Pacific Journal of Mathematics

HEREDITARILY ODD–EVEN AND COMBINATORIAL ISOLS

Joseph Barback

Volume 206 No. 1 September 2002

PACIFIC JOURNAL OF MATHEMATICS
Vol. 206, No. 1, 2002

HEREDITARILY ODD–EVEN AND COMBINATORIAL ISOLS

Joseph Barback

In this paper we study some of the arithmetic structure that is found in a special kind of semi-ring in the isols. These are the semi-rings $[D(Y), +, \cdot]$ that were introduced by J.C.E. Dekker, and that were later shown by E. Ellentuck to model the true universal recursive statements of arithmetic when Y is a regressive isol and is hyper-torre (= hereditarily odd-even = HOE). When Y is regressive and HOE, we further reflect on the structure of $D(Y)$. In addition, a new variety of regressive isol is introduced, called combinatorial. When Y is such an isol, then it is also HOE, and more, and the arithmetic of $D(Y)$ is shown to have a richer structure.

1. Introduction.

One of the nice directions in the theory of isols deals with modelling familiar arithmetic properties of the nonnegative integers in collections of isols. In this paper we continue in that direction. We study two particular properties for regressive isols; the property of *hereditarily odd-even* (HOE), and the property of being *combinatorial* (COMB). The first of these was studied in [**1**] and [**10**]; its significance is based on the work of E. Ellentuck on hyper-torre isols in [**8**], and on the fact that being HOE gives an arithmetic characterization to being regressive and hyper-torre. Being a combinatorial isol is a new notion introduced here. It is also defined by an arithmetic feature of nonnegative numbers which is extended to the isols.

Both HOE and COMB isols are studied in the context of the (Dekker) semi-ring of regressive isols that they generate. In [**6**] J.C.E. Dekker introduced the study of a special sort of semi-ring $D(Y)$ in the isols. Associated with each isol Y, the algebraic system $[D(Y), +, \cdot]$ provides a natural and interesting extension of the arithmetic of the nonnegative integers. Just how much of the structure of familiar arithmetic is present in these systems is known to depend upon the choice of the isol Y. It is the classical result of E. Ellentuck in [**8**] that if Y is a hyper-torre regressive isol, equivalently HOE, then the system $[D(Y), +, \cdot]$ is a model for the true universal recursive statements of arithmetic. By some recent work of T.G. McLaughlin in [**11**], and our own in [**4**], we know that there are new varieties of regressive isols

Y for which one can find a richer structure for the isols in $D(Y)$. It is in that way that the notion of a combinatorial isol evolved.

The principal concepts of our paper are the two properties of HOE and COMB for regressive isols, and the special semi-ring of isols $D(Y)$. These notions are defined in the following way:

Definition D1. An isol Y is said to be *hereditarily odd-even* (HOE) if every predecessor of Y has parity (i.e., is either an even isol or an odd isol). Thus Y is HOE if every $U \leq Y$ has the form $U = 2S$ or $U = 2S + 1$, for some isol S.

Definition D2. An isol Y is said to have *direct comparability of summands* (DCS), if whenever $Y = A + B$, then either $A \leq B$ or $B \leq A$.

Definition D3. An isol Y is said to be *combinatorial* (COMB) if $g^*(Y)$ has DCS, whenever g is any recursive combinatorial function, and g^* is the extension of g to the isols.

Definition D4. Let Y be any isol, and let RCF denote the collection of all recursive combinatorial functions of one variable. Then:

$$D(Y) = (U \leq g^*(Y) \mid \text{for any } g \text{ in RCF}).$$

These notions have been defined when Y is any isol, but in our study we shall almost always be assuming in addition that Y is regressive. This approach is significant because there is a rich collection of known results that we can then apply. Our main results in the paper concern the system of isols $[D(Y), +, \cdot]$ when Y is regressive and HOE, and when Y is regressive and COMB.

Regressive isols that are infinite and HOE have been studied in [**1**], [**4**] and [**10**]. Their existence was first shown by L. Harrington; the result and its proof are given in [**10**, Theorem 20.15]. The existence of infinite regressive isols that are combinatorial comes indirectly from the presentation in [**4**]. There, isols called *completely torre* were introduced and studied. These isols are infinite and regressive, and, based upon [**4**, Theorems 4.1 and 4.2], can be shown to be combinatorial. That there are completely torre isols was proved in [**4**, Theorem 5.1]. When Y is a regressive isol and HOE, we show that the members of $D(Y)$ are all comparable by the $\leq^*$ relation, and that there is present a weak form of Euclidean division. When Y is a regressive isol and COMB, it is shown that the members of $D(Y)$ are all comparable by the $\leq$ relation, and that now there is present the familiar form of Euclidean division.

2. Preliminaries.

We shall assume that the reader is familiar with basic concepts and results in the theory of isols and regressive isols. The main results that are applied appear in [1], [4], and [10]. It is very useful in the paper to be able to apply the classical metatheorem of A. Nerode that permits certain universal Horn sentences that are true in the nonnegative integers to be extended to sentences that are true in the isols.

