known (and can anyway be computed from the formula, or by representation theoretic arguments).

Type of R	h	J	$d(\varpi_i), i \in J$
A_r	$r + 1$	$\{1, 2, \dots, r\}$	$\binom{r+1}{i}$
$B_r, r \geq 2$	$2r$	$\{r\}$	2^r
$C_r, r \geq 2$	$2r$	$\{1\}$	$2r$
$D_r, r \geq 4$	$2r - 2$	$\{1, r - 1, r\}$	$2r, 2^{r-1}, 2^{r-1}$ respectively
E_6	12	$\{1, 6\}$	27, 27
E_7	18	$\{7\}$	56

From this table, one can list all pairs (R, λ) for which R has Coxeter number $l - 1$ and λ is minuscule. It is a simple matter to see that $d(\lambda) < l$ only when (R, λ) is as claimed. \square

3. The algebraic groups.

3.1. Let k be an algebraically closed field of characteristic $p > 0$, and let G be a connected, simply connected semisimple algebraic k -group. The non-0 weights of a maximal torus $T \leq G$ on $\mathfrak{g} = \text{Lie}(G)$ form a root system R of rank $r = \dim T$ in the character group $X = X^*(T)$. Since G is simply connected, X identifies with the full weight lattice of R as in Section 2. We fix a choice of simple roots S and positive roots R^+ . The dominant weights are denoted X^+ . The group G is assumed to be *quasisimple*; i.e., the root system R is indecomposable.

3.2. For each dominant weight $\lambda \in X^+$, the space of global sections of the corresponding line bundle on the flag variety affords an indecomposable rational G -module $H^0(\lambda)$ with simple socle. The modules $L(\lambda) = \text{soc } H^0(\lambda)$ comprise all of the simple rational modules for G (and are pairwise non-isomorphic).

The character of each $H^0(\lambda)$ is the same as in characteristic 0; hence in particular $\dim_k H^0(\lambda)$ is given by the Weyl degree formula, whose value at λ we denote $d(\lambda)$ as in 2.3.

3.3. Any dominant λ may be written as a finite sum $\sum_{i \geq 0} p^i \lambda_i$ with each λ_i a *restricted* weight. Recall that a dominant weight μ is restricted if $\langle \mu, \alpha^\vee \rangle < p$ for all simple roots α . Steinberg’s tensor product theorem says:

$$L(\lambda) \simeq L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes L(\lambda_2)^{[2]} \otimes \dots$$

where for a G -module V , $V^{[m]}$ stands for the m -th Frobenius twist of V .

For $d \geq 1$, let G_d be the d -th Frobenius kernel of G . Let V be a rational G -module and $m \geq 1$. If there is a rational G module W with $W^{[m]} \simeq V$, we regard W as the Frobenius *untwist* $W = V^{[-m]}$ of V . Now regard V as a module for G_d . Since G_d is a normal subgroup scheme, G acts on V^{G_d} ; since G_d acts trivially on this G -module, there is an untwisted rational G -module $(V^{G_d})^{[-d]}$. It follows that there is an untwist $H^i(G_d, V)^{[-d]}$ for all $i \geq 0$.

Consider now two G -modules V_1 and V_2 , and form $W = V_1 \otimes V_2^{[d]}$. The Frobenius kernel G_d acts trivially on $V_2^{[d]}$, so that

$$(1) \quad H^i(G_d, W)^{[-d]} \simeq H^i(G_d, V_1)^{[-d]} \otimes V_2$$

as G -modules for every $i \geq 0$.

3.4. . Let $W_p \leq \widehat{W}_p$ be as in Section 2 (for $l = p$), let $C = C_p \cap X^+$ denote the dominant weights in the lowest alcove, and let $\overline{C} = \overline{C}_p \cap X^+$ (\overline{C}_p is the closure in $X_{\mathbb{R}}$).

Proposition. *Let $\lambda \in X^+$.*

- (a) *If $H^i(G, L(\lambda)) \neq 0$ for some $i \geq 0$, then $\lambda \in W_p \bullet 0$.*
- (b) *If $H^i(G_1, L(\lambda)) \neq 0$ for some $i \geq 0$, then $\lambda \in \widehat{W}_p \bullet 0$.*
- (c) *$H^i(G, H^0(\lambda)) = 0$ for all $i > 0$.*
- (d) *If $\lambda \in \overline{C}$, then $L(\lambda) = H^0(\lambda)$; in particular, $\dim L(\lambda) = d(\lambda)$.*

Proof. (a) follows from the *linkage principle* for G [Jan87, Cor. II.6.17], and (b) from the linkage principle for G_1 [Jan87, Lemma II.9.16]. (c) follows from [Jan87, II.4.12]. (d) follows from [Jan87, II.6.13, II.5.10]. □

4. The Lie algebra and the cohomology of G_1 .

We want to describe explicitly the cohomology $H^*(G_1, k)$ in degree ≤ 2 . For this, we need some information on the Lie algebra \mathfrak{g} .

