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Let v be a valuation of the quotient field of a noetherian local domain R. Assume that v is centered at R. This paper studies the structure of the value semigroup of v, S. Ideals defining toric varieties can be defined from the graded algebra $K[T]$ of cancellative commutative finitely generated semigroups such that $T \cap (-T) = \{0\}$. The value semigroup of a valuation S need not be finitely generated but we prove that $S \cap (-S) = \{0\}$ and so, the study in this paper can also be seen as a generalization to infinite dimension of that of toric varieties.

In this paper, we prove that $K[S]$ can be regarded as a module over an infinitely dimensional polynomial ring A_v. We show a minimal graded resolution of $K[S]$ as A_v-module and we give an explicit method to obtain the syzygies of $K[S]$ as A_v-module. Finally, it is shown that free resolutions of $K[S]$ as A_v-module can be obtained from certain cell complexes related to the lattice associated to the kernel of the map $A_v \to K[S]$.

1. Introduction.

Let (R, m) be a noetherian local domain. Denote by F its quotient field and by K its residue field. A valuation of F centered at R (a valuation in the sequel) is a mapping v of the multiplicative group of F onto a totally ordered commutative group G, such that the following conditions are satisfied:

1. $v(xy) = v(x) + v(y)$;
2. $v(x + y) \geq \min\{v(x), v(y)\}$;
3. v is nonnegative on R and strictly positive on m.

G is called to be the value group of the valuation v. The set $S := \{v(f) | f \in R \setminus \{0\}\}$ is a commutative semigroup called the *value semigroup* of the valuation v. Note that when $\dim R = 2$, the most known case, S contains a lot of information about v. Our aim, in this paper, is to study the structure of S by extending methods of the toric geometry.

Section 2 of the paper provides some basic properties of the semigroup S. S need not be finitely generated. However, it satisfies an interesting property: S is combinatorially finite, i.e., the number of decompositions of any element in S as a finite sum of others in S is finite. When S is finitely generated, this property is equivalent to $S \cap (-S) = \{0\}$. There exists an extensive literature [**5**, **2**, **1**, **4**], which studies the graded algebra $K[S]$ of cancellative commutative finitely generated semigroups S such that $S \cap (-S) = \{0\}$ (this study includes ideals defining toric varieties). Therefore, we devote Section 3 to extend to the S-graded algebra $K[S]$, now S being the value semigroup of a valuation, the ideas of the toric case. Essentially we use the fact that S is combinatorially finite, so our study can also be seen as a generalization to infinite dimension of that of toric varieties.

Subsection 3.1 deals with $K[S]$ regarded as a module over an infinitely dimensional, in general, polynomial ring A_v. Both $K[S]$ and A_v are S-graded. We construct a minimal graded resolution of $K[S]$ as A_v-module and prove that an explicit isomorphism can be given between the (finitely dimensional) vector space of degree α syzygies ($\alpha \in S$) and the vector space of augmented homology of a simplicial complex $\triangle_\alpha$ introduced in [**3**]. Furthermore, we give a combinatoric method, adapting the one in [**4**], that allows us to obtain $\widetilde{H}_i(\triangle_\alpha)$ explicitly from vector space complexes associated to directed graphs. These directed graphs are associated to partitions of certain finite subsets of a generating set of S. The most interesting situation arises when the partitions are induced by the value subsemigroup of S defined by a subring T of R such that v is also centered at T.

Subsection 3.2 is divided in two parts, their nexus being the fact that the kernel I_0 of the mapping $A_v \to K[S]$, which gives to $K[S]$ structure of A_v-module, is spanned by binomials. In 3.2.1, we characterize, by means of a graphic condition, when a set of binomials constitutes a minimal homogeneous generating set of I_0. On the other hand, I_0 is spanned by a set of binomials satisfying that the difference between their exponents is in a lattice L. L is intimately related to the value group G of the valuation (see the beginning of Section 3). In 3.2.2, we associate to L an A_v-module M_L and we show how suitable cell complexes on minimal generating sets of the A_v-module M_L give rise to free resolutions of M_L, called cellular ones, and how some of these resolutions allow us to get free resolutions of $K[S]$ as A_v-module.

2. The value semigroup of a valuation.

Let S be a commutative semigroup with a zero element. S is said to be a cancellative semigroup if it satisfies a cancellative law, i.e., if $\alpha, \beta, \gamma \in S$ and $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$. Associated to S, we can consider an abelian group $G(S)$ and a semigroup homomorphism $i : S \to G(S)$ satisfying the following universal property: If H is a commutative group and $j : S \to H$ a semigroup homomorphism, then there exists a unique group homomorphism

$h : G(S) \to H$ with $h \circ i = j$. Moreover, i is injective if, and only if, S is cancellative.

Consider the functions $l : S \to \mathbf{N} \cup \{\infty\}$ and $t : S \to \mathbf{N} \cup \{\infty\}$ given by

$$l(\alpha) := \sup \left\{ n \in \mathbf{N} | \alpha = \sum_{i=1}^{n} \alpha_i, \text{ where } \alpha_i \in S \setminus \{0\} \right\}$$

and

$$t(\alpha) := \operatorname{card} \left\{ \{\alpha_i\}_{i=1,2,\ldots,n} \text{ finite subset of } S \setminus \{0\} \mid \alpha = \sum_{i=1}^{n} \alpha_i \right\}.$$

It is clear that if $\alpha, \beta \in S$ then, $l(\alpha + \beta) \geq l(\alpha) + l(\beta)$, and also that $t(\alpha + \beta) \geq t(\alpha) + t(\beta)$.

Definition 1. A commutative semigroup with a zero element S is said to be combinatorially finite (C.F.) if $t(\alpha) < \infty$ for each α in S.