We let $E = (0, 1, 2, \dots)$ whose members are called *numbers*, and let Λ be the collection of *isols*, and Λ_R the collection of *regressive isols*. If g is a recursive function in any number of variables, then g^* will denote the extension of g to the isols. A function $g : E \to E$ is said to be *increasing*, if $x \le y$ implies $g(x) \le g(y)$, for all numbers x and y. Combinatorial functions of one variable are increasing functions. The very useful property about recursive increasing functions that we use is: If g is a recursive increasing function (of one variable), then g^* maps regressive isols to regressive isols. Throughout the paper the set notations E and $D(Y)$ are also used to denote the algebraic systems $[E, +, \cdot]$ and $[D(Y), +, \cdot]$, respectively.

The following are some of the properties that HOE regressive isols are known to possess ([10]). Let Y be such an isol. Then every predecessor of Y is HOE, and so also is $g^*(Y)$ when g is any increasing recursive function. In addition, Y satisfies the following comparability of summands property (CS): Whenever $Y = A + B$, then either $A \le^* B$ or $B \le^* A$.

The relation of $\le^*$ was introduced by J.C.E. Dekker in [6]. It is defined in the following way: Let A and B be any isols. Then $A \le^* B$ is true just when (a) A and B are finite and $A \le B$, or (b) A is finite and B is infinite, or (c) both A and B are infinite and there are sets α and β belonging to A and B respectively, and a partial recursive function p, such that p is defined on all numbers in α, $p(\alpha) = \beta$, and p is one-one on α. It is one of the fundamental results ([6]), that when A and B are any isols, then $A \le^* B$ and $B \le^* A$ together imply that $A = B$.

One aim in [6] was to associate with any pair of regressive isols A and B a new regressive isol, written as $\min(A, B)$, called the *minimum of A and B*. A variety of results were proved which established a close similarity of $\min(A, B)$ in relation to $\le^*$, as one would find for the familiar *minimum* function for numbers of E in relation to $\le$. For example,

$$\min(A, B) = \min(B, A), \quad \min(A, B) \le^* A, \text{ and}$$
$$\min(A, B) = A \quad \text{if and only if } A \le^* B.$$

Later it was shown ([10]) that $\min(A, B) = \min^*(A, B)$ where $\min^*$ denotes the extension to the isols of the familiar minimum function on E^2 and A and B are any regressive isols. In addition, $\le^*$ is also the same as $\le_\Lambda$, being the extension to the isols of the familiar $\le$ relation, when it is retricted to regressive isols.

By combining these results with the Nerode metatheorem for extending statements to the isols we will be able to obtain some useful consequences that relate to functions and regressive isols.

Let A and B be regressive isols. It is a well-known property that $A \leq B$ implies that $A \leq^* B$, and also that the converse is not true ([6]). Thus if Y is a regressive isol that has DCS, then Y has CS. Even more is also true. Assume that Y is a regressive isol that has DCS. Then, based on results in [4], each predecessor of Y also has DCS, and then Y is HOE. It was also shown that $2Y$, as well as any finite multiple of Y, has DCS.

Assume that g is a recursive combinatorial function and let Y be a regressive isol. While it is known that if Y is HOE then $g^*(Y)$ is also HOE, it is presently not known whether if when Y has DCS then $g^*(Y)$ also has DCS. It is for the last reason, and because of our special interest in $D(Y)$, that the notion of a combinatorial isol was introduced in the manner it was.

From the results cited in our comments above we may directly obtain the following result.

Proposition 2.1. *Let Y be a regressive isol that is combinatorial and let X be any member of $D(Y)$. Then X has the following properties:* DCS, CS *and* HOE.

Let g be a recursive combinatorial function, and let X and Y be any isols. It follows from an early classical result of J. Myhill ([12]) that if $X \leq Y$ then $g^*(X) \leq g^*(Y)$. Based on this fact we may establish the following result.

Proposition 2.2. *Let Y be a regressive isol that is combinatorial and let $X \leq Y$. Then X is also regressive and combinatorial.*

Proof. The regressiveness of X follows directly from the fact that it is a predecessor of a regressive isol. To show that X is combinatorial, let g be any recursive combinatorial function. Then $X \leq Y$ implies $g^*(X) \leq g^*(Y)$, by the theorem of J. Myhill. Then, $g^*(Y)$ has DCS since Y is combinatorial, and $g^*(X)$ has DCS since it is a predecessor of a regressive isol that has DCS. Hence, it follows that X is a combinatorial isol, and it completes our proof.

It is one of the basic facts about recursive functions and their extension to the isols that if f and g are recursive combinatorial functions, then their composition $h = f \circ g$ is also recursive combinatorial, and $h^*(X) = f^*(g^*(X))$ for all isols X. Based on this fact we have the following result.

Proposition 2.3. *Let Y be a regressive and combinatorial isol. Then:*

(1) *$g^*(Y)$ is regressive and combinatorial for every recursive combinatorial function g, and*
(2) *every member of $D(Y)$ is regressive and combinatorial.*

Proof. In each part the regressive property of the isol is clear. We shall then only deal with the property of being combinatorial.

