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FIXED POINT SPACES IN ACTIONS OF EXCEPTIONAL ALGEBRAIC GROUPS

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Let G be a simple algebraic group of exceptional type acting transitively on an algebraic variety. We provide estimates for the dimensions of the subvarieties of fixed points of elements of G. These translate into estimates for the dimensions of intersections of conjugacy classes of G with closed subgroups.

Introduction.

In this paper we consider actions of simple algebraic groups of exceptional Lie type over algebraically closed fields. Let G be such a group, so that G is of type G_2, F_4, E_6, E_7 or E_8 over an algebraically closed field K of arbitrary characteristic. Suppose that M is a closed subgroup of G, and denote by Ω the coset variety G/M on which G acts transitively. For $x \in G$, the fixed point space

$$\mathrm{fix}_\Omega(x) = \{\omega \in \Omega : \omega x = \omega\}$$

is a subvariety of Ω, and we are interested in investigating its codimension, which we denote by

$$f(x, \Omega) = \dim \Omega - \dim \mathrm{fix}_\Omega(x).$$

Theorems 1 and 2 below provide lower bounds for $f(x, \Omega)$ for all x, Ω as above.

There are a number of motivations for studying this problem. The value of $f(x, \Omega)$ gives some measure of how much of the space Ω is fixed by x, and of course if we know $\dim \Omega$ then lower bounds for $f(x, \Omega)$ give corresponding upper bounds for $\dim \mathrm{fix}_\Omega(x)$. Moreover, in Proposition 1.14 below we prove that if $x \in M$ and x^G denotes the conjugacy class of x in G, then

$$f(x, G/M) = \dim x^G - \dim(x^G \cap M)$$

(note that $x^G \cap M$ is open in $\overline{x^G} \cap M$, hence is a variety and has a dimension — see [**16**]). Hence our bounds for $f(x, G/M)$ translate into bounds for the dimensions of intersections of conjugacy classes of exceptional algebraic groups with closed subgroups.

Originally, though, our motivation came from a problem about finite groups. If X is a finite group acting on a set Δ, then for $x \in X$ the quantity analogous to $-f(x,\Omega)$ is the *fixed point ratio*

$$\mathrm{fpr}(x,\Delta) = \frac{\mathrm{fix}_\Delta(x)}{|\Delta|},$$

the proportion of points fixed by x. Fixed point ratios of finite groups of Lie type, particularly for classical groups, have been studied in a number of papers, and have been applied to a variety of problems (see for example [**12**, **15**, **20**]). A sequel [**19**] to this paper contains bounds for fixed point ratios of elements of finite exceptional groups of Lie type in their transitive actions. A crucial part of the proof in [**19**] is to use the dimension estimates of Theorem 2 below, passing from algebraic to finite groups via a Frobenius morphism.

Using Proposition 1.14 (already mentioned), it is clear that if $M \leq N \leq G$, then $f(x, G/M) \geq f(x, G/N)$. Thus for the purpose of obtaining lower bounds for $f(x, G/M)$ it suffices to consider the case where M is maximal in G. Observe also that if $x = su$, where s is the semisimple part of x, and u the unipotent part, then for $g \in G$ we have $x \in M^g$ if and only if both $s \in M^g$ and $u \in M^g$, and hence $\mathrm{fix}_{G/M}(x) = \mathrm{fix}_{G/M}(s) \cap \mathrm{fix}_{G/M}(u)$. Hence it also suffices to consider only cases where x is a semisimple or unipotent element of G.

Theorem 2 contains our strongest result on lower bounds for $f(x,\Omega)$, but as its statement is rather involved and requires reference to some tables at the end of the paper, we first state the following somewhat simplified version.

Theorem 1. *Let G be a simple adjoint exceptional algebraic group over an algebraically closed field, let P be a maximal parabolic subgroup of G, and M a maximal closed subgroup of G which is not parabolic. If u is a nonidentity unipotent element of G, and s a nonidentity semisimple element of G, then*

$$f(u, G/P) \geq c_G, \quad f(s, G/P) \geq d_G,$$

$$f(u, G/M) \geq e_G, \quad \text{and} \quad f(s, G/M) \geq f_G,$$

where c_G, d_G, e_G, f_G are as in Table 1 below.

In particular, for any nonidentity element $g \in G$, and any closed subgroup X of G,

$$f(g, G/X) \geq c_G.$$

G	c_G	d_G	e_G	f_G	e'_G	h_G
E_8	12	24	24	48	40	48
E_7	6	11	12	22	20	22
E_6	4	6	6	12	10	12
F_4	4	4	4	8	6	6
G_2	2	2	2	3	4	4

Table 1.

Remarks.

(1) The bounds in Theorem 1 are sharp, in the sense that there exist a parabolic P and a unipotent element u such that $f(u, G/P) = c_G$, and so on. Nevertheless it is possible to improve the bounds greatly by subdividing the possibilities for u, s, P, M into a larger number of cases, and this we do in Theorem 2 below.

(2) As observed above, Proposition 1.14 shows that for $x \in M$ we have $f(x, G/M) = \dim x^G - \dim(x^G \cap M)$, so the bounds in Theorem 1 (and Theorem 2) also give information about how conjugacy classes of G intersect with a maximal subgroup.

Now we state Theorem 2, our strongest result concerning upper bounds for $f(x, \Omega)$ for exceptional algebraic groups, of which Theorem 1 is an immediate consequence. The conclusion refers to a number of tables which can be found in Section 7 at the end of the paper.

According to [**22**, **31**], the maximal closed subgroups of positive dimension in G can be classed as follows:

(1) parabolic subgroups,
(2) reductive subgroups of maximal rank (i.e., containing a maximal torus of G),
(3) a few other isomorphism types (mostly of small dimension compared to $\dim G$).

The conclusion of Theorem 2 is accordingly divided into three parts.

We need a little standard notation for the statement. Let P_i denote the standard parabolic subgroup of G which corresponds to deleting the i^{th} node from the Dynkin diagram. Let α be a long root in the root system of G, and, when $G = F_4$ or G_2, let β be a short root. Let U_α, U_β be corresponding root subgroups of G, and u_α, u_β nonidentity elements of U_α, U_β respectively. We call u_α a long root element of G, and u_β a short root element. Observe that when $(G, p) = (F_4, 2)$ or $(G_2, 3)$, the elements u_α and u_β are conjugate by a graph automorphism of G. This accounts for some of the parenthetical exclusions in the statement of Theorem 2.

A number of constants are referred to in the statement. The numbers $c_{G,i,\alpha}, c_{G,i,\beta}$ and $c'_{G,i}$ are defined in Tables 7.1 and 7.2 at the end of the paper, the numbers $d_{G,i,D}$ are defined in Table 7.3, and the numbers $f_{G,M,D}$ are defined in Table 7.4. Finally, the numbers e_G, e'_G, h_G are defined in Table 1 above.

Theorem 2. *Let G be a simple adjoint exceptional algebraic group over an algebraically closed field, and let M be a maximal closed subgroup of G. Let u be a nonidentity unipotent element of G, and s a nonidentity semisimple element; write $D = C_G(s)$.*

(I) *Suppose $M = P_i$ is a maximal parabolic subgroup. Then:*
   (a) *We have*

$$f(u_\alpha, G/P_i) = c_{G,i,\alpha}, \quad f(u_\beta, G/P_i) = c_{G,i,\beta}, \quad \text{and} \quad f(u, G/P_i) \geq c'_{G,i}$$

   if u is not a long or short root element.
   (b) *For D as in column 2 of Table 7.3, we have $f(s, G/P_i) \geq d_{G,i,D}$.*

(II) *Suppose M is reductive of maximal rank. Then:*
   (a) *$f(u, G/M) \geq e_G$, and moreover $f(u, G/M) \geq e'_G$, provided u is not a long root element (or a short root element when $(G,p) = (F_4, 2)$ or $(G_2, 3)$).*
   (b) *$f(s, G/M) \geq f_{G,M,D}$.*

(III) *Suppose M is neither parabolic nor reductive of maximal rank. Then $f(s, G/M) \geq h_G$, $f(u, G/M) \geq e_G$, and moreover $f(u, G/M) \geq e'_G$ provided u is not a long root element (or a short root element when $(G,p) = (F_4, 2)$ or $(G_2, 3)$).*

The layout of the paper is as follows. Section 1 consists of various preliminary results taken from the literature, mostly concerning properties of unipotent elements, semisimple elements and subgroups of exceptional algebraic groups. In Section 2 we start the proof of Theorem 2, proving Part (I)(a), the case of unipotent elements in parabolic subgroups. Section 3 contains the proof of Part (I)(b), semisimple elements in parabolics. In Sections 4 and 5 we prove Part (II), the cases of unipotent and semisimple elements in maximal rank subgroups. Finally, Section 6 contains the proof of Part (III), and Section 7 has Tables 7.1-7.4 referred to in the statement of Theorem 2.

1. Preliminaries.

In this section we present various results from the literature which we shall need, most of which concern properties of unipotent and semisimple elements in exceptional algebraic groups.

Throughout, G is a simple algebraic group over an algebraically closed field K of characteristic p (allowing $p = 0$).

A. Semisimple elements and subsystems.

A (not necessarily connected) reductive subgroup of G which contains a maximal torus is called a *subsystem subgroup*. The root system of such a subgroup is a subsystem of the root system of G.

Proposition 1.1. *For G of exceptional type, the maximal subsystem subgroups M of G are as follows:*

G	M^0	M/M^0
E_8	$A_1E_7, D_8, A_8, A_2E_6, D_4D_4,$ $A_4A_4, A_2^4, A_1^8, T_8$	$1, 1, Z_2, Z_2, S_3 \times Z_2,$ $Z_4, GL_2(3), AGL_3(2), 2.O_8^+(2)$
E_7	$T_1E_6, A_1D_6, A_7, A_2A_5,$ $A_1^3D_4, A_1^7, T_7$	$Z_2, 1, Z_2, Z_2,$ $S_3, L_3(2), 2 \times Sp_6(2)$
E_6	$T_1D_5, T_2D_4, A_1A_5, A_2^3, T_6$	$1, S_3, 1, S_3, O_6^-(2)$
F_4	$A_1C_3, B_4, C_4(p = 2), D_4, A_2A_2$	$1, 1, 1, S_3, Z_2$
G_2	A_1A_1, A_2	$1, Z_2$

Proof. This is immediate from Tables A,B in [**21**, p. 302]. □

If s is a semisimple element of G then $C_G(s)$ is a subsystem subgroup. Complete lists of the subsystems occurring are available (see for example [**9**] for types E_7, E_8). In the next result we record the subsystem subgroups which can occur as centralizers of semisimple elements of orders 2 and 3. This result is well-known (see for example [**14**, Tables 4.3.1, 4.7.1]).

Proposition 1.2. *Let G be adjoint and of exceptional type. The centralizers in G of semisimple involutions and elements of order* 3 *are as follows* (*where for $G = E_6$ we only include elements of order* 3 *which lift to elements of order* 3 *in the simply connected group $\hat{E}_6$*)*:*

G	*involution centralizers*	*centralizers of elements of order* 3
E_8	A_1E_7, D_8	$A_8, A_2E_6, E_7T_1, D_7T_1$
E_7	$A_1D_6, (A_7).2, (T_1E_6).2$	$A_2A_5, E_6T_1, D_6T_1, A_6T_1, A_1D_5T_1$
E_6	A_1A_5, D_5T_1	$A_5T_1, (D_4T_2).3, (A_2^3).3$
F_4	A_1C_3, B_4	A_2A_2, B_3T_1, C_3T_1
G_2	A_1A_1	A_2, A_1T_1

Further, the involutions in E_7 with centralizers $A_1D_6, (A_7).2, (T_1E_6).2$ lift to elements of orders $2, 4, 4$, respectively, in the simply connected group $\hat{E}_7$.

Next we record an elementary fact about conjugacy of semisimple elements. For M a reductive (not necessarily connected) subgroup of G, let T_M be a maximal torus of M^0, and let T be a maximal torus of G containing T_M. Define $W(M) = N_M(T_M)/T_M$. From the Bruhat decomposition of elements of G we see that there is a subgroup of $W(G) = N_G(T)/T$ which induces $W(M)$ on T_M. With abuse of notation, we refer to this subgroup also as $W(M)$.

Proposition 1.3. *Let M be a reductive subgroup of G, and let $P = QL$ be a parabolic subgroup of G with unipotent radical Q and Levi subgroup L. If s is a semisimple element of G, and $D = C_G(s)^0$, then:*

(i) *The number of M-conjugacy classes contained in $s^G \cap M^0$ is at most $|W(D)\backslash W(G)/W(M)|$, the number of $(W(D), W(M))$-double cosets in $W(G)$.*

(ii) *The number of P-conjugacy classes contained in $s^G \cap P$ is at most $|W(D)\backslash W(G)/W(L)|$.*

Proof. For (i) we may take $s \in T_M$. Every element of $s^G \cap M^0$ is M-conjugate to an element of T_M. Moreover, if two elements of T are G-conjugate then they are $W(G)$-conjugate ([**36**, II, 3.1]), and if two elements of T_M are M-conjugate then they are $W(M)$-conjugate. Part (i) follows. Part (ii) follows likewise, since every element of $s^G \cap P$ is P-conjugate to an element of L. □

The next two results concern the dimensions of centralizers of certain types of elements.

Proposition 1.4. *Let τ be either an involutory graph automorphism of E_6 or A_n, or a graph automorphism of D_4 of order 3.*

(i) *There are 2 classes of involutions in the coset $E_6\tau$; these have centralizers F_4, C_4 in E_6 if $p \neq 2$, and centralizers $F_4, C_{F_4}(t)$ if $p = 2$, where t is a long root involution in F_4.*

(ii) *There are $(2, n+1)$ classes of involutions in the coset $A_n\tau$; if $n = 2m$ is even the class has centralizer B_m, and if $n = 2m-1$ is odd the classes have centralizers C_m, D_m if $p \neq 2$, and centralizers $C_m, C_{C_m}(t)$ if $p = 2$, where t is a long root involution in C_m.*

(iii) *There are 2 classes of elements of order 3 in the coset $D_4\tau$; these have centralizers G_2, A_2 if $p \neq 3$, and centralizers $G_2, C_{G_2}(t)$ if $p = 3$, where t is a long root element of G_2.*

Proof. See [**14**, Tables 4.3.1, 4.7.1] for the cases where $|\tau| \neq p$, and [**2**, §19] and [**13**, 9.1] for the cases where $|\tau| = p = 2$ and $|\tau| = p = 3$, respectively. □

Proposition 1.5. *Let D be a connected reductive algebraic group.*

(i) *If t is an automorphism of D (as algebraic group) of order 2, then*

$$\dim C_D(t) \geq |\Sigma^+(D)| + \mathrm{rank}(D) - \mathrm{rank}(D'),$$

where $\Sigma^+(D)$ denotes the set of positive roots in the root system $\Sigma(D)$ of D.

(ii) *If $v \in D$ is a semisimple element of order* 3*, then*

$$\dim C_D(v) \geq \frac{2}{3}|\Sigma^+(D)| + \mathrm{rank}(D) - \mathrm{rank}(D').$$

Proof. It suffices to prove this for D simple, in which case it follows easily from the proofs of [**26**, 4.1, 4.3 and 4.4]. (A simple, uniform proof of the bound for $\dim C_D(t)$ with $p \neq 2$ and $t \in D$ can be found in [**32**, 2.1].) □

B. Unipotent elements and parabolics.

The classes of unipotent elements in exceptional algebraic groups can be found in [**6**, p. 401] over $\mathbb{C}$, and in [**7**, **11**, **27**, **28**, **33**, **34**] for arbitrary characteristics. Convenient notation and tables of all unipotent classes can be found in [**17**], where the Jordan canonical forms of all such elements on various G-modules are given. We adopt the notation of [**17**].

The following result is taken from [**6**, 5.9.6]. It is stated there for large primes, and was extended to all good primes in [**29**, **30**].

Proposition 1.6. *Let G be a simple algebraic group in characteristic p, and suppose p is not a bad prime for G. Then the unipotent classes in G are in bijective correspondence with G-classes of pairs $(L, P_{L'})$, where L is a Levi subgroup of G and $P_{L'}$ is a distinguished parabolic subgroup of L'. An element in the class corresponding to $(L, P_{L'})$ is a distinguished unipotent element of L'.*

The distinguished parabolic subgroups of simple algebraic groups are described in [**6**, p. 174], and this gives rise to the labelling of unipotent classes in [**6**], [**17**], etc. In particular, for type A_l only the Borel subgroups are distinguished, and accordingly, the only distinguished unipotent elements of A_l are the regular unipotent elements. Thus in the (fairly common) case that the Levi subgroup L has L' a product of factors of type A_l, there is just one corresponding unipotent class in G, consisting of elements which are regular in each factor.

When p is a bad prime, the labelling of unipotent classes of G given by Proposition 1.6 remains valid, except that there are a few extra classes, as summarised in [**17**] for G of exceptional type, and in [**6**, p. 180] for G of classical type. An interesting consequence of the unipotent class determination is that, excluding the extra classes, $\dim C_G(u)$ depends only on the label of the unipotent element u, and not on the characteristic. These numbers are tabulated in [**6**, pp. 401-407] for exceptional types.

The next result contains some consequences of the unipotent classification for exceptional groups.

Proposition 1.7. *Let G be an exceptional algebraic group, and let $1 \neq u \in G$ be a unipotent element such that $\dim C_G(u) > l_G$, where l_G is as in Table 2 below.*

Then u belongs to one of the conjugacy classes listed in the table; also given are the dimensions of $R = R_u(C^0)$ (where $C = C_G(u)$), the type of C^0/R, the dimension of the variety $\mathcal{B}_u$ of Borel subgroups of G containing u, and the order of C/C^0. When $p = 2$, the involution classes in G are those labelled kA_1 for some k (also $3A_1'', 3A_1'$ in E_7, and $\widetilde{A}_1, \widetilde{A}_1^{(2)}$ in F_4, G_2).

G	l_G	u with $\dim C_G(u) > l_G$	$\dim u^G$	$\dim R$	C^0/R	$\dim \mathcal{B}_u$	$\lvert C/C^0\rvert$
E_8	80	$A_1, 2A_1, 3A_1,$	58,92,112,	57,78,81,	$E_7, B_6, A_1F_4,$	91,74,64,	1,1,1,
		$A_2, 4A_1, A_2+A_1,$	114,128,136,	56,84,77,	$E_6, C_4, A_5,$	63,56,52,	2,1,2,
		$A_2+2A_1, A_3, A_2+3A_1,$	146,148,154,	78,45,77,	$A_1B_3, B_5, A_1G_2,$	47,46,43,	1,1,1,
		$2A_2, 2A_2+A_1,$	156,162,	64,69,	$G_2^2, A_1G_2,$	42,39,	2,1,
		$A_3+A_1, D_4(a_1)$	164,166	60,54	A_1B_3, D_4	38,37	1,6
E_7	41	$A_1, 2A_1, 3A_1'',$	34,52,54,	33,42,27,	$D_6, A_1B_4, F_4,$	46,37,36,	1,1,1,
		$3A_1', A_2, 4A_1,$	64,66,70,	45,32,42,	$A_1C_3, A_5, C_3,$	31,30,28,	1,2,1,
		$A_2+A_1, A_2+2A_1, A_3,$	76,82,84,	41,42,25,	$A_3T_1, A_1^3, A_1B_3,$	25,22,21,	2,1,1,
		$2A_2, A_2+3A_1,$	84,84,	32,35,	$A_1G_2, G_2,$	21,21,	1,1,
		$(A_3+A_1)'', 2A_2+A_1$	86,90	26,37	B_3, A_1^2	20,18	1,1
E_6	26	$A_1, 2A_1, 3A_1,$	22,32,40,	21,24,27,	$A_5, B_3T_1, A_1A_2,$	25,20,16,	1,1,1,
		$A_2, A_2+A_1,$	42,46,	20,23,	$A_2A_2, A_2T_1,$	15,13,	2,1,
		$2A_2, A_2+2A_1$	48,50	16,24	G_2, A_1T_1	12,11	1,1
F_4	18	$A_1, \widetilde{A}_1(p=2), \widetilde{A}_1(p\neq 2),$	16,16,22,	15,15,15,	$C_3, B_3, A_3,$	16,16,13,	1,1,2,
		$\widetilde{A}_1^{(2)}(p=2), A_1\widetilde{A}_1,$	22,28,	20,18,	$B_2, A_1A_1,$	13,10,	1,1,
		$A_2, \widetilde{A}_2(p\neq 2), \widetilde{A}_2(p=2)$	30,30,30	14,8,14	A_2, G_2, A_2	9,9,9	2,1,2
G_2	4	$A_1, \widetilde{A}_1(p=3),$	6,6,	5,5,	$A_1, A_1,$	3,3,	1,1,
		$\widetilde{A}_1(p\neq 3), \widetilde{A}_1^{(3)}(p=3)$	8,8	3,6	$A_1, 1$	2,2	1,1

Table 2.

Remark. In fact for $G = F_4$, the groups C^0/R are not explicitly given in the references [**33**, **34**], but the entries in Table 2 giving these groups are easily verified.

The following is another consequence of the unipotent class classification.

