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To Professor Jacques Faraut on his sixtieth birthday

Let $\mathcal{D}$ be a bounded symmetric domain of tube type and $\underline{G}$ its group of holomorphic automorphisms. In this paper, we describe explicitly the Plancherel Theorem of weighted Bergman spaces on $\mathcal{D}$ under the action of certain symmetric subgroups of $\underline{G}$.

1. Introduction.

Let $\underline{G}$ be a noncompact connected real semi-simple Lie group with finite center and Lie algebra $\underline{\mathfrak{g}}$. Let θ be a Cartan involution of $\underline{G}$ and $\underline{K} = \{g \in \underline{G} \mid \theta(g) = g\}$. We use the same letter θ to denote the differential of θ. Then, we have a direct sum decomposition $\underline{\mathfrak{g}} = \underline{\mathfrak{k}} \oplus \underline{\mathfrak{p}}$ in eigenspaces with respect to θ. We assume that $\underline{G}$ is hermitian, then there exists an element Z_0 in the center $\mathfrak{c}(\underline{\mathfrak{k}})$ of $\underline{\mathfrak{k}}$ such that $\mathfrak{c}(\underline{\mathfrak{k}}) = \mathbb{R}Z_0$.

Let σ be an involutive automorphism of $\underline{G}$. We may assume that σ commutes with θ and $\underline{\mathfrak{g}} = \underline{\mathfrak{h}} \oplus \underline{\mathfrak{q}}$ is the decomposition of the Lie algebra $\underline{\mathfrak{g}}$ with respect to σ. Since $\sigma^2 = \mathrm{id}$, there are two exclucive possibilites. Either $\sigma(Z_0) = Z_0$ and σ acts holomorphically on the symmetric domain $\mathcal{D} := \underline{G}/\underline{K}$, or $\sigma(Z_0) = -Z_0$ and σ acts anti-holomorphically on $\mathcal{D}$. In this paper we consider the case where σ is holomorphic. The case where σ is anti-holomorphic is considered by Yu. A. Neretin (cf. [**22**], [**23**]). See also [**8**] and [**30**].

Let $\mathcal{H}^2_\ell(\mathcal{D})$ be the ordinary Bergman space of $\mathcal{D}$ where $\mathcal{D}$ is of tube type. For $\nu > \ell - 1$, we consider a weighted Bergman space $\mathcal{H}^2_\nu(\mathcal{D})$ of holomorphic functions on $\mathcal{D}$. The universal covering $\widetilde{\underline{G}}$ of $\underline{G}$ can be realized as the set of pairs (g, φ) with $g \in \underline{G}$ and φ a holomorphic function on $\mathcal{D}$ where $e^{\varphi(z)} = \det(Dg(z))$. Here $Dg(z)$ denote the differential of the map $z \mapsto g \cdot z$. The group $\widetilde{\underline{G}}$ acts in $\mathcal{H}^2_\nu(\mathcal{D})$ by

$$(U_\nu(\widetilde{g})f)(z) = e^{\nu\varphi(z)} f(g \cdot z), \qquad \widetilde{g}^{-1} = (g, \varphi).$$

The representation U_ν is a unitary and irreducible representation.

Let p be the universal map of $\widetilde{\underline{G}}$ in $\underline{G}$, and let G be a symmetric subgroup of $\underline{G}$. In this work we study the decomposition of the restriction of

U_ν to the subgroup $\widetilde{G} := p^{-1}(G)$ of $\underline{\widetilde{G}}$. By [15] (see also [16], [17]) the restriction $U_{\nu}{}_{|\widetilde{G}}$ is decomposed multiplicity-free and discretely into irreducible representations $(\pi_\mu, \mathcal{H}_\mu)$ of $\widetilde{G}$ such that $\mathcal{H}_\mu \subset \mathcal{H}^2_\nu(\mathcal{D})$.

Let $\mathcal{S}$ be the Shilov boundary of $\mathcal{D}$. The action of the group G on $\mathcal{S}$ admits open orbits. We consider one of the orbits which is a causal symmetric space G/H of compact type. Moreover G/H is a symmetric Makarevič space. The geometry and analysis of the domain $\mathcal{D}$ and the Makarevič space G/H can be described using Jordan algebras.

To study the decomposition of $\mathcal{H}^2_\nu(\mathcal{D})$, we consider a G-invariant domain Ξ in the complexification $G_{\mathbb{C}}/H_{\mathbb{C}}$ of G/H introduced by J. Hilgert, B. Ørsted and G. Ólafsson (cf. [14]). A geometric descripition of the domain Ξ is given by W. Bertram. The domain Ξ can be realized as $\mathcal{D} \setminus \Sigma$ where Σ is an analytic set (cf. [3]).

We consider a covering $\widetilde{\Xi}$ of Ξ with infinite order. We show that there is a unitary isomorphism of $\mathcal{H}^2_\nu(\mathcal{D})$ onto a weighted Bergman space $\mathcal{H}^2_\nu(\widetilde{\Xi})$. It is a Hilbert space of holomorphic functions on $\widetilde{\Xi}$, which satisfy a monodromy condition and are square integrable with respect to a G-invariant measure on Ξ.

To describe explicitly the decomposition of $\mathcal{H}^2_\nu(\widetilde{\Xi})$ into irreducible subspaces we study the holomorphic discrete series of the universal covering $\widetilde{G}$. Our approach is based on the spherical Laplace transform associated with the ordered symmetric space G^c/H dual of G/H. See [1] for $G/H \simeq U(p,q)$ and [2] for G/H of Cayley type.

This paper is organized as follows: In Section 2, we give a geometric descripition of the covering $\widetilde{\Xi}$ of Ξ using the theory of Jordan algebras. In Section 3, we study the Bergman space $\mathcal{H}^2_\nu(\widetilde{\Xi})$ and its reproducing kernel and we establish a unitary isomorphism of $\mathcal{H}^2_\nu(\widetilde{\Xi})$ onto $\mathcal{H}^2_\nu(\mathcal{D})$. To describe explicitly the spectrum of $\mathcal{H}^2_\nu(\widetilde{\Xi})$ and to express its reproducing kernel as series of spherical functions associated with the ordered symmetric spaces G^c/H, we study in Section 4 the holomorphic discrete series of $\widetilde{G}$. In particular, we obtain a necessary condition for π_μ to appear in the Plancherel formula. In Section 5, we compute explicitly the L^2-norm of matrix coefficient associated with an H-spherical unitary highest weight representation. Then, we can state an explicit Plancherel Theorem. The case $G = \underline{K}$ is due to W. Schmid (cf. [28]). See also [11], [12], [26] and [29].

2. Geometric realization of the covering $\widetilde{\Xi}$.

Let $\mathbb{V}$ be a Euclidean Jordan algebra, and let Ω be the associated symmetric cone. We denote the dimension of $\mathbb{V}$ by n, the rank by r, and the unit element by e. A Euclidean Jordan algebra is said to be simple if it has no nontrivial ideal (cf. [12], Chapter II).

Let $\mathcal{D}$ be the unit disc of $\mathbb{V}_{\mathbb{C}} := \mathbb{V} + i\mathbb{V}$ with respect to the spectral norm

$$\mathcal{D} := \{\, z \in \mathbb{V}_{\mathbb{C}} \mid e - z \,\square\, \overline{z} \gg 0 \} ,$$

where $z \,\square\, w := L(zw) + \Big[L(z), L(w)\Big]$. Here $L(z)$ denotes the endomorphism of $\mathbb{V}_{\mathbb{C}}$ defined by $L(z)w = zw$.

Let $\underline{G}$ be the group of holomorphic automorphisms of $\mathcal{D}$ and let $\underline{K}$ be the isotropy subgroup of 0 in $\underline{G}$. It is a maximal compact subgroup of $\underline{G}$. The Lie algebra $\underline{\mathfrak{g}}$ of $\underline{G}$ is consists of vector fields of the form

$$X(z) = w + Tz - P(z)\overline{w},$$

where $w \in \mathbb{V}_{\mathbb{C}}, T \in \underline{\mathfrak{k}} := \mathrm{Lie}(\underline{K})$ and $P(z) := 2L(z)^2 - L(z^2)$. The application P is called a quadratic representation associated with $\mathbb{V}_{\mathbb{C}}$. We identify a vector field X with the triplet $(w, T, \overline{w})$.

Let α be an involutive automorphism of the Jordan algebra $\mathbb{V}$. Denote also α its $\mathbb{C}$-linear extention to $\mathbb{V}_{\mathbb{C}}$. The Jordan algebra $\mathbb{V}$ and its complexification $\mathbb{V}_{\mathbb{C}}$ decompose into eigenspaces with respect to the involution α

$$\mathbb{V} = \mathbb{V}^+ \oplus \mathbb{V}^-, \quad \mathbb{V}_{\mathbb{C}} = \mathbb{V}_{\mathbb{C}}^+ \oplus \mathbb{V}_{\mathbb{C}}^-.$$

We say that the pair $(\mathbb{V}, \alpha)$ is irreducible if it is not possible to write

$$(\mathbb{V}, \alpha) = (\mathbb{V}_1 \oplus \mathbb{V}_2, \alpha_1 \oplus \alpha_2).$$

We show that if $(\mathbb{V}, \alpha)$ is irreducible then either $\mathbb{V}$ is simple, or $\mathbb{V} = \mathbb{V}_\circ \times \mathbb{V}_\circ$ where $\mathbb{V}_\circ$ is a simple Euclidean Jordan algebra and $\alpha(x, y) = (-y, -x)$. We note that $\mathbb{V}^+$ is either simple or a direct sum of two simple algebras.

Let $\{c_1, \dots, c_r\}$ be a Jordan frame of $\mathbb{V}$. It is a complete system of orthogonal primitive idempotent elements. The algebra $R := \bigoplus_{j=1}^{r} \mathbb{R}c_j$ is a maximal associative subalgebra of $\mathbb{V}$. Assume α is given such that $\alpha(R) = R$, then $R = R^+ \oplus R^-$ is the decomposition of R into eigenspaces with respect to α. We note $r^+ := \dim R^+$.

Theorem 2.1 (cf. [4])**.** *Let $\mathbb{V}$ be a Euclidean Jordan algebra and let α be an involutive automorphism of $\mathbb{V}$.*

(1) *The rank of the Euclidean Jordan algebra $\mathbb{V}^+$ is equal to r^+.*
(2) *Either $R = R^+$ and $r = r^+$, or $r = 2r^+$ and* $\dim R^+ = \dim R^-$.

Let

$$\underline{G}^{(-\alpha)} := \{ g \in \underline{G} \;\mid\; (-\alpha) \circ g \circ (-\alpha) = g \} ,$$

and let G be its connected identity component. In particular if $\alpha = \mathrm{id}_{\mathbb{V}}$ then $G = \underline{K}$.

The Lie algebra $\mathfrak{g}$ of G is consists of vector fields X on $\mathbb{V}_{\mathbb{C}}$ such that $(-\alpha)\circ X\circ(-\alpha)=X$. Then $\mathfrak{g}$ is isomorphic to the set of triplets

$$\left\{(w,T,\overline{w})\ \mid\ w\in\mathbb{V}_{\mathbb{C}}^{-},\ T\in\underline{\mathfrak{k}}\text{ and }\alpha\circ T\circ\alpha=T\right\}.$$

We write, for $z\in\mathbb{V}_{\mathbb{C}}$, $j(z):=z^{-1}$ the inverse of z in the Jordan algebra $\mathbb{V}_{\mathbb{C}}$, and τ the conjugation of $\mathbb{V}_{\mathbb{C}}$ with respect to the real form $\mathbb{V}$. The application $\theta: g\mapsto(-j\tau)\circ g\circ(-j\tau)$ is a Cartan involution of the Lie algebra $\mathfrak{g}$ (cf. [3]). Then

$$\mathfrak{k}:=\mathfrak{g}^{\theta}=\{(0,T,0)\ \mid\ T\in\underline{\mathfrak{k}}\text{ and }\theta\circ T\circ\theta=T\},$$
$$\mathfrak{p}:=\mathfrak{g}^{-\theta}=\left\{(w,0,\overline{w})\ \mid\ w\in\mathbb{V}_{\mathbb{C}}^{-}\right\}.$$

Let H be the stabilizer of the base point ie in G,

$$H:=\{g\in G\ \mid\ g\cdot(ie)=ie\}.$$

Proposition 2.1. *The pair (G,H) is a symmetric pair.*

Proof. Let σ be the involution of G defined by

$$\sigma(g)=(-j)\circ g\circ(-j),$$

which commutes with the Cartan involution θ defined before. The differential of σ, also denoted by σ, is given by

$$\sigma(w,T,\overline{w})=(-\overline{w},-T',-w),$$

where T' denote the adjoint of T with respect to the scalar product on $\mathbb{V}$ defined by the trace. By definition of H, its Lie algebra $\mathfrak{h}$ is given by

$$\mathfrak{h}=\left\{(iw,T,iw)\ \mid\ w\in\mathbb{V}^{-},\ T\in\operatorname{Der}(\mathbb{V}^{+})\right\},$$

where $\operatorname{Der}(\mathbb{V}^{+})$ is the derivation algebra of $\mathbb{V}^{+}$. Then

$$\mathfrak{h}=\mathfrak{g}^{\sigma}:=\{X\in\mathfrak{g}\ \mid\ \sigma(X)=X\}.$$

□

The pseudo-Riemannian symmetric space G/H is the open orbit $G\cdot ie$ in the Shilov boundary of $\mathcal{D}$. It is a compactly causal symmetric space. Moreover G/H is a Makarevič symmetric space (cf. [3], [21]). With respect to the involution σ, the Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$ where $\mathfrak{q}:=\mathfrak{g}^{-\sigma}=\{(w,iL(v),w)\ \mid\ w\in\mathbb{V}^{-},\ v\in\mathbb{V}^{+}\}$.