Re (1). Let g be a recursive combinatorial function. To show that $g^*(Y)$ is combinatorial, let f be any recursive combinatorial function and consider

$f^*(g^*(Y))$. We wish to see that it has DCS. Setting $h = f \circ g$, we then have that h is recursive combinatorial and $f^*(g^*(Y)) = h^*(Y)$. Then $f^*(g^*(Y))$ has DCS, since Y is a combinatorial isol.

Re (2). Each member X of $D(Y)$ is predecessor of a $g^*(Y)$, for some recursive combinatorial function g. Then $g^*(Y)$ is a combinatorial isol by part (1), and then X is also combinatorial, by Proposition 2.2. That completes our proof.

Lastly, we mention that it is an open question at the present time whether the properties of HOE and COMB are distinct.

3. Characterizations for $D(Y)$ when Y is HOE.

Throughout this section we will assume that Y is a regressive isol. Our interest is to establish a variety of ways that the system $D(Y)$ may be characterized when Y is HOE.

A function $s : E \to E$ is called *elementary* if for each number x the value of $s(x)$ is 0 or 1. An elementary function is therefore the characteristic function of a subset of E, and a recursive elementary function corresponds to a recursive subset of E. Our interest is in recursive elementary functions and in establishing some properties about their extension to the isols, as: When Y is HOE, then the value of $s^*(Y)$ is an isol and is equal to 0 or 1. The extension to the isols of a recursive elementary function need not map each regressive isol to an isol. That fact follows from a result of E. Ellentuck which states that the extension of a recursive function $f : E \to E$ maps regressive isols to the isols if and only if f is eventually increasing, meaning that there is a number k such that the function $g(x) = f(x+k)$ is increasing. Thus, when applied to a recursive elementary function s, it follows that s^* maps Λ_R into Λ only in the case that s is eventually constant.

Proposition 3.1. *Let s be a recursive elementary function. Let U and V be isols such that $V = s^*(U)$. Then V is equal to 0 or 1.*

Proof. If V is equal to 0, then we are done. Let us now assume that V is positive, and let $V = W + 1$. Then $s^*(U) = W + 1$.

Since s is an elementary function, the statement

$$s(u) = w + 1 \to w = 0 \tag{1}$$

is true in the nonnegative integers. We would like to extend it to the isols. Though (1) is a Horn sentence (expressed with recursive functions), we shall first give an equivalent statement that is a Horn sentence with recursive combinatorial functions. Then, to the new sentence we apply the Nerode metatheorem.

Let s^+ and s^- be recursive combinatorial functions such that for all numbers x, $s(x) = s^+(x) - s^-(x)$. Then $s^*(U) = (s^+)^*(U) - (s^-)^*(U)$. We may then express (1) by

$$s^+(u) = s^-(u) + w + 1 \;\rightarrow\; w = 0. \tag{2}$$

And, then (2) extends to the isols by the Nerode metatheorem, which gives in particular, that

$$(s^+)^*(U) = (s^-)^*(U) + W + 1 \;\rightarrow\; W = 0. \tag{3}$$

Since $s^*(U) = (s^+)^*(U) - (s^-)^*(U)$, it follows from (3) that

$$s^*(U) = W + 1 \;\rightarrow\; W = 0.$$

Hence, from our given that $s^*(U) = W + 1$ it follows that $W = 0$, and also $V = 1$. That completes our proof.

Proposition 3.2. *Let Y be a regressive isol that is* HOE*. Let s be a recursive elementary function. Then $s^*(Y)$ is an isol.*

Proof. Let s^+ and s^- be recursive combinatorial functions such that for all x, $s(x) = s^+(x) - s^-(x)$. Then both $(s^+)^*(Y)$ and $(s^-)^*(Y)$ are HOE regressive isols, so each is either even or odd. We have by definition that $s^*(Y) = (s^+)(Y) - (s^-)(Y)$ is an element of Λ^* (the ring of "isolic integers"); and we would like to show that in fact $s^*(Y)$ is an element of Λ. Now for any number x, if $s^+(x)$ and $s^-(x)$ are both even or both odd, then $s(x) = 0$; otherwise $s(x) = 1$. Thus:

$$s^+(y) + s^-(y) = 2u \;\rightarrow\; s^+(y) = s^-(y) \tag{1}$$

and

$$s^+(y) + s^-(y) = 2u + 1 \;\rightarrow\; s^+(y) = s^-(y) + 1. \tag{2}$$

Since (1) and (2) are universal Horn formulas with only recursive combinatorial functions appearing, they both extend to Λ, and so we have

$$(s^+)^*(Y) + (s^-)^*(Y) = 2U \;\rightarrow\; (s^+)^*(Y) = (s^-)^*(Y) \tag{1'}$$

and

$$(s^+)^*(Y) + (s^-)^*(Y) = 2U + 1 \;\rightarrow\; (s^+)^*(Y) = (s^-)^*(Y) + 1. \tag{2'}$$

Either the antecedent of (1′) or that of (2′) must hold for some U, since $(s^+)^*(Y)$ and $(s^-)^*(Y)$ both have parity. But if either the conclusion of (1′) or that of (2′) holds, then $(s^-)^*(Y) \leq (s^+)^*(Y)$, and this implies that $(s^+)^*(Y) - (s^-)^*(Y)$ is an element of Λ. That completes our proof.