Proposition 1.8. *Upper bounds for the numbers $u(G)$ of classes of unipotent elements in exceptional algebraic groups G are as follows:*

$$u(E_8) \leq 74,\ u(E_7) \leq 46,\ u(E_6) \leq 21,\ u(F_4) \leq 20,\ u(G_2) \leq 6.$$

Next we record a result of Spaltenstein [**35**]:

Proposition 1.9. *If u is a unipotent element of the simple algebraic group G, and B is a Borel subgroup of G, then*

$$\dim(u^G \cap B) = \frac{1}{2}\dim u^G.$$

Moreover, if P is a parabolic subgroup of G, and $\mathcal{B}_P$ the variety of Borel subgroups of P, then

$$\dim(u^G \cap P) \leq \frac{1}{2}\dim u^G + \dim \mathcal{B}_P.$$

Finally, if $\mathcal{B}_u$ is the variety of Borel subgroups of G containing u, then

$$\dim \mathcal{B}_u = \frac{1}{2}(\dim C_G(u) - \mathrm{rank}(G)).$$

Proof. The first and last statements are in [**35**, p. 54] (see also [**6**, 5.10.2]). For the second statement, let $B \leq P$ and consider the surjective map $(u^G \cap B) \times P \to u^G \cap P$, sending $(u_1, x) \to u_1^x$. The preimage of u_1^x contains $\{(u_1^{b^{-1}}, bx) : b \in B\}$. So all fibres have dimension at least $\dim B$. The result follows. □

We shall also need information in the following proposition concerning unipotent classes in classical groups.

Proposition 1.10. *Let G be a classical group $GL_n(K)$, $GSp_n(K)$ or $GO_n(K)$, where K is an algebraically closed field of characteristic p, and let u be a nonidentity unipotent element in G. Suppose for each i, the Jordan canonical form for u has n_i Jordan blocks of size i.*

(i) *If $G = GL_n(K)$, then*

$$\dim C_G(u) = 2\sum_{i<j} i n_i n_j + \sum_i i n_i^2.$$

(ii) *If $G = GSp_n(K)$ with $p \neq 2$, then n_i is even whenever i is odd, and*

$$\dim C_G(u) = \sum_{i<j} i n_i n_j + \frac{1}{2}\sum_i i n_i^2 + \frac{1}{2}\sum_{i \text{ odd}} n_i.$$

(iii) *If $G = GO_n(K)$ with $p \neq 2$, then n_i is even whenever i is even, and*

$$\dim C_G(u) = \sum_{i<j} i n_i n_j + \frac{1}{2}\sum_i i n_i^2 - \frac{1}{2}\sum_{i \text{ odd}} n_i.$$

(iv) *Let $G = GO_n(K)$ with $p = 2$, and set $m = [n/2]$. Involutions in G are represented by elements a_{m-k}, c_{m-k} ($0 \leq k \leq m$ and $m-k$ even), b_{m-k} ($0 \leq k \leq m$ and $m-k$ odd), where each of $a_{m-k}, b_{m-k}, c_{m-k}$*

has $m-k$ Jordan blocks of size 2 and the rest of size 1. Further, if $n=2m+1$, then

$$\dim C_G(a_{m-k}) = m^2+m+k^2,$$
$$\dim C_G(b_{m-k}) = \dim C_G(c_{m-k}) = m^2+k^2+k;$$

and if $n=2m$ then

$$\dim C_G(a_{m-k}) = m^2+k^2-k,$$
$$\dim C_G(b_{m-k}) = \dim C_G(c_{m-k}) = m^2-m+k^2,$$

and a_{m-k}, c_{m-k} lie in $G' = SO_n(K)$, while $b_{m-k} \in G-G'$.

Proof. Parts (i)-(iii) follow from [**37**, pp. 34-39]. (For $K=\mathbb{C}$ the same results can be found in [**6**, p. 398].) Part (iv) follows from [**2**, Sections 7, 8]. □

Next we give some information concerning parabolic subgroups of the simple algebraic group G. Recall first that the highest root in the root system $\Sigma(G)$ of G is the root $\alpha_0 = \sum c_i\alpha_i$ with $\sum c_i$ maximal (where α_i are the fundamental roots). The highest roots of the simple root systems are as follows, where we use the notation of [**5**, p. 250], and denote $\sum_1^l c_i\alpha_i$ by the sequence $c_1c_2\dots c_l$:

$$\begin{array}{ll}
G=A_l: & \alpha_0 = 111\dots1\\
G=B_l: & \alpha_0 = 122\dots2\\
G=C_l: & \alpha_0 = 22\dots21\\
G=D_l: & \alpha_0 = 122\dots211\\
G=G_2: & \alpha_0 = 23\\
G=F_4: & \alpha_0 = 2342\\
G=E_6: & \alpha_0 = 122321\\
G=E_7: & \alpha_0 = 2234321\\
G=E_8: & \alpha_0 = 23465432
\end{array}$$

As usual, denote by $P_{i,j,\dots}$ the standard parabolic subgroup of G corresponding to deletion of nodes $i,j,\dots$ from the Dynkin diagram.

Proposition 1.11. *Suppose the Dynkin diagram of G is simply laced, and let $\alpha_0 = \sum c_i\alpha_i$ be the highest root. Then the nilpotence class of the unipotent radical $R_u(P_{i,j,\dots})$ is equal to $c_i+c_j+\cdots$.*

Proof. This is [**3**, Lemma 4]. □

Proposition 1.12. *Denote by U_{α_0} the long root subgroup of G corresponding to α_0, and let $1 \neq u_{\alpha_0} \in U_{\alpha_0}$. Then $P = N_G(U_{\alpha_0})$ is a parabolic subgroup of G, as in Table 3; we also give* $\dim R_u(P)$ *(and also* $\dim u_{\alpha_0}^G$*, since* $\dim u_{\alpha_0}^G = \dim R_u(P)+1$*).*

Proof. The appropriate parabolic is obtained by deleting those nodes adjacent to α_0 in the extended Dynkin diagram of G (see [**5**, p. 250]). For the last equality, observe that

$$\dim u_{\alpha_0}^G = \dim(G/C_G(u_{\alpha_0})) = \dim(G/P) + 1 = \dim R_u(P) + 1.$$

G	$P = N_G(U_{\alpha_0})$	$\dim R_u(P)$
A_l	P_{1l}	$2l-1$
B_l	P_2	$4l-5$
C_l	P_1	$2l-1$
D_l	P_2	$4l-7$
G_2	P_1	5
F_4	P_1	15
E_6	P_2	21
E_7	P_1	33
E_8	P_8	57

Table 3.

We conclude this subsection with a few further properties of long root elements.

Proposition 1.13. *Let u_α be a long root element of the simple algebraic group G. Then:*

(i) *If P is a parabolic subgroup, then the number of P-classes in $u_\alpha^G \cap P$ is finite, with representatives given by long root elements u_α for α in a fixed root system of P.*

(ii) *Let D be a connected semisimple subgroup of G containing u_α. Then u_α lies in a simple factor D_0 of D. Moreover, either u_α is a long root element of D_0, or $p = 2$, $D_0 = B_n$ lying in a subsystem subgroup A_{2n} of G, and u_α is a short root element of D_0.*

(iii) *Let M be a connected reductive subgroup of G, and suppose that u_α normalizes M but does not induce an inner automorphism on M. Then $p = 2$ and $M = XY$, a commuting product, where u_α centralizes Y and $X = D_n$ or T_1. Moreover, if $X = D_n$ then $C_X(u_\alpha) = B_{n-1}$.*

Proof. (i) Set $P_0 = N_G(U_\alpha)$, a parabolic subgroup. We may assume that P and P_0 contain a common Borel subgroup and maximal torus T. Then the double coset space $P\backslash G/P_0$ is finite, with double coset representatives lying in $N_G(T)$. Replacing P_0 by $P_0' = C_G(u_\alpha)$, the number of double cosets does not change since $P_0 = P_0'T$.

(ii) Say $D = D_1 \times D_2$, where each D_i is a product of simple factors. Set $u = u_\alpha$, and suppose we can write $u = u_1u_2$, with $1 \neq u_i \in D_i$. By

[1, 2.1], there exists $d_1 \in D_1$ such that $u_1^{-1}u_1^{d_1}$ is not a p-element. Then $u^{-1}u^{d_1} = u_1^{-1}u_1^{d_1}$ is not a p-element. So $J = \langle U_\alpha, U_\alpha^{d_1}\rangle$ is a group of type A_1 (see [23, 1.1]) and $a = u^{-1}u^{d_1}$ can be computed in J. It follows that a is a p'-element, not of order 2. Hence by [23, 1.2], $C_G(a)' = C_G(J)$. However, D_2 centralizes a, but does not centralize $u \in J$, a contradiction.

This shows that u_α is contained in a simple factor of D. The last statement follows from [23, 2.2, 3.2 and 3.3].

(iii) We first assert that if u_α normalizes but does not centralize a torus T in G, then $p = 2$ and u_α centralizes a sub-torus of codimension 1 in T. To see this, pick $t \in T$ such that $a = [u_\alpha, t] \neq 1$. Then $a \in T$, so $J = \langle U_\alpha, U_\alpha^t\rangle$ is a fundamental SL_2 in G (see [23, 1.1]). Moreover, u_α normalizes $T_1 = T \cap J$. Since $|N_J(T_1)/T_1| = 2$, it follows that $p = 2$ and $|a| > 2$. By [23, 1.2], $C_G(a) = C_G(T_1) = T_1D$, where $D = C_G(J)$. Since $T \leq C_G(T_1)$, it follows that u_α centralizes $T \cap D$, a torus of codimension 1 in T. The assertion is proved.

Now $M = M'Z$, where $Z = Z(M)^0$ and M' is semisimple. Now M' is a product of simple factors. Some may be permuted by u_α in orbits of size p: Let the product of these factors be $H = H_1 \ldots H_a$. Some may be fixed by u_α but have u_α inducing an outer automorphism on them: Let the product of these be $L = L_1 \ldots L_b$. The rest are fixed by u_α and have u_α inducing an inner automorphism: Call the product of these S. Then $M = HLSZ$.

Suppose that $H \neq 1$, and say u_α permutes the simple factors $H_1, \ldots, H_p$ cyclically. If T_0 is a maximal torus of H_1, then u_α normalizes the torus $T = T_0T_0^{u_\alpha} \ldots T_0^{u_\alpha^{p-1}}$ of $H_1 \ldots H_p$. By the earlier assertion, $p = 2$ and u_α centralizes a sub-torus of codimension 1 in T, whence $\dim T_0 = 1$ and $H = H_1H_2$ with $H_1 \cong H_2 \cong A_1$. Thus in this case $p = 2$ and $H \cong D_2$, which is a configuration allowed for in the conclusion of the proposition.

Now consider a factor L_i of L. If $p \neq 2$, then $L_i \cong D_4$ and $p = 3$. By 1.4, there are two classes of graph automorphisms of D_4, represented by τ and τt, where $C_{D_4}(\tau) = G_2$, and t is a long root element of this G_2. Both these automorphisms normalize a subgroup $(A_1)^3$ of D_4, permuting the 3 factors cyclically. Hence by the previous paragraph, neither can be induced by a root element of G. Therefore $p = 2$, and now it follows from [23, 3.3] that $L_i \cong D_n$ and $C_{L_i}(u_\alpha) \cong B_{n-1}$. Moreover, if T_{n-1} is a maximal torus of this B_{n-1}, then $C_{D_n}(T_{n-1}) = T_n$, a maximal torus of D_n normalized by u_α.

Because of the assertion in the first paragraph, if $H \neq 1$ then $L = 1$; if $L \neq 1$ then $H = 1$ and $L = L_1$ is simple; and if $H = L = 1$ then $Z \neq 1$ and u_α does not centralize Z, in which case $p = 2$ and u_α induces a reflection on Z.

We have established that $p = 2$ and $M = XY$, where $X = D_n$ or T_1 and u_α induces an inner automorphism on Y. Now arguing as in the proof of Part (ii), we deduce that u_α centralizes Y, completing the proof. □

C. Fixed point spaces and conjugacy classes.

We finish the section with a result relating fixed points to conjugacy classes.

Proposition 1.14. *Let G be an algebraic group, and let H be a closed subgroup. Write Ω for the coset variety G/H. Then for $x \in H$,*

$$f(x,\Omega) = \dim x^G - \dim(x^G \cap H).$$

Proof. Define

$$V = \{(g,\omega) \in G \times \Omega : \omega g = \omega\}.$$

If $\pi, \phi : G \times \Omega \to \Omega$ are the morphisms defined by

$$(g,\omega)\pi = \omega, \quad (g,\omega)\phi = \omega g,$$

then $V = \{(g,\omega) \in G \times \Omega : (g,\omega)\pi = (g,\omega)\phi\}$, and hence V is a closed subvariety of $G \times \Omega$.

For $x \in H$, define

$$V_x = \{(x^g,\omega) : g \in G, \omega \in \Omega, \omega x^g = \omega\}.$$

Then V_x is a variety, and the map $V_x \to x^G$ given by $(x^g,\omega) \to x^g$ has fibres of dimension $\dim \mathrm{fix}_\Omega(x)$, so

$$\dim V_x = \dim x^G + \dim \mathrm{fix}_\Omega(x).$$

On the other hand, the map $V_x \to \Omega$ given by $(x^g,\omega) \to \omega$ has fibres of dimension $\dim(x^G \cap H)$, so

$$\dim V_x = \dim \Omega + \dim(x^G \cap H).$$

The conclusion follows. □

2. Proof of Theorem 2, Part (I)(a): Unipotent elements in parabolics.

In this section we prove Part (I)(a) of Theorem 2. Thus let G be a simple algebraic group of exceptional type over an algebraically closed field K of characteristic p (allowing $p = 0$), and let P_i be a maximal parabolic subgroup of G. Write $P_i = Q_iL_i$, where Q_i is the unipotent radical and L_i a Levi subgroup. Let u be a nonidentity unipotent element of P_i, u_α a long root element, and u_β a short root element (in the cases where these exist). If $p > 0$ we take u to be of order p (as we may, for the purpose of proving Theorem 2).

We first establish:

Lemma 2.1. *We have*

$$\dim u_\alpha^G - \dim(u_\alpha^G \cap P_i) = c_{G,i,\alpha}$$

where $c_{G,i,\alpha}$ is as in Table 7.1. Moreover, if $(G,p) = (F_4, 2)$ or $(G_2, 3)$, then $\dim u_\beta^G - \dim(u_\beta^G \cap P_i) = c_{G,i,\beta}$, where $c_{G,i,\beta}$ is as in Table 7.2.

Proof. Observe that the last statement concerning $(F_4, 2)$ and $(G_2, 3)$ follows from the first part of the proposition, as can be seen by applying a graph automorphism of G. Hence we just need to prove the first statement.

Write $u = u_\alpha$. By 1.13(i), we can take $\dim(u^G \cap P_i) = \dim u^{P_i}$ with u lying in either Q_i or L_i.

Suppose first that $u \in L_i$. Let Q_i^- be the unipotent radical of the parabolic opposite to P_i. Now $Q_iL_iQ_i^-$ is an open dense subset of G, hence $Q_iL_iQ_i^- \cap C_G(u)$ is open dense in the connected group $C_G(u)$, and it follows that

$$\dim C_G(u) = \dim C_{Q_i}(u) + \dim C_{L_i}(u) + \dim C_{Q_i^-}(u).$$

Moreover, if w_0 is the longest element of the Weyl group, then w_0 (or, for $G = E_6$, $w_0\tau$ with τ a graph automorphism) interchanges Q_i with Q_i^- and normalizes an L_i-conjugate of U_α (a root group containing u), whence $\dim C_{Q_i^-}(u) = \dim C_{Q_i}(u)$. Since $\dim u^{P_i} = \dim u^{Q_i} + \dim u^{L_i}$, it follows that

$$\dim u^G = \dim u^{P_i} + \dim u^{Q_i}.$$

Therefore

$$\dim u^{Q_i} = \frac{1}{2}(\dim u^G - \dim u^{L_i}),$$

and hence

$$\dim u^G - \dim u^{P_i} = \dim u^{Q_i} = \frac{1}{2}(\dim u^G - \dim u^{L_i}).$$

The right hand side of this equation is easily calculated using 1.12, and is equal to $c_{G,i,\alpha}$.

Finally, if $u \in Q_i$ then $u^{P_i} \subseteq B$ for each Borel subgroup B of P_i, and hence by 1.9, $\dim u^{P_i} \le \frac{1}{2}\dim u^G$, whence

$$\dim u^G - \dim u^{P_i} \ge \frac{1}{2}\dim u^G,$$

which is larger than $c_{G,i,\alpha}$.

This completes the proof, except for those cases where L_i contains no conjugate of u. This occurs only when $G = G_2$ and $i = 1$. In this case, by 1.13(i) we may take $u \in Q_1 \backslash Z(Q_1)$, and $\dim u^G \cap P_1 = \dim u^{P_1}$. When $p \ne 3$, we have $\dim Z(Q_1) = 1$, and $Q_1/Z(Q_1)$ has the structure of an irreducible module for $L_1' \cong A_1$ of high weight $3\lambda_1$, with $uZ(Q_1)$ a maximal vector. It follows that $\dim C_{Q_1}(u) + \dim C_{L_1}(u) = 4+2$, whence $\dim u^{P_1} = 3$ and $\dim u^G - \dim u^{P_1} = 6 - 3 = c_{G,1,\alpha}$. And when $p = 3$, we again have $\dim C_{Q_1}(u) + \dim C_{L_1}(u) = 4 + 2$, giving the conclusion. □

Define $\mathcal{B}_i$ to be the variety of all Borel subgroups of G lying in P_i, and $\mathcal{P}_{i,u}$ the variety of all conjugates of P_i which contain u. For $P \in \mathcal{P}_{i,u}$, let $\mathcal{B}_{P,u}$ be the variety of Borel subgroups in P which contain u, and define

$$N_{i,u} = \min \{\dim \mathcal{B}_{P,u} : P \in \mathcal{P}_{i,u}\},$$

$$b_i = \dim \mathcal{B}_i.$$

Define also $\mathcal{B}_u$ to be the variety of Borel subgroups of G containing u.

Lemma 2.2. *We have*

$$\dim u^G - \dim(u^G \cap P_i) = f(u, G/P_i) \geq \dim(G/P_i) - \dim \mathcal{B}_u + N_{i,u}.$$

Proof. Let $\psi : \mathcal{B}_u \to \mathcal{P}_{i,u}$ be the surjective morphism sending a Borel subgroup B to the unique conjugate of P_i containing B. The fibres of ψ have dimension at least $N_{i,u}$, and hence

$$\dim \mathcal{P}_{i,u} = \dim \operatorname{Im} \psi \leq \dim \mathcal{B}_u - N_{i,u}.$$

Since

$$f(u, G/P_i) = \dim(G/P_i) - \dim \operatorname{fix}_{G/P_i}(u) = \dim(G/P_i) - \dim \mathcal{P}_{i,u},$$

the result follows. □

In view of the preceding lemma, it is desirable to obtain good lower bounds on the numbers $N_{i,u}$. The following result will be useful in this respect.

Lemma 2.3. *Let $P = QL$ be a parabolic subgroup of G with unipotent radical Q and Levi subgroup L, and let $x \in P$. If $v \in L$ is such that $xQ = vQ$, then $\dim C_{L(G)}(x) \leq \dim C_{L(G)}(v)$.*

Proof. Consider a P-filtration of $L(G)$ compatible with a direct sum decomposition under the action of L (for weight spaces of the central torus $Z(L)^0$). The unipotent radical Q is trivial on successive factors, so the dimension of the centralizer in $L(G)$ of x is certainly bounded above by the sum of the dimensions of its centralizers in each of the weight spaces. But this sum is just the dimension of the centralizer in $L(G)$ of v, giving the conclusion. □

Lemma 2.4. *If $\dim u^G \geq 2(b_i + c'_{G,i})$, then the conclusion of Theorem 2(I)(a) holds (i.e., $f(u, G/P_i) \geq c'_{G,i}$).*

Proof. By 1.9, $\dim(u^G \cap P_i) \leq \frac{1}{2} \dim u^G + b_i$, and hence

$$f(u, G/P_i) = \dim u^G - \dim(u^G \cap P_i) \geq \frac{1}{2} \dim u^G - b_i.$$

The conclusion follows. □

Lemma 2.5. *The conclusion of Theorem 2(I)(a) holds if $G = E_8$.*

Proof. Suppose $G = E_8$. For convenience we record the values of $\dim Q_i$, b_i and $c_i = c'_{E_8,i}$ below:

$i =$	1	2	3	4	5	6	7	8
$L'_i =$	D_7	A_7	A_1A_6	$A_1A_2A_4$	A_3A_4	A_2D_5	A_1E_6	E_7
$\dim Q_i =$	78	92	98	106	104	97	83	57
$b_i =$	42	28	22	14	16	23	37	63
$c_i =$	28	34	36	40	39	36	30	20

By 2.1 and 2.4, we may suppose that $\dim u^G < 2(b_i + c_i)$ and u is not a long root element. Hence using 1.7 we see that u belongs to one of the following classes of unipotent elements in G:

i	$u \in$ one of
1	$2A_1, 3A_1, A_2, 4A_1, A_2 + A_1$
$2, 3, 6$	$2A_1, 3A_1, A_2$
$4, 5$	$2A_1$
7	$2A_1, 3A_1, A_2, 4A_1$
8	$2A_1, 3A_1, A_2, 4A_1, \ldots, A_3 + A_1$

(In the last row, the list is ordered as in 1.7.)

By 2.2 we have $f(u, G/P_i) \geq \dim(G/P_i) - \dim \mathcal{B}_u + N_{i,u}$, where

$$N_{i,u} = \min \{\dim \mathcal{B}_{P,u} : P \in \mathcal{P}_{i,u}\}.$$

The values of $\dim \mathcal{B}_u$ are given by 1.7. We now establish lower bounds for $N_{i,u}$.

Let $v \in L_i$ be such that $uQ_i = vQ_i$. By 2.3, $\dim C_{L(G)}(u) \leq \dim C_{L(G)}(v)$, so we see from 1.7 and the dimensions of $C_{L(G)}(u)$ which are given in [**17**], that v is either in the same class as u, or in a class which occurs earlier in the list of classes above (but including the classes A_1 and $\{1\}$). For example, if $i = 1$ and u lies in class A_2, then v lies in one of the classes $1, A_1, 2A_1, 3A_1, A_2$. Note also that as L_i is a Levi subgroup, the label for v as an element of L_i is the same as its label as an element of G.