The Lie algebra $\underline{\mathfrak{g}}$ is semisimple and hermitien. By a theorem of Vinberg and Kostant, there is a regular $\underline{G}$-invariant cone (i.e., convex, closed, proper, and with nonempty interior) in $\underline{\mathfrak{g}}$. Let $C_{\max}$ be a maximal regular $\underline{G}$-invariant cone in $\underline{\mathfrak{g}}$ containing $(0,iI,0)$. By [25],

$$\Gamma(C_{\max}):=\underline{G}\exp(iC_{\max})=\left\{g\in\underline{G}_{\mathbb{C}}\ \mid\ g\cdot\overline{\mathcal{D}}\subset\mathcal{D}\right\}.$$

In the complexified space $G_{\mathbb{C}}/H_{\mathbb{C}}$ of G/H we consider the complex domain

$$\Xi:=\Gamma(C^{0})\cdot ie$$

where C^0 is the interior of $C := i\,(C_{\max} \cap \mathfrak{g})$. (This domain is introduced by J. Hilgert, B. Ørsted, and G. Ólafsson in [14].)

The domain Ξ can be realized as $\mathcal{D} \setminus \Sigma$ where Σ is the analytic set given by

$$\Sigma = \{z \in \mathcal{D} \mid \det(P(z + \alpha z)) = 0\}, \tag{1}$$

where the notation "det" denotes the determinant with respect to $\mathbb{V}$ (cf. [3]).

The domain Ξ can also be realized as a subset of the imaginary tangent bundle of G/H

$$\Xi \simeq G \times C^{\mathfrak{q}}/\sim,$$

where $C^{\mathfrak{q}} := C^0 \cap i\mathfrak{q}$, and $G \times C^{\mathfrak{q}}/\sim$ is the quotient of $G \times C^{\mathfrak{q}}$ by the equivalence relation: $(g_1, X_1) \sim (g_2, X_2)$ if and only if there exists $h \in H$ such that

$$g_2 = g_1 h \quad \text{and} \quad X_2 = \mathrm{Ad}(h^{-1})X_1, \tag{2}$$

(cf. [14]).

The open set Ξ is connected since it is homeomorphic to $\mathcal{D} \setminus \Sigma$, observing that $\mathcal{D}$ is connected and $\mathrm{codim}_{\mathbb{R}}(\Sigma) = 2$.

Let

$$\widetilde{\Xi} := \left\{(z, \zeta) \in \Xi \times \mathbb{C} \mid e^{\frac{2n}{r^+}\zeta} = \det(P(z + \alpha z))\right\}.$$

Note that $\dfrac{2n}{r^+}$ is an integer.

Theorem 2.2. *The set $\widetilde{\Xi}$ is a connected covering of infinite order of the domain Ξ.*

Proof. Let p be the map defined by

$$p : \widetilde{\Xi} \longrightarrow \Xi,$$
$$(z, \zeta) \mapsto z.$$

Then p is surjective. In fact for $z \in \Xi$, we have $\det(P(z + \alpha z)) \neq 0$, then there exists $\zeta \in \mathbb{C}$ such that $e^{\frac{2n}{r^+}\zeta} = \det(P(z + \alpha z))$. Let $z_0 \in \Xi$, we can find an open neighbourhood U of z_0 such that $p^{-1}(U)$ is homeomorphic to $U \times \mathbb{Z}$. In fact, since p is surjective, there exists $(z_0, \zeta_0) \in \widetilde{\Xi}$ such that $p(z_0, \zeta_0) = z_0$. We consider a determination of $\log\Big(\det(P(z + \alpha z))\Big)$ in the neighbourhood U of z_0, we can define a homeomorphism of $U \times \mathbb{Z}$ in $p^{-1}(U)$ as

$$(z, m) \mapsto \Big(z,\ \log\left(\det(P(z + \alpha z))\right) + 2\pi i m\Big).$$

Hence $\widetilde{\Xi}$ is a covering of infinite order of Ξ.

Let $\{e_1, e_2, \dots, e_{r^+}\}$ be a Jordan frame of $\mathbb{V}^+$. An element z of the form $z = \sum_{j=1}^{r^+} z_j e_j$ belongs to Ξ if and only if $0 < |z_j| < 1$. Let

$$z(t) = \sum_{j=1}^{r^+-1} e_j + e^{2\pi i t} e_{r^+} \in \overline{\Xi},$$

and

$$z_0 = \sum_{j=1}^{r^+} z_j e_j \in \Xi.$$

The curve $\varphi(t) := z(t) z_0$ belongs to Ξ and satisfies $\varphi(0) = \varphi(1) = z_0$. Let $\widetilde{\varphi}$ be the lifting of φ to $\widetilde{\Xi}$,

$$\begin{aligned} \widetilde{\varphi} : [0,1] &\longrightarrow \widetilde{\Xi}, \\ t &\mapsto (\varphi(t), \zeta(t)). \end{aligned}$$

Using the fact that $e_j = c_j$ if $r = r^+$ and $e_j = c_j + c_{j+r^+}$ if $r = 2r^+$, for all $1 \leq j \leq r^+$, we deduce that

$$\begin{aligned} e^{\frac{2n}{r^+}\zeta(t)} &= \det(P(z(t) + \alpha z(t))) \\ &= C(z_1, \dots, z_{r^+}) \left(e^{2\pi i t}\right)^{\frac{2n}{r^+}}, \end{aligned}$$

where $C(z_1, \dots, z_{r^+})$ is a nonzero constant depending on $z_1, \dots, z_{r^+}$. There exists $\hbar \in \mathbb{C}^*$ such that $C(z_1, \dots, z_{r^+}) = e^{\hbar}$ and $\frac{2n}{r^+}(\zeta(t) - 2\pi i t) = \hbar + 2\pi i \kappa(t)$. Here $\kappa(t)$ is an integer valued continuous function on $[0,1]$, therefore constant. Thus $\zeta(1) - \zeta(0) = 2\pi i$ and if $\widetilde{\varphi}(0) = (z_0, \zeta_0)$, then $\widetilde{\varphi}(1) = (z_0, \zeta_0 + 2\pi i)$. Thus if z_0 is an element of $\Xi \cap \bigoplus_{j=1}^{r^+} \mathbb{C} e_j$ and if (z_0, ζ_0^1) and (z_0, ζ_0^2) are two points of $\widetilde{\Xi}$, there exists a curve $\widetilde{\varphi}$ such that $\widetilde{\varphi}(0) = (w_0, \zeta_0^1)$ and $\widetilde{\varphi}(1) = (z_0, \zeta_0^2)$.

Let (z_1, ζ_1) and (z_2, ζ_2) be two points of $\widetilde{\Xi}$. Since Ξ is connected, there exists a curve φ_1 (resp. φ_2) such that

$$\varphi_1(0) = z_0, \quad \varphi_1(1) = z_1 \quad (\text{resp.} \quad \varphi_2(0) = z_0, \quad \varphi_2(1) = z_2).$$

Let $\widetilde{\varphi_1}$ (resp. $\widetilde{\varphi_2}$) be the lifting of φ_1 (resp. φ_2) to $\widetilde{\Xi}$ such that

$$\begin{aligned} \widetilde{\varphi_1}(0) &= (z_0, \zeta_0^1) \qquad & \widetilde{\varphi_1}(1) &= (z_1, \zeta_1) \\ \widetilde{\varphi_2}(0) &= (z_0, \zeta_0^2) \qquad & \widetilde{\varphi_2}(1) &= (z_2, \zeta_2). \end{aligned}$$

Using the fact that $z_0 \in \Xi \cap \bigoplus_{j=1}^{r^+} \mathbb{C} e_j$, we deduce that $\widetilde{\Xi}$ is connected. $\square$

Let

$$\top_\alpha(z) = \det(P(z+\alpha z)).$$

With respect to Lebesgue measure, the restriction to Ξ of the $G_{\mathbb{C}}$-invariant measure of $G_{\mathbb{C}}/H_{\mathbb{C}}$ is given by

$$d\xi = \frac{d\lambda(z)}{|\top_\alpha(z)|}.$$

Let

$$\top_\alpha(\widetilde{z})^{\frac{r}{4n}\nu} := e^{\frac{r}{2r^+}\nu\zeta}, \qquad \widetilde{z} = (z,\zeta) \in \widetilde{\Xi}. \tag{3}$$

The realization of the domain Ξ inside the imaginary tangent bundle of G/H will be used below which permits to show an integral formula.

Let $\mathfrak{a}$ be a Cartan subalgebra in $\mathfrak{k}\cap\mathfrak{q}$. Let $\triangle$ be the root system $\triangle(\mathfrak{g}_{\mathbb{C}}, i\mathfrak{a})$, and $\mathfrak{a}^+$ a positive Weyl chamber. Let $\triangle^+$ be the positive root system with respect to $i\mathfrak{a}^+$.

Theorem 2.3. *For an integrable function f on Ξ,*

$$\int_\Xi f(\xi)d\xi = c_0 \int_G \int_{C^+} f\left(g\exp(X)\cdot ie\right) \prod_{\beta\in\triangle^+} (\mathrm{sh}\,\langle\beta, 2X\rangle)^{m_\beta} dgdX,$$

where $C^+ := C^0 \cap i\mathfrak{a}^+$ and $m_\beta = \dim(\mathfrak{g}_\beta)$.

Proof. Let $Z := Z_H(i\mathfrak{a})$ be the centralizer subgroup of $i\mathfrak{a}$ in H. The map

$$\begin{aligned} \varphi: \quad G/Z\times C^+ &\longrightarrow \Xi, \\ (g\cdot Z, X) &\mapsto g\exp(X)\cdot ie \end{aligned}$$

is a diffeomorphism onto its open image. In fact, let $g_1\exp(X_1)\cdot ie$ and $g_2\exp(X_2)\cdot ie$ be two elements of Ξ such that $g_1\exp(X_1)\cdot ie = g_2\exp(X_2)\cdot ie$. Since the group G acts on $G/Z\times C^+$ (resp. Ξ) by

$$g_0\cdot(g\cdot Z, X) = (g_0 g\cdot Z, X), \quad \Big(\text{resp. } g_0\cdot(g\exp(X)\cdot ie) = g_0 g\exp(X)\cdot ie\Big),$$

we may assume that $g_1 = 1$. But since X_1 and X_2 are regular and in the same positive Weyl chamber, by the equivalence relation (2), we deduce that $X_1 = X_2$. Thus establishing injectivity.

We will compute the differential of φ. For this we consider the commutative diagram defined by

$$\begin{array}{ccc} G\times C^+ & \xrightarrow{\Phi} & G^{\mathbb{C}} \\ \phi\downarrow & & \downarrow\Psi \\ G/Z\times C^+ & \xrightarrow{\psi} & G^{\mathbb{C}}/H^{\mathbb{C}} \end{array}$$

$$\begin{array}{ccc} (g,X) & \xrightarrow{\Phi} & g\exp(X) \\ \phi\downarrow & & \downarrow\Psi \\ (g\cdot Z,X) & \xrightarrow{\psi} & g\exp(X)\cdot H_{\mathbb{C}}. \end{array}$$

Then $\mathrm{d}(\psi\circ\phi)(1,X)=\mathrm{d}(\Psi\circ\Phi)(1,X)$.

