Pacific Journal of Mathematics

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Volume 206 No. 1 September 2002

PACIFIC JOURNAL OF MATHEMATICS
Vol. 206, No. 1, 2002

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Let (M^m, g) be a compact Riemannian manifold isometrically immersed in a simply connected space form (euclidean space, sphere or hyperbolic space). The purpose of this paper is to give optimal upper bounds for the first nonzero eigenvalue of the Laplacian of (M^m, g) in terms of r-th mean curvatures and scalar curvature. As consequences, we obtain some rigidity results. In particular, we prove that if (M^n, g) is a compact hypersurface of positive scalar curvature immersed in $\mathbb{R}^{n+1}$ and if g is a Yamabe metric, then (M^n, g) is a standard sphere.

1. Introduction.

Let (M^m, g) be a compact, connected m-dimensional Riemannian manifold without boundary isometrically immersed into a simply connected space form $N^n(\kappa)$ ($\kappa = 0, 1$ or -1 respectively for Euclidean space, sphere or hyperbolic space) whose canonical metric will be denoted by h. A well-known inequality gives an extrinsic upper bound for the first nonzero eigenvalue $\lambda_1(M)$ of the Laplacian of (M^m, g) in terms of the square of the length of the mean curvature, denoted by $|H|^2$. Indeed, we have

$$\lambda_1(M)V(M) \leq m \int_M (|H|^2 + \kappa) dv_g \tag{1}$$

where dv_g and $V(M)$ denote respectively the Riemannian volume element and the volume of (M^m, g). Moreover the equality holds if and only if (M^m, g) is minimally immersed in a geodesic sphere of $N^n(\kappa)$. For $\kappa = 0$, this inequality was proved by Reilly ([**16**]) and can easily be extended to the spherical case $\kappa = 1$ by considering the canonical embedding of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ and by applying the inequality (1) for $\kappa = 0$ to the obtained immersion of (M^m, g) in $\mathbb{R}^{n+1}$. For immersions of (M^m, g) in the hyperbolic space, Heintze ([**10**]) first proved an L_∞ equivalent of (1) and conjectured (1) which was finally obtained by El Soufi and Ilias in [**7**]. In [**16**], Reilly has shown estimates of the $\lambda_1(M)$ of orientable manifolds (M^m, g) isometrically immersed in $\mathbb{R}^n$ in terms of more general invariants called r-th mean curvatures. Let us first define these invariants. Let B be the second fundamental form of the immersion, which is normal-vector valued, and let (B_{ij}), be its matrix with respect to an orthonormal frame $(e_i)_{1\leq i\leq m}$ at a point x of (M^m,g). For any integer $r\in\{1,\dots,m\}$, the r-th mean curvature of the immersion is the quantity, if r is even

$$H_r=\begin{pmatrix} m \\ r \end{pmatrix}^{-1}\frac{1}{r!}\sum_{\substack{i_1\dots i_r\\ j_1\dots j_r}}\epsilon\begin{pmatrix} i_1\dots i_r \\ j_1\dots j_r \end{pmatrix}h(B_{i_1j_1},B_{i_2j_2})\dots h(B_{i_{r-1}j_{r-1}},B_{i_rj_r})$$

and if r is odd

$$H_r=\begin{pmatrix} m \\ r \end{pmatrix}^{-1}\frac{1}{r!}\sum_{\substack{i_1\dots i_r\\ j_1\dots j_r}}\epsilon\begin{pmatrix} i_1\dots i_r \\ j_1\dots j_r \end{pmatrix}$$
$$\cdot\, h(B_{i_1j_1},B_{i_2j_2})\dots h(B_{i_{r-2}j_{r-2}},B_{i_{r-1}j_{r-1}})B_{i_rj_r}$$