Suppose first that $i = 1$. Then u lies in one of the classes $2A_1, 3A_1, A_2, 4A_1, A_2 + A_1$. Consider $u \in 2A_1$. Then v lies in one of the classes $1, A_1, 2A_1$. The minimal value of $\dim \mathcal{B}_{P,u}$ will be attained when v lies in class $2A_1$. There are two such classes in $L'_1 = D_7$: One corresponding to $2A_1$ acting as SO_4 on the usual 14-dimensional module V_{14}, and the other corresponding to $2A_1$ lying in an A_6 subgroup. For v in the SO_4-type class, with $p \neq 2$, v acts as $J_3 \oplus J_1^{11}$ on V_{14} (where J_i denotes a Jordan block of size i), and it follows from 1.10 that $\dim C_{D_7}(v) = 67$, whence by 1.9,

$$\dim \mathcal{B}_{P,u} = \frac{67 - 7}{2} = 30.$$

And for v in the SO_4-type class with $p = 2$, we have $v = c_2$ in the notation of 1.10, and 1.10 gives $\dim C_{D_7}(v) = 67$ again. For v in the other $2A_1$ class of D_7, with $p \neq 2$, v acts on V_{14} as $J_2^4 \oplus J_1^6$, and 1.10 gives $\dim C_{D_7}(u) = 55$; the same holds for $p = 2$, in which case $v = a_4$ in the notation of 1.10. Hence by 1.9,

$$\dim \mathcal{B}_{P,u} = 24.$$

It follows that for $u \in 2A_1$ we have $N_{1,u} = 24$. Also $\dim \mathcal{B}_u = 74$, so by 2.2,

$$f(u, G/P_1) \geq \dim G/P_1 - \dim \mathcal{B}_u + N_{1,u} = 78 - 74 + 24 = 28 = c'_{G,1},$$

as required.

This handles the case where $u \in 2A_1$. For the other possibilities for the class of u we argue in the same way: Use 1.10 to calculate the possibilities for $\dim C_{D_7}(v)$ - the minimum occurs when v is in the same class as u; for each such possibility we calculate $\dim \mathcal{B}_{P,u}$ using 1.9; hence we work out $N_{1,u}$, and finally application of 2.2 gives the required bound. The numbers which come out are given in the following table:

class of u, v	$\dim C_{D_7}(v)$	$\dim \mathcal{B}_{P,u}$	$N_{1,u}$
$3A_1$	49 or 51	21 or 22	21
A_2	49	21	21
$4A_1$	43	18	18
$A_2 + A_1$	39	16	16

This completes the proof when $i = 1$.

For $i = 2, 3$ or 6 we argue in the same way, obtaining the following information:

class of u, v	$\dim C_{A_7}(v)$ $(i = 2)$	$\dim C_{A_1A_6}(v) \geq$ $(i = 3)$	$\dim C_{A_2D_5}(v) \geq$ $(i = 6)$
$2A_1$	39	$3 + 28$	$8 + 25$
$3A_1$	33	$3 + 24$	$8 + 21$
A_2	37	$3 + 26$	$8 + 19$

Using 1.9 we deduce that $N_{i,2A_1} = 16, 12, 13$ according as $i = 2, 3, 6$ respectively; likewise $N_{i,3A_1} = 13, 10, 11$ and $N_{i,A_2} = 15, 11, 10$. The conclusion follows, using 2.2.

When $i = 4$ or 5, we have $u \in 2A_1$ and the above arguments yield $N_{4,u} = 8, N_{5,u} = 9$, again giving the result by 2.2.

Next consider $i = 7$, so $L_7' = A_1E_6$. Here $u \in 2A_1, 3A_1, A_2$ or $4A_1$, and in the first three cases the minimal value of $\dim \mathcal{B}_{P,u}$ is achieved when v lies in E_6, in the class of E_6 with the same label as u. Hence in these cases $N_{7,u} = 1 + \dim \mathcal{B}_v^{E_6}$, where $\dim \mathcal{B}_v^{E_6}$ is the value of $\dim \mathcal{B}_v$ regarding v as an element of E_6 (i.e., the dimension of the variety of Borels of E_6 containing u). And when $u \in 4A_1$ the minimal value is achieved when v projects to an element in the class $3A_1$ of E_6, and $N_{7,u} = \dim \mathcal{B}_v^{E_6}$. These values are given by 1.7, so we have

$u \in$	$N_{7,u}$
$2A_1$	21
$3A_1$	17
A_2	16
$4A_1$	16

The conclusion follows from 2.2.

Finally, the case where $i = 8$ is entirely similar: The minimal value of $\dim \mathcal{B}_{P,u}$ is achieved when v lies in $L_8' = E_7$, in the class of E_7 with the same label as u; so $N_{8,u} = \dim \mathcal{B}_v^{E_7}$, which is given by 1.7. In all cases 2.2 gives the required bound. □

Lemma 2.6. *The conclusion of Theorem 2(I)(a) holds if $G = E_7$.*

Proof. The argument is very similar to that of the previous proposition, and we just give a sketch. The values of $\dim Q_i$, b_i and $c_i = c'_{E_7,i}$ are as follows:

$i =$	1	2	3	4	5	6	7
$L_i' =$	D_6	A_6	A_1A_5	$A_1A_2A_3$	A_2A_4	A_1D_5	E_6
$\dim Q_i =$	33	42	47	53	50	42	27
$b_i =$	30	21	16	10	13	21	36
$c_i =$	12	16	18	21	20	16	10

Again we may suppose that $\dim u^G < 2(b_i + c_i)$ and u is not a long root element, so by 1.7 u belongs to one of the following classes:

i	$u \in$ one of
1	$2A_1, \dots, A_2 + 2A_1$
2, 6	$2A_1, 3A_1'', 3A_1', A_2, 4A_1$
3	$2A_1, 3A_1'', 3A_1', A_2$
4	$2A_1, 3A_1''$
5	$2A_1, 3A_1'', 3A_1'$
7	$2A_1, \dots, 2A_2 + A_1$

Let $v \in L_i$ with $uQ_i = vQ_i$. As in the previous proof, we find that the minimal value of $\dim \mathcal{B}_{P,u}$ is realised when v is in the class of L_i having the same label as that of u (when such a class exists in L_i).

For $i = 1$, we use 1.10 to calculate $\dim C_{L_1'}(v) = \dim C_{D_6}(v)$:

class of u, v	$\dim C_{D_6}(v)$	$N_{1,u}$
$2A_1$	38 or 46	16
type $3A_1$	36 or 34	15 or 14
A_2	32	13
$4A_1$	30	12
$A_2 + A_1$	26	10
$A_2 + 2A_1$	24	9

The only slightly subtle point to note concerns the classes of type $3A_1$. There are three such classes in D_6. One, represented by v_1 say, corresponds to a $3A_1$ subgroup of type $SO_4 \times A_1$, and hence acts on the usual module V_{12} as $J_3 \oplus J_2^2 \oplus J_1^5$ (when $p \neq 2$) and as c_4 (when $p = 2$ - notation of 1.10). As $V_{56} \downarrow D_6 = V_{12}^2 \oplus V_{D_6}(\lambda_5)$, it follows that v_1 has J_1 blocks on V_{56}, and hence by [**17**, Table 7], v_1 lies in the class $3A_1'$ of E_7. The other classes of type $3A_1$ in D_6 correspond to $3A_1$ subgroups of D_6 lying in an A_5 Levi, and have centralizer in D_6 of dimension 36.

The conclusion now follows for $i = 1$, using 2.2.

For $i = 2$, we have $L_2' = A_6$ and 1.10 gives:

$v \in$	$\dim C_{A_6}(v)$	$N_{2,u}$
$2A_1$	28	11
$3A_1$	24	9
A_2	26	10

As above, the action of a $3A_1$ element of A_6 on V_{56} shows that it is in the class $3A_1'$ of G. Now the conclusion follows from 2.2.

The argument for $i = 3, 4, 5, 6$ is similar. And for $i = 7$, $L_7' = E_6$, as at the end of the previous proposition we have $N_{7,u} = \dim \mathcal{B}_u^{E_6}$, which is given by 1.7, and now 2.2 gives the required bound. □

Lemma 2.7. *The conclusion of Theorem 2(I)(a) holds if $G = E_6$.*

Proof. The argument is entirely similar to that of the previous propositions, and is left to the reader. □

Lemma 2.8. *The conclusion of Theorem 2(I)(a) holds if $G = F_4$ or G_2.*

Proof. Suppose first that $G = F_4$ and $p \neq 2$. As usual choose $v \in L_i$ with $uQ_i = vQ_i$. For $i = 1$ we may assume $\dim u^G < 2(b_1 + c'_{F_4,1}) = 34$, so by 1.7, u lies in one of the classes $\widetilde{A}_1, A_1\widetilde{A}_1, \widetilde{A}_2, A_2$. In the usual way, the minimal value of $\mathcal{B}_u$ is realised when v is in the same class as u. When $u, v \in \widetilde{A}_1$, v acts as $J_2^2 \oplus J_1^2$ on the natural module V_6 for $L_1' = C_3$, so $\dim C_{C_3}(v) = 11$ by 1.10. Therefore $N_{1,u} = 4$, and 2.2 gives

$$f(u, G/P_1) \geq 15 - 13 + 4 = 6 = c_{F_4,1,\beta},$$

as required. When $u, v \in A_1\widetilde{A}_1$, v acts as J_2^3 and 1.10 gives $\dim C_{C_3}(v) = 9$, whence $N_{1,u} = 3$ and $f(u, G/P_1) \geq 8 = c'_{F_4,1}$, as required. If $u, v \in \widetilde{A}_2$ then v acts as J_3^2, $\dim C_{C_3}(v) = 7$ so $N_{1,u} = 2$, giving the result by 2.2. And if $u \in A_2$ then no conjugate of u lies in C_3, so v lies in one of the "earlier" classes $1, A_1, \widetilde{A}_1, A_1\widetilde{A}_1, \widetilde{A}_2$, and the result follows from previous calculations.

This completes the argument for $i = 1$. The remaining values of i (with $p \neq 2$) are handled very similarly, and we leave this to the reader.

Now consider $G = F_4, p = 2$. By 1.7 this group has 4 classes of involutions, namely $A_1, \widetilde{A}_1, A_1^{(2)}, A_1\widetilde{A}_1$. By 2.1 we may assume that u lies in one of the latter two classes. Both are fixed by a graph automorphism of G, so we only need to deal with $i = 1$ or 2. For $i = 1$, the class $A_1^{(2)}$ is represented by $u = x_{\alpha_3}(1)x_{\alpha_2+2\alpha_3}(1)$ (see [**17**, Table A]). The roots $\alpha_3, \alpha_2 + 2\alpha_3$ span a C_2 subsystem, and hence a conjugate v of u lies in $C_3 = L'_1$, and is in the class of c_2 (in the notation of 1.10). Then $\dim C_{C_3}(v) = 11$, $N_{1,u} = 4$, and 2.2 gives $f(u, G/P_1) \geq 6 = c'_{F_4,1}$, as required. The class $A_1\widetilde{A}_1$ is represented by $b_3 \in C_3$, and a similar argument gives the conclusion for this class when $i = 1$.

Now suppose $i = 2$. The group $L'_2 = A_1\widetilde{A}_2$ has three involution classes, with representatives in the classes $A_1, \widetilde{A}_1$ and $A_1\widetilde{A}_1$ of G. Hence for $u \in A_1^{(2)}$ we must have $v \in \{1\}, A_1$ or $\widetilde{A}_1$, whence $N_{2,u} \geq 2$, giving $f(u, G/P_2) \geq 9$ by 2.2. And for $u \in A_1\widetilde{A}_1$, we have $\dim \mathcal{B}_u = 10$, so 2.2 gives $f(u, G/P_2) \geq 10$ immediately.

Finally, the proof for $G = G_2$ is carried out in similar fashion, and we leave it to the reader. □

This completes the proof of Theorem 2(I)(a).

3. Proof of Theorem 2, Part (I)(b): Semisimple elements in parabolics.

In this section we prove Part (I)(b) of Theorem 2. Continue to assume that G is an exceptional algebraic group over the algebraically closed field K, and that $P_i = Q_iL_i$ is a maximal parabolic subgroup of G with unipotent radical Q_i and Levi subgroup L_i.

Let s be a nonidentity semisimple element of G lying in P_i, and write $D = C_G(s)$. By [**36**, II, 4.1], D^0 is reductive, and D/D^0 is isomorphic to a subgroup of the fundamental group of G, which has order 1, 2 or 3 (2 for E_7, 3 for E_6, 1 otherwise).

By 1.3(ii), $s^G \cap P_i$ consists of finitely many P_i-classes. Hence, replacing s by a suitable conjugate, we may assume that $\dim(s^G \cap P_i) = \dim s^{P_i}$.

Lemma 3.1. *The intersection $D \cap P_i$ is a parabolic subgroup of D. Moreover, $R_u(D \cap P_i) \leq Q_i$, and*

$$\dim s^G - \dim s^{P_i} = \dim Q_i - \dim R_u(D \cap P_i) = \dim s^{Q_i}.$$

Proof. Observe that s lies in a maximal torus T of P_i. Clearly $D = C_G(s)$ contains T, and hence $T \leq D \cap P_i$.

We now argue that $D \cap P_i$ is a parabolic subgroup of D. The T-root groups in G all lie in Q_i, L_i or Q_i^- (the unipotent radical of the parabolic opposite to P_i). Note that if $U_\alpha \leq C_G(s)$ then also $U_{-\alpha} \leq C_G(s)$. If $C_{Q_i}(s) = 1$ then $C_G(s)^0 = C_{L_i}(s)^0 = C_{P_i}(s)^0$, so $(D \cap P_i)^0 = D^0$. And if $C_{Q_i}(s) \neq 1$, by [**4**] we can embed $C_{P_i}(s)$ in a parabolic subgroup P of D such that $C_{Q_i}(s) \leq R_u(P)$. Then $D \cap P_i = C_{P_i}(s) = P$: For otherwise, there is a T-root group U_α such that $U_\alpha \leq P$ but $U_\alpha \not\leq C_{P_i}(s)$; then $U_\alpha \leq Q_i^-$, which forces $U_{-\alpha} \leq C_{Q_i}(s)$, whereas $\langle U_\alpha, U_{-\alpha}\rangle \cong SL_2$, a contradiction.

Thus $D \cap P_i$ is a parabolic subgroup of D. Moreover, $D \cap P_i = C_{P_i}(s) = C_{Q_i}(s)C_{L_i}(s)$ and $C_{L_i}(s)$ is reductive, so $R_u(D \cap P_i) = C_{Q_i}(s) \leq Q_i$. Finally,

$$\dim s^G - \dim s^{P_i} = \dim(G/P_i) - \dim(D/D \cap P_i),$$

and the last part follows, as $\dim(G/P_i) = \dim Q_i$ and $\dim(D/D \cap P_i) = \dim R_u(D \cap P_i)$. □

The preceding lemma shows that, for a given P_i, in order to bound $f(s, G/P_i)$ below it suffices to bound $\dim R_u(D \cap P_i)$ above. We shall see that it is possible to obtain the required bounds by using arguments involving root systems, in particular exploiting the fact that the root system of $R_u(D \cap P_i)$ must embed in that of Q_i. Throughout the remainder of this section, let G have simple roots $\alpha_1, \dots, \alpha_n$ and highest root α_0. Let D have simple factors $D_1, D_2, \dots$ in order of decreasing dimension, and assume that the positive roots of D are a subset of those of G. Let D_1 have simple roots $\beta_1, \dots, \beta_m$ and highest root β_0, and D_2 (if it exists) have simple roots $\gamma_1, \dots, \gamma_\ell$ and highest root γ_0. If $\alpha = \sum m_j \alpha_j$, the height of α with respect to P_i will mean m_i; similarly the height of $\sum n_j \beta_j$ with respect to the parabolic $P_{i_1 i_2 \dots}(D_1)$ will mean $n_{i_1} + n_{i_2} + \dots$, and so on. If X is a product of root groups, we write $\Phi(X)$ for the set of roots with root groups in X.

Lemma 3.2. *The conclusion of Theorem 2(I)(b) holds if $G = E_8$.*

Proof. Suppose $G = E_8$. Inspection of the lists given in [**9**] of subsystems occurring in centralizers of semisimple elements shows that either D has a factor E_7 or D_8, or D is contained in a group E_6A_2, D_5A_3, A_8, A_7A_1, $A_5A_2A_1$, ${A_4}^2$, D_7T_1 or $D_6A_1T_1$. For the purposes of this proof we shall say that D is 'small' if it has no E_7 or D_8 factor.

For convenience we record the dimension and nilpotence class of Q_i (the latter being given by 1.11):

	i							
	1	2	3	4	5	6	7	8
$\dim Q_i$	78	92	98	106	104	97	83	57
$\text{class}(Q_i)$	2	3	4	6	5	4	3	2

By 3.1, we may assume that $\dim R_u(D \cap P_i) > \dim Q_i - d_{G,i,D}$, and in particular that D has more than $\dim Q_i - d_{G,i,D}$ positive roots; if D is small, this number is 30, 34, 36, 39, 38, 36, 31 or 21 according as $i = 1$, 2, 3, 4, 5, 6, 7 or 8. Writing X_j for a subgroup of A_j containing a maximal torus, we see that the possibilities for D small are as follows:

i	D
4	D_7T_1
$3, 5, 6$	D_7T_1 or E_6X_2
$2, 7$	D_7T_1, E_6X_2 or A_8
1	D_7T_1, E_6X_2, A_8 or $D_6A_1T_1$
8	$D_7T_1, E_6X_2, A_8, D_6X_1T_1, D_5X_3, A_7X_1$ or $A_6A_1T_1$

First suppose $i = 1$. By 3.1, we have $R_u(D \cap P_1) \leq Q_1$, whence

$$\text{class}\,(R_u(D \cap P_1)) \leq \text{class}\,(Q_1) = 2.$$

If $D = E_7X_1$, then by 1.11, $E_7 \cap P_1 = P_j(E_7)$ for $j \in \{1, 2, 6, 7\}$, and so $\dim R_u(E_7 \cap P_1) \leq 42$, whence $\dim R_u(D \cap P_1) \leq 43$; thus

$$f(s, G/P_1) \geq \dim Q_1 - 43 = 35 = d_{G,1,E_7}$$

as required. If $D = D_8$ then $D \cap P_1 \neq P_5(D_8)$ or $P_6(D_8)$ because the Levi factor A_4A_3 of $P_5(D_8)$ or $A_5A_1^2$ of $P_6(D_8)$ does not embed in $L_1' = D_7$; thus $D \cap P_1$ is $P_j(D_8)$ for $j \in \{1, 2, 3, 4, 7, 8\}$ or $P_{jk}(D_8)$ for $j, k \in \{1, 7, 8\}$, giving $\dim R_u(D \cap P_1) \leq 38$. Likewise, if $D = D_7T_1$ then $D_7 \cap P_1$ is $P_j(D_7)$ for $1 \leq j \leq 7$ or $P_{jk}(D_7)$ for $j, k \in \{1, 6, 7\}$, giving $\dim R_u(D \cap P_1) \leq 30$; if $D = E_6X_2$ then $E_6 \cap P_1$ is $P_j(E_6)$ for $j \neq 4$ or $P_{16}(E_6)$, giving $\dim R_u(E_6 \cap P_1) \leq 25$, whence $\dim R_u(D \cap P_1) \leq 28$; if $D = A_8$ then $D \cap P_1$ is $P_j(A_8)$ or $P_{jk}(A_8)$ and $\dim R_u(D \cap P_1) \leq 27$; and if $D = D_6A_1T_1$ then $D_6 \cap P_1$ is $P_j(D_6)$ for $1 \leq j \leq 6$ or $P_{jk}(D_6)$ for $j, k \in \{1, 5, 6\}$, giving $\dim(R_u(D_6 \cap P_1)) \leq 22$, whence $\dim R_u(D \cap P_1) \leq 23$. In all cases, the bounds required for Theorem 2(I)(b) follow.

The arguments for $i = 2$, 3, 4 and 5 are all similarly straightforward; the only point to note is that if $D = D_8$ then $D \cap P_3 \neq P_{36}(D_8)$, because the Levi factor $A_2^2A_1^2$ does not embed in $L_3' = A_6A_1$. Thus the conclusion of Theorem 2(I)(b) holds in these cases.

Now let $i = 6$. The arguments for the cases where D is small are straightforward. Suppose $D = E_7X_1$; we must show that $\dim R_u(D \cap P_6)$ is at most $\dim Q_6 - d_{G,6,E_7} = 97 - 44 = 53$. Since $\text{class}(R_u(D \cap P_6)) \leq 4$, the only possibility requiring consideration is that of $D \cap P_6 = P_4(E_7)P_1(A_1)$. As both $R_u(P_4(E_7))$ and Q_6 have precisely three roots of height 4 (with respect to $P_4(E_7)$ and P_6 respectively), these roots must be equal; thus

$$\beta_0 = \alpha_0, \qquad \beta_0 - \beta_1 = \alpha_0 - \alpha_8, \qquad \beta_0 - \beta_1 - \beta_2 = \alpha_0 - \alpha_7 - \alpha_8,$$

whence $\beta_1 = \alpha_8$ and $\beta_2 = \alpha_7$ by subtraction. Now as γ_1 is orthogonal to β_0 it must be of the form $\sum m_j\alpha_j$ with $m_8 = 0$; as it is also orthogonal to β_1 and β_2 we must have $m_7 = 0$ and $m_6 = 0$—but then $\gamma_1 \notin \Phi(Q_6)$. Thus the case $D \cap P_6 = P_4(E_7)P_1(A_1)$ cannot occur. Similarly suppose $D = D_8$; we require $\dim R_u(D \cap P_6) \leq 97 - 50 = 47$, and the only case to be considered is that where $D \cap P_6 = P_{36}(D_8)$. Again both $R_u(P_{36}(D_8))$ and Q_6 have precisely three roots of height 4, so we must have

$$\beta_0 = \alpha_0, \qquad \beta_0 - \beta_2 = \alpha_0 - \alpha_8, \qquad \beta_0 - \beta_1 - \beta_2 = \alpha_0 - \alpha_7 - \alpha_8,$$

whence $\beta_1 = \alpha_7$ and $\beta_2 = \alpha_8$. As the coefficients of β_2 in β_0 and α_8 in α_0 are equal, each β_k for $k > 2$ must be of the form $\sum m_j\alpha_j$ with $m_8 = 0$, and must be orthogonal to α_7. However, β_3 cannot then have $m_6 = 1$; thus the case $D \cap P_6 = P_{36}(D_8)$ cannot occur. We have therefore shown that the conclusion of Theorem 2(I)(b) holds if $i = 6$.