Let λ_g (resp. Λ_g) be the left translation by g in G (resp. $G_{\mathbb{C}}/H_{\mathbb{C}}$), then for $(Y,U)\in\mathfrak{g}\times i\mathfrak{a}$,

$$\begin{aligned} &\mathrm{d}(\Psi\circ\Phi)(1,X)(Y,U)\\ &=\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\Psi\Big(\exp(tY)\exp(X+tU)\Big)\\ &=\mathrm{d}\Psi(\exp(X)).\mathrm{d}\lambda_{\exp(X)}(1).\left(e^{-\mathrm{ad}(X)}.Y+\frac{1-e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}.U\right)\\ &=\mathrm{d}\Lambda_{\exp(X)}\Psi(1).\mathrm{d}\Psi(1).\Big(e^{-\mathrm{ad}(X)}.Y+U\Big), \end{aligned}$$

where $\mathrm{d}\Psi(1)$ sends $\mathfrak{g}_{\mathbb{C}}$ onto $\mathfrak{q}_{\mathbb{C}}$. Note $\mathrm{P}_{\mathfrak{q}_{\mathbb{C}}}$ the projection of $\mathfrak{g}_{\mathbb{C}}$ onto $\mathfrak{q}_{\mathbb{C}}$ along $\mathfrak{h}_{\mathbb{C}}$, then

$$\begin{aligned} &\mathrm{P}_{\mathfrak{q}_{\mathbb{C}}}\Big(e^{-\mathrm{ad}(X)}Y+U\Big)\\ &=\frac{e^{-\mathrm{ad}(X)}Y+U-e^{\mathrm{ad}(X)}\sigma(Y)+U}{2}\\ &=\frac{e^{-\mathrm{ad}(X)}Y-e^{\mathrm{ad}(X)}\sigma(Y)}{2}+U\\ &=\mathrm{sh}\,(-\mathrm{ad}(X))\left(\frac{Y+\sigma(Y)}{2}\right)+\mathrm{ch}\,(\mathrm{ad}(X))\left(\frac{Y-\sigma(Y)}{2}\right)+U. \end{aligned}$$

Using the fact that $\sigma(\mathfrak{g}_\beta)=\mathfrak{g}_{-\beta}$, the Lie algebra $\mathfrak{g}$ can be written as

$$\mathfrak{g}=\mathfrak{z}\oplus\sum_{\beta\in\triangle^+}(1+\sigma)\mathfrak{g}_\beta\oplus\mathfrak{a}\oplus\sum_{\beta\in\triangle^+}(1-\sigma)\mathfrak{g}_\beta,$$

where $\mathfrak{z} := \mathrm{Lie}(Z)$. Then for all $Y \in \mathfrak{g}$ and $Y_\beta \in \mathfrak{g}_\beta$,

$$\mathrm{P}_{\mathfrak{q}_{\mathbb{C}}}\Big(e^{-\mathrm{ad}(X)}Y+U\Big) = \mathrm{sh}\,(-\beta(X))\;(Y_\beta+\sigma(Y_\beta))+\mathrm{ch}\,(\beta(X))\;(Y_\beta-\sigma(Y_\beta))+U.$$

Let ω be the volume form on Ξ which defines an invariant Haar measure on Ξ. Again, the volume form $\varphi^*\omega$ on $G/Z \times C^+$ is given by

$$\varphi^*\omega = c_0 \prod_{\beta\in\triangle^+} (\mathrm{sh}\,2\beta(X))^{m_\beta}\;\omega_1 \otimes \omega_2,$$

where ω_1 is a volume form on G/Z which defines an invariant measure, and ω_2 is a volume form on $i\mathfrak{a}$ which defines a Lebesgue measure. Using the fact that Z is compact, the integral formula holds. □

3. Weighted Bergman spaces and reproducing kernels.

In this section we introduce the Bergman space $\mathcal{H}^2_\nu(\widetilde{\Xi})$ associated with the covering $\widetilde{\Xi}$. We establish a unitary isomorphism of $\mathcal{H}^2_\nu(\mathcal{D})$ on $\mathcal{H}^2_\nu(\widetilde{\Xi})$. Then we compute the explicit expression of the reproducing kernel of $\mathcal{H}^2_\nu(\widetilde{\Xi})$.

For a real ν, let $\mathcal{O}_\nu(\widetilde{\Xi})$ be the space of holomorphic functions F on $\widetilde{\Xi}$ which satisfy

$$F(z,\zeta+2\pi i) = e^{2\pi i\frac{r}{2r^+}\nu}F(z,\zeta).$$

This condition will be called a monodromy condition. Remark that the function $\top_\alpha{}^{\frac{r}{4n}\nu}$ belongs to $\mathcal{O}_\nu(\widetilde{\Xi})$.

For $\nu > \dfrac{2n}{r} - 1$, let $\mathcal{H}^2_\nu(\widetilde{\Xi})$ be the Hilbert space of functions $F \in \mathcal{O}_\nu(\widetilde{\Xi})$ such that

$$\|F\|^2_\nu := \int_\Xi |F(\widetilde{\xi})|^2\; p_\nu(\xi)d\xi < \infty,$$

where

$$p_\nu(\xi) = \det(B(\xi,\overline{\xi}))^{\frac{r}{2n}\nu-1}|\top_\alpha(\xi)|^{-\frac{r}{2n}\nu+1},$$

and $B(z,w)$ is the Bergman operator defined by $B(z,w) := \mathrm{id}_{\mathbb{V}_{\mathbb{C}}} - 2z \,\square\, w + P(z)P(w)$.

Proposition 3.1. *Let z and w be two invertible elements of $\mathbb{V}_{\mathbb{C}}$. Thus*

$$\det(B(z,w)) = \Delta(z)^{\frac{2n}{r}}\Delta(z^{-1}-w)^{\frac{2n}{r}},$$

where Δ is the determinant polynomial associated with $\mathbb{V}$.

Proof. By definition $\det(B(z,w)) = \det(\mathrm{id}_{\mathbb{V}_{\mathbb{C}}} - 2z\square w + P(z)P(w))$. According to [**18**] Proposition 4.13,

$$z \,\square\, w = P(w^{-1},z)P(w),$$

where $P(z,w) = \dfrac{1}{2}\Big(P(z+w) - P(z) - P(w)\Big)$. Then

$$\mathrm{id}_{\mathbb{V}_{\mathbb{C}}} - 2z \,\square\, w + P(z)P(w) = \mathrm{id}_{\mathbb{V}_{\mathbb{C}}} - 2P(w^{-1},z)P(w) + P(z)P(w).$$

Moreover

$$\begin{aligned} P(w^{-1}-z) &= P(w)^{-1} + 2P(w^{-1}, -z) + P(z) \\ &= P(w^{-1}) - 2P(w^{-1}, z) + P(z). \end{aligned}$$

Hence we deduce the following equalities

$$-2P(w^{-1}, z)P(w) = P(w^{-1}-z)P(w) - P(z)P(w) - \mathrm{id}_{V_{\mathbb{C}}},$$

and

$$\mathrm{id}_{\mathbb{V}_{\mathbb{C}}} - 2z \,\square\, w + P(z)P(w) = P(w^{-1}-z)P(w).$$

Then, we have

$$\det(\mathrm{id}_{\mathbb{V}_{\mathbb{C}}} - 2z \,\square\, w + P(z)P(w)) = \det(P(w^{-1}-z))\det(P(w)),$$

where $\det(P(w)) = \Delta(w)^{\frac{2n}{r}}$. Finally

$$\det(B(z,w)) = \Delta(w)^{\frac{2n}{r}} \Delta(w^{-1}-z)^{\frac{2n}{r}} = \Delta(z)^{\frac{2n}{r}} \Delta(z^{-1}-w)^{\frac{2n}{r}}.$$

□

The universal covering $\widetilde{G}$ of G can be realized as the set of pairs (g,φ) with $g \in G$ and φ a holomorphic function on $\mathcal{D}$ defined by

$$e^{\varphi(z)} = \det(Dg(z)),$$

where $Dg(z)$ is the differential of the map $z \mapsto g\cdot z$. The product on $\widetilde{G}$ is given by

$$(g_1,\varphi_1)\cdot(g_2,\varphi_2) = (g_1g_2,\varphi_3),$$

where $\varphi_3(z) = \varphi_1(g_2\cdot z) + \varphi_2(z)$. For $\widetilde{g} = (g,\varphi) \in \widetilde{G}$, and $\kappa \in \mathbb{R}$, we will write

$$\det(Dg(z))^{\kappa} := e^{\kappa\varphi(z)}.$$

Let $\widetilde{\Gamma(C)} := \widetilde{G}\,\widetilde{\exp}(C)$ be the semigroup associated with the covering $\widetilde{G}$ where $\widetilde{\exp} : \mathfrak{g} \to \widetilde{G}$. We denote by $\widetilde{\Gamma(C^0)}$ the interior of $\widetilde{\Gamma(C)}$. The linear action of $\widetilde{G}$ on the space $\mathcal{H}^2_\nu(\widetilde{\Xi})$ is given by

$$(\pi_0(\widetilde{g})F)(\widetilde{\xi}) = F(g\cdot\widetilde{\xi}), \qquad \widetilde{g}^{-1} = (g,\varphi),$$

where $g\cdot\widetilde{\xi} = (g\cdot\xi,\zeta')$ and $e^{\frac{2n}{r^+}\zeta'} = \det(P(g\cdot\xi + \alpha(g\cdot\xi)))$ (cf. [**19**] Lemma 5.1).

The representation π_0 extends to a continuous representation of $\widetilde{\Gamma(-C)}$ and a holomorphic one of $\widetilde{\Gamma(-C^0)}$ (cf. [**25**]).

We recall that the Bergman space $\mathcal{H}^2_\nu(\mathcal{D})$ is the Hilbert space of holomorphic functions f on $\mathcal{D}$ such that

$$\|f\|^2 = \int_{\mathcal{D}} |f(z)|^2 \det\left(B(z,\overline{z})\right)^{\frac{r}{2n}\nu - 1} d\lambda(z) < \infty,$$

where λ denote the Lebesgue measure (cf. [12]). The action of $\widetilde{G}$ on $\mathcal{H}^2_\nu(\mathcal{D})$ is given by

$$(\pi_\nu(\widetilde{g})f)(z) = e^{\frac{r}{2n}\nu\varphi(z)} f(g\cdot z), \qquad \widetilde{g}^{-1} = (g,\varphi).$$

The unitary representation π_ν extends to a continuous representation of $\widetilde{\Gamma(-C)}$ and a holomorphic one of $\widetilde{\Gamma(-C^0)}$.

Let $\mathcal{A}_\nu$ be the operator given by

$$\begin{aligned}\mathcal{A}_\nu : \ \mathcal{H}^2_\nu(\mathcal{D}) &\longrightarrow \mathcal{O}_\nu(\widetilde{\Xi}),\\ f \quad &\mapsto \mathcal{A}_\nu(f) = \top_\alpha{}^{\frac{r}{4n}\nu}\, f.\end{aligned}$$

Since $\top_\alpha{}^{\frac{r}{4n}\nu} \in \mathcal{O}_\nu(\widetilde{\Xi})$, the operator $\mathcal{A}_\nu$ is well defined.

Theorem 3.1. *The operator $\mathcal{A}_\nu$ is a unitary isomorphism of $\mathcal{H}^2_\nu(\mathcal{D})$ onto $\mathcal{H}^2_\nu(\widetilde{\Xi})$ intertwining the representations π_ν and π_0.*

Proof. Since Σ is an analytic set of measure zero and $d\xi = \dfrac{d\lambda(z)}{|\top_\alpha(z)|}$,

$$\begin{aligned}\|\mathcal{A}_\nu(f)\|^2_\nu &= \int_\Xi |\mathcal{A}_\nu(f)(\widetilde{\xi})|^2\, p_\nu(\xi)d\xi\\ &= \int_\Xi |f(\xi)|^2 |\top_\alpha(\widetilde{\xi})|^{\frac{r}{2n}\nu} \det(B(\xi,\overline{\xi}))^{\frac{r}{2n}\nu-1} \frac{d\xi}{|\top_\alpha(\widetilde{\xi})|^{\frac{r}{2n}\nu-1}}\\ &= \int_\mathcal{D} |f(z)|^2\ \det(B(z,\overline{z}))^{\frac{r}{2n}\nu-1} d\lambda(z) = \|f\|^2.\end{aligned}$$

If f belongs to $\mathcal{H}^2_\nu(\mathcal{D})$ then $\mathcal{A}_\nu(f) \in \mathcal{O}_\nu(\widetilde{\Xi})$ and

$$\int_\Xi |\mathcal{A}_\nu f(\widetilde{\xi})|^2\, p_\nu(\xi)d\xi < \infty.$$

Hence the image of $\mathcal{A}_\nu$ is containd in $\mathcal{H}^2_\nu(\widetilde{\Xi})$ and $\mathcal{A}_\nu$ is isometric.

Moreover $\mathcal{A}_\nu$ is surjective. In fact, let $F \in \mathcal{H}^2_\nu(\widetilde{\Xi})$, then in particular $F \in \mathcal{O}_\nu(\widetilde{\Xi})$. Since $\det(P(\widetilde{z}+\alpha\widetilde{z}))^{-\frac{r}{4n}\nu} \in \mathcal{O}_{-\nu}(\widetilde{\Xi})$, the function

$$f(z) := \top_\alpha(\widetilde{z})^{-\frac{r}{4n}\nu} F(\widetilde{z}), \qquad \widetilde{z} = (z,\zeta) \in \widetilde{\Xi},$$

is holomorphic on $\Xi = \mathcal{D}\setminus\Sigma$. Moreover, the function f belongs to $\mathcal{H}^2_\nu(\mathcal{D})$. In fact

$$\begin{aligned}\|f\|^2 &= \int_\mathcal{D} |F(\widetilde{z})|^2\ |\top_\alpha(z)|^{-\frac{r}{2n}\nu} \det(B(z,\overline{z}))^{\frac{r}{2n}\nu-1} d\lambda(z)\\ &= \int_\Xi |F(\widetilde{\xi})|^2 |\top_\alpha(\xi)|^{-\frac{r}{2n}\nu+1} \det(B(\xi,\overline{\xi}))^{\frac{r}{2n}\nu-1} d\xi = \|F\|^2_\nu < \infty.\end{aligned}$$

Then f is a holomorphic function on $\mathcal{D}\setminus\Sigma$ and belongs to $L^2_\nu(\mathcal{D})$. Hence f extends to a holomorphic function on $\mathcal{D}$. This is the content of the following lemma.