4.1. . Recall that the prime p is *bad* [=not good] for the indecomposable root system R if one of the following holds: $p = 2$ and R is not of type A_r ; $p = 3$ and R is of type G_2, F_4 , or E_r ; $p = 5$ and R is of type E_8 .

The prime p is *very good* if it is not bad, and in case $R = A_r$, if also p does not divide $r + 1$. Notice that if $p > h$, then p is very good.

Application of the summary in [Hum95, 0.13] yields the following:

Lemma A. *Assume that p is very good. Then \mathfrak{g} is a simple Lie algebra. The adjoint G -module is simple, self-dual, and isomorphic with $L(\tilde{\alpha})$ where $\tilde{\alpha}$ is the dominant long root.*

Lemma B. *Assume that $p \geq h$. If W is any G -module, then $\text{Hom}_G(\mathfrak{g}, W^{[d]}) = 0$ for $d \geq 1$.*

Proof. When $p > h$ this follows since by the previous lemma \mathfrak{g} is a simple \mathfrak{g} -module with restricted highest weight. When $p = h$, we have $R = A_{p-1}$. Since G is simply connected, we have $\mathfrak{g} = \mathfrak{sl}_p$. Thus \mathfrak{g} is an indecomposable G -module with unique simple quotient $L(\tilde{\alpha})$, and the lemma follows. \square

4.2. . Let B be a Borel subgroup of G , and let \mathfrak{u} be the nilradical of $\text{Lie}(B)$. Regarding \mathfrak{u}^* as a B -module, we get a vector bundle on G/B which we also write as \mathfrak{u}^* . According to [AJ84, 3.8], the formal character of the G -module $H^0(G/B, \mathfrak{u}^*)$ is $\chi(\tilde{\alpha}) = \text{ch}(\mathfrak{g}^*)$.

Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone. There is by [AJ84, 3.9] an injective homomorphism of graded algebras $k[\mathcal{N}] \rightarrow H^0(G/B, \mathfrak{u}^*)$.

Lemma. *For simply connected, quasisimple algebraic groups G , $\mathfrak{g}^* \simeq k[\mathcal{N}]_1 \simeq H^0(G/B, \mathfrak{u}^*)$.*

Proof. Let $I(\mathcal{N}) \triangleleft k[\mathfrak{g}] = S\mathfrak{g}^*$ be the (homogeneous) defining ideal of the variety \mathcal{N} . We need to show that $I(\mathcal{N})_1 = 0$. If not, then $\mathcal{N} \subset V \subset \mathfrak{g}$ for some proper G -submodule V . A look at the summary in [Hum95, 0.13] shows that, since G is simply connected, the only G -submodules of \mathfrak{g} have dimension 0 or 1. On the other hand, by [Hum95, Theorem 6.19], the variety \mathcal{N} has codimension $\text{rank}(G)$ in \mathfrak{g} and so clearly can't be contained in a 1 dimensional linear subspace! \square

Remarks.

- (1) Here is a fancier result which implies the lemma if we assume that the prime p is good for G . Since G is simply connected and p is good, the Springer resolution

$$\varphi : \tilde{\mathcal{N}} = G \times^B \mathfrak{u} \rightarrow \mathcal{N}$$

given by $(g, X) \mapsto \text{Ad}(g)(X)$ is a *desingularization*, hence in particular a birational map; see [Hum95, Theorem 6.3 and Theorem 6.20]. Again since G is simply connected and p is good, the variety \mathcal{N} is normal ([Hum95, Theorem 4.24]). Standard arguments then yield an isomorphism of graded algebras $k[\mathcal{N}] \xrightarrow{\varphi^*} \Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}})$. Finally, the projection $\tilde{\mathcal{N}} \rightarrow G/B$ is an affine morphism, so that $\Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}) = H^0(G/B, \mathfrak{u}^*)$ as a graded algebra.