Next let $i = 7$; note that $\Phi(Q_7)$ has just two roots of height 3 with respect to P_7, namely α_0 and $\alpha_0 - \alpha_8$. In the case where D is small we require $\dim R_u(D \cap P_7) \leq 31$. If $D = A_8$ this bound is easily seen to be satisfied. If $D = D_7T_1$, the result is clear provided $\text{class}(R_u(D \cap P_7)) < \text{class}(P_7) = 3$, so assume $\text{class}(R_u(D \cap P_7)) = 3$; since $\Phi(R_u(D \cap P_7))$ can have at most two roots of height 3, we must have $D \cap P_7 = P_{13}(D_7)$ or $P_{2j}(D_7)$ for $j \in \{1, 6, 7\}$, and the bound follows. Similarly if $D = E_6X_2$ the result is clear unless $\text{class}(R_u(D \cap P_7)) = 3$, when consideration of roots of height 3 rules out $E_6 \cap P_7 = P_{15}(E_6)$ or $P_{36}(E_6)$, leaving just the case $D \cap P_7 = P_4(E_6)P_{12}(A_2)$ to be treated; here we must have

$$\beta_0 = \alpha_0, \qquad \beta_0 - \beta_2 = \alpha_0 - \alpha_8,$$

giving $\beta_2 = \alpha_8$. Since γ_1 and γ_2 are then orthogonal to α_8 and must have nonzero α_7-coefficient, they must both be of the form $\sum m_j\alpha_j$ with $m_7 = 2$ and $m_8 = 1$; but then $\gamma_0 = \gamma_1 + \gamma_2$ is not a root, which contradiction shows that $D \cap P_7 = P_4(E_6)P_{12}(A_2)$ cannot occur.

Now assume D is not small. If $D = D_8$ we must show that $\dim R_u(D \cap P_7) \leq 83 - 43 = 40$. The parabolic $D \cap P_7$ of D_8 cannot have Levi factor of type ${A_3}^2$, D_4A_2 or A_4A_2, since these do not embed in $L_7' = E_6A_1$; we cannot have $D \cap P_7 = P_{16}(D_8)$ or $P_{178}(D_8)$, since their unipotent radicals have too many roots of height 3; all other possibilities for $D \cap P_7$ satisfy the bound. If instead $D = E_7X_1$, we require $\dim R_u(D \cap P_7) \leq 83 - 36 = 47$: Both $R_u(P_5(E_7))$ and $R_u(P_{27}(E_7))$ have too many roots of height 3; if $E_7 \cap P_7 = P_3(E_7)$, equating roots of height 3 shows that $\beta_0 = \alpha_0$ and $\beta_1 = \alpha_8$, and then any root orthogonal to both β_0 and β_1 must be of the form $\sum m_j\alpha_j$

with both $m_8 = 0$ and $m_7 = 0$, and so lies outside $\Phi(Q_7)$—so we cannot have $D \cap P_7 = P_3(E_7)P_1(A_1)$; again, all other possibilities for $D \cap P_7$ satisfy the bound. Thus the conclusion of Theorem 2(I)(b) holds if $i = 7$.

Finally let $i = 8$; we have $\mathrm{class}(Q_8) = 2$, and α_0 is the only root of height 2 with respect to P_8. Thus if $\mathrm{class}(R_u(D_j \cap P_8)) = 2$ for any simple factor D_j of D, then α_0 must be the unique root of $\Phi(R_u(D_j \cap P_8))$ of height 2 with respect to $D_j \cap P_8$, and any root in $\Phi(D_k \cap P_8)$ for $k \neq j$ must be orthogonal to α_0 and hence outside $\Phi(Q_8)$. These considerations quickly show that all possibilities for $D \cap P_8$ satisfy the relevant bound, namely $\dim R_u(D \cap P_8) \leq 33$, 28 or 21 according as $D = E_7X_1$, $D = D_8$ or D is small. This completes the proof that the conclusion of Theorem 2(I)(b) holds if $G = E_8$. □

Lemma 3.3. *The conclusion of Theorem 2(I)(b) holds if $G = E_7$.*

Proof. Suppose $G = E_7$. In this proof we say that D is small if it has no factor E_6, D_6 or A_7; inspection of the lists in [**9**] as in the previous lemma shows that if D is small then D^0 is contained in a group A_5A_2, $A_3^2A_1$, $D_5A_1T_1$, $D_4A_1^2T_1$, A_6T_1 or $A_5A_1T_1$.

Using 1.11, we record the dimension and class of Q_i:

	i						
	1	2	3	4	5	6	7
$\dim Q_i$	33	42	47	53	50	42	27
$\mathrm{class}(Q_i)$	2	2	3	4	3	2	1

By 3.1, we may again assume that D has more than $\dim Q_i - d_{G,i,D}$ positive roots; if D is small, this number is 13, 16, 17, 19, 18, 16 or 10 according as $i = 1$, 2, 3, 4, 5, 6 or 7. Writing X_j for a subgroup of A_j containing a maximal torus again, we see that the possibilities for D small are as follows:

i	D^0
$4, 5$	A_6T_1 or $D_5X_1T_1$
$2, 3, 6$	$A_6T_1, D_5X_1T_1$ or A_5A_2
1	$A_6T_1, D_5X_1T_1, A_5X_2$ or $D_4A_1^2T_1$
7	$A_6T_1, D_5X_1T_1, A_5X_2, A_4X_2T_1, D_4X_1^2T_1$ or $A_3^2X_1$

First suppose $i = 1$. As with the case $i = 8$ for $G = E_8$, we see that if $\mathrm{class}(R_u(D_j \cap P_1)) = 2$ for any simple factor D_j of D, then α_0 must be the unique root of $\Phi(R_u(D_j \cap P_1))$ of height 2 with respect to $D_j \cap P_1$, and any root in $\Phi(D_k \cap P_1)$ for $k \neq j$ must be outside $\Phi(Q_1)$. These considerations quickly show that all possibilities for $D \cap P_1$ satisfy the relevant bound, namely $\dim R_u(D \cap P_1) \leq 21$, 17, 16 or 13 according as $D \triangleright E_6$, $D \triangleright D_6$, $D^0 = A_7$ or D is small.

Next suppose $i = 2$. The arguments here are all straightforward; we merely need note that if $D \triangleright D_6$ then $D_6 \cap P_2 \neq P_4(D_6)$, because the Levi factor $A_3A_1^2$ does not embed in $L_2' = A_6$, and for the same reason if $D^0 = A_7$

then $A_7 \cap P_2$ does not have Levi factor $A_3A_1^2$ or $A_2^2A_1$. Thus the conclusion of Theorem 2(I)(b) holds in this case.

Now let $i = 3$; note that $\Phi(Q_3)$ has just two roots of height 3 with respect to P_3, namely α_0 and $\alpha_0 - \alpha_1$. In the case where D is small, we require $\dim R_u(D \cap P_3) \leq 17$. If $D^0 = A_5A_2$ this bound is easily seen to be satisfied. If $D^0 = A_6T_1$, the result is clear provided $\text{class}(R_u(D \cap P_3)) < \text{class}(P_3) = 3$, so assume $\text{class}(R_u(D \cap P_3)) = 3$; since $\Phi(R_u(D \cap P_3))$ can have at most two roots of height 3, the bound follows immediately unless $A_6 \cap P_3 = P_{135}(A_6)$ or $P_{246}(A_6)$. As these cases are equivalent under a graph automorphism of A_6, it suffices to treat the former possibility; here we must have $\beta_0 = \alpha_0$ and $\beta_0 - \beta_6 = \alpha_0 - \alpha_1$, so that $\beta_6 = \alpha_1$—but then as β_1 is orthogonal to β_6 it cannot be of the form $\sum m_j\alpha_j$ with $m_3 = 1$, and thus cannot be of height 1 with respect to P_3, a contradiction. Thus the bound is satisfied if $D^0 = A_6T_1$. If instead $D^0 = D_5X_1T_1$, we cannot have $D_5 \cap P_3 = P_{145}(D_5)$ as this would require more than two roots of height 3; the bound is then clear unless $D' \cap P_3 = P_{13}(D_5)P_1(A_1)$. In this case we must have $\beta_0 = \alpha_0$ and $\beta_0 - \beta_2 = \alpha_0 - \alpha_1$, so that $\beta_2 = \alpha_1$; but then γ_1 must be orthogonal to both α_0 and α_1, which forces it to be of the form $\sum m_j\alpha_j$ with $m_1 = m_3 = 0$, contrary to $\gamma_1 \in \Phi(Q_3)$. Thus the bound is satisfied in all cases where D is small.

Now assume D is not small. If $D \rhd E_6$ the argument is straightforward; we require $\dim R_u(D \cap P_3) \leq 29$, and as $\text{class}(R_u(D \cap P_3)) \leq 3$ the minimal possibilities for the Levi factor of $E_6 \cap P_3$ are $A_2^2A_1$, A_3A_1 and A_4, each of which means that the bound is satisfied. If instead $D \rhd D_6$, we require $\dim R_u(D \cap P_3) \leq 24$; here the condition that $\Phi(R_u(D_6 \cap P_3))$ should contain at most two roots of height 3 with respect to $D_6 \cap P_3$ means that we cannot have $D_6 \cap P_3 = P_{14}(D_6)$, $P_{35}(D_6)$, $P_{36}(D_6)$ or $P_{156}(D_6)$, and the required bound follows. If $D^0 = A_7$, the requirement is $\dim R_u(D \cap P_3) \leq 22$; arguing as before with the nilpotence class and the number of roots of height 3, we see that we need only consider the possibility that $A_7 \cap P_3 = P_{247}(A_7)$ or $P_{257}(A_7)$ (up to equivalence under the graph automorphism of A_7). We must then have $\beta_0 = \alpha_0$ and $\beta_0 - \beta_1 = \alpha_0 - \alpha_1$, so that $\beta_1 = \alpha_1$—but then as β_7 is orthogonal to β_1 it cannot be of height 1 with respect to P_3, a contradiction. Thus the conclusion of Theorem 2(I)(b) holds if $i = 3$.

The arguments for $i = 4$ or 5 are all straightforward; the only point to note is that if $D^0 = A_7$ we cannot have $D \cap P_5 = P_{246}(A_7)$, because the Levi factor A_1^4 does not embed in $L_5' = A_4A_2$.

Now let $i = 6$; we have $\text{class}(Q_6) = 2$. The arguments for D small are all straightforward, and we obtain $\dim R_u(D \cap P_6) \leq 16$ as required. If $D \rhd E_6$ we require $\dim R_u(D \cap P_6) \leq 25$, and again this is immediate. If $D^0 = A_7$ we require $\dim R_u(D \cap P_6) \leq 20$; here $A_7 \cap P_6$ cannot have Levi factor $A_2^2A_1$ as this does not embed in $L_6' = D_5A_1$, and the bound follows. Lastly if $D \rhd D_6$ we require $\dim R_u(D \cap P_6) \leq 22$, which is satisfied unless $D = D_6A_1$

and $D \cap P_6 = P_4(D_6)P_1(A_1)$. We shall show that this is impossible; this is the most complicated of the cases to be treated.

Thus assume $D \cap P_6 = P_4(D_6)P_1(A_1)$. Since both Q_6 and $R_u(D_6 \cap P_6)$ have nilpotence class 2, the root β_4 must have α_6-coefficient 1, while each root β_k for $k \neq 6$ must have α_6-coefficient 0. As the coefficient of α_7 in α_0 is 1, the root β_4 must have α_7-coefficient 0. As D_6 is not a subsystem of E_6, not all the β_k can have α_7-coefficient 0, so we must have $\beta_k = \alpha_7$ for some k; since α_7 is not orthogonal to β_4, and β_3 appears with coefficient 2 in β_0, we may assume (after interchanging β_5 and β_6 if necessary) that $\beta_6 = \alpha_7$. Now if we let Ξ be the set of roots of the form $\sum m_j\alpha_j$ with $m_6 = 2$, then we must have $\gamma_1 \in \Xi$ as it is orthogonal to β_6 and has nonzero α_6-coefficient. However, γ_1 is orthogonal to both β_0 and $\beta_0 - \beta_2$, which also lie in Ξ; but any root in Ξ is orthogonal to precisely one other root in Ξ. This contradiction shows that we cannot have $D \cap P_6 = P_4(D_6)P_1(A_1)$.

Finally let $i = 7$; we have class$(Q_7) = 1$. Moreover $\Phi(Q_7)$ consists of 27 roots; of these, given two which are orthogonal there is exactly one other orthogonal to both. (This is easily seen by using the Weyl group to move the first root of an orthogonal pair to α_7; the 10 roots in $\Phi(Q_7)$ orthogonal to α_7 are those of the form $\sum m_j\alpha_j$ with $m_6 = 2$ and $m_7 = 1$, and these fall into five orthogonal pairs.) For D small, we require $\dim R_u(D \cap P_7) \leq 10$: If $D^0 = A_5A_2$ and $A_5 \cap P_7 = P_3(A_5)$, then we already have the three mutually orthogonal roots β_3, $\beta_2 + \beta_3 + \beta_4$ and β_0 in $\Phi(Q_7)$, so we cannot have $\gamma_j \in \Phi(Q_7)$ for $j = 1$ or 2; if $D^0 = A_6T_1$ we cannot have $A_6 \cap P_7 = P_3(A_6)$ or $P_4(A_6)$, as the Levi factor A_3A_2 does not embed in $L_7' = E_6$; if $D^0 = D_5A_1T_1$ we cannot have $D_5A_1 \cap P_7 = P_j(D_5)P_1(A_1)$ for $j \in \{4, 5\}$, since then β_j and γ_1 would be orthogonal to the three roots β_0, $\beta_0 - \beta_2$ and $\beta_0 - \beta_1 - \beta_2$ in $\Phi(Q_7)$; in all other cases the arguments are straightforward. If $D \triangleright E_6$ we require $\dim R_u(D \cap P_7) \leq 16$, and this is immediate. If $D \triangleright D_6$ we need $\dim R_u(D \cap P_7) \leq 15$; here we may use the argument just given for the case $D^0 = D_5A_1T_1$ to see that we cannot have $D' \cap P_7 = P_j(D_6)P_1(A_1)$ for $j \in \{5, 6\}$, and the bound follows. Lastly if $D^0 = A_7$ we must show that $\dim R_u(D \cap P_7) \leq 12$; here we cannot have $A_7 \cap P_7 = P_j(A_7)$ for $j \in \{3, 4, 5\}$, as the Levi factor A_3^2 or A_4A_2 does not embed in $L_7' = E_6$, and again the bound follows. This completes the proof that the conclusion of Theorem 2(I)(b) holds if $G = E_7$. □

Lemma 3.4. *The conclusion of Theorem 2(I)(b) holds if $G = E_6, F_4, G_2$.*

Proof. The proof is carried out using the methods of the previous lemmas; the only points which need mentioning are as follows. Firstly, let $G = E_6$. If $i = 1$ or 6, then $\Phi(Q_i)$ does not contain three pairwise orthogonal roots. Thus if $D = A_5A_1$ we cannot have either $A_5 \cap P_i = P_3(A_5)$ (as then $\Phi(R_u(D \cap P_i))$ would contain β_3, $\beta_2 + \beta_3 + \beta_4$ and β_0) or $D \cap P_i = P_j(A_5)P_1(A_1)$ for $j \in \{2, 4\}$ (as then $\Phi(R_u(D \cap P_i))$ would contain $\beta_2 + \beta_3 + \beta_4$, β_0 and γ_1);

and similarly if $D = A_4A_1T_1$ we cannot have $D' \cap P_i = P_j(A_4)P_1(A_1)$ for $j \in \{2,3\}$ (as then $\Phi(R_u(D \cap P_i))$ would contain $\beta_2 + \beta_3$, β_0 and γ_1). If instead $i = 3$ or 5, then Q_i has nilpotence class 2, and of the five roots of height 2 with respect to P_i no two are orthogonal; thus if $D = A_5A_1$ we cannot have $A_5 \cap P_i = P_{24}(A_5)$ (as then $\Phi(R_u(D \cap P_i))$ would contain $\beta_2 + \beta_3 + \beta_4$ and β_0).

Secondly, let $G = F_4$; here we exploit the distinction between long and short roots. If $i = 1$ then $\Phi(Q_1)$ contains just 6 short roots, of which no two sum to a short root and none is orthogonal to α_0; also α_0 is the unique root of height 2 with respect to P_1. Thus if $D = A_2\widetilde{A}_2$ we cannot have $\widetilde{A}_2 \cap P_i = P_{12}(\widetilde{A}_2)$; and if $D = A_3\widetilde{A}_1$ we cannot have $D \cap P_i = P_{jk}(A_3)P_1(\widetilde{A}_1)$ for $j, k \in \{1,2,3\}$. If $i = 3$ then $\Phi(Q_i)$ contains only 9 long roots, of which no three may be added to form a root (since their α_3-coefficients are all 2 or 4); thus if $D = B_4$ or $A_3\widetilde{A}_1$ the number of long roots in $\Phi(D) \setminus \Phi(R_u(D \cap P_3))$ must be at least 3 or 1 respectively. If $i = 4$, the following is true of $\Phi(Q_i)$: it contains only 6 long roots, of which no three are pairwise orthogonal, no two sum to a root and none is orthogonal to $\widetilde{\alpha}_0 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$, the unique short root of height 2 with respect to P_4; also, given any two of its long roots which are orthogonal, none of its short roots is orthogonal to both. Thus if $D = B_4$ we cannot have $D \cap P_i = P_j(B_4)$ for $j = 2$ or 3, as then $\Phi(R_u(P_j(B_4)))$ would contain 9 long roots; if $D = C_3A_1$ we cannot have $D \cap P_i = P_2(C_3)P_1(A_1)$, as then $\Phi(R_u(D \cap P_i))$ would contain the pairwise orthogonal roots β_0, $2\beta_2 + \beta_3$ and γ_1; if $D = B_3T_1$ we cannot have $B_3 \cap P_i = P_2(B_3)$, as then $\Phi(R_u(D \cap P_i))$ would contain $\beta_1 + \beta_2$ and $\beta_2 + 2\beta_3$, whose sum is a root; if $D = A_2\widetilde{A}_2$ we cannot have $A_2 \cap P_i = P_{12}(A_2)$, while $\widetilde{A}_2 \cap P_i = P_{12}(\widetilde{A}_2)$ would force $\widetilde{\alpha}_0 \in \Phi(R_u(D \cap P_i))$, whence $\Phi(A_2) \cap \Phi(R_u(D \cap P_i)) = \emptyset$; and if $D = A_3\widetilde{A}_1$ we cannot have either $A_3 \cap P_i = P_{jk}(A_3)$ for $j, k \in \{1,2,3\}$ or $D \cap P_i = P_2(A_3)P_1(\widetilde{A}_1)$.

All other cases, including those in which $G = G_2$, are straightforward, and may be left to the reader. □

This completes the proof of Theorem 2(I)(b). In fact all the bounds listed in Table 7.3 are sharp; for each entry $d_{G,i,D}$, it is possible to find an appropriate D for which $\dim Q_i - \dim R_u(D \cap P_i)$ takes the value given, as may be verified by using a computer to form all W-translates of $\Phi(D)$ and taking intersections with $\Phi(Q_i)$.

4. Proof of Theorem 2, Part (II)(a): Unipotent elements in maximal rank subgroups.

In this section we prove Theorem 2(II)(a). Thus let G be an exceptional algebraic group over the algebraically closed field K of characteristic p, and let M be a maximal closed reductive subgroup of G of maximal rank (that is, containing a maximal torus of G). The possibilities for M are given by 1.1.

We begin by handling elements in $M - M^0$. For convenience we deal with both semisimple and unipotent elements in this case:

Lemma 4.1. *Let $x \in M - M^0$ be an element of prime order, and let $D = C_G(x)$. Then $\dim x^G - \dim(x^G \cap (M - M^0))$ satisfies the bounds of Theorem 2(II): That is,*

$$\dim x^G - \dim(x^G \cap (M - M^0)) \geq \begin{cases} e_G, & x \text{ a long root element} \\ e'_G, & x \text{ unipotent, not a long root element} \\ f_{G,M,D}, & x \text{ semisimple.} \end{cases}$$

Proof. First consider $G = E_8$. The non-connected possibilities for M are those with $M^0 = A_8, A_2E_6, D_4D_4, A_4A_4, A_2^4, A_1^8$ or T_8. Using 1.4, we see that if $M^0 = A_8$ then $\dim(x^G \cap (M - M^0)) = \dim(A_8/B_4) = 44$; if $M^0 = A_2E_6$ then $\dim(x^G \cap (M - M^0)) \leq \dim(A_2E_6/A_1C_4) = 47$; if $M^0 = D_4D_4$ then $\dim(x^G \cap (M - M^0)) \leq \dim(D_4D_4/A_2A_2) = 40$; and in the other cases, $\dim(x^G \cap (M - M^0)) \leq 28$. The conclusion now follows if x is semisimple, because then $\dim x^G \geq 112$, 128 or 156 according as $D \triangleright E_7$, $D = D_8$ or D has no E_7 or D_8 factor. It also follows if x is not a root element (since then by 1.7, $\dim x^G \geq 92$), or if $M^0 \neq A_8$, A_2E_6, D_4D_4. However, if M^0 is one of the latter three subgroups, then x is not a root element by 1.13(iii).