where $\epsilon\begin{pmatrix} i_1\dots i_r \\ j_1\dots j_r \end{pmatrix}$ is zero if $\{i_1,\dots,i_r\}\neq\{j_1,\dots,j_r\}$ or if there exists p and q such that $i_p=i_q$, and in the contrary case $\epsilon\begin{pmatrix} i_1\dots i_r \\ j_1\dots j_r \end{pmatrix}$ is the signature of the permutation of $\begin{pmatrix} i_1\dots i_r \\ j_1\dots j_r \end{pmatrix}$. By convention, we put $H_0=1$ and $H_{m+1}=0$. Note that H_1 is nothing but the usual mean curvature vector and for submanifolds of $\mathbb{R}^n$, H_2 is up to a multiplicative coefficient the scalar curvature. If the codimension is 1 and if (M^m,g) is oriented by a normal vector field ν, it is convenient to work with the real valued second fundamental form b by: $b(X,Y)=h(B(X,Y),\nu)$. Therefore, the r-th mean curvatures of odd order can be defined as real valued (we replace in this case the vector field H_r by the scalar $h(H_r,\nu)$). Choosing an orthonormal frame at x such that $b_x(e_i,e_j)=\mu_i\delta_{ij}$, we get the following unified formulae, for any integer $r\in\{1,\dots,m\}$

$$H_r=\begin{pmatrix} m \\ r \end{pmatrix}^{-1}\sum_{i_1<\dots<i_r}\mu_{i_1}\dots\mu_{i_r}. \tag{2}$$

In [**16**], Reilly proved a sharp bound for $\lambda_1(M)$ of manifolds immersed in a Euclidean space, in terms of r-th mean curvatures. Recall this result:

Theorem 1.1 (see Reilly [**16**], Theorem A)**.** *Let (M^m,g) be a compact, orientable m-dimensional Riemannian manifold isometrically immersed by ϕ into $\mathbb{R}^n$.*

1. *If $m < n-1$ and if r is an even integer such that $r \in \{0,\dots,m-1\}$, then*

$$\lambda_1(M)\left(\int_M H_r dv_g\right)^2 \leq mV(M)\int_M |H_{r+1}|^2 dv_g.$$

Moreover if H_{r+1} doesn't vanish identically and if equality holds, then ϕ immerses (M^m, g) minimally into some hypersphere in $\mathbb{R}^n$.

2. *If $m = n-1$ and $r \in \{0,\dots,m-1\}$, then*

$$\lambda_1(M)\left(\int_M H_r dv_g\right)^2 \leq mV(M)\int_M H_{r+1}^2 dv_g.$$

Moreover if H_{r+1} doesn't vanish identically, equality holds if and only if ϕ immerses (M^m, g) as a hypersphere in $\mathbb{R}^n$.

Note that, if $m < n-1$ and r is odd, there is no inequality, because in the proof it is necessary that H_r can be viewed as a real quantity.

The purpose of this paper is to find similar upper bounds for submanifolds of the other space forms. In a first part, we extend Reilly's result to the sphere and the hyperbolic space (Theorems 2.1 and 2.2). In a second part, as a consequence of such estimates and using a different approach, we obtain for hypersurfaces of a simply connected space form upper bounds of $\lambda_1(M)$ in terms of the scalar curvature (Corollary 3.1 and Theorem 3.1). Moreover, these estimates allow us to obtain rigidity results (Remark 3.1). In particular, we prove that if (M^n, g) is a compact hypersurface of positive scalar curvature immersed in the Euclidean space and if g is a Yamabe metric, then (M^n, g) is a standard sphere (Corollary 3.2).

2. Upper bounds of $\lambda_1(M)$ in terms of r-th mean curvatures.

Let (M^m, g) be an orientable m-dimensional Riemannian manifold isometrically immersed by ϕ in an n-dimensional Riemannian manifold (N^n, h) of constant sectional curvature. Let B be the second fundamental form associated to ϕ. Before stating our results, we need some definitions. Let $(e_i)_{1\leq i\leq m}$ be an orthonormal frame at $x \in M$, $(e_i^\star)_{1\leq i\leq m}$ its dual coframe and (B_{ij}) the matrix of B with respect to the frame $(e_i)_{1\leq i\leq m}$. We define the following $(0,2)$-tensors T_r for $r \in \{1,\dots,m\}$:

- If r is even, we set

$$T_r = \frac{1}{r!}\sum_{\substack{i,i_1\dots i_r\\ j,j_1\dots j_r}} \epsilon\begin{pmatrix} i\; i_1\dots i_r\\ j\; j_1\dots j_r\end{pmatrix} h(B_{i_1j_1}, B_{i_2j_2})\dots h(B_{i_{r-1}j_{r-1}}, B_{i_rj_r}) e_i^\star\otimes e_j^\star.$$