- (2) On the other hand, if $G = PGL_r$, and $p|r$, one can find a linear form on \mathfrak{g} that vanishes on \mathcal{N} , hence there can be no isomorphism $k[\mathcal{N}]_1 \rightarrow H^0(G/B, \mathfrak{u}^*)$ (compare formal characters). So the lemma can fail when G is not simply connected. [Note that φ is not birational in this example. One can show that there is a G_{sc} -isomorphism $\psi : \tilde{\mathcal{N}}_{sc} \rightarrow \tilde{\mathcal{N}}$ (using some obvious notations). We get therefore a commuting

diagram:

$$\begin{array}{ccc}
 \tilde{\mathcal{N}} & \xrightarrow{\varphi_{sc} \circ \psi^{-1}} & \mathcal{N}_{sc} \\
 & \searrow \varphi & \downarrow \gamma \\
 & & \mathcal{N}
 \end{array}$$

The map $\varphi_{sc} \circ \psi^{-1}$ is birational. Since $\gamma^*k(\mathcal{N}) \subset k(\mathcal{N}_{sc})$ is a proper purely inseparable extension, so too is $\varphi^*k(\mathcal{N}) \subset k(\tilde{\mathcal{N}})$.

Proposition.

- (1) If $p \neq 2$ or if R is not of type C_r , then $H^1(G_1, k) = 0$.
- (2) Assume that $p \geq h$. Then $H^2(G_1, k)^{[-1]} \simeq \mathfrak{g}^*$ as G -modules.

Proof. For (1) see [Jan87, Lemma II.12.1]. For (2), first suppose that $p > h$. By [AJ84, 3.7, 3.9], there is a G -equivariant isomorphism of graded rings $k[\mathcal{N}]' \simeq H^*(G_1, k)^{[-1]}$ where $k[\mathcal{N}]'$ is again the graded coordinate ring of \mathcal{N} , but with the linear functions on \mathfrak{g} given degree 2. The claim now follows from the lemma.

When $p = h$, apply [AJ84, Cor. 6.3] to see that $H^2(G_1, k)^{[-1]} \simeq H^0(G/B, \mathfrak{u}^*)$; the claim follows again from the lemma in this case. □

5. Low dimensional modules for G .

5.1. . We recall first some facts about low dimensional modules established in [Jan96] and [Ser94].

Proposition. *Let L be a simple nontrivial restricted G module with highest weight λ . Suppose that $\dim L \leq p$.*

- (a) $\lambda \in \bar{C}$.
- (b) $\lambda \in C$ if and only if $\dim_k L < p$.
- (c) $h \leq p$. If moreover $\dim L < p$, then $h < p$.
- (d) If R is not of type A and $\dim L = p$, then $h < p$. If $p = h$ and $\dim L = p$, then $R = A_{p-1}$ and $\lambda = \varpi_i$ with $i \in \{1, p - 1\}$.

Proof. (a) follows from [Jan96, Lemma 1.4], and (b) from [Jan96, 1.6], see also [Ser94]. For (c), note first that (a) implies $\dim L = d(\lambda)$ by Proposition 3.4(d). If $\lambda \in \bar{C} \setminus C$, then (a) and (b) imply that $\dim L = p$, whence $p = h$ follows from Weyl’s degree formula. (c) now follows since C is empty if $p < h$ and $C = \{0\}$ if $p = h$.

In [Jan96, 1.6], Jantzen made a list of all simple restricted modules for G with dimension p . Inspecting that list yields (d). □

5.2. Vanishing results when \mathfrak{g} acts nontrivially. Let L be a simple module for G .

Proposition. *If G_1 (equivalently: \mathfrak{g}) acts nontrivially on L and $\dim L \leq p$, then $H^i(G, L) = 0$ for all $i \geq 0$.*

Proof. Write the highest weight of L as $\lambda = \mu_1 + p\mu_2$ with μ_1 restricted. Since $L^{\mathfrak{g}} = 0$, we have $\mu_1 \neq 0$. Since $p \geq \dim L \geq \dim L(\mu_1)$, Proposition 5.1 implies that $\mu_1 \in \bar{C}$ and that $h \leq p$. We have in particular that $L(\mu_1) = H^0(\mu_1)$, hence the proposition will follow from Proposition 3.4 if we show that μ_2 is 0.