Next let $G = E_7$. Here $M^0 = T_1E_6, A_7, A_2A_5, A_1^3D_4, A_1^7$ or T_7.

Suppose $M^0 = T_1E_6$. If $p \neq 2$ then by 1.4, $C_{M^0}(x) = F_4$ or C_4; and from the proof of [8, 2.15], $C_G(x)^0 = T_1E_6$ or A_7, respectively. Therefore $\dim x^G - \dim(x^G \cap (M - M^0)) = 54 - 27$ or $70 - 43$, which is equal to 27 in both cases, giving the conclusion. If $p = 2$ then again by 1.4, $\dim(x^G \cap (M - M^0)) = 27$ or 43. By [24, §2], $V_G(\lambda_7) \downarrow E_6 = V(\lambda_1) \oplus V(\lambda_6) \oplus 0^2$; since x interchanges the first two spaces, it has at least 27 Jordan blocks of size 2 on $V_G(\lambda_7)$, and hence by [17] lies in class $3A_1''$ or $4A_1$ in G. These classes have dimensions 54 and 70. We need to show that $\dim x^G - \dim(x^G \cap (M - M^0)) \geq e'_G = 20$. This will follow provided we show that when $C_{M^0}(x) = C_{F_4}(t)$ (in the notation of 1.4), x lies in the class $4A_1$ rather than $3A_1''$. To see this, let $u \in M - M^0$ be an involution with $C_{M^0}(u) = F_4$. This F_4 contains a subgroup $D_4\langle s\rangle$, where s is an element of order 3 inducing a triality automorphism of D_4. Moreover, $C_G(D_4) = (A_1)^3$, with s permuting the 3 factors. Since u is centralized by s, it must lie in a diagonal subgroup of this $(A_1)^3$. Now taking the element t to be a root element in the D_4, we see that tu lies diagonally in a subgroup $4A_1$. This $4A_1$ is a Levi subgroup of G: For the $3A_1$ is a Levi of type $SO_4 \times A_1$ in a Levi D_6 of G, and inspection of the Dynkin diagram of G shows that the fourth A_1 can be chosen to make a

Levi $4A_1$ subgroup with this. Therefore tu is in the class $4A_1$ of G. Since tu is a conjugate of x, this finishes the proof in this case.

Now suppose $M^0 = A_7$. If $p \neq 2$ then by 1.4, $C_{M_0}(x) = C_4$ or D_4, and in the latter case $C_G(x)^0 = A_7$ (see the proof of [8, 2.15]). Hence $\dim x^G - \dim(x^G \cap (M - M^0)) \geq 27$, 37 or 35, according as $D^0 = E_6T_1$, D_6A_1 or A_7. If $p = 2$ then 1.4 gives $\dim(x^G \cap (M - M^0)) = 27$ or 35. Also, if $V_{56} = V_G(\lambda_7)$, then $V_{56} \downarrow A_7 = V(\lambda_2) \oplus V(\lambda_6)$ (see [24, §2]). As x interchanges $V(\lambda_2)$ and $V(\lambda_6)$, it acts on V_{56} as J_2^{28} (where J_2 is a Jordan block of size 2). Therefore by [17], x is in class $3A_1''$ or $4A_1$ and the required bound follows provided we show that when $C_{M^0}(x) = C_{C_4}(t)$ (in the notation of 1.4), x lies in the class $4A_1$ rather than $3A_1''$. To see this, let v be an involution in $M - M^0$ such that $C_{M^0}(v) = C_4$. By 1.7, v must lie in the class $3A_1''$. Let J be a fundamental subgroup A_1 lying in this C_4, and take t to be an involution in J. Then $J < C_G(v)$, and by [23, 2.3], J lies in a Levi subgroup of a parabolic of G containing $C_G(v)$. Therefore vt is in the same class as the element ut of the previous paragraph, namely $4A_1$, as desired.

Now let $M^0 = A_2A_5$. By [31, 1.8],

$$L(E_7) \downarrow A_2A_5 = (V(\lambda_1) \otimes V(\lambda_2)) \oplus (V(\lambda_2) \otimes V(\lambda_4)) \oplus L(A_2A_5).$$

By 1.4, $\dim(x^G \cap (M - M^0)) = 19$ or 25. We know by 1.13(iii) that x is not a root element. If $p = 2$ then by 1.7, $\dim x^G \geq 52$, and the conclusion follows. And if $p \neq 2$ then x interchanges the first two spaces in the above restriction, whence we see that $\dim C_{L(G)}(x) = 69$ or 63. Hence $\dim x^G \geq 64$ and the conclusion follows.

Of the remaining cases, $M^0 = A_1^3D_4$ is dealt with by the same methods, and A_1^7, T_7 are trivial to handle. This completes the case where $G = E_7$.

Next consider $G = E_6$. Here $M^0 = T_2D_4$, A_2^3 or T_6.

Suppose $M^0 = T_2D_4$. By [24, §2],

$$L(E_6) \downarrow D_4 = V(\lambda_1)^2 \oplus V(\lambda_3)^2 \oplus V(\lambda_4)^2 \oplus L(D_4T_2).$$

If $|x| = 3$ then by 1.4, $\dim(x^G \cap (M - M^0)) = 16$ or 22. When $p \neq 3$ the above restriction implies $\dim C_G(x) = 30$ or 24, and the required bounds follow. And when $p = 3$, x has at least 16 Jordan blocks of size 3 on $L(E_6)$, so by [17], $x \notin A_1, 2A_1, 3A_1$, whence $\dim x^G \geq 42$ by 1.7, giving the result. A similar argument gives the result when $|x| = 2$; note that if $p \neq 2$ and $D^0 = D_5T_1$ then $\dim(D \cap M)^0 \geq \dim B_3T_1 = 22$, whence $\dim x^G - \dim(x^G \cap (M - M^0)) \geq 24$.

When $M^0 = A_2^3$, we have

$$V_G(\lambda_1) \downarrow A_2^3 = (V(\lambda_1) \otimes V(\lambda_2) \otimes 0) \oplus (V(\lambda_2) \otimes 0 \otimes V(\lambda_1)) \oplus (0 \otimes V(\lambda_1) \otimes V(\lambda_2)),$$

(see [**24**, Section 2]), from which we check that elements of order 2 or 3 in $M\backslash M^0$ do not have centralizer of type D_5; now the argument of the previous paragraph gives the conclusion. Finally the case where $M^0 = T_6$ is trivial.

The cases $G = F_4, G_2$ are entirely similar and left to the reader. □

Let u be a nonidentity unipotent element of M, of order p if $p > 0$. By the previous lemma we may ignore $u^G \cap (M - M^0)$: In other words, to prove Theorem 2(II)(a) it suffices to prove the lower bounds in the statement for $\dim u^G - \dim(u^G \cap M^0)$. In particular we can assume $u \in M^0$.

Since M^0 has finitely many unipotent classes (see 1.8), replacing u by a suitable conjugate we may take $\dim(u^G \cap M^0) = \dim u^{M^0}$ (i.e., u^{M^0} is an M^0-class of maximal dimension in $u^G \cap M^0$). Write

$$D = C_G(u).$$

Lemma 4.2. *The conclusion of Theorem 2(II)(a) holds if u is a long root element of G (or a short root element if $(G, p) = (F_4, 2)$ or $(G_2, 3)$).*

Proof. Suppose u is a long root element. By 1.13(ii), u lies in a simple factor M_0 of M^0, and is a root element therein. Therefore $\dim u^G - \dim u^M = \dim u^G - \dim u^{M_0}$. The possibilities for M_0 are given by 1.1, and the dimensions of u^G, u^{M_0} are given by 1.12. It follows from these results that

$$\dim u^G - \dim u^{M^0} \geq e_G,$$

(where e_G is as in Table 1 in the Introduction), as required. Finally, if u is a short root element and $(G, p) = (F_4, 2)$ or $(G_2, 3)$), application of a graph automorphism of G now gives the conclusion. □

Lemma 4.3. *The conclusion of Theorem 2(II)(a) holds if*

$$\dim M + \dim D \leq \dim G + \mathrm{rank}(G) - e'_G,$$

where e'_G is as in Table 1 (in the Introduction).

Proof. We have $\dim u^G - \dim u^{M^0} = \dim G - \dim D - \dim M + \dim C_M(u)$, and the last term is at least $\mathrm{rank}(G)$. The result follows. □

In view of 4.2, 4.3, we assume from now on that u is not a long root element (or a short root element if $(G, p) = (F_4, 2)$ or $(G_2, 3)$), and that

$$\dim M + \dim D > \dim G + \mathrm{rank}(G) - e'_G.$$

Lemma 4.4. *The possibilities for M are as follows:*

G	M^0
E_8	A_1E_7, D_8, A_8, A_2E_6
E_7	$T_1E_6, A_1D_6, A_7, A_2A_5$
E_6	T_1D_5, A_1A_5, T_2D_4
F_4	A_1C_3, B_4, D_4
G_2	A_2

Proof. Since u is not a long root element (or a short root element when $(G,p)=(F_4,2)$ or $(G_2,3)$), we see from 1.7 that $\dim D$ is at most 156, 81, 46, 30, 6, according as G is E_8, E_7, E_6, F_4, G_2, respectively. Since $\dim M > \dim G - \dim D + \mathrm{rank}(G) - e'_G$, the result now follows from 1.1. □

Observe that by 1.2, with one exception each of the possibilities for M listed in 4.4 is the centralizer in G of an element of order 2 or 3 (except when $p=2$ or 3 respectively); the exception is $M^0 = D_4 < F_4$. We shall deal with the various cases using this observation.

The involution centralizers are of the following types:

$$(*)\qquad \begin{array}{rl} G=E_8: & M=A_1E_7, D_8 \\ G=E_7: & M^0=T_1E_6, A_1D_6, A_7 \\ G=E_6: & M=T_1D_5, A_1A_5 \\ G=F_4: & M=B_4, A_1C_3. \end{array}$$

Lemma 4.5. *Assume $M = C_G(t)$ for some involution t. Then the conclusion of Theorem 2(II)(a) holds.*

Proof. Here M is as in $(*)$ above, with $p \neq 2$. We have

$$\begin{aligned} \dim u^G - \dim u^M &= \dim G - \dim D - \dim M + \dim M \cap D \\ &= \dim t^G - \dim t^D. \end{aligned}$$

Write $R = R_u(D^0)$ and $\overline{D} = D^0/R$. Choose a maximal unipotent subgroup E of $\overline{D}$ normalized by t, and let V be the preimage of E in D. Then V is also normalized by t; choose a maximal unipotent subgroup U of G containing V and normalized by t. Now $C_U(t)$ is a maximal unipotent subgroup of the reductive group $C_G(t)$. It follows that

$$\dim t^G = 2\dim t^U, \;\; \dim t^{\overline{D}} = 2\dim t^E.$$

We have $\dim t^R \leq \dim t^V - \dim t^E \leq \dim t^U - \dim t^E = \frac{1}{2}(\dim t^G - \dim t^{\overline{D}})$. It follows that

$$\dim t^G - \dim t^D = \dim t^G - \dim t^{\overline{D}} - \dim t^R \geq \frac{1}{2}(\dim t^G - \dim t^{\overline{D}}).$$

Consequently it is sufficient to prove that

$$(\dagger)\qquad \dim t^G - \dim t^{\overline{D}} \geq 2e'_G.$$

For $G = E_8$, $e'_G = 40$ and $\dim t^G \geq 112$, so we are done unless $\dim t^{\overline{D}} > 32$. A glance at 1.7 shows that the inequalities $\dim t^{\overline{D}} > 32$ and $(\dagger)$ are simultaneously possible only if u is in one of the classes $2A_1, A_2$, with $C_G(t) = A_1E_7$. Write $u = u_0u_1$ with $u_0 \in A_1, u_1 \in E_7$. Now

$$L(G) \downarrow A_1E_7 = L(A_1E_7) \oplus (V(\lambda_1) \otimes V(\lambda_7))$$

(see [**24**, Section 2]). If u lies in class $2A_1$ then by [**17**, Table 9], u acts on $L(G)$ as $J_3^{14} \oplus J_2^{64} \oplus J_1^{78}$ (where J_i denotes a Jordan block of size i). Hence from [**17**, Table 8] we see that u_1 must be in class A_1 or $2A_1$ of E_7. Hence by 1.7 we have

$$\dim u^G - \dim u^M \geq \dim u^G - 52 = 92 - 52 = e'_G,$$

as required. Now consider u in class A_2. The Jordan form of u on $L(G)$ is given in [**17**, Table 9]; and the possible Jordan forms of u_1 on $L(E_7)$ and $V_{E_7}(\lambda_7)$ are given in [**17**, Tables 7, 8]. From this we deduce that u_1 must lie in class $3A_1''$ or A_2 of E_7, whence by 1.7,

$$\dim u^G - \dim u^M \geq 114 - 66 > e'_G.$$

Next consider $G = E_7$. By 1.7 together with (†), we are done unless u lies in class $2A_1, 3A_1''$ or A_2 (with $C_G(t) = T_1E_6$ in the first and last cases). If $C_G(t) = T_1E_6$, then since this is a Levi subgroup of G, u lies in class $2A_1, 3A_1$ or A_2 of the E_6 factor, respectively (see 1.6). In fact we see from [**17**] that the $3A_1$ class in E_6 lies in the $3A_1'$ class of G, not the $3A_1''$ class. Hence by 1.7 we have

$$\dim u^G - \dim u^M \geq 20 = e'_G$$

(with equality for the $2A_1$ class).

This leaves $M^0 = C_G(t)^0 = A_1D_6$ or A_7 to consider. Here $u \in 3A_1''$. Now u lies in a subgroup $A_1^3T_4$ of G, so $T_4 \leq C_G(u)$. Also $t \in C_G(u)$, which is connected (see 1.7), so $u \in C_M(T_4)$. It follows that u lies in a Levi subgroup of M^0 of rank at most 3. Since $u \in 3A_1''$, this Levi subgroup is of type A_1^3. If $M^0 = A_7$, this implies that $L(A_7) \downarrow u$ has Jordan blocks of size 2, whereas by [**17**], elements $3A_1''$ have no such blocks on $L(E_7)$, a contradiction. And if $M^0 = A_1D_6$, observe that the two unipotent classes of type $3A_1$ in D_6 have actions $J_3 \oplus J_2^2 \oplus J_1^5$ and J_2^6 on the usual module, and hence by 1.10,

$$\dim u^G - \dim u^M \geq 54 - 32 > e'_G.$$

Next, if $G = E_6$ then (†) and 1.7 give the conclusion, except if $u \in 2A_1$ and $C_G(t) = T_1D_5$. As in a previous case, this is a Levi subgroup of G, so u has type $2A_1$ in D_5, with action $J_3 \oplus J_1^7$ or $J_2^4 \oplus J_1^2$ on the usual module. Then 1.10 gives $\dim u^G - \dim u^M \geq 32 - 20 > e'_G$.

Now suppose $G = F_4$. When $M = A_1C_3$ it is easy to see that the result holds, using (†). So suppose $M = B_4$. Since $p \neq 2$, the unipotent classes of M are labelled by Levi subgroups of B_4 (see 1.7). For such Levi subgroups which are also Levi subgroups of G, the corresponding unipotent element u has the same label as an element of F_4; the dimension of u^G is given by [**6**, p. 401], and that of u^M by 1.10, and we check that in all cases $\dim u^G - \dim u^M \geq 8 > e'_G$, as required. This leaves the Levi subgroups of B_4 which are not Levi subgroups of F_4; these are A_1A_1, A_3, A_1B_2 and B_4. By (†) we may assume that $\dim t^{\overline{D}} \geq 5$. From the list of possible $\overline{D}$ (see 1.7, with [6] to complete the list), we see that this implies that $u \in T_k E < G$, where T_k is a torus of rank k and E a semisimple group of rank $4-k \leq 2$. Hence u lies in such a subgroup of B_4, and it follows that u lies in the class A_1A_1 of B_4. Then u centralizes $C_G(A_1A_1) = C_2$, and it follows that u lies in the class $\widetilde{A}_1$ of G. Thus $\dim u^G - \dim u^M = 22 - 16 = e'_G$, as required. □

Lemma 4.6. *Assume M is as in $(*)$ above, with $p = 2$. Then the conclusion of Theorem 2(II)(a) holds.*

Proof. Consider first $G = E_8$. Recall that we may take u to have prime order, hence have order 2. Therefore by 1.7, u belongs to one of the classes $2A_1, 3A_1, 4A_1$. Moreover, $\dim u^M \leq 72$ by 1.5, so we may assume that $\dim u^G < 72 + e'_G = 112$, and hence that $u \in 2A_1$ (again by 1.7).

Let $M = A_1E_7$. Then $u = u_1$ or u_0u_1, where $1 \neq u_0 \in A_1$ and $u_1 \in E_7$ is in one of the involution classes $A_1, 2A_1, 3A_1'', 3A_1', 4A_1$ of E_7. By [31, 1.8],

$$L(G) \downarrow A_1E_7 = (\lambda_1 \otimes \lambda_7) \oplus (L(A_1E_7)).$$

The Jordan forms of the various possibilities for u_1 acting on $L(E_7)$ and on $V(\lambda_7) = V_{56}$ are given by [17], and hence we can calculate the Jordan forms of u_1 and u_0u_1 on $L(G)$, hence determining the classes of these elements in G (again using [17]). The outcome is as follows:

class of u_1 in E_7	class of u_1 in E_8	class of u_0u_1 in E_8
A_1	A_1	$2A_1$
$2A_1$	$2A_1$	$3A_1$
$3A_1''$	$3A_1$	$3A_1$
$3A_1'$	$3A_1$	$4A_1$
$4A_1$	$4A_1$	$4A_1$

It follows from this and the class dimensions in 1.7 that $\dim u^G - \dim u^M \geq e'_G$, as required.

Now let $M = D_8$. The involution classes in M are given by 1.10: Representatives are a_{2l}, c_{2l} $(l = 1, 2, 3, 4)$. The representatives a_{2l} lie in Levi subgroups lA_1 of an A_7 in M; and c_{2l} lies in a Levi subgroup $SO_4 \times (l-1)A_1$ (see [2, Section 8]). Inspecting the extended Dynkin diagram of G, we see that all but one of these Levi subgroups of D_8 are also Levi subgroups of G; the exception is $SO_4 \times 3A_1$. Excluding this exception for the time being, it follows that u has the same label in E_8 as in D_8. The dimensions of u^G and u^M are therefore given by 1.7 and 1.10 respectively, from which we check that $\dim u^G - \dim u^M \geq 40 = e'_G$ in all cases. Finally, consider u in the class $5A_1 = SO_4 \times 3A_1$ of D_8. Now $L(G) \downarrow D_8 = L(D_8) \oplus V(\lambda_8)$ by [31, 1.8]. We count Jordan blocks J_2 for u on $L(G)$. The action on the spin module $V(\lambda_8)$ gives 64 such blocks. Also u lies in a subgroup of type $SO_4 \times D_6$ of D_8, and the tensor product of natural modules $V_4 \otimes V_{12}$ is a summand of $L(D_8)$ restricted to this subgroup, which gives a further 24 blocks J_2 for u.

Finally, the projection of u to D_6 lies in a subgroup A_5, and the action on $L(A_5)$ gives another 18 J_2 blocks for u. Hence u has at least 106 blocks J_2 on $L(E_8)$. But this means that u is not in class $2A_1$ by [**17**], a contradiction. This completes the proof for $G = E_8$.

Next let $G = E_7$. Here $M^0 = T_1E_6, A_1D_6$ or A_7, and by 1.5, $\dim u^M \leq 43$. Hence we can assume that $\dim u^G < 43 + e'_G = 63$, so by 1.7, u lies in one of the involution classes $2A_1, 3A_1''$ of G.

Let $M^0 = T_1E_6$. As $u \in M^0$, u lies in class $2A_1$ or $3A_1$ of E_6. Also M^0 is a Levi subgroup of G, so u correspondingly lies in class $2A_1$ or $3A_1', 3A_1''$ of G; and in fact when u lies in class $3A_1$ of E_6, it lies in $3A_1'$ of E_7, as can be seen by considering the action of u on $V_{56} = V_{E_7}(\lambda_7)$ and using [**17**]. Now we check using 1.7 that $\dim u^G - \dim u^M \geq 20 = e'_G$.

Now consider $M^0 = A_7$. Involutions in M^0 have labels lA_1 $(l = 1, 2, 3, 4)$, and for $l \neq 4$ these are also Levi subgroups of G, whence $\dim u^G - \dim u^M \geq \dim(lA_1)^G - \dim(lA_1)^M$ (where for $l = 3$, lA_1 stands for either $3A_1'$ or $3A_1''$ in G), and this is at least e'_G by 1.7 and 1.10. For u in class $4A_1$ of M^0, we calculate the Jordan form of u on $V_{56} = V_G(\lambda_7)$ using $V_{56} \downarrow A_7 = V(\lambda_2) \oplus V(\lambda_6)$ (see [**24**, Section 2]); this Jordan form is $J_2^{24} \oplus J_1^8$, whence by [**17**], u is in class $3A_1'$ of G, and the conclusion again follows using 1.7 and 1.10.