Lemma 3.2 (cf. [6], [27]). *Let U be a domain in $\mathbb{C}^n$ and let A be an analytic set such that* $\operatorname{codim}_{\mathbb{R}}(A) \geq 1$. *If $f \in \mathcal{O}(U \setminus A)$ and if $f \in L^2(U)$, then f extends to a holomorphic function on U.*

It remains to show that $\mathcal{A}_\nu$ intertwining the representations π_0 and π_ν. In fact it follows from [3] 1.3 (9) that $\top_\alpha(g \cdot \widetilde{\xi})^{\frac{r}{4n}\nu} = e^{\frac{r}{2n}\nu\varphi(\xi)} \top_\alpha(\widetilde{\xi})^{\frac{r}{4n}\nu}$. Hence

$$\begin{aligned}
\mathcal{A}_\nu(\pi_\nu(\widetilde{g})f)(\widetilde{\xi}) &= \top_\alpha(\widetilde{\xi})^{\frac{r}{4n}\nu}\, e^{\frac{r}{2n}\nu\varphi(\xi)} f(g\cdot\xi) \qquad \Big(\widetilde{g}^{-1} = (g,\varphi)\Big) \\
&= e^{-\frac{r}{2n}\nu\varphi(\xi)} \top_\alpha(g\cdot\widetilde{\xi})^{\frac{r}{4n}\nu}\, e^{\frac{r}{2n}\nu\varphi(\xi)} f(g\cdot\xi) \\
&= (\pi_0(\widetilde{g})\mathcal{A}_\nu f)(\widetilde{\xi}).
\end{aligned}$$

□

Proposition 3.3. *The reproducing kernel of the Bergman space $\mathcal{H}^2_\nu(\widetilde{\Xi})$ is equal to*

$$K_\nu(\widetilde{\xi}_1, \widetilde{\xi}_2) = c_\nu \top_\alpha(\widetilde{\xi}_1)^{\frac{r}{4n}\nu} \det(B(\xi_1, \overline{\xi}_2))^{-\frac{r}{2n}\nu}\, \overline{\top_\alpha(\widetilde{\xi}_2)}^{\frac{r}{4n}\nu},$$

where c_ν is the positive constant

$$c_\nu = \frac{1}{\pi^n} \prod_{j=1}^{r} \frac{\Gamma\Big(\nu - (j-1)\frac{n-r}{r(r-1)}\Big)}{\Gamma\Big(\nu - \frac{n}{r} - (j-1)\frac{n-r}{r(r-1)}\Big)}. \tag{4}$$

The definition of $\overline{\top_\alpha(\widetilde{\xi}_2)}^{\frac{r}{4n}\nu}$ is similar to that given on (3).

Proof. The reproducing kernel of $\mathcal{H}^2_\nu(\mathcal{D})$ is given by

$$K^{\mathcal{D}}_\nu(z, z') = c_\nu \det(B(z, \overline{z}'))^{-\frac{r}{2n}\nu}.$$

From the definition of $\mathcal{A}_\nu$, the reproducing kernel of $\mathcal{H}^2_\nu(\widetilde{\Xi})$ is equal to

$$\begin{aligned}
K_\nu(\widetilde{\xi}, \widetilde{\xi}') &= \top_\alpha(\widetilde{\xi})^{\frac{r}{4n}\nu} K^{\mathcal{D}}_\nu(\xi, \xi') \overline{\top_\alpha(\widetilde{\xi}')}^{\frac{r}{4n}\nu} \\
&= c_\nu \top_\alpha(\widetilde{\xi})^{\frac{r}{4n}\nu} \det(B(\xi, \overline{\xi'}))^{-\frac{r}{2n}\nu} \overline{\top_\alpha(\widetilde{\xi}')}^{\frac{r}{4n}\nu}.
\end{aligned}$$

□

4. Holomorphic discrete series of $\widetilde{G}$.

Recall that $\{e_1, \ldots, e_{r^+}\}$ is the Jordan frame of R^+ and $\mathfrak{k} \cap \mathfrak{q} = \Big\{(0, iL(v), 0) \,\Big|\, v \in \mathbb{V}^+\Big\}$.

Let $\mathfrak{a}$ be the Cartan subalgebra in $\mathfrak{k} \cap \mathfrak{q}$ defined by

$$\mathfrak{a} = \left\{ \left(0, i\sum_{j=1}^{r^+} t_j L(e_j), 0\right) \,\middle|\, t_j \in \mathbb{R} \right\}.$$

We denote by $\triangle$ the root system $\triangle(\mathfrak{g}_{\mathbb{C}}, i\mathfrak{a})$, $\triangle^+$ the positive system with respect to the positive Weyl chamber $(i\mathfrak{a})^+$ defined by

$$(i\mathfrak{a})^+ = \left\{ \left(0, \sum_{j=1}^{r^+} t_j L(e_j), 0\right) \;\middle|\; 0 < t_1 < \cdots < t_{r^+} \right\}. \tag{5}$$

Let $X_0 := (0, I, 0) \in \mathfrak{g}_{\mathbb{C}}$. The eigenvalues of $\mathrm{ad}(X_0)$ are $1, 0$, and -1. Let $\triangle_0 := \{\alpha \in \triangle^+ \mid \alpha(X_0) = 0\}$ and $\triangle_1 := \{\alpha \in \triangle^+ \mid \alpha(X_0) = 1\}$. Then $\triangle^+ = \triangle_0 \cup \triangle_1$. The roots belonging to $\triangle_0$ are called compact and the roots belonging to $\triangle_1$ noncompact. Let $\rho := \frac{1}{2} \sum_{\alpha \in \triangle^+} m_\alpha \alpha$ be one half of the positive roots weighted by the dimension m_α of the root spaces. For the description and computation of ρ we refer to [**4**]. See also [**8**].

Let π be a unitary representation of the Lie group G on a Hilbert space $\mathcal{H}$, and let C be an invariant and regular cone in $i\mathfrak{g}$. The representation π is called C-positive if for all $X \in C$ and for all $\mathcal{C}^\infty$ vector v,

$$\frac{\mathrm{d}}{\mathrm{d\,t}}_{|\mathrm{t}=0} \langle\, \pi(\exp(tX))\, v \,|\, v \,\rangle \leq 0.$$

Let $\mathcal{R}$ be the set of the weights $\mu = (\mu_1, \mu_2, \ldots, \mu_{r^+}) \in \mathbb{R}^{r^+}$ such that

$$\mu_i - \mu_{i+1} \in \mathbb{N}, \qquad 1 \leq i \leq r^+ - 1.$$

(If $\mathbb{V}^+$ is a direct sum of two simple algebras with ranks p and q such that $p + q = r^+ (= r)$, then $i \neq p$.)

For $\mu \in \mathcal{R}$ and β a noncompact positive root. By [**14**], the "Harish-Chandra" condition $\langle \rho - \mu, \beta \rangle \leq 0$ can be written as

$(\star)$	$\mathbb{V}^+$ is simple	$\mu_{r^+} > \frac{n}{2r} - \frac{d+1}{8}$	if	$r = r^+$, $\left(d := \frac{2(n-r)}{r(r-1)}\right)$
		$\mu_{r^+} > \frac{n}{r} - \frac{1}{2}$	if	$r = 2r^+$.
$(\star\star)$	$\mathbb{V}^+$ is not simple	$\mu_1 + \mu_{r^+} > -2dp$	where	$r = r^+$.

If $\mu \in \mathcal{R}$ and satisfies the "Harish-Chandra" condition, then we can associate to μ a unitary and C-positive representation $(\pi_\mu, \mathcal{W}_\mu)$ of $\widetilde{G}$ with highest weight μ. This representation extends to a continuous representation of $\widetilde{\Gamma(C)}$ which is holomorphic on $\widetilde{\Gamma(C^0)}$.

Let $A := \exp \mathfrak{a}$, and $\mathfrak{g}_+^{\mathbb{C}} := \sum_{\beta \in \triangle^+} \mathfrak{g}_\beta^{\mathbb{C}}$.

Definition 4.1. A holomorphic function Φ in $\widetilde{\Xi}$ will be called a conical function if there exists a continuous character χ_μ of A such that

$$\begin{aligned} \mathcal{I}(a)\Phi &= \chi_\mu(a)\Phi, \quad (a \in A), \\ d\mathcal{I}(X)\Phi &= 0, \quad (X \in \mathfrak{g}_+^{\mathbb{C}},) \end{aligned}$$

where $(\mathcal{I}(g)F)(\widetilde{\xi}) = F(g^{-1} \cdot \widetilde{\xi})$.

For all $\mathbf{s} = (s_1, s_2, \dots, s_r) \in \mathbb{C}^r$ and z in $\mathbb{V}_{\mathbb{C}}$, we write

$$\Delta_{\mathbf{s}}(z) := \Delta_1(z)^{s_1-s_2}\, \Delta_2(z)^{s_2-s_3}\ \dots\ \Delta_r(z)^{s_r},$$

where Δ_j is the principal minor of order j (cf. [12]).

For $\mu = (\mu_1, \mu_2, \dots, \mu_{r^+})$, let,

$$\Phi_\mu(\widetilde{z}) := \Delta_\mu\left(\frac{\widetilde{z} + \alpha(\widetilde{z})}{2}\right).$$

The function Φ_μ satisfies the monodromy condition.

Proposition 4.2. *The function Φ_μ is conical, and any conical function is proportional to Φ_μ.*

The proof is similar to that given for Proposition XI.2.1 in [12].

Note that $\mathcal{W}_\mu^{\infty}$ (resp. $\mathcal{W}_\mu^{-\infty}$) is the vector space of $\mathcal{C}^\infty$ (resp. distribution) vectors of $\mathcal{W}_\mu$, and $(\mathcal{W}_\mu^{-\infty})^H$ the vector space of H-invariant distribution vectors of $\mathcal{W}_\mu$. Let $\mathcal{R}_H$ be the subset of highest weight $\mu \in \mathcal{R}$ such that $(\mathcal{W}_\mu^{-\infty})^H \neq \{0\}$.

For $\mu \in \mathcal{R}_H$, we denote ψ_μ an H-invariant distribution vector. For all element $w \in \mathcal{W}_\mu$, the holomorphic mapping $\mathcal{F} : \mathcal{W}_\mu \longrightarrow \mathcal{O}(\widetilde{\Xi})$, $w \mapsto \mathcal{F}(w)(\widetilde{\xi}) := \langle \pi_\mu(\widetilde{\gamma}_1^{-1})\, w | \psi_\mu\rangle$ where $\widetilde{\xi} = \widetilde{\gamma} \cdot H$, is a continuous embedding. Then the representation π_μ is realized on a Hilbert space $\mathcal{H}_\mu$ of holomorphic functions on $\widetilde{\Xi}$. In the case where $w = v_\mu$, a normalized highest weight vector, we denote $\mathcal{F}_\mu(\widetilde{\xi}) := \langle \pi_\mu(\widetilde{\gamma}^{-1}) v_\mu \,|\, \psi_\mu\rangle$. The function $\mathcal{F}_\mu$ is a conical function.

Let $\mathcal{R}_{\nu,H}$ be the highest weight subset of $\mathcal{R}_H$ such that $\mathcal{F}_\mu \in \mathcal{O}_\nu(\widetilde{\Xi})$.

Proposition 4.3. *The function $\mathcal{F}_\mu$ satisfies the monodromy condition, i.e., $\mathcal{F}_\mu \in \mathcal{O}_\nu(\widetilde{\Xi})$, if and only if*

$$\mu_i \in \mathbb{Z} + \frac{\nu}{2}, \quad (1 \le i \le r^+) \quad \textit{if}\ \ r = r^+$$

and

$$\mu_i \in \mathbb{Z} + \nu, \quad (1 \le i \le r^+) \qquad \textit{if}\ \ r = 2r^+.$$

Proof. Since $\mathcal{F}_\mu$ is conical, then it is proportional to Φ_μ. If $\mathbb{V}^+$ is simple, then $\top_\alpha$ is proportional to the Jordan determinant Δ of $\mathbb{V}^+$. In fact, $\top_\alpha$ is homogeneous of degree $2n$ and Δ is homogeneous of degree r^+, then $\top_\alpha(\widetilde{z}) = \Delta(\widetilde{z} + \alpha(\widetilde{z}))^{\frac{2n}{r^+}}$ and $\mathcal{F}_\mu \in \mathcal{O}_\nu(\widetilde{\Xi})$ if and only if $\mu_{r^+} \in \mathbb{Z} + \dfrac{r}{2r^+}\nu$. Using the fact that $\mu_i - \mu_{i+1} \in \mathbb{N}$, the result holds for $\mathbb{V}^+$ simple.

If $\mathbb{V}^+$ is a direct sum of two simple Jordan algebras $\mathbb{V}_1^+$ of rank p and $\mathbb{V}_2^+$ of rank q such that $p+q = r^+(=r)$, then there exist $z_1 \in \mathbb{V}_{1,\mathbb{C}}^+$ and $z_2 \in \mathbb{V}_{2,\mathbb{C}}^+$ such that $z + \alpha(z) = z_1 + z_2$ and $\top_\alpha(\widetilde{z}) = \Delta^{(1)}(\widetilde{z}_1)^{\frac{2n}{r}}\Delta^{(2)}(\widetilde{z}_2)^{\frac{2n}{r}}$ where $\Delta^{(1)}$ (resp. $\Delta^{(2)}$) is the Jordan determinant of $\mathbb{V}_1^+$ (resp. $\mathbb{V}_2^+$). Hence $\mathcal{F}_\mu \in \mathcal{O}_\nu(\widetilde{\Xi})$

if and only if $\mu_p \in \mathbb{Z} + \frac{\nu}{2}$ and $\mu_r \in \mathbb{Z} + \frac{\nu}{2}$. The assertion follows from the fact that $\mu_i - \mu_{i+1} \in \mathbb{N}$ for all $i \neq p$. □

Remark. In [**1**] we consider the case $G/H \simeq U(p,q)$ and we establish another isomorphism between $\mathcal{H}^2_\nu(\mathcal{D})$ and $\mathcal{H}^2_\nu(\widetilde{\Xi})$. The correspondence between the present isomorphism $\mathcal{A}_\nu$ and the one used in [**1**] is given by $f \mapsto \det(A)^{\frac{\nu}{2}} \det(D)^{-\frac{\nu}{2}} f(z)$ for all $z = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{D}$. This correspondence explains the shift between the highest weight μ shown in [**1**] and the present form of μ.