If $\dim L = p$, Steinberg’s tensor product theorem gives $\mu_2 = 0$. If $\dim L < p$ then 5.1 shows that $p < h$ and $\mu_1 \in C$. If $H^i(G, L) \neq 0$ for some i , then $\lambda \in W_p \bullet 0$ by the linkage principle, whence $\mu_1 \in W \bullet 0 + pX = \widehat{W}_p \bullet 0$. Now Proposition 2.4 applies; it shows that $\dim L(\mu_1) = p - 1$ whence we have $\mu_2 = 0$ by another application of Steinberg’s theorem. \square

5.3. Second cohomology. Here we prove our main tool for describing second cohomology; first we require the following:

Lemma. *Let $E_2^{p,q} \implies H^{p+q}$ be a convergent, first quadrant spectral sequence.*

- (1) *If $E_2^{0,1} = E_2^{1,1} = E_2^{0,2} = 0$, then $H^2 \simeq E_2^{2,0}$.*
- (2) *If $E_2^{1,0} = E_2^{1,1} = E_2^{2,0} = 0$, then $H^2 \simeq E_2^{0,2}$.*

Proof. We verify (1), the argument for (2) is the same. We must show that $E_\infty^{2,0} \simeq E_2^{2,0}$; first note that $E_3^{2,0}$ is the cohomology of the sequence

$$E_2^{0,1} \rightarrow E_2^{2,0} \rightarrow E_2^{4,-1}$$

from which we get $E_3^{2,0} \simeq E_2^{2,0}$. For any first quadrant spectral sequence one has (by similar reasoning) that $E_a^{2,0} \simeq E_{a+1}^{2,0}$ for $a > 2$, so we get the desired isomorphism. \square

Theorem. *Suppose that $p \geq h$. Let V be a G -module for which $H^i(G, V) = 0$ for $i = 1, 2$, and let $d \geq 1$.*

- (1) *$H^1(G, V^{[d]}) = 0$, and*
- (2) *$H^2(G, V^{[d]}) \simeq \text{Hom}_G(\mathfrak{g}, V)$.*

Proof. The Frobenius kernel G_1 is a normal subgroup of G ; thus there is a Lyndon-Hochschild-Serre spectral sequence computing $H^i(G, V^{[d]})$ which in view of 3.3 (1) has the form

$$E_2^{s,t} = H^s(G, H^t(G_1, V^{[d]})^{[-1]}) = H^s(G, H^t(G_1, k)^{[-1]} \otimes V^{[d-1]}).$$

If $t = 1$, $E_2^{s,t} = 0$ by Proposition 4.2(1).

There is an exact sequence of the form [Jan87, I.4.1(4)]

$$0 \rightarrow E_2^{1,0} \rightarrow H^1(G, V^{[d]}) \rightarrow E_2^{0,1} = 0.$$

Thus $H^1(G, V^{[d]}) \simeq E_2^{1,0} \simeq H^1(G, V^{[d-1]})$. We get now (1) by induction on d .

Proposition 4.2(2) shows now that $H^2(G_1, k) \simeq \mathfrak{g}^*$. Thus, the only possible non-0 E_2 terms of total degree 2 are

$$E_2^{0,2} = H^0(G, \mathfrak{g}^* \otimes V^{[d-1]}) = \text{Hom}_G(\mathfrak{g}, V^{[d-1]})$$

$$E_2^{2,0} = H^2(G, V^{[d-1]}).$$

For $d > 1$, we apply Lemma 4.1(B) to see that $E_2^{0,2} = 0$ whence $H^2(G, V^{[d]}) \simeq E_2^{2,0} = H^2(G, V^{[d-1]})$ by part (1) of the lemma; thus (2) will follow provided it holds for $d = 1$. In that case, we have $E_2^{2,0} = 0$ by assumption, and the result just proved in part (1) shows that $E_2^{1,0} = 0$. Thus part (2) of the lemma applies; it shows that $H^2(G, V^{[1]}) \simeq E_2^{0,2} = \text{Hom}_G(\mathfrak{g}, V)$ as desired. \square

5.4. The second cohomology of small modules. Let $L = L(\lambda)$ be a simple G -module, and suppose that $\dim L \leq p$. Proposition 5.2 showed that the vanishing of cohomology for L is a consequence of the linkage principle when $\lambda \notin pX$. However, if $\lambda \in p\mathbb{Z}R$, λ is linked to 0, so the linkage principle does not yield vanishing. The following result shows that, despite the linkage of λ and 0 in this case, the second cohomology is usually 0.