Proposition 1. *Let S be a* C.F. *semigroup. Then the following statements hold.*

i) *For each $\alpha \in S$, there is no infinite sequence $\{\alpha_i\}_{i=1}^{\infty}$ of elements in $S \setminus \{0\}$ such that $\alpha - \sum_{i=1}^{n} \alpha_i \in S$ whenever $n \geq 1$.*
ii) *$S \cap (-S) = \{0\}$, where $-S = \{-x \in G(S) | x \in S\}$.*

Proof.
i) If we had a sequence $\{\alpha_i\}_{i=1}^{\infty}$ as above, then S would not be a C.F. semigroup since $t(\alpha)$ would equal ∞.

ii) Assume that there exists $\alpha \in S \cap (-S)$, $\alpha \neq 0$. Write $\alpha_i = \alpha$ if i is an even number and $\alpha_i = -\alpha$ whenever α is an odd number. Then, the sequence $\{\alpha_i\}_{i=1}^{\infty}$ contradicts i).

Corollary 1. *Assume that S is a* C.F. *semigroup, then:*

i) $t(\alpha) = 0$ *if, and only if,* $\alpha = 0$.
ii) $l(\alpha) = 0$ *if, and only if,* $\alpha = 0$.

Proof. It is clear that $t(\alpha)$ and $l(\alpha)$ are not equal to 0 whenever $\alpha \neq 0$. Conversely, $t(0) \neq 0$ (or $l(0) \neq 0$) implies $0 = \sum_{i=1}^{n} \alpha_i$, $\alpha_i \in S \setminus \{0\}$, $n \geq 2$ and therefore $S \cap (-S) \neq \{0\}$ which contradicts Proposition 1.

Remark. Statement ii) in Proposition 1 allows us to prove the existence of a function $h : S \to \mathbf{N}$ satisfying $h(\alpha + \beta) = h(\alpha) + h(\beta)$ for $\alpha, \beta \in S$ and $h(\alpha) = 0$ if, and only if, $\alpha = 0$. When S is finitely generated, the above condition implies S combinatorially finite. As a consequence, both the existence of h and statements i) and ii) in Proposition 1 can be taken as a definition of C.F. finitely generated semigroup.

On the other hand, we can not interchange the functions t and l in Definition 1, since although $l(\alpha) < \infty$ for all $\alpha \in S$ holds whenever S be a

C.F. semigroup, the converse is not true. To see it, consider the additive semigroup $S = \mathbf{Z}_1 \bigoplus \mathbf{Z}$, where $\mathbf{Z}_1 = \{x \in \mathbf{Z} | x \geq 1\}$. Pick $\alpha = (x, y) \in S$, it is clear that the number of sums in a decomposition of α as a sum of elements in S is x or less. Therefore $l(\alpha) < \infty$. However, S is not a C.F. semigroup, because, for instance, $(2, 0) = (1, m) + (1, -m)$ for all $m \in \mathbf{Z}$.

Now, consider the value semigroup S of a valuation. Next theorem gives some interesting properties of S.

Theorem 1. *Let v be a valuation of F centered at R and denote by S (G) the value semigroup (group) of v. Then:*

i) *The groups $G(S)$ and G are equal. Therefore $G(S)$ is ordered.*
ii) *S is a cancellative ordered commutative semigroup which is torsion free.*
iii) *S is a* C.F. *semigroup.*

Proof.
i) G contains S and, since F is the quotient field of R, we have $G \subseteq G(S)$. Therefore $G = G(S)$.

ii) Denote by $R_v = \{f \in F \setminus \{0\} | v(f) \geq 0\}$ the valuation ring of v. R_v is a local ring and $m_v := \{f \in F \setminus \{0\} | v(f) > 0\}$ is its maximal ideal. Let $f \in F \setminus \{0\}$ be such that $v(f) \neq 0$. Then $v(f)$ (or $v(1/f)) > 0$, so f (or $1/f) \in m_v$ and thus f^p (or $1/f^p) \in m_v$ whenever $p \in \mathbf{N} \setminus \{0\}$. As a consequence, $v(f) \neq 0$ implies $v(f^p) \neq 0$. This proves that G is a torsion-free group. Finally, all the properties of S given in ii) are clear since S is a subsemigroup of G.

iii) Recall that the Krull dimension of R_v is usually called the rank of v $(rk(v))$ and that a v-ideal of R is the intersection of R with an ideal of R_v. R is a noetherian ring, therefore $rk(v) < \infty$ (see [**6**, App. 2]) and each v-ideal a is spanned by finitely many elements in R, i.e., $a = \langle h_1, h_2, \ldots, h_r \rangle$, $h_i \in R$ $(1 \leq i \leq r)$. If $\alpha = \min\{v(h_i) | i = 1, 2, \ldots, r\}$, then it is straightforward that $a = P_\alpha := \{f \in R | v(f) \geq \alpha\}$. So, the family $F = \{P_\alpha\}_{\alpha \in S}$ consists of all v-ideals of R.

To prove that S is C.F., we first assume that $rk(v) = 1$. Then F forms a simple infinite descending chain under inclusion [**6**, Lemma 3, App. 3] and therefore, the elements in S form a simple infinite ascending chain under the ordering in S. So S is C.F. Now, apply induction on the rank of v and assume that S is not C.F. Then, we can express $\alpha = \alpha_{1i} + \alpha_{2i}$, $\alpha, \alpha_{1i}, \alpha_{2i} \in S$ and the sets $\{\alpha_{1i}\}_{i=1}^{\infty}$ and $\{\alpha_{2i}\}_{i=1}^{\infty}$ are infinite. S is well-ordered since the set of v-ideals so is [**6**, App.3]. Consequently, rearranging the sets $\{\alpha_{1i}\}_{i=1}^{\infty}$ and $\{\alpha_{2i}\}_{i=1}^{\infty}$, we obtain that one of them constitutes a simple infinite descending chain. To show that this fact is not possible, we only need to observe that v can be written $v = u \circ w$, where u is of rank $rk(v) - 1$ and w is a rank one valuation of the residue field of u and then, apply induction and the corollary of [**6**, App. 3], which asserts that if $b_2 \subset b_1$ are two consecutive u-ideals, then the v-ideals a such that $b_2 \subset a \subset b_1$ are either finite in number or form a simple descending infinite sequence.