- If r is odd, we set

$$T_r = \frac{1}{r!} \sum_{\substack{i,i_1 \dots i_r \\ j,j_1 \dots j_r}} \epsilon \begin{pmatrix} i\, i_1 \dots i_r \\ j\, j_1 \dots j_r \end{pmatrix} h(B_{i_1 j_1}, B_{i_2 j_2}) \dots$$
$$h(B_{i_{r-2} j_{r-2}}, B_{i_{r-1} j_{r-1}}) B_{i_r j_r} \otimes e_i^\star \otimes e_j^\star.$$

By convention $T_0 = g$. As for the r-th mean curvatures, we have an unified formulae if the codimension of (M^m, g) is 1 (i.e., $m = n-1$); indeed, choosing a unit normal field ν and a g-orthonormal frame $(e_i)_{1 \leq i \leq m}$ at a point $x \in M$ which diagonalizes the scalar valued second fundamental form b (i.e., $b_x(e_i, e_j) = \mu_i \delta_{ij}$), the tensors T_r can be viewed as scalar valued $(0,2)$-tensors (if r is odd we replace T_r by the tensor $h(T_r(.,.), \nu)$) and we have at x

$$T_r = \begin{pmatrix} m \\ r \end{pmatrix}^{-1} \sum_{\substack{i_1 < \dots < i_r \\ i_j \neq i}} \mu_{i_1} \dots \mu_{i_r} e_i^\star \otimes e_i^\star. \tag{3}$$

We first prove a lemma which is well-known in codimension 1:

Lemma 2.1. *Let (M^m, g) be a n-dimensional Riemannian manifold isometrically immersed in a n-dimensional Riemannian manifold of constant sectional curvature. Let $r \in \{1, \dots, m\}$, and if $m < n-1$, assume that r is even. Then we have*

$$\mathrm{div}_M T_r = 0.$$

Proof. The proof is known when $m = n - 1$ (see for instance [**17**]). Assume that $m < n-1$ and r is even and let ∇^M denote the Riemannian connection of (M^m, g). Let $x \in M$ and $(e_i)_{1 \leq i \leq m}$ be an orthonormal parallel frame at x, then we have

$$\begin{aligned}
&\mathrm{div}_M T_r(e_j) \\
&= \frac{1}{r!} \sum_i \nabla_{e_i} T_r(e_i, e_j) \\
&= \frac{1}{(r-1)!} \sum_{\substack{i_1 \dots i_r \\ j_1 \dots j_r \\ i}} \epsilon \begin{pmatrix} i\, i_1 \dots i_r \\ j\, j_1 \dots j_r \end{pmatrix} h((\nabla_{e_i} B)_{i_1 j_1}, B_{i_2 j_2}) \dots h(B_{i_{r-1} j_{r-1}}, B_{i_r j_r}) \\
&= \frac{1}{(r-1)!} \sum_{\substack{i_1 \dots i_r \\ j_1 \dots j_r}} \epsilon \begin{pmatrix} i\, i_1 \dots i_r \\ j\, j_1 \dots j_r \end{pmatrix} h((\nabla_{e_{i_1}} B)_{i j_1}, B_{i_2 j_2}) \dots h(B_{i_{r-1} j_{r-1}}, B_{i_r j_r})
\end{aligned}$$

where we used in the last equality the Codazzi equation and the fact that the sectional curvature of (N^n, h) is constant. Therefore

$$\begin{aligned}\mathrm{div}_M T_r(e_j) &= \frac{1}{(r-1)!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r\\ i}} \epsilon\begin{pmatrix} i_1\, i \ldots i_r\\ j\, j_1 \ldots j_r\end{pmatrix} h((\nabla_{e_i}B)_{i_1j_1}, B_{i_2j_2})\ldots \\ &\qquad\qquad h(B_{i_{r-1}j_{r-1}}, B_{i_rj_r}) \\ &= -\frac{1}{(r-1)!}\sum_{\substack{i_1\ldots i_r\\ j_1\ldots j_r\\ i}} \epsilon\begin{pmatrix} i\, i_1 \ldots i_r\\ j\, j_1 \ldots j_r\end{pmatrix} h((\nabla_{e_i}B)_{i_1j_1}, B_{i_2j_2})\ldots \\ &\qquad\qquad h(B_{i_{r-1}j_{r-1}}, B_{i_rj_r}) \\ &= -\mathrm{div}_M T_r(e_j).\end{aligned}$$

This completes the proof. □

In the following lemma, we give some relations between the r-th mean curvatures and the tensors T_r. These relations are also well-known in codimension 1 (see for instance [**17**]).