Comment. The classical metatheorem of A. Nerode permits one to extend universal Horn sentences that are built up from recursive combinatorial functions that are true in the nonnegative integers to sentences that are true in the isols. In that way the metatheorem was applied in the proofs of Propositions 3.1. and 3.2. Sometimes in our paper when we wish to apply metatheorem our reasoning leads us first to a Horn sentence that is expressed in terms of recursive functions, but not necessarily recursive combinatorial. So an application of the metatheorem is not directly appropriate, and what we then do is convert the statement into an equivalent expression that is a Horn sentence and which is expressed in terms of recursive combinatorial functions (as in the proof of Proposition 3.1). The metatheorem is then applied to extend the latter statement to the isols.

In our paper the conversion of any such Horn sentence to another may always be accomplished by the familiar method of replacing a recursive function g, that may not be combinatorial, by a difference $g^{+}-g^{-}$ of two recursive combinatorial functions, and then appropriately arranging the sides of an equation so that only recursive combinatorial functions, and their sums and products, are represented. In what follows we shall sometimes apply the metatheorem to Horn statements that are simply expressed in terms of recursive functions. It is done so with the belief that no unnecessary problems would occur for the reader.

Proposition 3.3. *Let Y be a regressive isol that is* HOE*. Let s be any recursive elementary function. Then s^* maps $D(Y)$ into $D(Y)$.*

Proof. Let V be a member of $D(Y)$. Then $V \leq g^*(Y)$ for some recursive combinatorial function g. Since Y is HOE, then $g^*(Y)$ is also since g is a recursive combinatorial function. Then V is also HOE, since it is the predecessor of a HOE isol. Lastly, $s^*(V)$ is an isol and equal to 0 or 1, by Propositions 3.1. and 3.2. Hence $s^*(V)$ is in $D(Y)$.

The converse of Proposition 3.3 is also true, and we shall now present some ideas that will lead us to that result.

Definition. Let k be any positive number. Then $\left[\frac{x}{k}\right]$ will be denote the recursive function of x whose value is the greatest integer that is present upon a division of x by k. Equivalently, $x = k\left(\left[\frac{x}{k}\right]\right) + r$ for a unique number r with $0 \leq r < k$.

For each value of a positive number k the function $\left[\frac{x}{k}\right]$ is recursive increasing. Its extension to the isols will therefore map regressive isols to regressive isols, and also regressive isols that are HOE to regressive isols that are HOE. We also note that by applying the Nerode metatheorem to some valid statements in the nonnegative integers, we may infer when Y is regressive, that Y is even if and only if $Y = 2\left[\frac{Y}{2}\right]^*$, and Y is odd if and only if $Y = 2\left[\frac{Y}{2}\right]^* + 1$.

Proposition 3.4. *Let Y be a regressive isol. Assume that $s^* : D(Y) \to D(Y)$, for every recursive elementary function s. Then Y is* HOE.

Proof. Let $V \leq Y$. Then V is a member of $D(Y)$. We wish to show that V has parity. Let s be the recursive elementary function defined by, $s(x) = 0$

if x is even, and $s(x) = 1$ if x is odd. Then $s^*(V)$ is in $D(Y)$, by our hypothesis, and also, its value is either 0 or 1, by Proposition 3.1. Because each of the statements,

$$s(x) = 0 \rightarrow x = 2\left[\frac{x}{2}\right], \quad \text{and} \quad s(x) = 1 \rightarrow x = 2\left[\frac{x}{2}\right] + 1,$$

is true for every number x, it follows (by the Nerode metatheorem) that if $s^*(V) = 0$ then $V = 2\left[\frac{V}{2}\right]^*$, and if $s^*(V) = 1$ then $V = 2\left[\frac{V}{2}\right]^* + 1$. Hence V is either even or odd, and it follows that Y is HOE.

We know if Y is HOE then Y has comparability of summands (CS), and then also every member of $D(Y)$ has CS, since each is HOE. This latter property of comparability of summands of members of $D(Y)$ actually implies that Y is HOE. We wish establish that fact, and we begin with following result. It is proved in [**1**] as Lemma L1, and a proof for it will be omitted here.

Proposition 3.5. *Let Y be a regressive isol such that $2Y+1$ has* CS*. Then Y has parity.*

Proposition 3.6. *Let Y be a regressive isol. Then the following statements are equivalent:*

(1) *Y is* HOE;
(2) *each member of $D(Y)$ has* CS;
(3) *all of the members of $D(Y)$ are $\leq^*$ comparable.*

Proof. For (1) $\rightarrow$ (2): This direction follows from our previous comments, being that: When Y is HOE, then each member of $D(Y)$ is also HOE, and hence has CS.

For (2) $\rightarrow$ (3): Assume (2), and let V and W be in $D(Y)$. Then the isol $V + W$ is also in $D(Y)$, and hence has CS. That implies that $V \leq^* W$ or $W \leq^* V$, which gives (3).