Finally, suppose $M = A_1D_6$. By 1.6, $\dim u^G \geq 52$, so we may assume that $\dim u^M > 52 - e'_G = 32$. Write $u = u_1$ or u_0u_1, where $1 \neq u_0 \in A_1$ and $u_1 \in D_6$. By 1.10, the dimension bound implies that u_1 is conjugate to c_6 or c_4 in D_6. Observe

$$L(G) \downarrow A_1D_6 = L(A_1D_6) \oplus (1 \otimes V(\lambda_5)),$$

and as we have seen before, the Jordan forms of c_4, c_6 on $V(\lambda_5)$ are both J_2^{32}. Hence we calculate the possible Jordan forms of u on $L(G)$, from which we deduce using [**17**] that u is in class $3A_1'$ or $4A_1$ of G. This means that $\dim u^G \geq 64$. Since $\dim u^M \leq \dim(u_0c_6)^M = 38$, the conclusion follows. This completes the proof for $G = E_7$.

When $G = E_6$ we have $M = T_1D_5$ or A_1A_5, so $\dim u^M \leq 25$ or 22 respectively, by 1.5. Therefore, assuming as we may that $\dim u^G < \dim u^M + e'_G$, we see using 1.7 that $M = T_1D_5$ and $u \in 2A_1$. As D_5 is a Levi subgroup this means that u lies in a class $2A_1$ of D_5, which, as shown in the proof of 2.5, has dimension at most 20. Thus $\dim u^G - \dim u^M \geq 32 - 20 > e'_G$.

Now let $G = F_4$. If $M = B_4$, involution classes of M and their centralizers are given by [**33**, 2.2], and contain the elements $y_1, y_2, y_3, y_6, y_7, y_8$ given there; the involution classes in G are given in [**33**, Theorem 2.1], containing the elements x_1, x_2, x_3, x_4 (where x_1, x_2 are short and long root elements, respectively). Moreover, y_1, y_2, y_3, y_6 are equal to x_1, x_2, x_3, x_4, respectively. And y_7 lies in a product A_1A_1 of two long root A_1's in G, whence from the restriction of $V_{26} = V_G(\lambda_4)$ to A_1A_1 we see that $V_{26} \downarrow y_7$ has Jordan block structure $J_2^{10} \oplus J_1^6$; therefore by [**17**], y_7 is G-conjugate to x_3. Similarly y_8 is G-conjugate to x_4. Thus we can now record the unipotent class dimensions in G and $M = B_4$:

u	$\dim u^{B_4}$	u conjugate to	$\dim u^G$
y_1	8	x_1	16
y_2	12	x_2	16
y_3	14	x_3	22
y_6	18	x_4	28
y_7	16	x_3	22
y_8	20	x_4	28

Thus for non-root elements, $\dim u^G - \dim u^M \geq 6 = e'_G$, giving the result in this case.

For $M = A_1C_3$ we have $\dim u^M \leq 14$ by 1.5, whence we can assume $\dim u^G < 14 + e'_G = 20$. By 1.7 this forces u to be a root element, which is not the case. □

The cases remaining to be considered are as follows:

$$(**) \qquad \begin{array}{rl} G = E_8: & M^0 = A_8, A_2E_6 \\ G = E_7: & M^0 = A_2A_5 \\ G = E_6: & M^0 = T_2D_4 \\ G = F_4: & M^0 = D_4 \\ G = G_2: & M^0 = A_2 \end{array}$$

Observe that by 1.2, when $p \neq 3$ and $G \neq F_4$, we have $M^0 = C_G(v)^0$ for some element $v \in G$ of order 3.

Lemma 4.7. *The conclusion of Theorem 2(II)(a) holds in the cases (∗∗) above.*

Proof. By the assumption just before 4.4,

$$\dim D > \dim G - \dim M + \mathrm{rank}(G) - e'_G$$

(where $D = C_G(u)$), and hence using 1.7 we see that u lies in one of the following classes in G:

$$\begin{array}{rl} G = E_8,\ M = A_2E_6: & 2A_1, 3A_1, A_2 \\ G = E_8,\ M = A_8: & 2A_1 \\ G = E_7: & 2A_1, 3A_1'' \\ G = E_6: & 2A_1 \\ G = F_4: & \widetilde{A}_1 (p \neq 2), \widetilde{A}_1^{(2)} (p = 2), A_1\widetilde{A}_1, A_2, \widetilde{A}_2 \\ G = G_2: & \widetilde{A}_1 (p \neq 3), \widetilde{A}_1^{(3)} (p = 3). \end{array}$$

Consider $G = E_8$. Suppose first that $M^0 = A_2E_6$ with $p \neq 3$. Write $M^0 = C_G(v)$ with v of order 3, as above, and set $u = u_0u_1$ with $u_0 \in A_2, u_1 \in E_6$.

If $u \in 2A_1$ then by 1.7, $D = D^0$ and $D/R_u(D) = B_6$. Since $u \in M^0$, v lies in D, and hence v centralizes a maximal torus T_6 of D. It follows that u lies in $C_{M^0}(T_6)$, a Levi subgroup of M^0 of semisimple rank at most 2. Consequently u_1 lies in class $A_1, 2A_1$ or A_2 of E_6. If $u_0 \neq 1$ then u_0u_1 lies in a Levi subgroup $2A_1, 3A_1$ or A_1A_2 of G, respectively, so has this as its label as these subsystems are unique up to conjugacy. Thus u_1 lies in class A_1 or $2A_1$ of E_6, and the result follows using 1.7. The same argument deals with the case where u lies in the class A_2. And if $u \in 3A_1$, then as above we see that v centralizes a rank 5 torus T_5. If this projects to a rank 3 torus in the factor E_6, then it projects to $T_2 < A_2$, so $u_0 = 1$ and hence $u = u_1$ must lie in class $3A_1$ of the Levi subgroup E_6, giving the result by 1.7. And if T_5 projects to a rank 4 torus in E_6, we use the previous argument again.

Continue to assume $M^0 = A_2E_6$, now with $p = 3$. Since $\dim u^G \geq 92$, we can assume that $\dim u^M > 92 - e'_G = 52$, hence that $\dim C_M(u) < 34$. Since $u_1 \in E_6$ has order 3, this implies that u_1 is in one of the classes $A_2 + A_1, 2A_2, A_2 + 2A_1, 2A_2 + A_1$ of E_6 (see 1.7 and [**6**, p. 402]). We can also assume that $\dim u^G < \dim u^M + e'_G \leq 6 + \dim(2A_2 + A_1)^{E_6} + 40 = 100$, whence u lies in class $2A_1$ of G by 1.7. However, by [**17**], on $L(E_6)$ each of the above classes u_1 has at least 22 Jordan blocks of size 3, whereas on $L(G)$, the class $2A_1$ has only 14 such blocks, a contradiction. This completes the proof for $M^0 = A_2E_6$.

Now suppose $M^0 = A_8$. Here u lies in class $2A_1$ of G. If $p \neq 3$, $M^0 = C_G(v)$, the above argument forces u to lie in class $A_1, 2A_1$ or A_2 of M^0. As each of these is a Levi in E_8, the class must in fact be $2A_1$ in M^0, which by 1.10 has dimension 28. Therefore $\dim u^G - \dim u^M \geq 92 - 28$. And when $p = 3$, we can assume $\dim u^M > \dim u^G - e'_G = 52$. By 1.10 this means that u has 3 Jordan blocks of size 3 on the usual 9-dimensional module for M^0. But then u has more than 14 Jordan blocks of size 3 on $L(A_8)$, whereas class $2A_1$ has only 14 such blocks on $L(G)$, a contradiction. The lemma is now proved for $G = E_8$.

The proof for $G = E_7$ or E_6 is very similar to the above, and is left to the reader.

Now let $G = F_4$. Here $M = D_4.S_3$, $u \in M^0$. Then $M^0 < B_4$, and we have already shown that $\dim u^G - \dim u^{B_4} \geq e'_G$, so there is nothing more to be done.

Finally, in $G = G_2$, the classes $\widetilde{A}_1 (p \neq 3), \widetilde{A}_1^{(3)}$ do not intersect $M^0 = A_2$, since the two unipotent classes in A_2 are those with labels A_1 and $G_2(a_1)$. $\square$

5. Proof of Theorem 2, Part (II)(b): Semisimple elements in maximal rank subgroups.

In this section we prove Theorem 2(II)(b). Continue to assume that G is an exceptional algebraic group over the algebraically closed field K of characteristic p, and let M be a maximal closed reductive subgroup of G of maximal rank. Let s be a nonidentity semisimple element of M. By 4.1, we need only prove the bounds in Theorem 2 for $\dim s^G - \dim(s^G \cap M^0)$. By 1.3(i), replacing s by a suitable conjugate we may take $s \in M^0$ and $\dim(s^G \cap M^0) = \dim s^{M^0}$. Write

$$D = C_G(s).$$

Now s lies in a maximal torus T of M^0, and clearly $T \leq D \cap M$. Thus taking roots with respect to T we have

$$\begin{aligned} \dim s^G - \dim s^M &= \dim G - \dim M - \dim D + \dim(D \cap M) \\ &= 2(|\Phi^+(G)| - |\Phi^+(M)| - |\Phi^+(D)| + |\Phi^+(D \cap M)|). \end{aligned}$$

As with the proof of Theorem 2(I)(b), we shall see that we may obtain the required bounds by using root system arguments. We note that conjugacy classes of subsystems of simple root systems were determined in [**10**]. We shall use the notation employed there; in particular we shall write D_2 for a subsystem of D_n which is orthogonal to a D_{n-2} subsystem, and distinguish the two classes of A_5A_1 subsystems in E_7 as $(A_5A_1)'$ and $(A_5A_1)''$.

Let Φ be a root system and Ψ be a subsystem of Φ. We shall use the following concept. If X is a type of root system, we say that Ψ is *X-dense in* Φ if every subsystem of Φ of type X meets Ψ. Observe that if Ψ is X-dense in Φ, then for any subsystem Φ_1 of Φ we have that $\Psi \cap \Phi_1$ is X-dense in Φ_1, while any subsystem of Φ containing Ψ is also X-dense in Φ. Note also that in the case where Φ has only one root length, a subsystem Ψ is A_2-dense precisely if $\Phi \setminus \Psi$ does not contain distinct roots α, β and $\alpha + \beta$; such subsystems are called anti-open in [**18**]. For convenience we repeat from [**18**] the list of all proper anti-open subsystems; note that a factor D_1 here is to be interpreted as $\emptyset$.

Lemma 5.1. *If Ψ is a proper subsystem of Φ, then Ψ is anti-open in Φ if and only if $(\Phi, \Psi) = (A_n, A_\ell A_{n-\ell-1})$, $(B_n, B_\ell D_{n-\ell})$, $(C_n, C_\ell C_{n-\ell})$, $(C_n, \widetilde{A}_{n-1})$, $(D_n, D_\ell D_{n-\ell})$, (D_n, A_{n-1}), (E_6, D_5), (E_6, A_5A_1), (E_7, E_6), (E_7, A_7), (E_7, D_6A_1), (E_8, D_8), (E_8, E_7A_1), (F_4, C_3A_1), (F_4, B_4) or $(G_2, A_1\widetilde{A}_1)$.*

The first part of the following lemma generalizes the trivial direction of Proposition 4.2 of [**18**]. Let X be a type of root system, and take a root system of type X with simple roots $\beta_1, \ldots, \beta_s$ and highest root $\sum m_j\beta_j$; we define the height of X to be $\sum m_j$. In the results which follow, we shall write $\alpha_1, \dots, \alpha_n$ for simple roots of the root system Φ and $\alpha_0 = \sum n_j \alpha_j$ for its highest root.

Lemma 5.2.

(a) *Let $r \in \mathbb{N}$, and let Ψ be a subsystem of the root system Φ whose Dynkin diagram is obtained from that of Φ by*

(i) *removing nodes $\alpha_{i_1}, \alpha_{i_2}, \dots$ with $n_{i_1} + n_{i_2} + \cdots = r - 1$, or*

(ii) *first extending and then removing nodes $\alpha_{i_1}, \alpha_{i_2}, \dots$ with $n_{i_1} + n_{i_2} + \cdots = r$.*

Then Ψ is X-dense in Φ for any type of root system X of height at least r.

(b) *If $(\Phi, \Psi) = (E_8, {D_4}^2)$ or $(E_7, D_4 {A_1}^3)$, then Ψ is A_4-dense in Φ.*

Proof. (a) If either (i) or (ii) holds, the positive roots outside Ψ are those of the form $\sum m_j \alpha_j$ with $m_{i_1} + m_{i_2} + \cdots \in \{1, \dots, r-1\}$; thus no sum of r or more positive roots outside Ψ (allowing repetitions) can be another positive root outside Ψ.

(b) In each case there is a single $W(\Phi)$-orbit of subsystems having the same type as Ψ, and the Dynkin diagram of Ψ is obtained from that of Φ by extending and deleting the α_1-node, then extending and deleting the α_6-node; since each node removed has label 2 in the relevant diagram, the roots in Ψ are those with m_1 and m_6 even. Let Φ' be any subsystem of Φ of type A_4, with simple system $\beta_1, \beta_2, \beta_3, \beta_4$. For $\beta \in \Phi'$, set $d_\beta = (k_1, k_6) \in {\mathbb{Z}_2}^2$, where for $i \in \{1, 6\}$ we set $k_i = 0$ or 1 according as the coefficient of α_i in β is even or odd; thus $d_{\beta+\beta'} = d_\beta + d_{\beta'}$, and we have an additive map $d : \Phi' \to {\mathbb{Z}_2}^2$. Assume if possible that $(0,0)$ is not in the image of d. By composing d with a suitable automorphism of ${\mathbb{Z}_2}^2$ we may assume firstly that $d_{\beta_1} = (0,1)$, and then that $d_{\beta_2} = (1,0)$ (since if $d_{\beta_2} = (0,1)$ then $d_{\beta_1+\beta_2} = (0,0)$, contrary to assumption). We cannot then have $d_{\beta_3} = (1,0)$ or $(1,1)$ (else either $d_{\beta_2+\beta_3}$ or $d_{\beta_1+\beta_2+\beta_3}$ would be $(0,0)$), so this forces $d_{\beta_3} = (0,1)$; but then any choice for d_{β_4} gives some root β with $d_\beta = (0,0)$. Hence at least one of the roots of Φ' lies in Ψ as required. □

It will also be useful to observe that certain subsystems are not X-dense. As already mentioned, if Ψ fails to be X-dense in Φ, then so does any subsystem of Ψ.

Lemma 5.3. *If Φ and Ψ are as follows, then Ψ is not A_3-dense in Φ:*

(i) $\Phi = A_n$, Ψ *of rank* $n-3$;

(ii) $\Phi = D_n$, $\Psi = A_{n-3}A_1$;

(iii) $\Phi = D_4$, $\Psi = D_2$;

(iv) $\Phi = D_5$, $\Psi = {D_2}^2$;

(v) $\Phi = D_6$, $\Psi = D_3A_1$ *or* ${A_2}^2$;

(vi) $\Phi = D_7$, $\Psi = A_3A_2$, D_3A_2, $D_3D_2A_1$ *or* $A_2{D_2}^2$;

(vii) $\Phi = E_6$, $\Psi = A_4$, $A_3{A_1}^2$ *or* ${A_2}^2A_1$;

(viii) $\Phi = E_7$, $\Psi = (A_5A_1)'$ *or* $(A_5A_1)''$.

Proof. In each case we exhibit a subsystem of Φ of type A_3 lying outside Ψ, by giving simple roots $\beta_1, \beta_2, \beta_3$.

(i) Let $\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}$ (with $i_1 < i_2 < i_3$) be the simple roots of Φ outside Ψ, and take $\beta_1 = \alpha_{i_1}$, $\beta_2 = \alpha_{i_1+1} + \cdots + \alpha_{i_2}$, $\beta_3 = \alpha_{i_2+1} + \cdots + \alpha_{i_3}$.

(ii) Let α_{n-2} and α_{n-1} be the simple roots of Φ outside Ψ, and take $\beta_1 = \alpha_{n-2}$, $\beta_2 = \alpha_{n-1}$, $\beta_3 = \alpha_{n-3} + \alpha_{n-2} + \alpha_n$.

(iii) Let Ψ have simple roots α_3, α_4 and take $\beta_1 = \alpha_2$, $\beta_2 = \alpha_1$, $\beta_3 = \alpha_2 + \alpha_3 + \alpha_4$.

(iv) Let Ψ have simple roots $\alpha_0, \alpha_1, \alpha_4, \alpha_5$ and take $\beta_1 = \alpha_3$, $\beta_2 = \alpha_2$, $\beta_3 = \alpha_3 + \alpha_4 + \alpha_5$.

(v) If $\Psi = D_3A_1$ with simple roots $\alpha_0, \alpha_4, \alpha_5, \alpha_6$, take $\beta_i = \alpha_i$ for $i = 1, 2, 3$. If $\Psi = {A_2}^2$ with simple roots $\alpha_1, \alpha_2, \alpha_4, \alpha_6$, take $\beta_1 = \alpha_3 + \alpha_4$, $\beta_2 = \alpha_5$, $\beta_3 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_6$.

(vi) If $\Psi = A_3A_2$ with simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_7$, take $\beta_1 = \alpha_4 + \alpha_5$, $\beta_2 = \alpha_6$, $\beta_3 = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7$. If $\Psi = D_3A_2$ with simple roots $\alpha_1, \alpha_2, \alpha_5, \alpha_6, \alpha_7$, or $\Psi = D_3D_2A_1$ with simple roots $\alpha_0, \alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_7$, or $\Psi = A_2{D_2}^2$ with simple roots $\alpha_0, \alpha_1, \alpha_3, \alpha_4, \alpha_6, \alpha_7$, take $\beta_1 = \alpha_2 + \alpha_3$, $\beta_2 = \alpha_4 + \alpha_5$, $\beta_3 = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$.

(vii) If $\Psi = A_4$ with simple roots $\alpha_3, \alpha_4, \alpha_5, \alpha_6$, take $\beta_1 = \alpha_2$, $\beta_2 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$, $\beta_3 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$. If $\Psi = A_3{A_1}^2$ with simple roots $\alpha_0, \alpha_1, \alpha_4, \alpha_5, \alpha_6$, take $\beta_1 = \alpha_3 + \alpha_4$, $\beta_2 = \alpha_2$, $\beta_3 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5$. If $\Psi = {A_2}^2A_1$ with simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6$, take $\beta_1 = \alpha_4$, $\beta_2 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$, $\beta_3 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$.

(viii) If $\Psi = (A_5A_1)'$ with simple roots $\alpha_0, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$, take $\beta_1 = \alpha_2$, $\beta_2 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$, $\beta_3 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$. If $\Psi = (A_5A_1)''$ with simple roots $\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7$, take $\beta_1 = \alpha_3$, $\beta_2 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$, $\beta_3 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$. □

Lemma 5.4. *If Φ and Ψ are as follows, then Ψ is not A_4-dense in Φ:*

(i) $\Phi = A_n$, Ψ *of rank* $n - 4$;
(ii) $\Phi = D_5$, $\Psi = {A_1}^2$;
(iii) $\Phi = D_6$, $\Psi = D_2{A_1}^2$;
(iv) $\Phi = E_6$, $\Psi = A_3A_1$ *or* ${A_1}^4$;
(v) $\Phi = E_7$, $\Psi = A_3{A_1}^3$ *or* ${A_1}^7$.

Proof. As with the previous result we exhibit a subsystem of Φ of type A_4 lying outside Ψ, by giving simple roots $\beta_1, \beta_2, \beta_3, \beta_4$.

(i) Let $\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}, \alpha_{i_4}$ (with $i_1 < i_2 < i_3 < i_4$) be the simple roots of Φ outside Ψ, and take $\beta_1 = \alpha_{i_1}$, $\beta_2 = \alpha_{i_1+1} + \cdots + \alpha_{i_2}$, $\beta_3 = \alpha_{i_2+1} + \cdots + \alpha_{i_3}$, $\beta_4 = \alpha_{i_3+1} + \cdots + \alpha_{i_4}$.

(ii) Let Ψ have simple roots α_0, α_5, and take $\beta_i = \alpha_i$ for $i = 1, 2, 3, 4$.

(iii) Let Ψ have simple roots $\alpha_0, \alpha_1, \alpha_3, \alpha_6$ and take $\beta_1 = \alpha_2$, $\beta_2 = \alpha_3 + \alpha_4$, $\beta_3 = \alpha_5$, $\beta_4 = \alpha_4 + \alpha_6$.

(iv) If $\Psi = A_3A_1$ with simple roots $\alpha_0, \alpha_4, \alpha_5, \alpha_6$, take $\beta_1 = \alpha_2$, $\beta_2 = \alpha_3 + \alpha_4$, $\beta_3 = \alpha_1$, $\beta_4 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$. If $\Psi = {A_1}^4$ with simple roots $\alpha_0, \alpha_1, \alpha_4, \alpha_6$, take $\beta_1 = \alpha_3$, $\beta_2 = \alpha_2 + \alpha_4$, $\beta_3 = \alpha_5$, $\beta_4 = \alpha_1 + \alpha_3 + \alpha_4$.

(v) Write ${\alpha_0}' = \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7$, and ${\alpha_0}'' = \alpha_2+\alpha_3+2\alpha_4+\alpha_5$. If $\Psi = A_3{A_1}^3$ with simple roots $\alpha_0, {\alpha_0}', \alpha_3, \alpha_4, \alpha_5, \alpha_7$, take $\beta_1 = \alpha_2$, $\beta_2 = \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6$, $\beta_3 = {\alpha_0}''$, $\beta_4 = \alpha_6+\alpha_7$. If $\Psi = {A_1}^7$ with simple roots $\alpha_0, {\alpha_0}', {\alpha_0}'', \alpha_2, \alpha_3, \alpha_5, \alpha_7$, take $\beta_1 = \alpha_2+\alpha_4+\alpha_5+\alpha_6$, $\beta_2 = \alpha_1+\alpha_3+\alpha_4+\alpha_5$, $\beta_3 = \alpha_6 + \alpha_7$, $\beta_4 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$. □

Lemma 5.5. *If $\Phi = D_7$, then Ψ is not X-dense in Φ in the following cases:*

(i) $\Psi = A_2{A_1}^2$ *or* $D_2{A_1}^2$, $X = A_5$;

(ii) $\Psi = D_3D_2$, $X = D_4$.