Since ν is very large $\left(\nu > \frac{2n}{r} - 1\right)$, the representation π_μ satisfies the Harish-Chandra condition for all $\mu \in \mathcal{R}_{\nu,H}$.

Let

$$C_\mu(\nu) = \int_\Xi |\langle \pi_\mu(\widetilde{\gamma}^{-1})\, v_\mu | \psi_\mu\rangle|^2 \, p_\nu(\xi) d\xi.$$

Proposition 4.4. *For $\mu \in \mathcal{R}_{\nu,H}$, the Hilbert space $\mathcal{H}_\mu$ belongs to $\mathcal{H}^2_\nu(\widetilde{\Xi})$ if and only if $C_\mu(\nu)$ is finite.*

In this case we denote $\mu \in \mathcal{R}'_{\nu,H}$.

Proof. This is proved in [**1**] Proposition 4.2. □

Hence the Plancherel Theorem can be written as:

Theorem 4.1 (cf. [**13**])**.** *The Bergman space $\mathcal{H}^2_\nu(\widetilde{\Xi})$ is decomposed multiplicity-free and discretely into irreducible Hilbert subspaces,*

$$\mathcal{H}^2_\nu(\widetilde{\Xi}) = \bigoplus_{\mu \in \mathcal{R}'_{\nu,H}} \mathcal{H}_\mu.$$

Moreover, the reproducing kernel can be written as

$$K_\nu(\widetilde{\xi}_1, \widetilde{\xi}_2) = \sum_{\mu \in \mathcal{R}'_{\nu,H}} \frac{1}{C_\mu(\nu)} \langle \pi_\mu(\widetilde{\gamma}_2^\sharp \widetilde{\gamma}_1)^{-1} \psi_\mu \,|\, \psi_\mu\rangle.$$

The series converges uniformly on compact subsets of $\widetilde{\Xi} \times \widetilde{\Xi}$.

5. Computation of the constant $C_\mu(\nu)$.

Let $\mathcal{M}$ be a differentiable manifold. A causal structure on $\mathcal{M}$ is a field of cones $\mathcal{M} \ni x \mapsto C_x \subset T_x\mathcal{M}$. The cones C_x are assumed to be closed, convex, proper, and with nonempty interior. Furthermore the cones C_x depend smoothly on x. A piecewise $\mathcal{C}^1$ curve $\gamma : [0,1] \longrightarrow \mathcal{M}$ is said to be causal if for all t, the derivative $\dot{\gamma}(t)$ belongs to the cone $C_{\gamma(t)}$. The causal structure is said to be global if there exists no nontrivial closed causal curve.

In that case one defines a partial ordering on $\mathcal{M}$ in the following way: One writes $x \leq y$ if there exists a causal curve from x to y.

Let $\mathfrak{g}$ be the Lie algebra defined in Section 2. Let σ be an involutive automorphism of $\mathfrak{g}$ that commutes with the Cartan involution θ where $\mathfrak{g} = \mathfrak{k} \bigoplus^{\theta} \mathfrak{p}$, and $\mathfrak{g} = \mathfrak{h} \bigoplus^{\sigma} \mathfrak{q}$.

Let $G^c := (\underline{G}^{(\alpha)})_0$ where $\underline{G}^{(\alpha)} = \{g \in \underline{G} \mid \alpha \circ g \circ \alpha = g\}$ and the subscript 0 means the identity component. The group G^c is the group of holomorphic automorphisms of the tube domain T_{Ω^+} associated with the involution α defined by

$$T_{\Omega^+} := \mathbb{V}^- + \Omega^+ = \left\{x + y \mid x \in \mathbb{V}^-, \quad y \in \Omega^+\right\}$$

where $\Omega^+ := \mathbb{V}^+ \cap \Omega$ and Ω is the symmetric cone associated with $\mathbb{V}$. (If $\mathbb{V}$ and $\mathbb{V}^+$ are simple, the cone Ω^+ coincides with the open cone associated with the Jordan algebra $\mathbb{V}^+$.) The group G^c is the c-dual group of G. We consider on $\mathcal{M} := G^c/H$ the causal structure defined by the field of cones

$$C_x = -\overline{\Omega}.$$

The noncompactly causal symmetric space $\mathcal{M}$ is an ordered symmetric space. By [**3**], the intersection $\mathcal{M} \cap \mathbb{V}$ is a union of connected components of the set $\{x \in \mathbb{V} \mid \det B_\alpha(x, x) \neq 0\}$ where $B_\alpha(x, y) := B(x, \alpha y)$ and $B(x, y)$ is the Bergman operator. In particular

$$(\mathcal{M} \cap \mathbb{V})_0 = \{x \in \mathbb{V} \mid \det B_\alpha(x, x) \neq 0\}_0 .$$

Let $\mathfrak{g}^c = \mathrm{Lie}(G^c)$ (the c-dual algebra of $\mathfrak{g}$). We denote also by σ the $\mathbb{C}$-linear extention of σ to the complexified algebra $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. The involution $\theta^c := \theta\sigma_{|\mathfrak{g}^c}$ is a Cartan involution of $\mathfrak{g}^c$. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}^c \cap i\mathfrak{q}$ where $\mathfrak{p}^c := (\mathfrak{g}^c)^{\theta^c}$ (note that $i\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{k} \cap \mathfrak{q}$), $\triangle$ the root system for the pair $(\mathfrak{g}^c, \mathfrak{a})$, and let $\triangle^+$ be the positive root system with respect to the positive Weyl chamber $\mathfrak{a}^+$ (see (5)).

Let

$$\mathfrak{n} := \bigoplus_{\beta \in \triangle^+} \mathfrak{g}^c_\beta, \qquad \overline{\mathfrak{n}} := \bigoplus_{\beta \in -\triangle^+} \mathfrak{g}^c_\beta,$$

$$N := \exp \mathfrak{n}, \qquad \overline{N} := \exp \overline{\mathfrak{n}}, \qquad A := \exp \mathfrak{a}.$$

Let $x_0 := e\,H$ be the base point of G^c/H. The map

$$N \times A \longrightarrow \mathcal{M}, \qquad (n, a) \mapsto na \cdot x_0,$$

is a diffeomorphism of $N \times A$ onto its open image $NA \cdot x_0$. For all $x = n \exp(X) \cdot x_0$ ($X \in \mathfrak{a}$), we write $X = A(x)$. We denote $a_H(x) := \exp A(x)$.

Let $\mathcal{M}^+$ be the subset of $\mathcal{M}$ defined by

$$\mathcal{M}^+ := \{x \in \mathcal{M} \mid x \geq x_0\},$$

called the future of x_0. By [**9**], $\mathcal{M}^+ \subset NA \cdot x_0$.

The spherical Laplace transform of an H-invariant function f is defined by

$$\widehat{f}(\lambda) = \int_{\mathcal{M}^+} f(x) a_H(x)^{-\lambda} dx, \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

Using the following integral formula,

$$\int_{\mathcal{M}^+} f(x)dx = \int_{-\mathfrak{a}^+} \int_H f(h \exp(X) \cdot x_0) dh \prod_{\beta \in -\triangle^+} (\mathrm{sh}\,\langle \beta, X \rangle)^{m_\beta}\, dX, \tag{6}$$

the spherical Laplace transform can be written as

$$\widehat{f}(\lambda) = c \int_{-\mathfrak{a}^+} f(\exp(X) \cdot x_0)\, \varphi_\lambda(\exp(X)) \prod_{\beta \in -\triangle^+} (\mathrm{sh}\,\langle \beta, X \rangle)^{m_\beta}\, dX,$$

where φ_λ is the spherical function of the ordered symmetric space $\mathcal{M}$, defined in the interior S^0 of $S := \{g \in G^c \mid g \cdot x_0 \geq x_0\ \} \subset NAH$,

$$\varphi_\lambda(g) = \int_H a_H(hg)^{-\lambda} dh$$

(cf. [10]). The c-function of the symmetric space $\mathcal{M}$, which we denote by $c_{\mathcal{M}}$, is defined by the integral

$$c_{\mathcal{M}}(\lambda) = \int_{\overline{N} \cap HAN} a_H(\overline{n})^{-(\lambda+\rho)} d\overline{n}.$$

Remark. From Theorem 2.3 and the integral formula (6), we obtain

$$\int_\Xi f(\xi) d\xi = c_0 \int_{G/H} \int_{\mathcal{M}^+} f(g \cdot i x^{\frac{1}{2}}) d\dot{g} dx.$$

This integral formula is a generalization of that given in Proposition X.3.4 of [12] where $G = \underline{K}$.

Let $\mathcal{W}_\mu^\omega$ (resp. $\mathcal{W}_\mu^{-\omega}$) be the space of analytic (resp. hyperfunction) vectors of $\mathcal{W}_\mu$. By [5] Theorem 1.1, $\left(\mathcal{W}_\mu^{-\omega}\right)^H = \left(\mathcal{W}_\mu^{-\infty}\right)^H$ where $\left(\mathcal{W}_\mu^{-\omega}\right)^H$ is the subspace of H-invariant hyperfunction vectors of $\mathcal{W}_\mu^{-\omega}$. Moreover, if the representation $(\pi_\mu, \mathcal{W}_\mu)$ satisfies the Harish-Chandra condition, the linear form

$$L_\mu(f) = \int_H \langle \pi_\mu(h) f |\, v_\mu \rangle\, dh, \qquad f \in \mathcal{W}_\mu^\omega$$

defines an H-invariant hyperfunction vector (cf. [20]). Using the fact that $\dim \left(\mathcal{W}_\mu^{-\infty}\right)^H \leq 1$, and we deduce that if ψ_μ is an H-invariant distribution vector, there exist a constant c_0 such that

$$\langle f | \psi_\mu \rangle = c_0 L_\mu(f), \qquad f \in \mathcal{W}_\mu^\infty. \tag{7}$$

In particular if $f = v_\mu$ then

$$\langle v_\mu | \psi_\mu \rangle = c_0 \int_H \langle \pi_\mu(h) v_\mu | v_\mu \rangle dh.$$

Using the integral formula for all functions $f \in L^1(H)$,

$$\int_H f(h)dh = \int_{\overline{N}\cap HAN} f(h(\overline{n}))a_H(\overline{n})^{-2\rho}d\overline{n},$$

(cf. [**24**]), we deduce that

$$\begin{aligned}
L_\mu(v_\mu) &= \int_{\overline{N}\cap HAN} \langle \pi_\mu(h(\overline{n}))v_\mu|v_\mu\rangle a_H(\overline{n})^{-2\rho}d\overline{n} \\
&= \int_{\overline{N}\cap HAN} \langle \pi_\mu(a_H(\overline{n})^{-1}\overline{n})v_\mu|v_\mu\rangle a_H(\overline{n})^{-2\rho}d\overline{n} \quad \Big(\pi_\mu(n(\overline{n}))v_\mu = v_\mu\Big) \\
&= \int_{\overline{N}\cap HAN} \langle \pi_\mu(\overline{n})v_\mu|v_\mu\rangle a_H(\overline{n})^{-2\rho-\mu}d\overline{n} \\
&= \int_{\overline{N}\cap HAN} \langle v_\mu|\pi_\mu(\overline{n})^* v_\mu\rangle a_H(\overline{n})^{-2\rho-\mu}d\overline{n} \\
&= \int_{\overline{N}\cap HAN} a_H(\overline{n})^{-(2\rho+\mu)}d\overline{n} \\
&= c_{\mathcal{M}}(\mu+\rho).
\end{aligned}$$

That implies,

$$c_0 = \frac{\langle v_\mu|\psi_\mu\rangle}{c_{\mathcal{M}}(\mu+\rho)}. \tag{8}$$

Lemma 5.1. *For all* $\gamma \in \Gamma(C)$,

$$\langle \pi_\mu(\gamma^{-1})\psi_\mu|\psi_\mu\rangle = \frac{|\langle \psi_\mu|v_\mu\rangle|^2}{c_{\mathcal{M}}(\mu+\rho)}\, \varphi_{-\mu}(\gamma^{-1}).$$

Proof. By [**10**], for all $\gamma \in \Gamma(C^0)\cap G^c$ there exist $\overline{n} \in \overline{N}, a_H(\gamma) \in A$, and $h \in H$, such that $\gamma = \overline{n}a_H(\gamma)h$. Hence for $\gamma \in \Gamma(C^0)\cap G^c$,

$$\langle \pi_\mu(\gamma^{-1})\psi_\mu|v_\mu\rangle = a_H(\gamma^{-1})^\mu \langle \psi_\mu|v_\mu\rangle.$$

The equalities (7) and (8) yield

$$\begin{aligned}
\langle \pi_\mu(\gamma^{-1})\psi_\mu|\psi_\mu\rangle &= \frac{\langle v_\mu|\psi_\mu\rangle}{c_{\mathcal{M}}(\mu+\rho)} \int_H \langle \pi_\mu(h\gamma^{-1})\psi_\mu|v_\mu\rangle dh \\
&= \frac{\langle v_\mu|\psi_\mu\rangle}{c_{\mathcal{M}}(\mu+\rho)} \int_H a_H(h\gamma^{-1})^\mu \langle \psi_\mu|v_\mu\rangle dh \\
&= \frac{|\langle \psi_\mu|v_\mu\rangle|^2}{c_{\mathcal{M}}(\mu+\rho)}\, \varphi_{-\mu}(\gamma^{-1}).
\end{aligned}$$

Now the assertion follows from the fact that the function $\gamma \mapsto \langle \pi_\mu(\gamma^{-1})\psi_\mu|\psi_\mu\rangle$ is holomorphic on $\Gamma(C^0)$ and coincides with $\frac{|\langle \psi_\mu|v_\mu\rangle|^2}{c_{\mathcal{M}}(\mu+\rho)}\, \varphi_{-\mu}(\gamma^{-1})$ in $\Gamma(C^0)\cap G^c$. □

Let $(\pi_\mu, \mathcal{W}_\mu)$ be an H-spherical unitary highest weight representation of $\widetilde{G}$ such that $(\pi_\mu, \mathcal{W}_\mu)$ belongs to the relative discrete series, and

$$\int_{G/H} |\langle \pi_\mu(g) v_\mu | \psi_\mu \rangle|^2 d\dot{g} = \frac{1}{\delta_\mu}$$

where δ_μ is the relative formal dimension calculated in [**20**], $\delta_\mu = d_\mu \cdot c_{\mathcal{M}}(\mu + \rho)$, with $d_\mu = \displaystyle\prod_{\beta \in \triangle^+} \frac{\langle \mu + \rho, \beta \rangle}{\langle \rho, \beta \rangle}$ the formal dimension of the representation π_μ.