For (3) $\rightarrow$ (1): Assume (3), and let $U \leq Y$. We wish to show that U is even or odd. Since Y is in $D(Y)$, then U and $2U + 1$ are also in $D(Y)$. In view of Proposition 3.5, it suffices for us to observe that $2U + 1$ has CS. If $2U+1 = V+W$, then V and W are in $D(Y)$, and hence are $\leq^*$ comparable. And that implies that $2U + 1$ has CS. It completes our proof.

Our final topic in this section is about division for isols in the system $D(Y)$. Unless one assumes some special property for Y to satisfy there is no chance that any familiar form of division, as in the arithmetic of E, would be true in $D(Y)$. For example, if Y is neither even nor odd, as would be the case if Y is infinite and multiple-free, then one does not even have $Y = 2V$ or $Y = 2V + 1$.

We shall see that certain familiar forms of division apply for the systems $D(Y)$ when Y is HOE, and when Y is COMB. It is the first of these that we shall now begin to study.

Definition. We will say that $D(Y)$ permits *E-divisibility* if for all positive numbers k, and all V in $D(Y)$, there exist unique members A and r of $D(Y)$ such that, $V = kA + r$ and $0 \le r < k$.

Comment. We would like to comment upon the uniqueness property that appears in the above definition. Because the familiar uniqueness property of Euclidean division in arithmetic will extend to the isols, that extension provides a corresponding uniqueness property in the isols. In our study of E-divisibility, we shall then only focus upon the appropriate existence of isols, as A and r, in the above definition, and not on their uniqueness. Later in our paper, in Section 5, we will present a complete description of how the uniqueness property of arithmetic may be extended to the regressive isols.

Proposition 3.7. *Let Y be a regressive isol. Then the following statements are equivalent:*

(1) *Y is* HOE;
(2) *$D(Y)$ permits E-divisibility.*

Proof. For (1) → (2): Assume that Y is HOE. Let V be in $D(Y)$. Let k be any positive number. Then V is HOE. We wish to show that

$$V = k\left(\begin{bmatrix} V \\ k \end{bmatrix}^*\right) + r \quad \text{for a number } r \text{ with } 0 \le r < k. \tag{1}$$

If we can show (1), then it is clear that our desired conclusion will follow, since $\left[\begin{smallmatrix} V \\ k \end{smallmatrix}\right]^*$ is an isol and is in $D(Y)$.

For each number i with $0 \le i < k$, let s_i be the recursive elementary function defined by: $s_i(x) = 1$ if $x = k\left(\left[\begin{smallmatrix} x \\ k \end{smallmatrix}\right]\right) + i$, and $s_i(x) = 0$ otherwise. Then, for each i with $0 \le i < k$,

$$s_i(x) = 1 \;\rightarrow\; x = k\left(\begin{bmatrix} x \\ k \end{bmatrix}\right) + i, \tag{2}$$

and

$$s_0(x) + \cdots + s_{k-1}(x) = 1, \tag{3}$$

for all numbers x. We can now consider the extension to the isols of these statements. It follows from (3) and its extension to the isols that

$$(s_0)^*(V) + \cdots + (s_{k-1})^*(V) = 1. \tag{4}$$

In view of Propositions 3.1 and 3.2, it follows that the value of just one of the summands in (4) is equal to 1, and each of the others is equal to 0. Let us assume $(s_j)^*(V) = 1$. Then, from the extension to the isols of (2) we may infer that

$$V = k\left(\begin{bmatrix} V \\ k \end{bmatrix}^*\right) + j,$$

and where $0 \leq j < k$. Since $\begin{bmatrix} V \\ k \end{bmatrix}^*$ is a member of $D(Y)$, we may therefore conclude statement (2).

For (2) $\rightarrow$ (1): Assume that $D(Y)$ permits E-divisibility. Let $V \leq Y$. Note that V is a member of $D(Y)$. We wish to show that V has parity. By our assumption it follows that $V = 2A$ or $V = 2A + 1$, for some isol A in $D(Y)$. Therefore V is even or odd, and that completes our proof.

We will conclude this section with a single result that contains all of the characterizations about $D(Y)$ that have been shown to be equivalent to Y being HOE.

Theorem 3.1. *Let Y be a regressive isol. Then the following statements are equivalent:*

1. Y *is* HOE;
2. *each member of $D(Y)$ has comparability of summands;*
3. *all the members of $D(Y)$ are $\leq^*$ comparable;*
4. s^* *maps $D(Y)$ into $D(Y)$, for every recursive elementary function* s;
5. $D(Y)$ *permits E-divisibility.*

Proof. We obtain the equivalence of (1), (2), and (3), by Proposition 3.6. Propositions 3.3 and 3.4 imply that (1) and (4) are equivalent. Lastly, Proposition 3.7 gives the equivalence of (1) and (5).