Theorem. *Let L be a simple G -module with $\dim L \leq p$. If $H^2(G, L) \neq 0$, then $L \simeq \mathfrak{g}^{[d]}$ for some $d \geq 1$.*

Proof. Let L' be such that $L \simeq (L')^{[d]}$ for $d \geq 0$, and such that \mathfrak{g} acts nontrivially on L' . We have by 5.1 that $p \geq h$. Also, we have by Proposition 5.2 that $H^i(G, L') = 0$ for $i \geq 1$. If $d = 0$, we are done. If $d > 1$, then Theorem 5.3 applies, and we get that

$$H^2(G, L) \simeq \text{Hom}_G(\mathfrak{g}, L').$$

We get by Proposition 5.1 that $p > h$ unless $R = A_{p-1}$ and $L' = L(\varpi_i)$ with $i \in \{1, p-1\}$. If $p > h$, then \mathfrak{g} is a simple G -module by Lemma 4.1(A). So if $\text{Hom}_G(\mathfrak{g}, L') \neq 0$ then $L' \simeq \mathfrak{g}$ whence $L \simeq \mathfrak{g}^{[d]}$ as claimed.

In the remaining case, one must just note that weight considerations yield $\text{Hom}_G(\mathfrak{g}, L(\varpi_i)) = 0$ for $i = 1, p-1$, whence $H^2(G, L) = 0$. \square

5.5. The second cohomology of twists of the adjoint module. The first assertion of Theorem C of the introduction follows from the following:

Proposition. *Assume that $p > h$. Then $H^1(G, \mathfrak{g}^{[d]}) = 0$ and $H^2(G, \mathfrak{g}^{[d]}) \simeq \text{End}_G(\mathfrak{g})$ has dimension 1 for $d \geq 1$.*

Proof. Since $p > h$, Lemma 4.1(A) shows that \mathfrak{g} is the simple module with highest weight $\tilde{\alpha}$. It follows that $\mathfrak{g} = H^0(\tilde{\alpha})$, and thus that $H^i(G, \mathfrak{g}) = 0$ for $i \geq 1$ by Proposition 3.4. The proposition now follows from Theorem 5.3. \square

Remark. Note that $\dim \mathfrak{g} > h$ (in fact, $\dim \mathfrak{g} = (h + 1)r$ where r is the rank of G). So we get also: If $\dim \mathfrak{g} \leq p$, then $\dim H^2(G, \mathfrak{g}^{[d]}) = 1$ for $d \geq 1$.

5.6. A second proof. Here we give a second proof of the non-vanishing of H^2 for twists of the adjoint module; the result proved here verifies the remaining assertion of Theorem C of the introduction. We have included the argument since it offers some “explanation” for the non-vanishing.

The group G arises by base change from a split reductive group scheme \mathbf{G} over \mathbb{Z} . Let \mathbb{Z}_p be the complete ring of p -adic integers, and let \mathbb{Q}_p be its field of quotients. For any finite field extension F of \mathbb{Q}_p , let \mathfrak{o} denote the integers in F . The residue field $\mathfrak{o}/\mathfrak{m}$ may be identified with the extension \mathbb{F}_q of \mathbb{F}_p .

Let K denote the group of points $\mathbf{G}(\mathfrak{o})$ regarded as a subgroup of $\mathbf{G}(F)$. Since \mathbf{G} is smooth, the reduction homomorphism $K \rightarrow \mathbf{G}(\mathbb{F}_q)$ is surjective (see [Tit79, 3.4.4]).

For $n \geq 1$, let $K_n \subset K$ be the kernel of the map $K \rightarrow \mathbf{G}(\mathfrak{o}/\mathfrak{m}^n)$. Note that $K/K_1 = \mathbf{G}(\mathbb{F}_q)$ acts by conjugation on each quotient K_n/K_{n+1} .

Proposition. (a) *There is for each $m \geq 1$ a canonical isomorphism $K_m/K_{m+1} \simeq \mathfrak{g}_{\mathbb{F}_q}$ as representations for $\mathbf{G}(\mathbb{F}_q)$, where $\mathfrak{g}_{\mathbb{F}_q}$ is the Lie algebra of $\mathbf{G}_{\mathbb{F}_q}$.*

(b) *If $H^2(\mathbf{G}(\mathbb{F}_q), \mathfrak{g}_{\mathbb{F}_q}) = 0$, the exact sequence of groups*

$$1 \rightarrow K_1 \rightarrow K \rightarrow \mathbf{G}(\mathbb{F}_q) \rightarrow 1$$

splits.