Theorem 5.1. *Let P_ν be the G-invariant function such that $P_\nu(\exp(2X) \cdot ie) := p_\nu(\exp(X) \cdot ie)$. The weight $\mu \in \mathcal{R}^{'}_{\nu,H}$ if and only if the spherical Laplace transform $\widehat{P}_\nu(-\mu)$ is finite. Moreover*

$$C_\mu(\nu) = \frac{1}{\delta_\mu}\, \widehat{P}_\nu(-\mu),$$

where δ_μ is the relative formal dimension.

Proof. We assume that the H-invariant distribution vector ψ_μ is normalized by $\langle \psi_\mu | v_\mu \rangle = 1$. By the integral formula of Theorem 2.3 and the fact that p_ν is G-invariant, we deduce that

$$\begin{aligned} C_\mu(\nu) &= \int_\Xi |\langle \pi_\mu(\gamma^{-1}) v_\mu | \psi_\mu \rangle|^2 \, p_\nu(\xi) d\xi \\ &= \int_G \int_{C^+} |\langle \pi_\mu(\gamma^{-1}) v_\mu | \psi_\mu \rangle|^2 \, p_\nu(\exp(X) \cdot ie) \prod_{\beta \in \triangle^+} \text{sh}\,(2\beta(X))^{m_\beta} \, dg dX \\ &= \frac{1}{d_\mu} \int_{C^+} \|\pi_\mu(\exp(X)) \psi_\mu\|^2 \, p_\nu(\exp(X) \cdot ie) \prod_{\beta \in \triangle^+} \text{sh}\,(2\beta(X))^{m_\beta} \, dX. \end{aligned}$$

According to the last lemma, this yields

$$\begin{aligned} &\int_\Xi |\langle \pi_\mu(\gamma) \psi_\mu | v_\mu \rangle|^2 p_\nu(\xi) d\xi \\ &= \frac{1}{\delta_\mu} \int_{C^+} p_\nu(\exp(X) \cdot ie) \varphi_{-\mu}(\exp(2X)) \prod_{\beta \in \Delta^+} \text{sh}\,(2\beta(X))^{m_\beta} \, dX \\ &= \frac{1}{\delta_\mu} \widehat{P}_\nu(-\mu). \end{aligned}$$

□

For any $x \in \mathbb{V}$ we have $x = x^+ + x^-$, where $x^+ := \dfrac{x + \alpha(x)}{2} \in \mathbb{V}^+$ and $x^- := \dfrac{x - \alpha(x)}{2} \in \mathbb{V}^-$. We denote $x := (x^+, x^-)$.

Let $\mathcal{J}$ be the bounded set in $\mathbb{V}$ defined by

$$\mathcal{J} = \Big\{x = (x^+, x^-) \in \mathbb{V} \mid x^+ \in \Omega,\ x^+ + x^- \in e - \overline{\Omega}\Big\} \subset \Big(\mathbb{V}^+ \cap \Omega\Big) \times \Big((e + \mathbb{V}^-) \cap \Omega\Big).$$

Proposition 5.2. *The set $\mathcal{J}$ coincides with*

$$\Big\{x \in \mathcal{M} \mid x \geq x_0\Big\},$$

the future of x_0.

Proof. Let $\varphi : [0,1] \longrightarrow \mathbb{V}$ be the curve defined by

$$\varphi(t) = tx + (1-t)e,$$

where $\varphi(0) = e$ and $\varphi(1) = x$. Since Ω is convex,

$$\varphi(t) + \alpha(\varphi(t)) = t(x + \alpha(x)) + (1-t)2e \in \Omega.$$

Thus $\varphi(t) \in \mathcal{M}$ for all $t \in [0,1]$. Moreover $\dot{\varphi}(t) = x - e \in -\overline{\Omega} \simeq C^c$, where

$$C^c = \Big\{(v, 0, -\alpha(v)) \mid v \in \overline{\Omega}\Big\} \subset \mathfrak{g}^c,$$

the regular cone in $\mathfrak{g}^c$ such that $C^c \cap \mathfrak{p}^c \neq \emptyset$, where $\mathfrak{p}^c = \{(v, L(w), -v) \mid v \in \mathbb{V}^+,\ w \in \mathbb{V}^-\}$ (cf. [3], p. 26). Then, φ is a nontrivial causal curve in $\mathcal{M}$ from x to x_0 and x belongs to the future of x_0.

Conversely, let $\varphi : [a,b] \longrightarrow \mathcal{M}$ be a causal curve. Assume that there exists $t > a$ such that $\varphi(t) \notin \mathcal{M} \cap \mathbb{V}$ and

$$\kappa = \inf \Big\{t \in [a,b] \mid \varphi(t) \notin \mathcal{M} \cap \mathbb{V}\Big\}.$$

Since $\mathcal{M} \cap \mathbb{V}$ is open in $\mathcal{M}$, then $\varphi(t) \in \mathcal{M} \cap \mathbb{V}$ if $t < \kappa$ and $\varphi(t) \notin \mathcal{M} \cap \mathbb{V}$ if not. Hence

$$\lim_{\substack{t \to \kappa \\ t < \kappa}} \|\varphi(t)\| = \infty.$$

Moreover the curve $\varphi : [a, \kappa[\longrightarrow \mathbb{V}$ is causal with respect to the causal structure defined by the cone $-\overline{\Omega}$. Then, $\varphi(t) \in e - \overline{\Omega}$ and for all $t \in [a, \kappa[$, $\varphi(t)$ belongs to the connected component of x_0 in $\mathcal{M} \cap \mathbb{V}$ given by

$$\Big\{x \in \mathbb{V} \mid x + \alpha(x) \in \Omega\Big\}.$$

Hence for all $t \in [a, \kappa[$, $\varphi(t) \in \mathcal{J}$ such that $\mathcal{J}$ is a bounded set. This leads to contradiction. ☐

Lemma 5.3. *Let $\mathbf{s} \in \mathbb{C}^{r^+}$ and $x = n \exp A(x) \cdot x_0 \in NA \cdot x_0$. To identify $\mathfrak{a}_{\mathbb{C}}$ with $\mathbb{C}^{r^+}$ we have*

$$e^{\langle \mathbf{s} | A(x) \rangle} = \Delta_{\mathbf{s}}\Big(\frac{x + \alpha(x)}{2}\Big).$$

Proof. The function $x \mapsto \Delta_{\mathbf{s}}\Big(\dfrac{x+\alpha(x)}{2}\Big)$ is N-invariant. Then, for all $x = n\exp A(x)\cdot x_0$,

$$\Delta_{\mathbf{s}}\Big(\frac{x+\alpha(x)}{2}\Big) = \Delta_{\mathbf{s}}(\exp A(x)\cdot x_0).$$

Since $A(x) \in \mathfrak{a}_{\mathbb{C}}$, there exists $(t_1, t_2, \dots, t_{r^+}) \in \mathbb{C}^{r^+}$ such that $A(x) = \sum_{j=1}^{r^+} t_j L(e_j)$, and $\exp A(x)\cdot x_0 = \sum_{j=1}^{r^+} e^{t_j} e_j$. Here $\{e_j\}_{1\le j\le r^+}$ is the Jordan frame of R^+. Thus for all $\mathbf{s} = (s_1, \dots, s_{r^+})$, we deduce that

$$\Delta_{\mathbf{s}}\left(\frac{x+\alpha(x)}{2}\right) = \Delta_{\mathbf{s}}\left(\sum_{j=1}^{r^+} e^{t_j} e_j\right) = \prod_{j=1}^{r^+} e^{t_j s_j} = e^{\langle \mathbf{s} | A(x)\rangle}.$$

□

To give an explicit formula of the spherical Laplace transform $\widehat{P}_\nu(-\mu)$ and an explicit descripition of the spectrum $\mathcal{R}'_{\nu,H}$, we recall some notations.

The Gindikin gamma function of the symmetric cone Ω is defined by the following integral, generalizing the classical Siegel integral

$$\begin{aligned}\Gamma_\Omega(\mathbf{s}) &= \int_\Omega e^{-\mathrm{tr}(x)} \Delta_{\mathbf{s}-\frac{n}{r}}(x)dx \\ &= \prod_{j=1}^{r} \Gamma\Big(s_j - \frac{d}{2}(j-1)\Big), \qquad d := \frac{2(n-r)}{r(r-1)}\end{aligned}$$

for $\mathbf{s} = (s_1, \dots, s_r)$ where $\Re s_j > (j-1)\dfrac{d}{2}$ for $j = 1, \dots, r$.

The Gindikin beta function of the symmetric cone Ω is defined by the following integral

$$\begin{aligned}\mathcal{B}_\Omega(\mathbf{s}, \mathbf{m}) &= \int_{\Omega\cap(e-\Omega)} \Delta_{\mathbf{s}-\frac{n}{r}}(x)\Delta_{\mathbf{m}-\frac{n}{r}}(e-x)dx \\ &= \frac{\Gamma_\Omega(\mathbf{s})\Gamma_\Omega(\mathbf{m})}{\Gamma_\Omega(\mathbf{s}+\mathbf{m})}.\end{aligned}$$

Case (I). $\mathbb{V}^+$ **is a simple Jordan algebra.**

Let $n^\pm := \dim(\mathbb{V}^\pm)$, and Γ_{Ω^+} (resp. $\mathcal{B}_{\Omega^+}$) be the Gindikin gamma (resp. beta) function of the symmetric cone $\Omega^+ := \Omega \cap \mathbb{V}^+$ of $\mathbb{V}^+$.

Lemma 5.4. *For* $\lambda \in \mathbb{C}$ *such that* $\Re(\lambda) > \dfrac{n}{r} - 1$,

$$\int_{(-e+\Omega)\cap\mathbb{V}^-} \Delta(e+v)^{\frac{r}{r^+}\lambda - \frac{n}{r^+}} dv = \frac{\Gamma_\Omega(\lambda)}{\Gamma_{\Omega^+}\Big(\frac{r}{r^+}\lambda\Big)}.$$

Proof. With respect to the decomposition of the Jordan algebra $\mathbb{V}$ as $\mathbb{V} = \mathbb{V}^+ \oplus \mathbb{V}^-$, the Gindikin gamma function of the symmetric cone Ω can be written as

$$\begin{aligned}\Gamma_\Omega(\lambda) &= \int_\Omega e^{-tr(x)} \Delta(x)^{\lambda - \frac{n}{r}} dx \\ &= \int_{\Omega^+} e^{-tr(x^+)} \left[\int_{\{x^- \mid x^+ + x^- \in \Omega\}} \Delta(x^+ + x^-)^{\lambda - \frac{n}{r}} dx^- \right] dx^+.\end{aligned}$$

Moreover $x^+ + x^- = P((x^+)^{\frac{1}{2}})(e+v)$, where P is the quadratic representation and $v = P((x^+)^{-\frac{1}{2}})x^-$. Hence

$$\begin{aligned}\Gamma_\Omega(\lambda) &= \int_{\Omega^+} e^{-tr(x^+)} \Delta(x^+)^{\frac{r}{r^+}\lambda - \frac{n}{r^+} + \frac{n^-}{r^+}} dx^+ \int_{(-e+\Omega)\cap\mathbb{V}^-} \Delta(e+v)^{\frac{r}{r^+}\lambda - \frac{n}{r^+}} dv \\ &= \Gamma_{\Omega^+}\Big(\frac{r}{r^+}\lambda\Big) \int_{(-e+\Omega)\cap\mathbb{V}^-} \Delta(e+v)^{\frac{r}{r^+}\lambda - \frac{n}{r^+}} dv.\end{aligned}$$

This proves that

$$\int_{(-e+\Omega)\cap\mathbb{V}^-} \Delta(e+v)^{\frac{r}{r^+}\lambda - \frac{n}{r^+}} dv = \frac{\Gamma_\Omega(\lambda)}{\Gamma_{\Omega^+}\Big(\frac{r}{r^+}\lambda\Big)}.$$