4. Combinatorial regressive isols.

In this section we present a collection of results that extend the ideas of the previous section to the setting of $D(Y)$ when Y is a regressive isol and combinatorial (COMB). It is interesting to observe the transition of certain properties as one goes from Y being regressive and HOE, to Y being regressive and COMB, and how these become expressed for members in $D(Y)$. We shall see that the following analogues appear:

Y HOE	Y COMB
$\leq^*$	$\leq$
CS	DCS
E-divisibility	I-divisibility

Here I-divisibility refers to a very general extension of Euclidean division that is present in $D(Y)$ when Y is regressive and COMB. Moreover, there is a very nice analogue of E. Ellentuck's Theorem: When Y is regressive and Hypertorre (= HOE), then $D(Y)$ is a model for the true universal-recursive statements of arithmetic. It follows from the recent work of T.G. McLaughlin in [**11**] that one has: When Y is regressive and COMB, then $D(Y)$ is a model for all the true AE-recursive statements of arithmetic.

In the following we shall assume that Y is a regressive isol.

Theorem 4.1. *The following statements are equivalent:*

(1) Y *is* COMB*;*
(2) *each member of* $D(Y)$ *has* DCS*;*
(3) *the members of* $D(Y)$ *are* $\leq$ *comparable.*

Proof. For (1) $\rightarrow$ (2): (We shall repeat an earlier observation.) Assume (1), and let V be a member of $D(Y)$. Then $V \leq g^*(Y)$, for some recursive combinatorial function g. Then $g^*(Y)$ has DCS, as Y is COMB, and then V has DCS, since it is a predecessor of a regressive isol that has DCS. Hence (2).

For (2) $\rightarrow$ (3): Assume (2). Let V and W be members of $D(Y)$. Then $V + W$ is also a member of $D(Y)$. Hence $V + W$ has DCS, and therefore $V \leq W$ or $W \leq V$.

For (3) $\rightarrow$ (1): Assume (3). Let g be any recursive combinatorial function of one variable. We wish to show that $g^*(Y)$ has DCS. We first observe that $g^*(Y)$ is a member of $D(Y)$, and therefore whenever $g^*(Y) = V + W$, then $V \leq W$ or $W \leq V$ since both V and W are in $D(Y)$, and the members of $D(Y)$ are $\leq$ comparable, by our assumption. Hence $g^*(Y)$ has DCS, and so Y is combinatorial. That completes our proof.

Definition. We will say that $D(Y)$ permits I-*divisibility* if for all members V and A of $D(Y)$, with A positive, there exist unique members Q and R of $D(Y)$, such that $V = QA + R$ and $0 \leq R < A$.

Comment. As in the case for E-divisibility that was introduced earlier, also here for I-divisibility, one has the feature that if there exist isols Q and R as in the previous definition then their values are unique. The reasoning that is appropriate to achieve the uniqueness feature is presented in the concluding section of our paper. For now we shall only consider the existence of isols as Q and R, and not their uniqueness.

Lemma 4.1. *Let* Y *be* COMB*, and let* V *and* W *be members of* $D(Y)$*. Assume that* $V \leq^* W$*. Then* $V \leq W$*.*

Proof. By Theorem 4.1 it follows that either $V \leq W$ or $W \leq V$. We may therefore assume that $W \leq V$. Then also $W \leq^* V$, and by combining that with the given fact that $V \leq^* W$, we have $V = W$. Then, here also, one has $V \leq W$.

Let $g : E \to E$ be recursive and combinatorial function. We would like to show that g^* maps $D(Y)$ into $D(Y)$. First, let us recall the result of J. Myhill ([**12**]) that if V and W are any isols, then $V \leq W$ implies that $g^*(V) \leq g^*(W)$. Now assume that V is a member of $D(Y)$. Then $V \leq f^*(Y)$, for some recursive and combinatorial function f. And therefore, $g^*(V) \leq g^*(f^*(Y))$. Here $g^*(f^*(Y)) = (g \circ f)^*(Y)$, where the composition function $g \circ f$ is also recursive and combinatorial. Hence, $g^*(V)$ is a member of $D(Y)$, and we conclude that g^* maps $D(Y)$ into $D(Y)$.

This result may be generalized for recursive functions $g : E^n \to E$ in the following two ways. First, if g is a recursive combinatorial function, then g^* maps $D(Y)^n$ into $D(Y)$. This result was proved by J.C.E. Dekker in [**6**]. In addition, we also have:

Lemma 4.2. *Let Y be regressive and* COMB*. Let $g : E^n \to E$ be any recursive function. Then g^* maps $D(Y)^n$ into $D(Y)$.*

Proof. To show that g^* maps $D(Y)^n$ into $D(Y)$, let g^+ and g^- be any recursive and combinatorial functions such that $g = g^+ - g^-$ on E^n. Then $g^* = (g^+)^* - (g^-)^*$ on Λ^n.

Let $V_1, \ldots, V_n$ be any members of $D(Y)$. Then

$$g^*(V_1, \ldots, V_n) = (g^+)^*(V_1, \ldots, V_n) - (g^-)^*(V_1, \ldots, V_n). \tag{1}$$

Because each of g^+ and g^- is recursive and combinatorial, then each of $(g^+)^*(V_1, \ldots, V_n)$ and $(g^-)^*(V_1, \ldots, V_n)$ is an element of $D(Y)$, and therefore these isols are $\leq$ comparable, by Theorem 4.1.