(c) *There is a p -power q_0 , depending only on the root system R of G , such that $H^2(\mathbf{G}(\mathbb{F}_q), \mathfrak{g}_{\mathbb{F}_q}) \neq 0$ whenever $q \geq q_0$.*

(d) *There is an integer $a_0 \geq 1$ such that $H^2(G, \mathfrak{g}^{[a]}) \neq 0$ whenever $a \geq a_0$.*

Proof. (a) Follows from [DG70, II.§4.3]. (b) Since K_1 is a pro- p group [PR94, Lemma 3.8], this follows from [Ser67, Lemma 3].

(c) Choose a \mathbb{Q}_p vectorspace V and a nontrivial faithful \mathbb{Q}_p -rational representation $\mathbf{G}_{\mathbb{Q}_p} \rightarrow \mathrm{GL}(V)$. For each extension F of \mathbb{Q}_p with integers \mathfrak{o} , the group $K = \mathbf{G}(\mathfrak{o})$ is a subgroup of (the group of F -points of) $\mathrm{GL}(V_F)$. If $H^2(\mathbf{G}(\mathbb{F}_q), \mathfrak{g}_{\mathbb{F}_q}) = 0$, the sequence in (b) is split and V_F is a nontrivial $F[\mathbf{G}(\mathbb{F}_q)]$ -module.

Since F has characteristic 0, it is well-known that the minimal dimension of a nontrivial $F[\mathbf{G}(\mathbb{F}_q)]$ module is bounded below by the value $f(q)$ of a polynomial $f \in \mathbb{Q}[x]$, depending only on G , for which $f(q) \rightarrow \infty$ as $q \rightarrow \infty$. We may choose q_0 such that $f(q) > \dim_{\mathbb{Q}_p} V$ for each $q > q_0$, and (c) follows at once.

(d) now follows from (c) and [CPSvdK77, Cor. 6.9]. □

6. Small simple modules for G_1 .

Combining results of [KLT99] with the results recorded in 2.4, we obtain some explicit results on G_1 cohomology of low dimensional simple modules:

Proposition. *Let L be a nontrivial simple G_1 module with $\dim \leq p$. Assume for some $i \geq 0$ that $H^i(G_1, L) \neq 0$. Then $\dim L = p - 1$. Moreover, there is a quadruple $(R, \lambda, i(0), V)$ in the following table for which R is the root system of G , λ the high weight of L , $i \geq i(0)$ and $H^{i(0)}(G_1, L)^{[-1]} \simeq V$ as G -modules.*

R	λ	$i(0)$	$H^{i(0)}(G_1, L)^{[-1]}$
A_1	$(p - 2)\varpi_1$	1	$L(\varpi_1)$
A_{p-2}	ϖ_1, ϖ_{p-2}	$p - 2$	$L(\lambda)$
$C_{(p-1)/2}$ p odd	ϖ_1	$p - 2$	$L(\lambda)$

Proof. By [Jan87, Prop. II.3.14], $L = \text{res}_{G_1}^G L(\lambda)$ for some restricted dominant weight $0 \neq \lambda$. Thus $L(\lambda)$ is a restricted, simple G module with dimension $\leq p$. It follows from Proposition 5.1 that $h \leq p$, that $\lambda \in \bar{C}$, and that $L = H^0(\lambda)$ as modules for G .

Suppose that $H^i(G_1, L) \neq 0$ for some i . By the linkage principle for G_1 (Proposition 3.4(b)), we must have $\lambda \in \widehat{W}_p \bullet 0$, hence $\lambda \in C$. This implies that $h < p$. Proposition 2.2 shows that $\lambda = (p - h)\varpi_i = w_0 w_i \bullet 0 + p\varpi_i$ for some $i \in J$, and Proposition 2.3 yields $\dim L = p - 1$ and lists the possible pairs (R, λ) .

For $h < p$, Kumar, Lauritzen and Thomsen [KLT99, Theorem 8] have extended a result of Andersen and Jantzen [AJ84, 3.7]; this result implies in particular that the minimal degree for which $H^*(G_1, L)$ is non-0 is $\ell(w_0 w_i)$, and that

$$H^{\ell(w_0 w_i)}(G_1, L)^{[-1]} \simeq H^0(\varpi_i).$$

It is straightforward to compute for each pair (R, λ) the length $\ell(w_0 w_i)$; one gets in this way the result. □

Remark. The Theorem implies the fact (used by Jantzen in the proof of [Jan96, Lemma 1.7]) that $H^1(G_1, L) = 0$ for all simple G_1 modules L with $\dim L \leq p$. The argument used by Jantzen there relied on the calculations of H^1 carried out in [Jan91].

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Received October 26, 2000. This work was supported by a grant from the National Science Foundation.

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