□

Proposition 5.5. *For* $\mu = \Big(m_1 + (r/2r^+)\nu, \ldots, m_{r^+} + (r/2r^+)\nu\Big)$, *the spherical Laplace transform* $\widehat{P}_\nu(-\mu)$ *is finite if and only if*

$$m_1 \geq m_2 \geq \cdots \geq m_{r^+} \geq 0.$$

Moreover

$$\widehat{P}_\nu(-\mu) = c_0 \Gamma_\Omega\Big(\nu - \frac{n}{r}\Big) \frac{\Gamma_{\Omega^+}\Big(\mu - \nu\frac{r}{2r^+} + \frac{n^+}{r^+}\Big)}{\Gamma_{\Omega^+}\Big(\mu + \nu\frac{r}{2r^+} - \frac{n^-}{r^+}\Big)},$$

where c_0 *is a positive constant.*

Proof. By Proposition 3.1 and for all $x = n \exp A(x) \cdot x_0$, the function p_ν is given by

$$p_\nu(x) = \det\Big(P(x + \alpha(x))\Big)^{-\frac{r}{2n}\nu + 1} \Delta(e - x^2)^{\nu - \frac{2n}{r}}.$$

Since $A(x) \in \mathfrak{a}_\mathbb{C}$, there exists $(t_1, \ldots, t_{r^+}) \in \mathbb{C}^{r^+}$ such that $A(x) = \sum_{j=1}^{r^+} t_j L(e_j)$, and

$$\exp\left(\frac{A(x)}{2}\right) \cdot x_0 = \sum_{j=1}^{r^+} e^{\frac{t_j}{2}} e_j.$$

Let $x = \sum_{j=1}^{r^+} e^{t_j} e_j$ and $u = \sum_{j=1}^{r^+} e^{\frac{t_j}{2}} e_j$. Using the fact that p_ν is G-invariant, we deduce that

$$p_\nu(u) = \Delta(x^+)^{-\nu\frac{r}{2r^+}+\frac{n}{r^+}} \Delta(e-x)^{\nu-\frac{2n}{r}} = P_\nu(x).$$

For $x \in \mathbb{V}$, let $d\delta(x)$ be the G-invariant measure on $\mathcal{M}$, where its restriction to $\mathcal{M} \cap \mathbb{V}$ is equal to $d\delta(x) = \Delta(x^+)^{-\frac{n}{r^+}} dx^+ dx^-$. Using Lemma 5.3, the spherical Laplace transformation of P_ν can be written as

$$\begin{aligned}
\widehat{P}_\nu(-\mu) &= \int_{x \geq x_0} e^{\langle \mu, A(x)\rangle} P_\nu(x) d\delta(x) \\
&= \int_{\Omega^+ \cap (e-\Omega^+)} \int_{\{x^- | e - x^+ - x^- \in \Omega\}} \Delta_\mu(x^+) \Delta(x^+)^{-\nu\frac{r}{2r^+}+\frac{n}{r^+}} \\
&\qquad \cdot \Delta(e-x)^{\nu-\frac{2n}{r}} \Delta(x^+)^{-\frac{n}{r^+}} dx^+ dx^- \\
&= \int_{\Omega^+ \cap (e-\Omega^+)} \Delta_{\mu-\nu\frac{r}{2r^+}}(x^+) \\
&\qquad \cdot \left[\int_{\{x^- | e - x^+ - x^- \in \Omega\}} \Delta(e - x^+ - x^-)^{\nu - \frac{2n}{r}} dx^- \right] dx^+.
\end{aligned}$$

Using the last lemma, and the fact that $e - x^+ - x^- = P((e-x^+)^{\frac{1}{2}})(e-v)$ where $v = P((e-x^+)^{-\frac{1}{2}})x^-$, we obtain

$$\begin{aligned}
\widehat{P}_\nu(-\mu) &= \int_{\Omega^+ \cap (e-\Omega^+)} \Delta_{\mu-\nu\frac{r}{2r^+}}(x^+) \Delta(e-x^+)^{\nu\frac{r}{r^+}-\frac{2n}{r^+}+\frac{n^-}{r^+}} dx^+ \\
&\qquad \cdot \int_{(e-\Omega)\cap\mathbb{V}^-} \Delta(e-v)^{\nu\frac{r}{r^+}-\frac{2n}{r^+}} dv \\
&= \frac{\Gamma_\Omega\left(\nu - \frac{n}{r}\right)}{\Gamma_{\Omega^+}\left(\nu\frac{r}{r^+} - \frac{n}{r^+}\right)} \mathcal{B}_{\Omega^+}\left(\mu - \nu\frac{r}{2r^+} + \frac{n^+}{r^+}, \nu\frac{r}{r^+} - \frac{n}{r^+}\right),
\end{aligned}$$

such that $\mu_j - \nu\dfrac{r}{2r^+} + \dfrac{n^+}{r^+} > (j-1)\dfrac{n^+ - r^+}{r^+(r^+-1)}$ for $1 \leq j \leq r^+$. By the fact that $\mu_j = m_j + \nu\dfrac{r}{2r^+}$, where $m_j \in \mathbb{Z}$ and $\mu_j - \mu_{j+1} \in \mathbb{N}$, this condition will be written as

$$m_1 \geq m_2 \geq \cdots \geq m_{r^+} \geq 0.$$

□

It follows from Proposition 5.5 that the spectrum of the Bergman space $\mathcal{H}^2_\nu(\widetilde{\Xi})$ is given by

$$\mathcal{R}'_{\nu,H} = \Big\{\mu = \Big(m_1 + (r/2r^+)\nu,\ m_2 + (r/2r^+)\nu, \ldots, m_{r^+} + (r/2r^+)\nu\Big)\Big| \\ m_1 \geq m_2 \geq \cdots \geq m_{r^+} \geq 0\Big\}.$$

Recall that the constant $c_\nu = \dfrac{1}{\pi^n}\,\dfrac{\Gamma_\Omega(\nu)}{\Gamma_\Omega\Big(\nu - \dfrac{n}{r}\Big)}$ (see (4)), the reproducing kernel $K_\nu(\widetilde{\xi}_1, \widetilde{\xi}_2)$ can be written as

$$\top_\alpha(\widetilde{\xi}_1)^{\frac{r}{4n}\nu} \det(B(\xi_1, \overline{\xi}_2))^{-\frac{r}{2n}\nu}\overline{\top_\alpha(\widetilde{\xi}_2)}^{\frac{r}{4n}\nu} \\ = \sum \frac{c_0 d_\mu}{\Gamma_\Omega(\nu)}\,\frac{\Gamma_{\Omega^+}\Big(\mathbf{m} + \nu\dfrac{r}{r^+} - \dfrac{n^-}{r^+}\Big)}{\Gamma_{\Omega^+}\Big(\mathbf{m} + \dfrac{n^+}{r^+}\Big)}\,\varphi_{-\mu}\Big((\widetilde{\gamma}_2^\sharp\widetilde{\gamma}_1)^{-1}\Big),$$

for all $\widetilde{\xi}_1 = \widetilde{\gamma}_1 \cdot H$ and all $\widetilde{\xi}_2 = \widetilde{\gamma}_2 \cdot H$ in $\widetilde{\Xi}$. The summation index runs over the integers of type $\mathbf{m} := (m_1, m_2, \ldots, m_{r^+})$ such that $m_1 \geq \cdots \geq m_{r^+} \geq 0$. We use $\mathbf{m} \geq 0$ to denote the summation index. The function

$$\nu \mapsto \top_\alpha(\widetilde{\xi}_1)^{\frac{r}{4n}\nu} \det(B(\xi_1, \overline{\xi}_2))^{-\frac{r}{2n}\nu}\overline{\top_\alpha(\widetilde{\xi}_2)}^{\frac{r}{4n}\nu},$$

is holomorphic on $\mathbb{C}$ and coincides with

$$\sum_{\mathbf{m}\geq 0} \frac{c_0 d_\mu}{\Gamma_\Omega(\nu)}\frac{\Gamma_{\Omega^+}\Big(\mathbf{m} + \nu\dfrac{r}{r^+} - \dfrac{n^-}{r^+}\Big)}{\Gamma_{\Omega^+}\Big(\mathbf{m} + \dfrac{n^+}{r^+}\Big)}\,\varphi_{-\mu}\Big((\widetilde{\gamma}_2^\sharp\widetilde{\gamma}_1)^{-1}\Big),$$

for $\nu \in \Big]\dfrac{2n}{r} - 1, +\infty\Big[$, therefore these two functions coincide everywhere. The following theorem holds.

Theorem 5.2. *Assume α is given such that $\mathbb{V}^+$ is a simple algebra. Then for all $\nu \in \mathbb{C}$ such that $\Re(\nu) > \dfrac{n}{r} - 1$,*

$$\top_\alpha(\widetilde{\xi}_1)^{\frac{r}{4n}\nu} \det(B(\xi_1, \overline{\xi}_2))^{-\frac{r}{2n}\nu}\overline{\top_\alpha(\widetilde{\xi}_2)}^{\frac{r}{4n}\nu} \\ = \sum_{\mathbf{m}\geq 0} \frac{c_0 d_\mu}{\Gamma_\Omega(\nu)}\frac{\Gamma_{\Omega^+}\Big(\mathbf{m} + \nu\dfrac{r}{r^+} - \dfrac{n^-}{r^+}\Big)}{\Gamma_{\Omega^+}\Big(\mathbf{m} + \dfrac{n^+}{r^+}\Big)}\varphi_{-\mu}\Big((\widetilde{\gamma}_2^\sharp\widetilde{\gamma}_1)^{-1}\Big).$$

The series converges uniformly on compact subsets of $\widetilde{\Xi} \times \widetilde{\Xi}$.

An application (Makarevič symmetric spaces of Cayley type)**.** Let $\mathbb{V}$ be a simple Euclidean Jordan algebra of dimension n and rank r. The bounded symmetric domain associated with $\mathbb{V}_{\mathbb{C}} \times \mathbb{V}_{\mathbb{C}}$ is the bidisc $\mathcal{D} \times \mathcal{D}$ where $\mathcal{D}$ is the unit disc of $\mathbb{V}_{\mathbb{C}}$. Let α be the involution on $\mathcal{D} \times \mathcal{D}$ defined by $\alpha(z,w) = (-w,-z)$. In this case, the domain Ξ is realized as $\mathcal{D} \times \mathcal{D} \setminus \Sigma$ where

$$\Sigma = \{(z,w) \in \mathcal{D} \times \mathcal{D} \mid \Delta(z-w) = 0\},$$

and Δ is the determinant polynomial associated with $\mathbb{V}$ (see (1)).

Let

$$\Big\{z, w; z', w'\Big\} := \frac{\Delta(z'-z)\Delta(w-w')}{\Delta(z-w)\Delta(z'-w')}$$

be the cross-ratio of four points z, w, z', and w' of $\mathbb{V}_{\mathbb{C}}$. This definition generalizes the classical cross-ratio-matrix and satisfies the G-invariance property $\Big\{g\cdot z, g\cdot w; g\cdot z', g\cdot w'\Big\} = \Big\{z, w; z', w'\Big\}$. The reproducing kernel of the Bergman space $\mathcal{H}^2_\nu(\widetilde{\Xi})$ is equal to

$$K_\nu(\widetilde{z}_1, \widetilde{w}_1; \widetilde{z}_2, \widetilde{w}_2) = c_\nu^2 \left\{\widetilde{z}_1, \widetilde{w}_1; \overline{\widetilde{z}_2}^{-1}, \overline{\widetilde{w}_2}^{-1}\right\}^\nu,$$

for $(\widetilde{z}_1, \widetilde{w}_1)$ and $(\widetilde{z}_2, \widetilde{w}_2)$ in $\widetilde{\Xi}$.

Using Theorem 5.2, we deduce a formula of a complex power of a cross-ratio of four points,

$$\left\{\widetilde{z}_1, \widetilde{w}_1; \overline{\widetilde{z}_2}^{-1}, \overline{\widetilde{w}_2}^{-1}\right\}^\nu = \sum_{\mathbf{m}\geq 0} \frac{c_0\, d_\mu}{\Gamma_\Omega(\nu)^2} \, \frac{\Gamma_\Omega\Big(\mathbf{m} + 2\nu - \frac{n}{r}\Big)}{\Gamma_\Omega\Big(\mathbf{m} + \frac{n}{r}\Big)} \, \varphi_{-\mu}\Big((\widetilde{\gamma}_2^\sharp \widetilde{\gamma}_1)^{-1}\Big),$$

for all $(\widetilde{z}_1, \widetilde{w}_1) = \widetilde{\gamma}_1 \cdot H$ and all $(\widetilde{z}_2, \widetilde{w}_2) = \widetilde{\gamma}_2 \cdot H$. We remark that this formula is a generalization of that given in Theorem 3.1 of [**7**].

Let $\mathcal{W}$ be the Wallach set

$$\mathcal{W} = \Big\{0, \frac{n-r}{r(r-1)}, \dots, \frac{n}{r} - 1\Big\} \cup \Big]\frac{n}{r} - 1, +\infty\Big[,$$

(cf. [**12**] p. 268), and let $\mathcal{K}_\nu(\widetilde{z}_1, \widetilde{w}_1; \widetilde{z}_2, \widetilde{w}_2) := \{\widetilde{z}_1, \widetilde{w}_1; \overline{\widetilde{z}_2}^{-1}, \overline{\widetilde{w}_2}^{-1}\}^\nu$. The kernel $\mathcal{K}_\nu$ is of positive type on $\widetilde{\Xi} \times \widetilde{\Xi}$ if and only if ν belongs to $\mathcal{W}$. Then $\mathcal{K}_\nu$ is a reproducing kernel of some Hilbert space $\mathcal{H}_\nu(\widetilde{\Xi})$. In particular if $\nu > \frac{2n}{r} - 1$, $\mathcal{H}_\nu(\widetilde{\Xi}) = \mathcal{H}^2_\nu(\widetilde{\Xi})$.