In the sequel, S will denote the value semigroup of a valuation. An element $\alpha \in S$ is said to be irreducible if $l(\alpha) = 1$. Then, we can state the following:

Corollary 2. *The semigroup S is generated by its irreducible elements. This set need not be finite.*

Proof. The first statement is clear since S is C.F. Now consider a valuation v centered at a regular 2-dimensional noetherian local ring. Assume that the rank and the rational rank of v equal 1 and that the transcendence degree of v is 0. Finally, suppose that the value group of v is not isomorphic to $\mathbf{Z}$, then, S has an infinite minimal system of generators. These generators are exactly the irreducible elements of S which concludes the proof.

3. The semigroup algebra of a valuation.

Let v be a valuation. Denote by S its value semigroup. The *semigroup algebra* of v is the semigroup K-algebra associated to S and it will be denoted by $K[S]$. $K[S]$ is the S-graded K-algebra $K[S] = \bigoplus_{\alpha\in S}(K[S])_\alpha$, $(K[S])_\alpha := K\alpha$.

Denote by Λ a minimal set of generators of S as semigroup. For instance, we can think of Λ as the set of irreducible elements in S. Λ is, in general, an infinite set. For a set $\mathbf{T}$, write $\mathbf{T}^{(\Lambda)} = \bigoplus_{\lambda\in\Lambda}\mathbf{T}_\lambda$ where $\mathbf{T}_\lambda = \mathbf{T}$. Consider the mapping $\psi : \mathbf{Z}^{(\Lambda)} \to G(S)$ given by $\psi(e_\lambda) = \lambda$, $\{e_\lambda\}_{\lambda\in\Lambda}$ being the standard basis of the $\mathbf{Z}$-module $\mathbf{Z}^{(\Lambda)}$. The ordering in $G(S)$ gives to $\mathbf{Z}^{(\Lambda)}$ an structure of lattice. The kernel of ψ, L, is a sublattice of $\mathbf{Z}^{(\Lambda)}$ whose intersection with $\mathbf{N}^{(\Lambda)}$ is the origin 0. This can be easily deduced from the fact that $S \cap (-S) = \{0\}$. The morphism ψ induces a surjective K-algebra homomorphism $\phi_0 : K[\mathbf{N}^{(\Lambda)}] \to K[S]$ which allows to regard $K[S]$ as a $K[\mathbf{N}^{(\Lambda)}]$-module. We shall use two approaches to study the semigroup algebra of v. Firstly, we shall construct a minimal free resolution of the $K[\mathbf{N}^{(\Lambda)}]$-module $K[S]$ and we shall study its syzygy modules by means of a concrete simplicial complex and secondly, we shall obtain minimal free resolutions of the former module from certain type of cell complexes on the lattice module $M_L = K[\mathbf{N}^{(\Lambda)} + L] \subseteq K[\mathbf{Z}^{(\Lambda)}]$. In particular, we shall get a more explicit free resolution of $K[S]$.

3.1. Syzygies of the semigroup algebra.

3.1.1. . For a start, we state a basic result for the development of this subsection. It holds for semigroups S satisfying $l(\alpha) < \infty$ for all nonzero element $\alpha \in S$. Thus, we can use it in our case: S is the value semigroup of a valuation. Let A be an S-graded ring $A = \bigoplus_{\alpha \in S} A_\alpha$ and $M = \bigoplus_{\alpha \in S} M_\alpha$ an S-graded A-module.

Proposition 2 (Graded Nakayama's Lemma). *Let A and M be as above. Denote by $m = \bigoplus_{\alpha \in S, \alpha \neq 0} A_\alpha$ the irrelevant ideal of A. If $mM = M$, then $M = 0$.*

Proof. If $M \neq 0$, then there exists an element $\beta \in S$ such that the degree β homogeneous component of M, M_β, does not vanish. Now $M_\beta = (mM)_\beta$ proves that β can be written $\beta = \delta + \gamma$; $\delta, \gamma \in S$ and $M_\gamma \neq 0$. Iterating, we conclude that $l(\beta)$ is not finite, which is a contradiction.

Now consider the K-algebra $K[\mathbf{N}^{(\Lambda)}]$ which, for the sake of simplicity, will be expressed as a polynomial ring $K[\{X_\lambda\}_{\lambda \in \Lambda}]$ with, possibly, infinitely many indeterminates and it will be denoted by A_v. A_v is S-graded if we give degree $\lambda \in S$ to the indeterminate X_λ and so, we can express $A_v = \bigoplus_{\alpha \in S} (A_v)_\alpha$, where $(A_v)_\alpha$ denotes the homogeneous component of degree α of A_v. $(A_v)_\alpha$ is a K-vector space. Note that, for any semigroup S, we have that S is C.F. if, and only if, $\dim_K (A_v)_\alpha < \infty$ and $l(\alpha) < \infty$ for all $\alpha \in S$. Denote by M_v the irrelevant ideal of A_v and by I_0 the kernel of ϕ_0. I_0 is a homogeneous ideal of A_v. Let B be a minimal homogeneous generating set of I_0 and denote by B_α the set of elements in B of degree α. Applying Proposition 2, it is straightforward to deduce that the set of classes in $I_0/M_v I_0$ of the elements of B_α is a basis of the vector space of the homogeneous component of degree α of $I_0/M_v I_0$. B_α is a finite set since $(A_v)_\alpha$ is a finite-dimensional vector space. Set $B_\alpha = \{Q_1, Q_2, \ldots, Q_{d(\alpha)}\}$ and $L_1 := \bigoplus_{\alpha \in S} (A_v)^{d(\alpha)}$. If $\phi_{1,\alpha} : (A_v)^{d(\alpha)} \to A_v$ is the A_v-module homomorphism given by $\phi_{1,\alpha}(a_1, a_2, \ldots, a_{d(\alpha)}) = \sum_{i=1}^{d(\alpha)} a_i Q_i$, then we have the A_v-module homomorphism $\phi_1 : L_1 \to A_v$, $\phi_1 = \sum_{\alpha \in S} \phi_{1,\alpha}$. We give degree α to the generators of $(A_v)^{d(\alpha)}$, thus L_1 is an S-graded free A_v-module and ϕ_1 a homogeneous homomorphism of degree 0. Repeating this procedure for each syzygy module $I_i := \mathrm{Ker}\phi_i$, we get a minimal free resolution of the S-graded A_v-module $K[S]$:

$$\cdots \to L_i \xrightarrow{\phi_i} L_{i-1} \to \cdots \to L_1 \xrightarrow{\phi_1} A_v \to K[S] \to 0.$$

Tensoring by K, we note that there exists a homogeneous degree 0 isomorphism of S-graded A_v-modules between the i-th *Tor* module $Tor_i^{A_v}(K[S], K)$ and $L_i \bigotimes_{A_v} K$, $i \geq 0$.

On the other hand, we can consider a generalized Koszul complex as follows:

$$\cdots \to \bigwedge^p A_v^{(\Lambda)} \xrightarrow{d_p} \bigwedge^{p-1} A_v^{(\Lambda)} \to \cdots \to A_v^{(\Lambda)} \xrightarrow{d_1} A_v \xrightarrow{d_0} K \to 0, \tag{1}$$

d_0 is the natural obvious epimorphism and if $\{e_\lambda\}_{\lambda\in\Lambda}$ is the standard basis of the A_v-module $A_v^{(\Lambda)}$, then we have

$$d_p(e_J)=\sum_{r=1}^{p}(-1)^r X_{\lambda_r} e_{J\setminus\{\lambda_r\}},$$

where $e_J=e_{\lambda_1}\wedge e_{\lambda_2}\wedge\cdots\wedge e_{\lambda_p}$ whenever $J=\{\lambda_1,\lambda_2,\ldots,\lambda_p\}\subseteq\Lambda$. $\bigwedge^p A_v^{(\Lambda)}$ can be regarded as an S-graded A_v-module by giving to e_J the degree $\sum_{r=1}^{p}\lambda_r$. Thus (1) is an S-graded free resolution where all the homomorphisms are homogeneous of degree 0.

We shall write $K[S].(\Lambda)$ for the complex obtained by tensoring (1) through with $K[S]$:

$$\cdots\to\bigwedge^{p}(K[S])^{(\Lambda)}\xrightarrow{e_p}\bigwedge^{p-1}(K[S])^{(\Lambda)}\to\cdots\to K[S]\xrightarrow{e_0}K\bigotimes_{A_v}K[S]\to 0.$$

The formula for e_p is the same one as d_p but replacing X_{λ_r} by λ_r. Furthermore the homomorphisms e_p are homogeneous of degree 0 under the induced gradings. As a consequence, taking into account the commutative property of the *Tor* functor, there exists a homogeneous degree 0 isomorphism of S-graded A_v-modules between the i-th *Tor* module $Tor_i^{A_v}(K,K[S])$ and the i-th homology module $H_i(K[S].(\Lambda))$.

Finally, for each $\alpha\in S$, we give a K-vector space complex isomorphic to that of homogeneous components of degree α in $K[S].(\Lambda)$. Denote by $P(\Lambda)$ the power set of Λ, $P(\Lambda)$ is an abstract simplicial complex. Set

$$\triangle_\alpha:=\left\{J\subseteq\Lambda | J \text{ is a finite subset of } \Lambda \text{ and } \alpha-\sum_J\in S\right\},$$

where $\sum_J=\sum_{\lambda\in J}\lambda$. $\triangle_\alpha$ is a simplicial subcomplex of $P(\Lambda)$. Associate to $\triangle_\alpha$, we consider the complex of vector spaces $C.(\triangle_\alpha)$ such that its vector spaces are $C_i(\triangle_\alpha)=\bigoplus_{J\in\triangle_\alpha,\mathrm{card}(J)=i+1}KJ$, $i\geq -1$ and its boundaries $\partial: C_i(\triangle_\alpha)\to C_{i-1}(\triangle_\alpha)$ are given by $\partial(J)=\sum_{\beta\in J}(-1)^{\eta_J(\beta)}J\setminus\{\beta\}$, where $\eta_J(\beta)$ denotes the number of place that β has among the elements in J. The homology of this complex will be called the augmented homology of $\triangle_\alpha$. This subsection can be summarized in the following:

Theorem 2. *For each $\alpha\in S$, there exists an explicit isomorphism of K-vector spaces between the vector space $(I_i)_\alpha/(M_vI_i)_\alpha$ of i-th syzygies of degree α of $K[S]$ as A_v-module and the i-th augmented homology vector space of the simplicial complex $\triangle_\alpha$, $\widetilde{H}_i(\triangle_\alpha)$.*

3.1.2. . We devote this subsection to show how bases for the homology $\widetilde{H}_i(\triangle_\alpha)$ can be explicitly computed from bases of the homology of vector space complexes associated to directed graphs which depend on the set Λ.

This will be done adapting the results by Campillo and Gimenez in the case of toric affine varieties [4].