Lemma 2.2. *For any integer $r \in \{1, \ldots, m\}$, we have*

$$\mathrm{tr}\,(T_r) = k(r)H_r.$$

Moreover, if r is even

$$\sum_{ij} T_r(e_i, e_j)B(e_i, e_j) = k(r)H_{r+1}$$

and if r is odd

$$\sum_{ij} h(T_r(e_i, e_j), B(e_i, e_j)) = k(r)H_{r+1}$$

where $k(r) = (m-r)\begin{pmatrix} m\\ r\end{pmatrix}$.

Proof. It follows easily from the definitions of T_r and H_r, so we will omit it. □

Now, we extend Theorem 1.1 of Reilly mentioned in the introduction to submanifolds of the sphere.

Theorem 2.1. *Let (M^m, g) be a compact, orientable m-dimensional Riemannian manifold isometrically immersed by ϕ into $\mathbb{S}^n$.*

1. *If $m < n-1$ and if r is an even integer such that $r \in \{0, \dots, m-1\}$, then*
$$\lambda_1(M)\left(\int_M H_r dv_g\right)^2 \leq mV(M)\int_M \left(|H_{r+1}|^2 + H_r^2\right) dv_g.$$
Moreover, if H_r doesn't vanish identically, and if equality holds then ϕ immerses M minimally into $\mathbb{S}^n$ or some geodesic hypersphere of $\mathbb{S}^n$.
2. *If $m = n-1$ and $r \in \{0, \dots, m-1\}$, then*
$$\lambda_1(M)\left(\int_M H_r dv_g\right)^2 \leq mV(M)\int_M \left(H_{r+1}^2 + H_r^2\right) dv_g. \tag{4}$$
If H_r doesn't vanish identically and if equality holds, then (M^m, g) is minimally immersed in $\mathbb{S}^n$ or $\phi(M)$ is a geodesic sphere. Moreover, if $\phi(M)$ is contained in a hemisphere, we have equality if and only if ϕ immerses (M^m, g) as a geodesic hypersphere of $\mathbb{S}^n$.

Remark 2.1. As in Theorem 1.1, the method used doesn't allow us to have an inequality if $m < n-1$ and r is odd.

On the other hand, this theorem can't be deduced from Theorem 1.1 of Reilly by considering the canonical embedding of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, but is a consequence of a more general result given in Proposition 2.1 below.

Let (M^m, g) be a compact m-dimensional Riemannian manifold isometrically immersed by ϕ in $\mathbb{R}^n$ and denotes by B its second fundamental form. We assume that (M^m, g) is endowed with a free divergence $(0,2)$-tensor T and we define a normal vector field H_T at a point $x \in M$, by
$$H_T(x) = \sum_{1 \leq i,j \leq n} T(e_i, e_j) B(e_i, e_j)$$
where $(e_i)_{1 \leq i \leq m}$ is an orthonormal basis of the tangent space of M at x. We have the following generalization of Theorem 1.1:

Proposition 2.1. *Let (M^m, g) be a compact, orientable m-dimensional Riemannian manifold isometrically immersed by ϕ into $\mathbb{R}^n$ and assume that (M^m, g) is endowed with a free divergence $(0,2)$-tensor T. Then, we have*
$$\lambda_1(M)\left(\int_M \mathrm{tr}(T) dv_g\right)^2 \leq mV(M)\left(\int_M |H_T|^2 dv_g\right). \tag{5}$$
Moreover, if H_T doesn't vanish identically and if equality holds, then (M^m, g) is minimally immersed into a geodesic hypersphere of $\mathbb{R}^n$.