Proof. Once more we exhibit a subsystem of Φ of type X lying outside Ψ, by giving simple roots β_i.

(i) If $\Psi = A_2{A_1}^2$ with simple roots $\alpha_1, \alpha_2, \alpha_4, \alpha_7$, take $\beta_1 = \alpha_2 + \alpha_3$, $\beta_2 = \alpha_4 + \alpha_5$, $\beta_3 = \alpha_6$, $\beta_4 = \alpha_5 + \alpha_7$, $\beta_5 = \alpha_3 + \alpha_4$. If $\Psi = D_2{A_1}^2$ with simple roots $\alpha_1, \alpha_3, \alpha_6, \alpha_7$, take $\beta_1 = \alpha_1+\alpha_2$, $\beta_2 = \alpha_3+\alpha_4$, $\beta_3 = \alpha_5+\alpha_6+\alpha_7$, $\beta_4 = \alpha_4 + \alpha_5$, $\beta_5 = \alpha_2 + \alpha_3$.

(ii) Let Ψ have simple roots $\alpha_0, \alpha_1, \alpha_2, \alpha_6, \alpha_7$ and take $\beta_1 = \alpha_3$, $\beta_2 = \alpha_4$, $\beta_3 = \alpha_5$, $\beta_4 = \alpha_5 + \alpha_6 + \alpha_7$. □

Lemma 5.6. *The conclusion of Theorem 2(II)(b) holds if $G = E_8$.*

Proof. Write $\Phi = \Phi(G)$. As in the proof of 3.2, we take the list of possibilities for D from [**9**]. Observe that if s_1 and s_2 are semisimple elements with centralizers D_1 and D_2, then if $D_1 > D_2$ we have $|\Phi^+(D_1)| - |\Phi^+(D_1 \cap M)| \geq |\Phi^+(D_2)| - |\Phi^+(D_2 \cap M)|$, and so $f(s_1, G/M) \leq f(s_2, G/M)$; thus it suffices to consider the cases $D = E_7A_1$, D_8, D_7T_1, E_6A_2, A_8, $D_6A_1T_1$, D_5A_3, A_7A_1, ${A_4}^2$ and $A_5A_2A_1$. Again as in 3.2, we say that D is small if it contains no E_7 or D_8 factor; thus $|\Phi^+(D)|$ is 64, 56 or at most 42 according as D is E_7A_1, D_8 or small.

The possibilities for M are listed in 1.1; note that if $M^0 = {A_2}^4$, ${A_1}^8$ or T_8 then $|\Phi^+(M)| \leq 12$, and hence $f(s, G/M) \geq 2(120 - 12 - |\Phi^+(D)|)$, which is 88, 104 or at least 132 according as D is E_7A_1, D_8 or is small. It therefore suffices to consider the cases $M^0 = E_7A_1$, D_8, A_8, E_6A_2, ${D_4}^2$ and ${A_4}^2$. We may assume that D has more than $120 - |\Phi^+(M)| - \frac{1}{2}f_{G,M,D}$ positive roots; if D is small, this number is 21, 24, 30, 30, 38 or 36 according as $M^0 = E_7A_1$, D_8, E_6A_2, A_8, ${D_4}^2$ or ${A_4}^2$. Thus the possibilities for D small are as follows:

M^0	D
${D_4}^2, {A_4}^2$	D_7T_1 or E_6A_2
A_8, E_6A_2	D_7T_1, E_6A_2, A_8 or $D_6A_1T_1$
E_7A_1, D_8	$D_7T_1, E_6A_2, A_8, D_6A_1T_1, D_5A_3$ or A_7A_1

First let $M^0 = {A_4}^2$, so that by 5.2 $\Phi(M)$ is both A_5-dense and D_4-dense in Φ. If $D = E_7A_1$ or D_8 then by 5.1 $\Phi(D)$ is A_2-dense in Φ, and so $\Phi(D\cap M)$ is A_2-dense in $\Phi(M)$; thus by 5.1 we see that the intersection of D with each A_4 factor must be A_4, A_3T_1 or $A_2A_1T_1$, so that $|\Phi^+(D\cap M)| \geq 8$. For $D = D_8$ it follows that $f(s, G/M) \geq 2(120-20-56+8) = 104$, while for $D = E_7A_1$ we have $f(s, G/M) \geq 2(120-20-64+8) = 88$. If $D = D_7T_1$ or E_6A_2 then by 5.2 $\Phi(D)$ is A_3-dense in Φ, so $\Phi(D \cap M)$ is A_3-dense in $\Phi(M)$. For $D = E_6A_2$ we see by 5.3 that $\Phi^+(D\cap M)$ must contain at least two roots from each A_4 factor, so $|\Phi^+(D\cap M)| \geq 4$; this gives $f(s, G/M) \geq 2(120-20-39+4) = 130$. For $D = D_7T_1$ we cannot have $\Phi(D \cap M) = {D_2}^2$, $D_2{A_1}^2$, A_2D_2 or $A_2{A_1}^2$, because by 5.5 the first and third are not D_4-dense in $\Phi(M)$ and the second and fourth are not A_5-dense there; thus we must have $|\Phi^+(D \cap M)| \geq 6$, whence $f(s, G/M) \geq 2(120 - 20 - 42 + 6) = 128$.

Next let $M^0 = {D_4}^2$. If $D = D_7T_1$ or E_6A_2, by 5.2 $\Phi(D)$ is A_3-dense in Φ; by 5.3 $\Phi^+(D \cap M)$ must contain at least three roots from each D_4 factor, so $|\Phi^+(D\cap M)| \geq 6$, giving $f(s, G/M) \geq 2(120-20-42+6) = 120$. If $D = E_7A_1$ or D_8 then $\Phi(D \cap M)$ is A_2-dense in $\Phi(M)$, and so by 5.1 the intersection of D with each D_4 factor is D_4, A_3T_1, D_3T_1 or ${D_2}^2$, whence $|\Phi^+(D\cap M)| \geq 2.4 = 8$; for $D = D_8$ this gives $f(s, G/M) \geq 2(120-24-56+8) = 96$, while for $D = E_7A_1$ we have $f(s, G/M) \geq 2(120-24-64+8) = 80$.

Now let $M^0 = A_8$, so that $\Phi(M)$ is A_3-dense in Φ by 5.2. If $D = E_7A_1$ or D_8 then $\Phi(D\cap M)$ is A_2-dense in $\Phi(M)$, and so must be A_8, A_7, A_6A_1, A_5A_2 or A_4A_3; for $D = D_8$ this gives $f(s, G/M) \geq 2(120 - 36 - 56 + 16) = 88$, while for $D = E_7A_1$ we have $f(s, G/M) \geq 2(120 - 36 - 64 + 16) = 72$. If $D = D_7T_1$, E_6A_2 or A_8 then $\Phi(D\cap M)$ is A_3-dense in both $\Phi(D)$ and $\Phi(M)$; the latter condition implies that it must have rank at least 6 by 5.3. Listing the subsystems of $\Phi(M)$ of rank at least 6 we find that only $A_3A_2A_1$ and ${A_2}^3$ have fewer than 12 positive roots. For $D = E_6A_2$ or A_8 we thus have $f(s, G/M) \geq 2(120-36-39+9) = 108$; for $D = D_7T_1$ neither $A_3A_2A_1$ nor ${A_2}^3$ is a subsystem of $\Phi(D)$, so $f(s, G/M) \geq 2(120 - 36 - 42 + 12) = 108$. If $D = D_6A_1T_1$ then by 5.3 $|\Phi^+(D \cap M)| \geq 3$, so $f(s, G/M) \geq 2(120-36-31+3) = 112$.

The cases where $M^0 = E_6A_2$ may all be treated in like fashion; we use the fact that $\Phi(D\cap M)$ is A_3-dense in $\Phi(D)$, and usually either A_2-dense or A_3-dense in $\Phi(M)$, to produce lower bounds for $|\Phi^+(D \cap M)|$, from which the required bounds on $f(s, G/M)$ follow. For example, if $D = E_7A_1$ then A_2-density in $\Phi(M)$ implies that $\Phi(D \cap M)$ must be YZ where $Y = E_6$, D_5 or A_5A_1 and $Z = A_2$ or A_1; by A_3-density in $\Phi(D)$ we cannot have $\Phi(D\cap M) = A_5{A_1}^2$, so $|\Phi^+(D\cap M)| \geq 19$, giving $f(s, G/M) \geq 2(120-39-64+19) = 72$. Similarly, in all cases where $M^0 = D_8$, the A_2-density of $\Phi(D\cap M)$ in $\Phi(D)$ immediately leads to the required bounds.

Finally, let $M = E_7A_1$, so that $\Phi(D\cap M)$ is A_2-dense in $\Phi(D)$; by taking those cases already treated in which $D = E_7A_1$, and interchanging the roles of D and M, we are left with the cases $D = E_7A_1$, D_7T_1, $D_6A_1T_1$, D_5A_3 and A_7A_1 to consider. A_2-density immediately disposes of the last three of these; for $D = D_7T_1$ we note that $\Phi(D \cap M)$ cannot be D_4D_3 since $\Phi(M)$ has no such subsystem, and all other A_2-dense possibilities satisfy the required bound. Thus we are left with $D = E_7A_1$; we seek to show that $f(s, G/M) \geq 48$, and so $|\Phi^+(D \cap M)| \geq 32$. The A_2-dense subsystems which do not satisfy this bound are A_7, A_7A_1 and D_6A_1; we may see that these do not occur as follows. If Φ' is any subsystem of Φ of type A_7 or D_6A_1, and Φ' lies in an E_7 subsystem Ψ, then $\mathbb{Z}\Phi' \cap \Phi = \Psi$; thus no such subsystem Φ' can lie in two distinct E_7 subsystems. It follows that $\Phi(D\cap M)$ cannot be A_7 or A_7A_1; and if the intersection of $\Phi(D)$ with the E_7 factor of $\Phi(M)$ is D_6A_1, then the A_1 cannot lie in the E_7 factor of $\Phi(D)$, so that the A_1 factor of $\Phi(D)$ lies in the E_7 factor of $\Phi(M)$—but now interchanging the roles of D and M shows that the A_1 factor of $\Phi(M)$ lies in $\Phi(D)$, and so $\Phi(D \cap M) = D_6{A_1}^2$. This concludes the proof that the conclusion of Theorem 2(II)(b) holds if $G = E_8$. ☐

Lemma 5.7. *The conclusion of Theorem 2(II)(b) holds if $G = E_7$.*

Proof. We proceed as in the previous proof, and write $\Phi = \Phi(G)$. The list of possibilities for D from [**9**] shows that it suffices to consider the cases $D^0 = E_6T_1$, D_6A_1, A_7, $D_5A_1T_1$, A_6T_1, A_5A_2, $A_5A_1T_1$, $D_4{A_1}^2T_1$ and ${A_3}^2A_1$. Again as in 3.3, we say that D is small if it contains no E_6, D_6 or A_7 factor; thus $|\Phi^+(D)|$ is 36, 31, 28 or at most 21 according as D^0 is E_6T_1, D_6A_1, A_7 or small.

The possibilities for M are listed in 1.1; note that if $M^0 = {A_1}^7$ or T_7 then $|\Phi^+(M)| \leq 7$, and hence $f(s, G/M) \geq 2(63 - 7 - |\Phi^+(D)|)$, which is 40, 50, 56 or at least 70 according as D^0 is E_6T_1, D_6A_1, A_7 or small. It therefore suffices to consider the cases $M^0 = E_6T_1$, D_6A_1, A_7, A_5A_2 and $D_4{A_1}^3$. We may assume that D has more than $63 - |\Phi^+(M)| - \frac{1}{2}f_{G,M,D}$ positive roots; if D is small, this number is 10, 12, 13, 16 or 19 according as $M^0 = E_6T_1$, D_6A_1, A_7, A_5A_2 or $D_4{A_1}^3$. Thus the possibilities for D small are as follows:

M^0	D^0
$D_4{A_1}^3$	$D_5A_1T_1$ or A_6T_1
A_5A_2	$D_5A_1T_1, A_6T_1$ or A_5A_2
A_7	$D_5A_1T_1, A_6T_1, A_5A_2, A_5A_1T_1$ or $D_4{A_1}^2T_1$
E_6T_1, D_6A_1	$D_5A_1T_1, A_6T_1, A_5A_2, A_5A_1T_1, D_4{A_1}^2T_1$ or ${A_3}^2A_1$

First let $M^0 = D_4{A_1}^3$; then by 5.2 $\Phi(M)$ is A_4-dense in Φ, and so $\Phi(D \cap M)$ is A_4-dense in $\Phi(D)$. By 5.4 we see that if $D^0 = A_6T_1$ or $D_5A_1T_1$ then $|\Phi^+(D \cap M)| \geq 3$, so $f(s, G/M) \geq 2(63 - 15 - 21 + 3) = 60$; similarly if $D^0 = A_7$ then $|\Phi^+(D\cap M)| \geq 4$, so $f(s, G/M) \geq 2(63-15-28+4) = 48$. If $D^0 = D_6A_1$ or E_6T_1 then $\Phi(D)$ is A_2-dense in Φ, so $\Phi(D \cap M)$ is A_2-dense in $\Phi(M)$, which forces the intersection of $\Phi(D)$ with the D_4 factor of $\Phi(M)$

to be D_4, D_3, A_3 or ${D_2}^2$. For $D^0 = D_6A_1$ it follows that $|\Phi^+(D \cap M)| \geq 4$, and so $f(s, G/M) \geq 2(63 - 15 - 31 + 4) = 42$. For $D^0 = E_6T_1$ we must have $\Phi(D \cap M) = D_4$, $A_3{A_1}^j$ for $0 \leq j \leq 2$ or ${A_1}^4$; since A_3A_1 and ${A_1}^4$ are not A_4-dense in $\Phi(D)$ by 5.4, it follows that $|\Phi^+(D \cap M)| \geq 8$, giving $f(s, G/M) \geq 2(63 - 15 - 36 + 8) = 40$.

Next let $M^0 = A_5A_2$, so that by 5.2 we have A_3-density of $\Phi(M)$ in Φ and hence of $\Phi(D \cap M)$ in $\Phi(D)$. By 5.3 it follows that if $D^0 = A_5A_2$ then $|\Phi^+(D \cap M)| \geq 3$; if $D^0 = A_6T_1$ then $\Phi(D \cap M)$ is not ${A_1}^3$ or A_2A_1, so $|\Phi^+(D \cap M)| \geq 5$; and if $D^0 = D_5A_1T_1$ then $\Phi(D \cap M)$ is not ${A_1}^4$ or A_2A_1, so $|\Phi^+(D \cap M)| \geq 5$. If $D^0 = E_6T_1$, D_6A_1 or A_7 then $\Phi(D \cap M)$ must also be A_2-dense in $\Phi(M)$, and thus must be either A_5A_k or $A_{4-j}A_jA_k$ for $0 \leq j \leq 2$ and $1 \leq k \leq 2$. Thus $|\Phi^+(D \cap M)| \geq 7$; and for $D^0 = E_6T_1$ we cannot have $\Phi(D \cap M) = {A_2}^2A_1$ or $A_3{A_1}^2$ by A_3-density in E_6, so $|\Phi^+(D \cap M)| \geq 9$. In all cases the required bound on $f(s, G/M)$ follows.

If $M^0 = A_7$, we have A_2-density of $\Phi(M)$ in Φ and hence of $\Phi(D \cap M)$ in $\Phi(D)$; in all cases the required lower bound on $|\Phi^+(D \cap M)|$ follows immediately from 5.1. Likewise A_2-density disposes of all cases with $M^0 = D_6A_1$ except those in which $D^0 = D_6A_1$ or $D_5A_1T_1$; these cases require further treatment. First assume $D^0 = D_6A_1$. By A_2-density we see that if $|\Phi^+(D \cap M)| \leq 14$ then we must have $\Phi(D \cap M) = D_4D_2$, ${D_3}^2A_1$ or ${D_3}^2$. Moreover, if the first of these holds then the A_1 factor of $\Phi(D)$ cannot be involved in the D_2 (otherwise the intersection of $\Phi(M)$ with the D_6 factor of $\Phi(D)$ would not be A_2-dense); the same is true of the A_1 factor of $\Phi(M)$. Thus the intersection of the two D_6 factors would have to be a D_4D_2 or ${D_3}^2$ subsystem. However, if Φ' is a D_4D_2 or ${D_3}^2$ subsystem of a D_6 subsystem Ψ of Φ, then $\mathbb{Z}\Phi' \cap \Phi = \Psi$; thus Ψ is the unique D_6 subsystem containing Φ', and so the intersection of two distinct D_6 subsystems cannot be either D_4D_2 or ${D_3}^2$. We therefore have $|\Phi^+(D \cap M)| \geq |\Phi^+(A_5)| = 15$ and so $f(s, G/M) \geq 2(63 - 31 - 31 + 15) = 32$. Now assume $D^0 = D_5A_1T_1$. Here A_2-density shows that the intersection of $\Phi(M)$ with the D_5 factor of $\Phi(D)$ must be D_5, D_4, A_4 or D_3D_2; we shall show that we cannot have $\Phi(D \cap M) = D_3D_2$, from which it will follow that $|\Phi^+(D \cap M)| \geq 9$ and so $f(s, G/M) \geq 2(63 - 31 - 21 + 9) = 40$. We know by 5.2 that $\Phi(D)$ is A_3-dense in Φ; by 5.3 neither A_3A_1 nor D_3A_1 is A_3-dense in D_6, so if we had $\Phi' = \Phi(D \cap M) = D_3D_2$ then Φ' would have to lie in both the D_6 factor of $\Phi(M)$ and the D_5 factor of $\Phi(D)$. This would imply that $\mathbb{Z}\Phi' \cap \Phi$ equals the D_5 factor of $\Phi(D)$; since each D_5 subsystem in Φ is orthogonal to a unique positive root, there is a unique positive root β of Φ orthogonal to Φ', namely that of the A_1 factor of $\Phi(D)$. However, the positive root of the A_1 factor of $\Phi(M)$ is orthogonal to the D_6 factor, and thus to Φ'; so it must be β, and $\Phi(D \cap M)$ is $D_3D_2A_1$ rather than D_3D_2.

Finally let $M^0 = E_6T_1$, so that $\Phi(D \cap M)$ is A_2-dense in $\Phi(D)$; by taking those cases already treated in which $D = E_6T_1$, and interchanging the roles of D and M, we are left with the cases $D^0 = E_6T_1$, $D_5A_1T_1$, A_6T_1, $A_5A_1T_1$, $D_4{A_1}^2T_1$ and ${A_3}^2A_1$ to consider. A_2-density immediately disposes of the last three of these. For $D^0 = A_6T_1$ we note that A_3A_2 is not a subsystem of $\Phi(M)$; for $D^0 = D_5A_1T_1$, similarly $A_3{A_1}^3$ is not a subsystem of $\Phi(M)$, and the A_3-density of $\Phi(D \cap M)$ in $\Phi(M)$ rules out the possibilities $\Phi(D \cap M) = A_4$ or $A_3{A_1}^2$; in either case we see that $|\Phi^+(D \cap M)| \geq |\Phi^+(A_4A_1)| = 11$ and $f(s, G/M) \geq 2(63-36-21+11) = 34$. If $D^0 = E_6T_1$ we cannot have $\Phi(D \cap M) = A_5A_1$, because if Φ' is an A_5A_1 subsystem of Φ which lies in an E_6 subsystem Ψ, then $\mathbb{Z}\Phi' \cap \Phi = \Psi$, and so Φ' cannot lie in two distinct E_6 subsystems of Φ; thus by A_2-density we must have $|\Phi^+(D \cap M)| \geq |\Phi^+(D_5)| = 20$ and $f(s, G/M) \geq 2(63-36-36+20) = 22$. This concludes the proof that the conclusion of Theorem 2(II)(b) holds if $G = E_7$. □

Lemma 5.8. *The conclusion of Theorem 2(II)(b) holds if $G = E_6, F_4, G_2$.*

Proof. The proof is carried out using the methods of the previous lemmas, so we only provide a sketch. For $G = E_6$ arguments based on A_2- and A_3-density alone suffice in all cases except that where M^0 and D^0 are both D_5T_1; here we note that the intersection of two D_5 subsystems cannot be D_3D_2, by the spanning argument used several times above, and so $|\Phi^+(D \cap M)| \geq |\Phi^+(A_4)| = 10$, giving $f(s, G/M) \geq 2(36-20-20+10) = 12$.