To start with, we describe the type of vector space complexes which we shall use to compute $\widetilde{H}_i(\triangle_\alpha)$. Assume that Γ is a subset of Λ, which is a finite set of generators of a semigroup T, and B a subset of T. We shall call the directed graph of T associated to the pair (Γ, B) to the directed graph $G_{\Gamma B}(T)$ (denoted $G_{\Gamma B}$ if it does not cause confusion) whose vertex set is $\{m \in T | m - \sum_L \in B$ for some subset $L \subseteq \Gamma\}$ and such that (m, m') is an edge iff $m' = m + \gamma$ for some $\gamma \in \Gamma$. A K-vector space complex $C.(G_{\Gamma B}(T, m))$ can be associated to the pair $(G_{\Gamma B}, m)$, m being a vertex of $G_{\Gamma B}$, if the following condition holds: Whenever $b \in B$ and $\lambda, \lambda' \in \Gamma$ satisfy $b + \lambda + \lambda' \in B$, then $b + \lambda \in B$ and $b + \lambda' \in B$. In such a case $G_{\Gamma B}$ is called to be a chain graph. Each vector space $C_i(G_{\Gamma B}(T, m))$, $i \geq -1$, is equal to $\bigoplus KL$ where the sum is over all subsets L of Γ of cardinality $i + 1$ such that $m - \sum_L \in B$. The boundaries are induced by those of the simplicial complex $P(\Lambda)$.

Next, we state the main result of this subsection.

Theorem 3. *The homology $\widetilde{H}_i(\triangle_\alpha)$ can be explicitly reached from finitely many homologies of K-vector space complexes of the type $C.(G_{\Gamma B}(T, m))$ for suitable T, Γ, B and m.*

To reach a homology from others means to obtain bases of the homology from bases of the others by means of exact sequences. Let's see how to reach $\widetilde{H}_i(\triangle_\alpha)$. Let $\overline{S}_\alpha = \{\alpha' \in S | \alpha - \alpha' \in S\}$. $\overline{S}_\alpha$ is finite since S is C.F. Denote by S_α the subsemigroup of S spanned by $\overline{S}_\alpha$. It is not difficult to prove that $\triangle_\alpha = \{J \subseteq \overline{S}_\alpha | \alpha - \sum_J \in S_\alpha\}$. Now, pick a partition of $\overline{S}_\alpha$, $\overline{S}_\alpha = \Omega_\alpha \cup \Pi_\alpha$, consider the Apery set of $\overline{S}_\alpha$ relative to Π_α:

$$A(\alpha) = A = \{a \in S_\alpha | a - e \notin S_\alpha \text{ for all } e \in \Pi_\alpha\}$$

and the related set

$$K_\alpha := \left\{L \subseteq \overline{S}_\alpha | L \cap \Pi_\alpha \neq \emptyset \text{ and } \alpha - \sum_L \in S_\alpha\right\} \cup \left\{L \subseteq \Omega_\alpha | \alpha - \sum_L \in S_\alpha \setminus A\right\}.$$

There is no loss of generality in assuming that α is a vertex of $G_{\Omega_\alpha A}(S_\alpha)$ and then, it is clear that the complex associate to $(G_{\Omega_\alpha A}, \alpha)$ makes sense. It will be denoted $C.(A(\alpha))$ and it is exactly the augmented relative simplicial complex $\widetilde{C}.(\triangle_\alpha, K_\alpha)$. Therefore, we can state the following long exact sequence, which allows to reach the homology $\widetilde{H}_i(\triangle_\alpha)$ from others.

$$(2) \quad \cdots \to H_{i+1}(A_\alpha) \to \widetilde{H}_i(K_\alpha) \to \widetilde{H}_i(\triangle_\alpha) \to H_i(A_\alpha) \to \widetilde{H}_{i-1}(K_\alpha) \to \ldots$$

$H_{i+1}(A_\alpha)$ and $H_i(A_\alpha)$ are as we desire. Let us see that $\widetilde{H}_i(K_\alpha)$ and $\widetilde{H}_{i-1}(K_\alpha)$ so are. Firstly, define the simplicial complex

$$\overline{K}_\alpha := K_\alpha \cup \left\{ L = I \cup J \;\middle|\; I \subseteq \Omega_\alpha, J \subseteq \Pi_\alpha, \mathrm{card}\,(J) \geq 2, \alpha - \sum_{I \cup J} \notin S_\alpha \right.$$
$$\left. \text{but} \;\; \alpha - \sum_I - e \in S_\alpha \;\; \text{for each} \;\; e \in J \right\}$$

and the subcomplexes of $\overline{K}_\alpha$,

$$K_\alpha(j) := K_\alpha \cup \{L = I \cup J \in \overline{K}_\alpha \setminus K_\alpha \,|\, \mathrm{card}\,(J) \leq j\},$$

$1 \leq j \leq \mathrm{card}\,(\Pi_\alpha)$. $\overline{K}_\alpha$ is acyclic and so $\widetilde{H}_{i+1}(\overline{K}_\alpha, K_\alpha) \cong \widetilde{H}_i(K_\alpha)$. Also $\widetilde{H}_i(\overline{K}_\alpha, K_\alpha) \cong \widetilde{H}_i(K_\alpha(\mathrm{card}\,(\Pi_\alpha)), K_\alpha(1))$. This last homology can be reached from $\widetilde{H}_i(K_\alpha(j), K_\alpha(j-1))$, $2 \leq j \leq \mathrm{card}\,(\Pi_\alpha)$, since the following exact sequence of vector space complexes

$$0 \to C.(K_\alpha(j), K_\alpha(i)) \to C.(K_\alpha(k), K_\alpha(i)) \to C.(K_\alpha(k), K_\alpha(j)) \to 0$$