Let us observe that

$$g^-(x_1, \ldots, x_n) \leq g^+(x_1, \ldots, x_n), \tag{2}$$

for all numbers $x_1, \ldots, x_n$; this is true because $g = g^+ - g^-$ on E^n and g is a recursive function. We may extend statement (2) to the isols, and since each of $(g^+)^*(V_1, \ldots, V_n)$ and $(g^-)^*(V_1, \ldots, V_n)$ is a regressive isol, we may then infer that

$$(g^-)^*(V_1, \ldots, V_n) \leq^* (g^+)^*(V_1, \ldots, V_n). \tag{3}$$

We also know that the isols in (3) are $\leq$ comparable. By Lemma 4.1, it then follows that

$$(g^-)^*(V_1, \ldots, V_n) \leq (g^+)^*(V_1, \ldots, V_n). \tag{4}$$

In view of (1) and (4), we may conclude that $g^*(V_1, \ldots, V_n)$ is an isol. Lastly, it is also a member of $D(Y)$, for from (1) we see that $g^*(V_1, \ldots, V_n) \leq (g^+)^*(V_1, \ldots, V_n)$, and the latter isol is a member of $D(Y)$.

We may therefore conclude that g^* maps $D(Y)^n$ into $D(Y)$, and it completes our proof.

Theorem 4.2. *Let Y be a regressive isol. Then the following statements are equivalent:*

(1) *Y is* COMB;
(2) *g^* maps $D(Y)^n$ into $D(Y)$, for every recursive function $g : E^n \to E$;*
(3) *$D(Y)$ permits I-divisibility.*

Proof. For the direction (1) → (2), it is the content of Lemma 4.2.

For (2) → (3): Assume (2), and let $q(v, a)$ and $r(v, a)$ be recursive functions that relate to Euclidean division in the following way: For all numbers v and positive numbers a,

$$v = q(v, a) \cdot a + r(v, a) \quad \text{and} \quad 0 \leq r(v, a) < a. \tag{1}$$

Statement (1) gives a familiar representation of Euclidean division. We will make another representation, it being more appropriate for our reasoning here. Let $m(v, a)$ be an additional recursive function such that

$$a = b + 1 \to (v = q(v, a) \cdot a + r(v, a) \quad \text{and} \quad r(v, a) + 1 + m(v, a) = a), \tag{2}$$

for all numbers v, a and b. It is easy to see that statement (2) also represents Euclidean division on E, and also this statement will extend to the isols by the Nerode metatheorem. Because each of the functions q, r and m is recursive, then each of their extensions to the isols will map $D(Y)^2$ into $D(Y)$, by our assumption.

To verify that $D(Y)$ permits I-divisibility, let V and A be members of $D(Y)$, with A positive. Then $A = B + 1$, for an isol B, and B is also a member of $D(Y)$. Let $Q = q^*(V, A)$, $R = r^*(V, A)$ and $M = m^*(V, A)$. Based on the extension of (2) to the isols, we have

$$V = QA + R \quad \text{and} \quad R + 1 + M = A. \tag{3}$$

Since each of the isols Q, R and M is a member of $D(Y)$, and, as then the second condition in (3) implies $0 \leq R < A$, our desired conclusion for I-divisibility in $D(Y)$ follows.

For (3) → (1): Assume (3). To verify (1), let $g : E \to E$ be any recursive combinatorial function, and let $V \leq g^*(Y)$. Then V is a member of $D(Y)$.

We wish to show that V has DCS. Let $V = A + B$. Then A and B are also members of $D(Y)$, since each is a predecessor of V.

If $A = 0$, then $A \leq B$. Let us now assume that A is positive. Then by the I-divisibility in $D(Y)$, there are members Q and R of $D(Y)$ such that

$$B = QA + R \quad \text{and} \quad 0 \leq R < A.$$

Case i. If $Q = 0$, then $B = R$, and then $B < A$. Then also $B \leq A$.

Case ii. If Q is positive, then $A \leq QA$, and then $A \leq B$.

We may therefore conclude that either $A \leq B$ or $B \leq A$, and it therefore follows that V has DCS. This completes our proof.

With Theorems 4.1 and 4.2 we obtain an analogue of Theorem 3.1 for regressive isols that are combinatorial. In these results it is interesting to see the increase of richness in the arithmetic of $D(Y)$ as one goes from Y being HOE to Y being combinatorial.

5. Concluding remarks.

(1) We would like to verify that the familiar uniqueness property for Euclidean division in the nonnegative extends also to the regressive isols. It is done here just for regressive isols because it was in that setting that the property was introduced, in relation to E-divisibility and I-divisibility for members of $D(Y)$, when Y is a regressive isol.