Now let $G = F_4$; here $M^0 = B_4$, D_4, C_3A_1 or $A_2\widetilde{A}_2$, and if $p = 2$ we may also have $M^0 = C_4$ or $\widetilde{D}_4$. If $M^0 = D_4$ then $\Phi(M)$ consists of all the long roots of Φ; so $\Phi(D \cap M)$ consists of the long roots of $\Phi(D)$, and the values $f(s, G/M)$ are clear. Applying the graph automorphism gives the values for $M^0 = \widetilde{D}_4$ when $p = 2$ (note that D cannot then be B_4). In all cases where $M^0 = C_3A_1$, the fact that $\Phi(D \cap M)$ is anti-open in $\Phi(D)$ immediately leads to the required bounds. If $M^0 = B_4$ then $\Phi(D \cap M)$ contains all long roots of $\Phi(D)$ and is anti-open in $\Phi(D)$; these considerations suffice in all cases (note that if $D = A_2\widetilde{A}_2$ then $\Phi^+(D \cap M)$ contains at least one short root in addition to all positive long roots of $\Phi^+(D)$). Again, applying the graph automorphism deals with the possibility $M^0 = C_4$ when $p = 2$. Allowing interchange of the roles of D and M, the only cases remaining to be treated are those where $M^0 = A_2\widetilde{A}_2$ and $D = B_3T_1$ or $A_3\widetilde{A}_1$. As far as long roots are concerned, those of $\Phi(G)$ form a D_4 system, while those of $\Phi(D)$ form an A_3 subsystem, which thus is anti-open and must meet the A_2 factor of $\Phi(M)$; this suffices to give the bound for $D = A_3\widetilde{A}_1$, so assume $D = B_3T_1$ and suppose if possible that $\Phi(D \cap M) = A_1$. There is a unique positive short root β such that the roots orthogonal to it are those of $\Phi(D)$; since by assumption the A_2 factor of $\Phi(M)$ does not lie in $\Phi(D)$, not all of its roots are orthogonal to β, and so β cannot lie in the $\widetilde{A}_2$ factor of $\Phi(M)$. Thus if $\gamma_1, \gamma_2, \gamma_3$ are the positive short roots of $\Phi(D)$, then $\beta, \gamma_1, \gamma_2, \gamma_3$ are four orthogonal short roots lying outside the $\tilde{A}_2$ factor of $\Phi(M)$; but this is impossible because the short roots of $\Phi(G)$ form a $\tilde{D}_4$ subsystem, in which any $\tilde{D}_2{}^2$ subsystem is anti-open. Thus we cannot have $\Phi(D \cap M) = A_1$, and so $|\Phi^+(D \cap M)| \geq 2$, from which the required bound follows.

Finally if $G = G_2$ the bounds are immediate from consideration of long and short roots. □

This completes the proof of Theorem 2(II)(b).

6. Completion of proof of Theorem 2: Part (III).

In this section we complete the proof of Theorem 2 by handling Part (III). Thus let M be a maximal closed subgroup of the exceptional algebraic group G, and suppose M does not contain a maximal torus of G. If M is finite then $\dim x^G \cap M = 0$ for all $x \in G$, and Theorem 2(III) obviously holds. Hence we may assume M has positive dimension. We shall make use of the information given about the possibilities for M in [**22**, Theorem 1]; this result gives the list of possibilities for M, excluding some unknown cases in small characteristics. Note that all the unknown cases in small characteristics have M of small dimension, and are quickly ruled out in the next lemma by dimension arguments.

Lemma 6.1. *Either the conclusion of Theorem 2(III) holds, or G, M are as follows:*

G	M
E_8	G_2F_4
E_7	A_1F_4
E_6	$F_4, C_4 (p \neq 2)$
F_4	$A_1G_2 (p \neq 2), G_2 (p = 7)$

Proof. If $M \neq M^0$ then M^0 possesses a graph automorphism, and we see from [**22**, Theorem 1] that G, M^0 are as in the following table:

G	M^0
E_8	$A_2, A_3, A_1G_2G_2$
E_7	A_2, D_4, A_1A_1
E_6	A_2

It follows easily using 1.4 that if $x \in M - M^0$ is of prime order then $\dim x^G - \dim(x^G \cap (M - M^0)) \geq e_G$, e'_G or h_G, according as x is a root element, a unipotent non-root element, or a semisimple element, respectively. Thus in order to prove Theorem 2(III), we need only show that $\dim x^G - \dim(x^G \cap M^0) \geq e_G$, e'_G or h_G in the respective cases just described.

First consider a semisimple element $s \in M^0$. As usual we can assume that $\dim(s^G \cap M^0) = \dim s^{M^0}$. By 1.1, we have $\dim s^G \geq k_G = 112, 54, 32, 16, 6,$

according as G has type E_8, E_7, E_6, F_4, G_2 respectively. Hence Theorem 2(III) holds unless $\dim s^{M^0} > k_G - h_G$. Thus we may assume that $\dim s^M > k_G - h_G$, whence $\dim M - \mathrm{rank}(M) > k_G - h_G$. Now [**22**, Theorem 1] shows that M is in the list in the conclusion.

Now consider a unipotent element $u \in M^0$. If u is a long root element then by 1.13(ii), u lies in a simple factor of M and is a long root element therein (note that by the maximality of M, the case in 1.13(iii) where u is a short root element in a subgroup B_n does not arise). Now we see that the conclusion of Theorem 2(III) holds, using [**22**, Theorem 1] and 1.12.

Thus we may assume u is not a long root element (or a short root element if $(G,p) = (F_4, 2)$ or $(G_2, 3)$). Then by 1.7, $\dim u^G \geq k'_G = 92, 52, 32, 22, 8$, according as $G = E_8, \ldots, G_2$. As above, we may suppose $\dim M - \mathrm{rank}(M) > k'_G - e'_G$, which again leads to the list in the conclusion. □

Lemma 6.2. *The conclusion of Theorem 2(III) holds if $G = E_6, M = F_4$ or C_4 $(p \neq 2)$.*

Proof. In this case, $M = C_G(\tau)$, where τ is an involutory graph automorphism of G (see 1.4).

Consider first a semisimple element $s \in M$. Letting $D = C_G(s)$, from 1.5 we have

$$\dim D \cap M = \dim C_D(\tau) \geq |\Sigma^+(D)| + \mathrm{rank}(D) - \mathrm{rank}(D'),$$

and therefore

$$\begin{aligned} \dim s^G - \dim s^M &= \dim G - \dim D - \dim M + \dim D \cap M \\ &\geq \dim G - \dim M - (|\Sigma^+(D)| + \mathrm{rank}(D')). \end{aligned}$$

Hence either the conclusion of Theorem 2(III) holds, or $M = F_4$ and $(|\Sigma^+(D)| + \mathrm{rank}\,(D')) > 14$, in which case $D^0 = D_5T_1, D_4T_2, A_5T_1, A_5A_1, A_4A_1T_1$ or A_2^3.

Assume then that $M = F_4$ and D is in this list. Observe that $M \cap D = C_D(\tau) = C_{F_4}(s)$, a reductive group, and τ induces an automorphism on D.

Suppose $D = T_1D_5$. When $p \neq 2$, D centralizes an involution t, so $M \cap D = C_{F_4}(t) = B_4$; and when $p = 2$ the fact that $C_D(\tau) = C_{F_4}(s)$ is reductive forces it to be B_4 again. Thus $\dim s^G - \dim s^M = 32 - 16 > h_G$, as required.

Likewise, if $D^0 = D_4T_2$ then $C_D(\tau)^0 = B_3T_1$ (note that $B_2B_1T_1$ is not possible, as this does not lie in a Levi subgroup of F_4); if $D^0 = A_5T_1$ or A_5A_1 then D centralizes an involution t when $p \neq 2$, so $C_D(\tau) \leq C_{F_4}(t)$, whence $C_D(\tau)^0 = C_3T_1$ or C_3A_1 respectively (and when $p = 2$ the fact that $C_D(\tau)$ is reductive forces the same conclusion); if $D = A_4A_1T_1$ then $C_D(\tau) = B_2A_1T_1$; and if $D^0 = A_2^3$ then $|s| = 3$, so $\dim C_M(s) \geq 16$ by 1.5. The required bounds follow.

Now consider a unipotent element $u \in M$ of order p. For $M = F_4$, unipotent class representatives for F_4, and the corresponding classes in E_6, are given in [**17**, Table A], and the required bound for $\dim u^G - \dim u^M$ is immediate from [**6**, pp. 401-2].

Now let $M = C_4$ ($p \neq 2$). If u is not in class A_1 or $2A_1$ of G, then $\dim u^G \geq 40$ by 1.7, hence we may assume that $\dim u^M > 40 - e'_G = 30$. By 1.10, the only such class in C_4 is that of a regular unipotent element, with a single Jordan block on the usual 8-dimensional C_4-module V_8. As u has order p, this implies that $p \geq 11$. If V_{27} denotes the 27-dimensional G-module $V_G(\lambda_1)$, then $V_{27} \downarrow C_4 = V_{C_4}(\lambda_2)$ by [**24**, 2.5]. One checks that on this module u has one Jordan block of size 13 if $p \geq 13$, and has 2 blocks of size 11 if $p = 11$. Hence by [**17**, Table 5], u lies in class $E_6(a_1)$ of G, giving $\dim u^G - \dim u^M = 70 - 32 > e'_G$.

If u is in class A_1 then u is a long root element in both G and C_4, so $\dim u^G - \dim u^M = 22 - 8$. Finally, suppose u is in class $2A_1$. By [**17**], $V_{27} \downarrow u = J_3 \oplus J_2^8 \oplus J_1^8$. Since $V_{27} \downarrow C_4 = V_{C_4}(\lambda_2)$, the only compatible possibility for $V_8 \downarrow u$ is $J_2^2 \oplus J_1^4$. Then by 1.10, $\dim u^M = 14$, while $\dim u^G = 32$, so the result holds in this case also. □

Lemma 6.3. *The conclusion of Theorem 2(III) holds if $G = E_7, M = A_1F_4$.*

Proof. We first handle unipotent elements u. If u is a root element of G, then by 1.13 and 1.12, $\dim u^G - \dim u^M \geq 33 - 16 = 17$.

Now suppose u is not a root element. Then by 1.7, $\dim u^G \geq 52$. If $p = 2$ then u is an involution, and from 1.7 we have $\dim u^M \leq 30$, giving $\dim u^G - \dim u^M \geq 52 - 30 > 20 = e'_G$, as required. So assume $p \neq 2$. We may assume that

$$\dim u^G < e'_G + \dim u^M \leq 20 + \dim M - \mathrm{rank}(M) = 70.$$

Hence by 1.7, u lies in class $2A_1, 3A_1'', 3A_1'$ or A_2 of G.

Write $u = u_0u_1$ with $u_0 \in A_1, u_1 \in F_4$. We may assume that $\dim u^M > \dim u^G - e'_G$, whence $\dim u^M > 32$ and $\dim u_1^{F_4} > 30$. Therefore u_1 is in one of the classes $A_2 + \widetilde{A}_1, B_2, \ldots, F_4$ of F_4 listed in order as in [**17**, Table 4]. By [**24**, 2.4],

$$L(G) \downarrow A_1F_4 = L(A_1F_4) \oplus (V(2) \otimes V(\lambda_4)).$$

Hence, using [**17**, Tables 3, 4], we can compute the possible Jordan forms of $u = u_0u_1$ on $L(G)$ with u_1 in one of the above classes. We find that none of these agrees with the Jordan form of any of the classes $2A_1, 3A_1'', 3A_1', A_2$, as given in [**17**, Table 8]. This completes the proof for unipotent elements.

Now consider a semisimple element $s \in G$. If s is an involution, then by 1.2, $\dim s^G \geq 54$ while $\dim s^M \leq 30$, giving $\dim s^G - \dim s^M \geq 24 > h_G = 22$, as required. So we may suppose s has odd (prime) order. We may also assume that $\dim s^G < 22 + \dim s^M$, whence $\dim s^G < 72$. By 1.1, this forces $C_G(s) = T_1D_6$ or T_1E_6. Now by [24, Section 2],

$$L(G) \downarrow A_1D_6 = L(A_1D_6) \oplus (V(1) \otimes V(\lambda_5)),$$
$$L(G) \downarrow T_1E_6 = L(T_1E_6) \oplus V(\lambda_1) \oplus V(\lambda_6).$$

It follows that if $C_G(s) = T_1D_6$ then for some root of unity δ, the eigenvalues of s on $L(G)$ are 1 (multiplicity 67), δ, δ^{-1} (multiplicity 32 each) and δ^2, δ^{-2} (multiplicity 1 each); and if $C_G(s)^0 = T_1E_6$ then the eigenvalues of s on $L(G)$ are 1 (multiplicity 79), δ, δ^{-1} (multiplicity 27 each).

Write $s = s_1s_2$ with $s_1 \in A_1, s_2 \in F_4$. The conclusion is clear if $s_2 = 1$, so assume $s_2 \neq 1$.

Suppose now that $s_1 \neq 1$ also, and consider the composition factor $V(2) \otimes V(\lambda_4)$ of $M = A_1F_4$ on $L(E_7)$. Since a maximal torus of M has nontrivial 0-weight spaces on each factor $V(2)$ and $V(\lambda_4)$ (of dimension 2 on $V(\lambda_4)$), we see that the eigenvalues δ, δ^{-1} both appear with positive multiplicity for s_1 on the 3-dimensional factor $V_{A_1}(2)$, and for s_2 on the 26-dimensional factor $V_{F_4}(\lambda_4)$. In particular, the eigenvalue δ^2 appears for s on the tensor product. It follows that if $\delta^3 \neq 1$, then $C_G(s) = T_1D_6$ and s_1, s_2 act on the two tensor factors as $\mathrm{diag}(\delta, \delta^{-1}, 1)$, $\mathrm{diag}(\delta, \delta^{-1}, 1^{24})$ respectively. But then $\dim C_{V(2)\otimes V(\lambda_4)}(s) = 26$, and so, as $\dim C_G(s) = 67$, we have $\dim C_{L(A_1F_4)}(s) = 41$. This forces $\dim C_{F_4}(s_2) \geq 38$, whereas there is no such semisimple element in F_4 (by 1.1 for example).

Therefore $\delta^3 = 1$ and s has order 3. On $V(2) \otimes V(\lambda_4)$ we have $s = s_1s_2 = \mathrm{diag}(\delta, \delta^{-1}, 1) \otimes \mathrm{diag}(\delta, \ldots, \delta, \delta^{-1}, \ldots, \delta^{-1}, 1^{26-2k})$ (where in the second factor δ, δ^{-1} appear with multiplicity k). Then $\dim C_{V(2)\otimes V(\lambda_4)}(s) = 26$ again, giving a contradiction as before.

We have now established that $s_1 = 1$, that is, $s \in F_4$. Also $C_G(s) = T_1D_6$ or T_1E_6, and $A_1 = C_G(F_4) \leq C_G(s) = C_G(T_1)$, whence $T_1 \leq C_G(A_1) = F_4$.

Suppose now that $p \neq 2$. If $C_G(s) = T_1D_6$, then the torus T_1 lies in a fundamental subgroup $J \cong SL_2$ centralizing D_6, and [25, Theorem 1] forces $J < F_4$. Thus J is a fundamental SL_2 in F_4, whence $C_{F_4}(s) = C_{F_4}(T_1) = T_1C_3$, and $\dim s^G - \dim s^M = 66 - 30$, giving the conclusion. And if $C_G(s) = T_1E_6$ then an involution in T_1 lifts to an element of order 4 in the simply connected cover of G, which is impossible as $T_1 < F_4$.

Finally, consider the case where $p = 2$. Here there is an element $t \in T_1$ of order 3, and $C_G(t) = C_G(s) = C_G(T_1)$. Moreover by 1.2, $C_{F_4}(t) = A_2A_2, T_1B_3$ or T_1C_3, and

$$\dim s^G - \dim s^M = \dim s^G - (\dim M - \dim C_{F_4}(t)).$$

The right hand side is greater than $h_G = 22$, except when $C_G(s) = T_1E_6$ and $C_{F_4}(t) = A_2A_2$. However, in the latter case, we deduce from

$$V_{F_4}(\lambda_4) \downarrow A_2A_2 = (V(\lambda_1) \otimes V(\lambda_1)) \oplus (V(\lambda_2) \otimes V(\lambda_2)) \oplus (0 \otimes V(\lambda_1 + \lambda_2))$$

(see [**24**, Section 2]), that $\dim C_{V(\lambda_4)}(t) = 8$, whence $\dim C_{V(2)\otimes V(\lambda_4)}(t) = 24$. This yields $\dim C_{L(G)}(t) < 79$, a contradiction (as $C_G(t) = C_G(s) = T_1E_6$ in this case). □

Lemma 6.4. *The conclusion of Theorem 2(III) holds if $G = E_8, M = G_2F_4$.*

Proof. For s semisimple, we have $\dim s^G - \dim s^M \geq 112 - \dim M + \text{rank}\,(M) > h_G$, as required.

Now consider unipotent elements u. If u is a root element of G then $\dim u^G - \dim u^M \geq 58 - 16 > e_G$. Otherwise, we may assume that

$$\dim u^G < \dim M - \text{rank}(M) + e'_G = 100.$$

Hence by 1.7, u lies in class $2A_1$ of G. Then $\dim u^G = 92$, so we can suppose that $\dim u^M > 92 - e'_G = 52$, whence $\dim u^{F_4} > 40$. As u has order p, we see from 1.7 that $p \neq 2$ or 3. By [**17**], the largest Jordan block of u on $L(G)$ has size 3. Hence if $u = u_0u_1$ with $u_0 \in G_2, u_1 \in F_4$, then by [**17**] again, u_1 lies in class A_1 or $\widetilde{A}_1$ of F_4. But then $\dim u_1^M < 40$, a contradiction. □

Lemma 6.5. *The conclusion of Theorem 2(III) holds if $G = F_4$.*

Proof. Here $M = A_1G_2\,(p \neq 2)$ or $G_2\,(p = 7)$. If u is a long root element of G, then use of 1.13 gives $\dim u^G - \dim u^M \geq 16 - 6 > e_G = 4$. And if u is not a long root element then by 1.7, $\dim u^G \geq 22$ and $\dim u^M \leq 14$, giving $\dim u^G - \dim u^M \geq 8 \geq e'_G$, as required.

Now consider a semisimple element $s \in M$. We can suppose $C_G(s) = B_4$, as otherwise by 1.1 we have $\dim s^G \geq 28$, giving the conclusion. Then $\dim s^G = 16$, while since s is an involution, $\dim s^M \leq 10$, giving the result. □

The proof of Theorem 2 is now complete.

7. The tables of bounds for Theorem 2.

This section consists of four tables which define the constants referred to in the statement of Theorem 2. The numbers $c_{G,i,\alpha}, c_{G,i,\beta}$ and $c'_{G,i}$ are defined in Tables 7.1 and 7.2; and the numbers $d_{G,i,D}$ and $f_{G,M,D}$ in Tables 7.3 and 7.4. In the latter tables, separate bounds are given for certain cases in which the subgroup D has a large normal factor, as indicated by the heading "$D \triangleright$" in the second column. For example, from Table 7.3 for $G = E_8$, we have $d_{G,2,E_7A_1} = 41$, $d_{G,4,A_7T_1} = 67$ (since here $D \triangleright E_7$, A_7 respectively); and from Table 7.4 for $G = E_6$, we have $f_{G,A_5A_1,D_5T_1} = 16$, $f_{G,N_G(D_4T_2),A_4A_1T_1} = 26$.

In Table 7.4, if $G = F_4$ then $M^0 = C_4$ or $\widetilde{D}_4$ only occurs for $p = 2$; likewise if $G = G_2$ then $M^0 = \widetilde{A}_2$ only occurs for $p = 3$.

i	$c_{E_8,i,\alpha}$	$c'_{E_8,i}$	$c_{E_7,i,\alpha}$	$c'_{E_7,i}$	$c_{E_6,i,\alpha}$	$c'_{E_6,i}$
1	18	28	8	12	4	6
2	22	34	11	16	6	8
3	23	36	12	18	7	10
4	25	40	14	21	9	12
5	25	39	13	20	7	10
6	22	36	10	16	4	6
7	18	30	6	10		
8	12	20				

Table 7.1. $G = E_8, E_7, E_6$.

i	$c_{F_4,i,\alpha}$	$c_{F_4,i,\beta}$	$c'_{F_4,i}$	$c_{G_2,i,\alpha}$	$c_{G_2,i,\beta}$	$c'_{G_2,i}$
1	5	$6 - 2\delta_{p,2}$	$8 - 2\delta_{p,2}$	3	$3 - \delta_{p,3}$	$4 - \delta_{p,3}$
2	7	$9 - 3\delta_{p,2}$	$11 - 2\delta_{p,2}$	2	3	$4 - \delta_{p,3}$
3	6	$9 - 2\delta_{p,2}$	$11 - 2\delta_{p,2}$			
4	4	$6 - \delta_{p,2}$	$8 - 2\delta_{p,2}$			

Table 7.2. $G = F_4, G_2$.

G	$D \triangleright$	i = 1	2	3	4	5	6	7	8
E_8	E_7	35	41	44	48	47	44	36	24
	D_8	40	48	51	55	54	50	43	29
	other	48	58	62	67	66	61	52	36
E_7	E_6	12	17	18	22	21	17	11	
	D_6	16	20	23	26	24	20	12	
	A_7	17	23	25	28	27	22	15	
	other	20	26	30	34	32	26	17	
E_6	D_5	6	8	10	12	10	6		
	A_5	8	11	13	15	13	8		
	other	10	12	16	18	16	10		
F_4	B_4	4	6	7	5				
	C_3	8	11	11	8				
	B_3	8	12	12	9				
	other	10	13	14	11				
G_2	A_2	2	3						
	other	3	3						

Table 7.3. Values of $d_{G,i,D}$.

G	$D \triangleright$	M^0					
E_8		E_7A_1	D_8	E_6A_2	A_8	${D_4}^2$	other
	E_7	48	56	65	68	72	84
	D_8	56	64	81	84	88	100
	other	70	80	102	108	116	128
E_7		E_6T_1	D_6A_1	A_7	other		
	E_6	22	24	27	36		
	D_6	24	32	32	39		
	A_7	27	32	35	45		
	other	34	40	44	58		
E_6		D_5T_1	A_5A_1	D_4T_2	${A_2}^3$	T_6	
	D_5	12	16	20	24	26	
	A_5	16	20	24	27	34	
	other	20	24	26	32	42	
F_4		B_4	C_4	D_4	$\widetilde{D}_4$	C_3A_1	$A_2\widetilde{A}_2$
	B_4	8	–	9	–	8	12
	C_3	8	10	12	16	14	18
	B_3	10	8	16	12	16	22
	other	12	12	16	16	18	24
G_2		A_2	$\widetilde{A}_2$	$A_1\widetilde{A}_1$			
	A_2	6	–	4			
	other	3	3	4			

Table 7.4. Values of $f_{G,M,D}$.

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