holds for sequences (i, j, k) equal to $(1, 2, 3), (1, 3, 4), \ldots,$ $(1, \mathrm{card}\,(\Pi_\alpha) - 1,$ $\mathrm{card}\,(\Pi_\alpha))$. As a consequence, we only need to show that the homology $\widetilde{H}_i(K_\alpha(j), K_\alpha(j-1))$ can be computed from finitely many homologies of complexes associated to chain graphs. Indeed, a subset $J \subseteq \Pi_\alpha$ with $\mathrm{card}\,(J) \geq 2$ is said to be associated to $d \in S_\alpha$, if $d - \sum_J \notin S_\alpha$ but $d - e \in S_\alpha$ for each $e \in J$. If we denote by D_α^J the set of elements d in S_α such that J is associated to d, then

$$\widetilde{H}_i(K_\alpha(j), K_\alpha(j-1)) \cong \bigoplus_{J \subseteq \Pi_\alpha, \mathrm{card}\,(J) = j} H_{i-j}\left(G_{\Omega_\alpha D_\alpha^J}(S_\alpha, \alpha)\right).$$

A further study leads us to obtain finite subsets of S_α, such that $\widetilde{H}_i(\triangle_\alpha)$ vanishes when α does not belong to them. In fact, for $-1 \leq l \leq \mathrm{card}\,(\Omega_\alpha)$ write

$$M_\alpha(l) := K_\alpha \cup \{L = I \cup J \in \overline{K}_\alpha \setminus K_\alpha \,|\, \mathrm{card}\,(I) \leq l\}.$$

As above,

$$\widetilde{H}_i(\overline{K}_\alpha, K_\alpha) \cong \widetilde{H}_i(M_\alpha(\mathrm{card}\,(\Omega_\alpha)), M_\alpha(-1)). \tag{3}$$

This last homology can be reached from $\widetilde{H}_i(M_\alpha(l), M_\alpha(l-1))$ and

$$\widetilde{H}_i(M_\alpha(l), M_\alpha(l-1)) \cong \bigoplus \widetilde{H}_{i-l}(\Theta_{\alpha - \sum_I}),$$

where the sum is over all subsets $I \subseteq \Omega_\alpha$ such that $\mathrm{card}\,(I) = l$ and $\alpha - \sum_I \in S_\alpha$, and where $\Theta_d = \{J \subseteq \Pi_\alpha \,|\, d - \sum_J \in S_\alpha\}$. Consequently, (2) and (3) prove that if we consider

$$C_i(\alpha) := \left\{ m \in S_\alpha \;\middle|\; m = a + \sum_I ; a \in A(\alpha), I \subseteq \Omega_\alpha \text{ and card}\,(I) = i+1 \right\}$$
$$\cup \left\{ m \in S_\alpha \mid \exists I \subseteq \Omega_\alpha, \text{card}\,(I) = l \leq i \text{ with } \widetilde{H}_{i-l}\left(\Theta_{m-\sum_I}\right) \neq 0 \right\},$$

then $\widetilde{H}_i(\triangle_\alpha) = 0$ if $\alpha \notin C_i(\alpha)$. The simplicity of the set Θ_d has an important consequence:

Proposition 3 (See [4, Pr. 6.2])**.** *The set $C_i(\alpha)$ is finite when we choose a suitable partition of the set $\overline{S}_\alpha$.*

A crucial fact in the above proposition is that S_α is finitely generated. A suitable partition of $\overline{S}_\alpha$ would be a convex partition, that is, a partition $\overline{S}_\alpha = \Omega_\alpha \cup \Pi_\alpha$ where the cone generated by S_α (in $V_{S\alpha} := G(S_\alpha) \bigotimes_{\mathbf{Z}} \mathbf{Q}$) is equal to the cone generated by Ω_α (in $V_{S\alpha}$) and card$\,(\Omega_\alpha)$ equals to the number of extremal rays of the cone spanned by S_α.

3.2. The defining ideal of the semigroup. The K-algebra $K[S]$ is isomorphic to A_v/I_0. The ideal I_0, usually called the defining ideal of S, is spanned by a set of binomials which are difference of two monomials of the same degree. This set need not be finite. In the first part of this subsection, we shall use [2] to give a method to compute a minimal homogeneous generating set of I_0, B, formed by binomials of the type described above. This method uses the structure of graph of the simplicial complex $\triangle_\alpha$. On the other hand, denote by $L_v = K[\{X_\lambda^{\pm 1}\}_{\lambda\in\Lambda}]$ the Laurent polynomial ring associate to the set Λ and write $X^a = \prod_{\lambda\in\Lambda'} X_\lambda^{a_\lambda} \in L_v$ whenever $a = \sum_{\lambda\in\Lambda'} a_\lambda e_\lambda \in \mathbf{Z}^{(\Lambda)}$, Λ' being a finite subset of Λ. Obviously, $A_v \subset L_v = K[\mathbf{Z}^{(\Lambda)}]$. Recalling the notation at the beginning of Section 3, we observe that

$$I_0 = \langle X^a - X^b \,|\, a - b \in L \rangle \subset A_v. \tag{4}$$

Following the ideas of [1], this fact will serve us, in the second part of this subsection, to obtain minimal free resolutions of $K[S]$ as A_v-module from suitable cell complexes on M_L.

3.2.1. Minimal generating sets of the defining ideal. A minimal homogeneous generating set of I_0, B, can be expressed $B = \cup_{\alpha\in S} B_\alpha$, where B_α is the set of elements in B of degree α. As a consequence of 3.1.1, we have that B_α is a finite set and card $B_\alpha = \dim_K \widetilde{H}_0(\triangle_\alpha)$. Moreover, $\triangle_\alpha$ is a graph which has $\dim_K \widetilde{H}_0(\triangle_\alpha) + 1$ connected components. If $a = \sum_{\lambda\in\Lambda'} a_\lambda e_\lambda \in \mathbf{N}^{(\Lambda)}$ ($a_\lambda \neq 0$), then $X^a \in A_v$, the support of X^a, Supp$\,(X^a)$, is the set Λ' and the degree of X^a, $\deg(X^a)$, is $\sum_{\lambda\in\Lambda'} a_\lambda \lambda \in S$.