We shall consider the expression for the familiar uniqueness property in E, then find an equivalent form, and then apply the Nerode metatheorem to extend the latter expression to the isols. Let $\Delta(v, q, a, r)$ denote the statement

$$v = qa + r \quad \text{and} \quad 0 \leq r < a,$$

for numbers v, q, a and r. Since the relation $0 \leq r$ is always true for numbers of E, it will be dropped. We also rewrite $r < a$ as $\min(r + 1, a) = r + 1$, where min is the familiar minimum function of E^2. With these changes then $\Delta(v, q, a, r)$ is equivalent to

$$v = qa + r \quad \text{and} \quad \min(r + 1, a) = r + 1. \tag{1}$$

We shall let $\Delta_0(v, q, a, r)$ denote the expression in (1). It is usual when the uniqueness property is introduced for it to have the form: For all numbers v, q, a, q_0, r and r_0, with a positive, then $\Delta(v, q, a, r)$ and $\Delta(v, q_0, a, r_0)$ together imply that $q = q_0$, and $r = r_0$. We shall use the following equivalent form in terms of (1): For all numbers b, v, q, a, q_0, r and r_0, we have

$$\big(a = b + 1 \;\text{ and }\; \Delta_0(v, q, a, r) \;\text{ and }\; \Delta_0(v, q_0, a, r_0)\big) \\ \rightarrow (q = q_0 \;\text{ and }\; r = r_0). \tag{2}$$

Then (2) represents a Horn sentence built up from equations between recursive functions. Because it is true in E, its extension to the isols is also true, by the metatheorem of A. Nerode. Let Δ_0^* denote the extension of Δ_0 (namely, statement (1)) to the isols, and consider $\Delta_0^*(V, Q, A, R)$ for regressive isols V, Q, A and R. It gives

$$V = QA + R \;\text{ and }\; \min{}^*(R + 1, A) = R + 1. \tag{3}$$

We note in (3) that the equation $\min^*(R + 1, A) = R + 1$ is equivalent to the relation $R + 1 \leq^* A$, since $R + 1$ and A are regressive isols.

When we combine the previous results and comments, we arrive at the following form for the uniqueness property of division in the regressive isols:

For all regressive isols V, Q, A, R, Q_0 and R_0, with A positive,

$$\text{(4)} \qquad \text{if } V = QA + R, \quad V = Q_0 A + R_0, \quad R + 1 \leq^* A \text{ and } R_0 + 1 \leq^* A, \text{ then } Q = Q_0 \text{ and } R = R_0.$$

In relation to the uniqueness property that was considered earlier in our paper, property (4) is stronger than what was needed then. It follows from the fact that when A and R are any regressive isols, then $A < R$ implies that $A + 1 \leq^* R$, since $A < R$ is equivalent to $A + 1 \leq R$.

(2) We would like to introduce a result about regressive combinatorial isols. It illustrates an interesting property based on the work in [**4**]. Let us call a regressive isol Y *special combinatorial*, if Y is infinite, combinatorial, and whenever $Y + A$ is a regressive isol, then $Y + A$ is also combinatorial. Such isols exist, for it follows from the work in [**4**] that each completely torre regressive isol is special combinatorial.

Acknowledgement. I am happy to express my appreciation to Dr. J.C.E. Dekker and Dr. T.G. McLaughlin for their helpful assistance in my study and preparation of the topics presented in the paper, and to the referee for the many suggestions to improve its clarity and presentation.

References

[1] J. Barback, *On hereditarily odd-even isols and a comparability of summands property*, Pacific J. Math., **118** (1985), 27-35, MR 86m:03076, Zbl 0583.03027.

[2] ———, *Regressive isols and the relation $ya \leq yb$*, J. London Math. Soc. (2), **41** (1990), 1-9, MR 91h:03060, Zbl 0718.03032.

[3] ———, *On regressive isols and comparability of summands and a theorem of R. Downey*, Math. Logic Quarterly, **43** (1997), 83-91, MR 98a:03070, Zbl 0864.03030.

[4] ———, *A fine structure in the theory of isols*, Math. Logic Quarterly, **44** (1998), 229-264, MR 99m:03090, Zbl 0903.03027.

[5] J.N. Crossley and J.B. Remmel, *The work of Anil Nerode: A retrospective*, in ' Logical Methods', Proceedings of a conference at Cornell University '92 (J. Crossley, ed.), Birkhauser, Boston, (1993), 1-91, MR 95g:01024a, Zbl 0824.03001.

[6] J.C.E. Dekker, *The minimum of two regressive isols*, Mathematische Zeitschrift, **83** (1964), 345-366, MR 28 #3927, Zbl 0122.01002.

[7] R. Downey, *On hyper-torre isols*, J. Symbolic Logic, **54** (1989), 1160-1166, MR 91b:03073, Zbl 0705.03025.

[8] E. Ellentuck, *Hyper-torre isols*, J. Symbolic Logic, **46** (1981), 1-5, MR 82f:03038, Zbl 0503.03020.

[9] J. Hirschfeld, *Models of arithmetic and recursive functions*, Israel J. Math., **20** (1975), 111-126, MR 52 #2858, Zbl 0311.02050.

[10] T.G. McLaughlin, *Regressive Sets and the Theory of Isols*, Marcel Dekker, New York, 1982, MR 83i:03073, Zbl 0484.03025.

[11] ———, *Combinatorial isols and the arithmetic of Dekker semirings*, Math. Logic Quarterly, **48** (2002), 323-342.

[12] J. Myhill, *Recursive equivalence types and combinatorial functions*, Bull. Amer. Math. Soc., **64** (1958), 373-376, MR 57 #2896, Zbl 0085.24801.

Received March 7, 2000 and revised June 7, 2000.

SUNY College
Buffalo NY 14222