This proposition will be a consequence of a generalization of the Hsiung-Minkowski formulas. For this purpose, let us first define a second order differential operator L_T on $C^\infty(M)$ by
$$L_T u = -\mathrm{div}_M(T^\sharp \nabla^M u)$$

where ∇^M is the gradient associated to the metric g and $T^\sharp$ is the symmetric endomorphism associated to T with respect to g (i.e., $g(T^\sharp X, Y) = T(X,Y)$). The differential operator L_T is self-adjoint because T is a free-divergence tensor, and it is easy to see that

$$L_T(u) = -\langle D^2 u, T\rangle \tag{6}$$

where D^2 and $\langle\ ,\ \rangle$ denote respectively the hessian operator and the inner product extended to tensors. Now, if $(\partial_i)_{1\le i\le n}$ and ϕ^i denote respectively the canonical basis of $\mathbb{R}^n$ and the component functions of ϕ in this basis, we set

$$L_T\phi = \sum_{i\le n} L_T\phi^i\partial_i.$$

Now, we can state:

Lemma 2.3. *We have*

$$L_T\phi = -H_T \tag{7}$$

and

$$\frac{1}{2}L_T|\phi|^2 = -\langle \phi, H_T\rangle - \mathrm{tr}\,(T). \tag{8}$$

Proof. The proof of (7) is similar to that of the well-known formula $\Delta\phi = -mH$ and Formula (8) is an immediate consequence of (7). □

Proof of Proposition 2.1. Doing a translation if necessary, we can assume that the center of mass of ϕ is at the origin; that is $\int_M \phi^i dv_g = 0$ for all $i \le n$. From the variational characterization of $\lambda_1(M)$, we have for any i

$$\lambda_1(M)\int_M (\phi^i)^2 dv_g \le \int_M |d\phi^i|^2 dv_g \tag{9}$$

and if the equality holds, then each ϕ^i is an eigenfunction of the Laplacian. From the above inequality and by applying Lemma 2.3 and using a Cauchy-Schwartz inequality, we obtain the following inequalities

$$\begin{aligned}\lambda_1(M)\left(\int_M \mathrm{tr}(T)dv_g\right)^2 &= \lambda_1(M)\left(\int_M \langle H_T,\phi\rangle dv_g\right)^2 \\ &\le \lambda_1(M)\left(\int_M |H_T|^2 dv_g\right)\left(\int_M |\phi|^2 dv_g\right) \\ &\le \left(\int_M |H_T|^2 dv_g\right)\left(\int_M \sum_i |d\phi^i|^2 dv_g\right) \\ &= mV(M)\left(\int_M |H_T|^2 dv_g\right).\end{aligned} \tag{10}$$

This proves the inequality (5) of Proposition 2.1.

Equality case. If (5) is an equality, then inequalities in (10) are equalities too. But since $|H_T|$ doesn't vanish identically on M, we deduce that

$$\lambda_1(M)\sum_i\int_M(\phi^i)^2dv_g=\sum_i\int_M|d\phi^i|^2dv_g$$

this implies with (9) that the functions ϕ_i are eigenfunctions of $\lambda_1(M)$. Hence by Takahashi's theorem ([**19**], Theorem 3) we deduce that ϕ is a minimal immersion of (M^m,g) into a hypersphere of radius $\sqrt{m/\lambda_1(M)}$. □

Proof of Theorem 2.1. The desired inequality can't be deduced from Theorem 1.1, but it will be a consequence of the generalized inequality (5) of Proposition 2.1. In fact, let T_r be the $(0,2)$-tensors associated to the second fundamental form B of ϕ and let i be the canonical embedding of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Then, as before the normal vector field H'_{T_r} associated to the second fundamental form B' of the isometric immersion $i\circ\phi$ is given at $x\in M$ by

$$H'_{T_r}=\sum_{1\le i,j\le n}T_r(e_i,e_j)B'(e_i,e_j)$$

where $(e_i)_{1\le i\le m}$ is an orthonormal basis of the tangent space of M at x. Now, it follows from (5) that

$$\lambda_1(M)\left(\int_M\operatorname{tr}(T_r)dv_g\right)^2\le mV(M)\left(\int_M|H'_{T_r}|^2dv_g\right)\tag{11}$$

now, it is easy to see that $B'=B-g\phi$ and then $H'_{T_r}=H_{T_r}-\operatorname{tr}(T_r)\phi$. This gives us

$$|H'_{T_r}|^2=|H_{T_r}|^2+\operatorname{tr}(T_r)^2$$

therefore, reporting this last relation in (11) we obtain

$$\lambda_1(M)\left(\int_M\operatorname{tr}(T_r)dv_g\right)^2\le mV(M)\int_M\left(|H_{T_r}|^2+\operatorname{tr}(T_r)^2\right)dv_g.\tag{12}$$

Now the inequalities of Theorem 2.1 follow by using Lemma 2.2 which gives us $|H_{T_r}|=k(r)|H_{r+1}|$ and $\operatorname{tr}(T_r)=k(r)H_r$, where $k(r)=(m-r)\begin{pmatrix}m\\r\end{pmatrix}$.