It is clear that I_0 is an ideal generated by the set of binomials $\mathcal{B} = \{X^a - X^b \,|\, \deg(X^a) = \deg(X^b)\}$. Let C be a subset of $\mathcal{B}$ whose binomials have a fixed degree α. We shall call graph associated to C to a graph whose vertex set is the set of connected components of $\triangle_\alpha$ which contain the support of a monomial belonging to a binomial in C. Two connected components, those associated to the monomials X^a and X^b, are adjacent by an edge whenever $X^a - X^b \in C$. C will be a *generating tree* for $\triangle_\alpha$ if the graph associated to C is, in fact, a tree.

Theorem 4. *A subset $B = \cup_{\alpha\in S} B_\alpha \subseteq \mathcal{B}$ is a minimal homogeneous generating set of I_0 if, and only if, B_α is a generating tree for $\triangle_\alpha$ whenever $\dim_K \widetilde{H}_0(\triangle_\alpha) \neq 0$ and $B_\alpha = \emptyset$, otherwise.*

This theorem is analogous to the stated in [**2**] for finitely generated semigroups and the proof runs similarly. It is based on the fact that two monomials M and M' of degree $\alpha \in S$ satisfy $M - M' \in (M_v I_0)_\alpha$ if, and only if, $\mathrm{Supp}\,(M)$ and $\mathrm{Supp}\,(M')$ are in the same connected component of $\triangle_\alpha$. Furthermore, it is possible to decide whether $\dim_K \widetilde{H}_0(\triangle_\alpha) \neq 0$ by a close method to that given in [**2**, Th. 3.11].

3.2.2. Cellular resolutions of $K[S]$. For a start, we establish a relation between the module $M_L = K[\mathbf{N}^{(\Lambda)} + L]$ and the semigroup algebra of v, $K[S]$. Denote by $A_v[L]$ the group algebra of L over A_v. $A_v[L]$ is the subalgebra of $K[\{X_\lambda\}_{\lambda\in\Lambda}, \{Z_\lambda^{\pm 1}\}_{\lambda\in\Lambda}]$ generated by the monomials $X^a Z^l$ where $a \in \mathbf{N}^{(\Lambda)}$ and $l \in L$. Thus, we can give a $\mathbf{Z}^{(\Lambda)}$-grading on $A_v[L]$ by writing $\deg(X^a Z^l) = a + l$. On the other hand, the morphism $h : A_v[L] \to M_L$, $X^a Z^l \to X^{a+l}$ gives to M_L an structure of $\mathbf{Z}^{(\Lambda)}$-graded $A_v[L]$-module. Moreover, if $J = \mathrm{Ker}(h)$, then the following equality chain holds,

$$M_L \bigotimes_{A_v[L]} A_v = A_v[L]/J \bigotimes_{A_v[L]} A_v = A_v/I_0 = K[S].$$

Next, we shall consider two equivalent categories $\mathcal{A}$ and $\mathcal{B}$. $\mathcal{A}$ contains M_L, and $K[S]$, viewed as A_v-module, is in $\mathcal{B}$. This shall give the desired relation between M_L and $K[S]$. $\mathcal{A}$ will be the category of $\mathbf{Z}^{(\Lambda)}$-graded $A_v[L]$-modules, where the morphisms are $\mathbf{Z}^{(\Lambda)}$-graded $A_v[L]$-module homomorphisms of degree 0, and $\mathcal{B}$ the category of $G(S)$-graded A_v-modules, where the morphisms are, also, of degree 0. Note that $K[S]$ is S-graded and therefore $G(S)$-graded. The functor $\pi : \mathcal{A} \to \mathcal{B}$ which gives the equivalence is $\pi(M) = M \bigotimes_{A_v[L]} A_v$. Notice that if $M \in \mathcal{A}$, $M = \bigoplus_{a\in\mathbf{Z}^{(\Lambda)}} M_a$, then π identifies as $\pi(M)_\alpha$, $\alpha \in G(S)$, all the vector spaces M_a such that $\psi(a) = \alpha$, where ψ is the mapping given at the beginning of Section 3. A complete proof of this equivalence is similar to that of the case of finitely generated semigroups [**1**, Th. 3.2] and we omit it.

Now, taking into account that the degrees of M_L are in $\mathbf{N}^{(\Lambda)} + L$, we can state:

Theorem 5. *Let $\pi : \mathcal{A} \to \mathcal{B}$ be the equivalence of categories above given. Then π transforms $\mathbf{Z}^{(\Lambda)}$-graded (minimal) free resolutions of M_L as $A_v[L]$-module into S-graded (minimal) free resolutions of $K[S]$ as A_v-module, and conversely.*

Finally, we shall see how to get free resolutions of M_L from regular cell complexes and, consequently, how to get free resolutions of $K[S]$. First at all, denote by $\leq$ the ordering in $\mathbf{Z}^{(\Lambda)}$ defined so: $a \leq b$ if, and only if, $b-a \in \mathbf{N}^{(\Lambda)}$. Also, set $\min(M_L) := \{X^a \in M_L \,|\, X^a/X_\lambda \notin M_L \text{ for all } \lambda \in \Lambda\}$.

Proposition 4. *The $\mathbf{Z}^{(\Lambda)}$-graded A_v-module M_L satisfies the following properties:*

i) *The set of monomials in M_L of degree $\leq a$ is finite for each $a \in \mathbf{Z}^{(\Lambda)}$.*
ii) *M_L is generated as A_v-module by the set* $\min(M_L)$.