We note that $\frac{n}{r} \in \frac{1}{2}\mathbb{N}$. In [**12**], J. Faraut and A. Korányi showed that the analytic continuation Bergman space $\mathcal{H}_{\frac{n}{r}}(\mathcal{D} \times \mathcal{D})$ ($\simeq \mathcal{H}_{\frac{n}{r}}(\widetilde{\Xi})$ by Theorem 3.1) coincides with the Hardy space $H^2(\mathcal{D} \times \mathcal{D})$. It is a Hilbert space of holomorphic functions f on $\mathcal{D} \times \mathcal{D}$ such that

$$\sup_{0<r_1,r_2<1} \int_{\mathcal{S}\times\mathcal{S}} |f(r_1 u_1, r_2 u_2)|^2 d\sigma(u_1) d\sigma(u_2) < \infty,$$

where $\mathcal{S}$ is the Shilov boundary of $\mathcal{D}$.

Using the same notations as in Section 3, the covering $\widetilde{G}$ of order 2 of G acts on $H^2(\mathcal{D}\times\mathcal{D})$ by the representation $\widetilde{\pi}_{\frac{n}{r}} := \pi_{\frac{n}{r}} \otimes \pi_{\frac{n}{r}}$: for $f \in H^2(\mathcal{D}\times\mathcal{D})$ and $\widetilde{g}^{-1} = (g,\varphi) \in \widetilde{G}$,

$$(\widetilde{\pi}_{\frac{n}{r}}(\widetilde{g})f)(z,w) = \left(\frac{\Delta(g\cdot z - g\cdot w)}{\Delta(z-w)}\right)^{\frac{n}{r}} f(g\cdot z, g\cdot w).$$

Using [4] Theorem 3.2.1, we can deduce that the representation $\widetilde{\pi}_{\frac{n}{r}}$ decomposes into a discrete sum of irreducible unitary representations π_μ of $\widetilde{G}$ with highest weights

$$\mu = \Big(m_1 + \frac{n}{r}, m_2 + \frac{n}{r}, \ldots, m_r + \frac{n}{r}\Big)$$

such that $m_1 \geq m_2 \geq \cdots \geq m_r \geq 0$. In particular, the decomposition of the Hardy space $H^2(\mathcal{D}\times\mathcal{D})$ holds. Hence, we find the decomposition of $H^2(\mathcal{D}\times\mathcal{D})$ shown in [7] when $\dfrac{n}{r}$ belongs to $\mathbb{N}$.

Case (II): $\mathbb{V}^+$ **is a direct sum of two simple algebras.**

Let $\mathbb{V}^+$ be a direct sum of two simple algebras $\mathbb{V}_1^+$ of rank p and $\mathbb{V}_2^+$ of rank q such that $p+q = r^+$ $(= r)$. Let $n_1 = p + \frac{1}{2}dp(p-1)$ denote the dimension of $\mathbb{V}_1^+$, and $n_2 = q + \frac{1}{2}dq(q-1)$ the dimension of $\mathbb{V}_2^+$. Then the involution α is given by $\alpha = P(w)$, where $w = c_1 + c_2 + \cdots + c_p - c_{p+1} - c_{p+2} - \cdots - c_r$. The cone Ω^+ is a direct sum of Ω_1^+ and Ω_2^+, where $\Omega_1^+ := \Omega^+ \cap \mathbb{V}_1^+$ and $\Omega_2^+ := \Omega^+ \cap \mathbb{V}_2^+$.
For $\mathbf{s} \in \mathbb{C}^p$ (resp. $\mathbb{C}^q$), we write $\Delta_{\mathbf{s}}^{(1)}$ (resp. $\Delta_{\mathbf{s}}^{(2)}$) for the generalized power function associated with Ω_1^+ (resp. Ω_2^+).

Using the same techniques as in Lemma 5.4, we show that for all $\lambda \in \mathbb{C}$ such that $\Re(\lambda) > \dfrac{n}{r} - 1$,

$$(9)\quad \int_{(-e+\Omega)\cap\mathbb{V}^-} \Delta(e+v)^{\lambda - \frac{n}{r}}\, dv = \frac{\Gamma_\Omega(\lambda)}{\Gamma_{\Omega_1^+}\Big(\lambda - \frac{n^+}{r} + \frac{n_1}{p}\Big)\Gamma_{\Omega_2^+}\Big(\lambda - \frac{n^+}{r} + \frac{n_2}{q}\Big)}.$$

Here $\Gamma_{\Omega_1^+}$ (resp. $\Gamma_{\Omega_2^+}$) denotes the Gindikin gamma function associated with the symmetric cone Ω_1^+ (resp. Ω_2^+).

Proposition 5.6. *For all* $\mu = \Big(m_1 + (\nu/2), \ldots, m_p + (\nu/2);\ m_{p+1} + (\nu/2), \ldots, m_r + (\nu/2)\Big) := (\underline{\mu}_1; \underline{\mu}_2)$, *the spherical Laplace transform* $\widehat{P}_\nu(-\mu)$ *is finite if and only if*

$$m_1 \geq \cdots \geq m_p \geq 0, \quad m_{p+1} \geq \cdots \geq m_r \geq 0.$$

Moreover

$$\widehat{P}_\nu(-\mu) = c_0\,\Gamma_\Omega\Big(\nu - \frac{n}{r}\Big) \cdot \frac{\Gamma_{\Omega_1^+}\Big(\underline{\mu}_1 - \frac{\nu}{2} + \frac{n_1}{p}\Big)\Gamma_{\Omega_2^+}\Big(\underline{\mu}_2 - \frac{\nu}{2} + \frac{n_2}{q}\Big)}{\Gamma_{\Omega_1^+}\Big(\underline{\mu}_1 + \frac{\nu}{2} - \frac{n^+}{r} - \frac{n}{r} + \frac{2n_1}{p}\Big)\Gamma_{\Omega_2^+}\Big(\underline{\mu}_2 + \frac{\nu}{2} - \frac{n^+}{r} - \frac{n}{r} + \frac{2n_2}{q}\Big)},$$

where c_0 is a positive constant.

Proof. As proved in Proposition 5.5, we have

$$\begin{aligned}\widehat{P}_\nu(-\mu) &= \int_{\Omega_1^+\cap(e-\Omega_1^+)} \Delta^{(1)}_{\underline{\mu}_1-\frac{\nu}{2}}(x_1^+)\ \Delta^{(1)}(e-x_1^+)^{\nu+\frac{n^-}{r}-\frac{2n}{r}}\,dx_1^+ \\ &\quad \int_{\Omega_2^+\cap(e-\Omega_2^+)} \Delta^{(2)}_{\underline{\mu}_2-\frac{\nu}{2}}(x_2^+)\ \Delta^{(2)}(e-x_2^+)^{\nu+\frac{n^-}{r}-\frac{2n}{r}}\,dx_2^+ \\ &\quad \int_{(e-\Omega)\cap\mathbb{V}^-} \Delta(e-v)^{\nu-\frac{2n}{r}}\,dv \\ &= \mathcal{B}_{\Omega_1^+}\Big(\underline{\mu}_1 - \frac{\nu}{2} + \frac{n_1}{p}, \nu - \frac{n^+}{r} - \frac{n}{r} + \frac{n_1}{p}\Big) \\ &\quad \mathcal{B}_{\Omega_2^+}\Big(\underline{\mu}_2 - \frac{\nu}{2} + \frac{n_2}{q}, \nu - \frac{n^+}{r} - \frac{n}{r} + \frac{n_2}{q}\Big) \\ &\quad \int_{(e-\Omega)\cap\mathbb{V}^-} \Delta(e-v)^{\nu-\frac{2n}{r}}\,dv.\end{aligned}$$

Then the assertion follows from formula (9). □

Let $\mathbf{m} := (\mathbf{m}_1, \mathbf{m}_2) = (m_1, \dots, m_p; m_{p+1}, \dots, m_r)$. We use $\mathbf{m} \geq 0$ to denote $m_1 \geq \cdots \geq m_p \geq 0$; $\quad m_{p+1} \geq \cdots \geq m_r \geq 0$. We apply the similar arguments as in Theorem 5.2, the following theorem holds.

Theorem 5.3. *Assume α is given such that $\mathbb{V}^+$ is a direct sum of two simple algebras. Then for all $\nu \in \mathbb{C}$ such that $\Re(\nu) > \frac{n}{r} - 1$,*

$$\begin{aligned}&\top_\alpha(\widetilde{\xi}_1)^{\frac{r}{4n}\nu} \det(B(\xi_1, \overline{\xi}_2))^{-\frac{r}{2n}\nu}\overline{\top_\alpha(\widetilde{\xi}_2)}^{\frac{r}{4n}\nu} \\ &= \sum_{\mathbf{m}\geq 0} \frac{c_0 d_\mu}{\Gamma_\Omega(\nu)} \\ &\quad \cdot \frac{\Gamma_{\Omega_1^+}\Big(\mathbf{m}_1 + \nu - \frac{n^+}{r} - \frac{n}{r} + \frac{2n_1}{p}\Big)\Gamma_{\Omega_2^+}\Big(\mathbf{m}_2 + \nu - \frac{n^+}{r} - \frac{n}{r} + \frac{2n_2}{q}\Big)}{\Gamma_{\Omega_1^+}\Big(\mathbf{m}_1 + \frac{n_1}{p}\Big)\Gamma_{\Omega_2^+}\Big(\mathbf{m}_2 + \frac{n_2}{q}\Big)} \varphi_{-\mathbf{m}-\frac{\nu}{2}} \\ &\quad \Big((\widetilde{\gamma}_2^\sharp\widetilde{\gamma}_1)^{-1}\Big).\end{aligned}$$

The series converges uniformly on compact subsets of $\widetilde{\Xi} \times \widetilde{\Xi}$.

Remark. If $p = 0$ or $q = 0$, then $\alpha = \mathrm{id}_{\mathbb{V}}$ (the compact case). The last series can be written as

$$\Delta(e-z)^{-\nu} = c_0 \sum_{\mathbf{m}\geq 0} d_{\mathbf{m}} \frac{(\nu)_{\mathbf{m}}}{(\frac{n}{r})_{\mathbf{m}}} \varphi_{\mathbf{m}}(z),$$

with $\mathbf{m} = (m_1, m_2, \ldots, m_r)$. Here $(\mathbf{s})_{\mathbf{m}}$ denote $(\mathbf{s})_{\mathbf{m}} = \dfrac{\Gamma_\Omega(\mathbf{s}+\mathbf{m})}{\Gamma_\Omega(\mathbf{s})}$, for all $\mathbf{s} \in \mathbb{C}^r$ and $\mathbf{m} \in \mathbb{N}^r$. Hence, we find the generalized binomial formula shown in [**12**]. See also [**2**].

We present here the table of Makarevič symmetric space G/H, and its dual symmetric space G^c/H.

$\mathbb{V}$	G/H	G^c/H
1) $\mathrm{Herm}(n,\mathbb{C})$	$U(p,q)$ (p+q=n) $SO^*(2n)/SO(n,\mathbb{C})$ $Sp(4n,\mathbb{R})/Sp(2n,\mathbb{C})$	$GL(n,\mathbb{C})/U(p,q)$ $SO(n,n)/O(n,\mathbb{C})$ $Sp(n,n)/Sp(2n,\mathbb{C})$
2) $\mathrm{Sym}(n,\mathbb{R})$	$U(p,q)/O(p,q)$ $Sp(2n,\mathbb{R})$	$GL(n,\mathbb{R})/O(p,q)$ $Sp(2n,\mathbb{C})/Sp(2n,\mathbb{R})$
3) $\mathrm{Herm}(n,\mathbb{H})$	$U(2p,2q)/Sp(p,q)$ $SO^*(2n)$	$U^*(2n)/Sp(p,q)$ $SO(2n,\mathbb{C})/O^*(2n)$
4) $\mathbb{R}\times\mathbb{R}^{n-1}$	$SO(p)\times SO(2,q)/$ $SO(p-1)\times SO(1,q)$	$SO(1,p-1)\times SO(1,q+1)/$ $SO(p-1)\times SO(1,q)$
5) $\mathrm{Herm}(3,\mathbb{O})$	$E_6\times U(1)/F_4$ $E_{6(-14)}\times U(1)/F_{4(-20)}$ $SU(6,2)/Sp(3,1)$	$E_6\times\mathbb{R}^+/F_4$ $E_{6(-14)}\times\mathbb{R}^+/F_{4(-20)}$ $SU^*(8)/Sp(3,1)$
$\mathbb{V}\times\mathbb{V}$, where $\mathbb{V}$ is either of the type 1), 2), 3), 4), or 5).	*Cayley type* $SU(n,n)/SL(n,\mathbb{C})\times\mathbb{R}^*_+$ $Sp(2n,\mathbb{R})/GL(n,\mathbb{R})$ $SO^*(4n)/SL(n,\mathbb{H})\times\mathbb{R}^*_+$ $SO(2,n)/SO(1,n-1)\times\mathbb{R}^*_+$ $E_{7(-25)}/E_{6(-26)}\times\mathbb{R}^*_+$	 $SU(n,n)/SL(n,\mathbb{C})\times\mathbb{R}^*_+$ $Sp(2n,\mathbb{R})/GL(n,\mathbb{R})$ $SO^*(4n)/SL(n,\mathbb{H})\times\mathbb{R}^*_+$ $SO(2,n)/SO(1,n-1)\times\mathbb{R}^*_+$ $E_{7(-25)}/E_{6(-26)}\times\mathbb{R}^*_+$

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