Proof.
i) Write $a = \sum_{\lambda \in \Lambda' \subset \Lambda} a_\lambda e_\lambda$ and set $a^+ = \sum_{\lambda \in \Lambda', a_\lambda > 0} a_\lambda e_\lambda$ and $a^- = \sum_{\lambda \in \Lambda', a_\lambda < 0} a_\lambda e_\lambda$. If d is the degree of a monomial in M_L, then $d = l + b^+$, where $l \in L$ and $b^+ \in \mathbf{N}^{(\Lambda)}$. It is clear that, as above, $l = l^+ + l^-$ where $\psi(l^+) = -\psi(l^-) \in S$. So, $d \leq a$ if, and only if, $l^+ + b^+ + l^- \leq a^+ + a^-$. As a consequence the set $\{l^+ \,|\, d \leq a\}$ is finite and so is the set $\{\psi(l^+) \,|\, d \leq a\} \subseteq S$. Finally, $\{l^- \,|\, d \leq a\}$ is also a finite set, since S is a C.F. semigroup.
ii) This is a straightforward consequence of the fact that, there is no infinite decreasing sequence under divisibility of monomials in M_L, which follows from i).

Put $\min(M_L) = \{X^a \,|\, a \in I \subset \mathbf{Z}^{(\Lambda)}\}$. I is, generally, an infinite set. Consider a regular cell complex X such that I is its set of vertices and ϵ an incidence function on pairs of faces. A typical example of a regular cell complex is the set of faces of a convex polytope.

Associated to X, a cellular complex of A_v-modules $M.(X)$ can be defined in the following way: The modules are $M_i(X) = \bigoplus_{J \in X, \dim J = i} A_v J$, $i \geq 0$, (we have identified the face J in X with its set of vertices) and the boundaries are given by

$$\partial J = \sum_{J' \in X, J' \neq \emptyset} \epsilon(J, J')(m_J/m_{J'})J',$$

where m_J is the least common multiple of the set $\{X^a \,|\, a \in J\}$. $M.(X)$ is $\mathbf{Z}^{(\Lambda)}$-graded, the degree of a face J being the exponent vector of m_J. When $M.(X)$ is a free resolution of M_L, it is called to be a *cellular resolution* of M_L. Set $\triangle = \{J \in P(I) \,|\, J \text{ is a finite set}\}$ and associate to $\triangle$ an incidence function as in the definition of $\triangle_\alpha$ (see 3.1.1). $\triangle$ is a cell complex and its associated cellular complex $M.(\triangle)$ is a cellular resolution of M_L called the *Taylor resolution* of M_L. This is an easy consequence of the fact that the subcomplex $\triangle_{\leq a}$ of $\triangle$ on the vertices of degree $\leq a$ is acyclic for all $a \in \mathbf{N}^{(\Lambda)}$.

We desire to apply Theorem 5 to get free resolutions of $K[S]$. In order to do it, we observe that the mapping $\bigoplus_{J\in\mathcal{R}} A_v[L]J \to M_i(X)$, $Z^l J \to J+l$ is an isomorphism of $\mathbf{Z}^{(\Lambda)}$-graded A_v-modules if X satisfies that

$$J + l \in X \text{ whenever } J \in X \text{ and } l \in L, \tag{5}$$

$\mathcal{R}$ being a set of representatives of the set of i-dimensional orbits defined by the action of L over X. Thus, we shall call to X *equivariant* if it satisfies (5) and $\epsilon(J, J') = \epsilon(J + l, J' + l)$ for all $l \in L$. If X is equivariant, it is straightforward that $M.(X)$ is a $\mathbf{Z}^{(\Lambda)}$-graded complex of $A_v[L]$-modules and that $M.(X)$ is exact over A_v if, and only if, it is exact over $A_v[L]$. In this case, $M.(X)$ is called an *equivariant cellular resolution* of M_L. Applying Theorem 5, we have proved the following:

Theorem 6. *Let S be the value semigroup of a valuation. If $M.(X)$ is a (minimal) equivariant cellular resolution of M_L, then $\pi(M.(X))$ is a (minimal) free resolution of $K[S]$ as A_v-module.*

$\triangle$ is an equivariant cell complex. Its simplicity allows us to give an explicit resolution of $K[S]$ as A_v-module. For each $\alpha \in S$, denote by $\mathrm{mon}(A_v)_\alpha$ the set of monomials in $(A_v)_\alpha$ and by $E_i(\alpha)$ the set of cardinality i subsets of $\mathrm{mon}(A_v)_\alpha$ whose greatest common divisor is 1. Now, if $F_i(a)$ denotes the set of cardinality i subsets of $\min(M_L)$ whose least common multiple is $a \in \mathbf{Z}^{(\Lambda)}$, it is clear, from the definition of $M.(\triangle)$, that $M_i(\triangle) = \bigoplus_{a\in\mathbf{N}^{(\Lambda)}+L} A_v F_i(a)$.

Regarding $M_i(\triangle)$ as $A_v[L]$-module and by Theorem 5, it is clear that π takes $F_i(a)$ bijectively to $E_i(\psi(a))$, $\pi(J) = \{X^a/X^c \mid X^c \in J\}$. As a consequence $\pi(M.(\triangle))$ can be expressed so: The A_v-modules are $\bigoplus_{\alpha\in S} A_v E_i(\alpha)$ and the boundaries are given by

$$\partial(I) = \sum_{X^c\in I} (-1)^{\eta_I(X^c)} \gcd(I \setminus \{X^c\})[I \setminus \{X^c\}],$$

where $I \in E_i(\alpha)$, η_I is defined as in 3.1.1 and $[I \setminus \{X^c\}]$ means to remove the common factor $\gcd(I \setminus \{X^c\})$ from $I \setminus \{X^c\}$.

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