Equality case. If we assume that H_r doesn't vanish identically, then it is also the case for H'_{T_r} and we can deduce as in the previous proof, that if equality holds then M is minimally immersed in a geodesic hypersphere of $\mathbb{R}^{n+1}$ with radius less or equal to 1. If the radius is equal to 1, then M is minimally immersed in $\mathbb{S}^n$ if not M is minimally immersed in a geodesic hypersphere of $\mathbb{S}^n$.

Conversely, if $m=n-1$ and if $\phi(M)$ is a geodesic hypersphere of $\mathbb{S}^n$, then $\lambda_1(M)=(n-1)(H_1^2+1)$. On the other hand $H_r=H_1^r$, and inequality (4) becomes an equality. □

These results are a consequence of a Hsiung-Minkowski formulae for submanifolds of $\mathbb{R}^n$ or $\mathbb{S}^n$. For submanifolds of the hyperbolic space, such a formulae exists but doesn't allow us to generalize these theorems in this case. However, using a different approach, we can obtain a partial result for hypersurfaces of $\mathbb{H}^{n+1}$.

Theorem 2.2. *Let (M^n, g) be a compact, orientable n-dimensional Riemannian manifold isometrically immersed by ϕ into $\mathbb{H}^{n+1}$. Let $r \in \{0, \dots, n-2\}$. If H_r is a positive constant and if ϕ is convex (i.e., its second fundamental form is semi definite), then we have*

$$\lambda_1(M)V(M)H_r^2 \leq n \int_M \left(H_{r+1}^2 - H_r^2\right) dv_g. \tag{13}$$

Moreover, the equality holds if and only if ϕ immerses M as a geodesic hypersphere in $\mathbb{H}^{n+1}$.

Proof. Here, (M^n, g) is isometrically immersed in $\mathbb{H}^{n+1}$ and we assume it to be oriented by a unit normal field ν. Therefore as noticed before, the r-th mean curvatures will be considered as scalar quantities (see (2)) defined over M. In a recent paper, using the fact that any space form $N^{n+1}(\kappa)$ is conformally embedded in $\mathbb{S}^{n+1}$, we establish a relation between r-th mean curvatures and the conformal factor ([**9**]). We recall this result in the case which we are interested in, that is when $\kappa = -1$. Let Π be a conformal embedding of $(\mathbb{H}^{n+1}, can_{\mathbb{H}})$ into $(\mathbb{S}^{n+1}, can_{\mathbb{S}})$ and let f be the function defined on $\mathbb{H}^{n+1}$ such that $\Pi^\star can_{\mathbb{S}} = e^f can_{\mathbb{H}}$. Then we have for any integer $r \in \{0, \dots, n-1\}$ (see Proposition 3.1 of [**9**])

$$\begin{aligned} H_{r+1}^2 - H_r^2 &\\ = (H_{r+1} - FH_r)^2 + e^{f\circ\phi}H_r^2 + \frac{1}{4}|\nabla^M(f\circ\phi)|^2 H_r^2 &\\ - \frac{1}{2k(r)}g(T_r\nabla^M(f\circ\phi), \nabla^M(f\circ\phi))H_r - \frac{1}{k(r)}H_r L_r(f\circ\phi) & \end{aligned} \tag{14}$$

where $L_r = L_{T_r}$, $F = (1/2)can_{\mathbb{H}}(\nabla^{\mathbb{H}^{n+1}} f, \nu) \circ \phi$, $\nabla^{\mathbb{H}^{n+1}}$ and ∇^M denote respectively the gradient of $\mathbb{H}^{n+1}$ and M. Furthermore, we have shown (see the proof of Theorem 1.1 of [**9**]) that for any integer $r \in \{0, \dots, n-2\}$ and under the assumption of the convexity of ϕ

$$\frac{1}{4}|\nabla^M(f\circ\phi)|^2 H_r^2 - \frac{1}{2k(r)}g(T_r\nabla^M(f\circ\phi), \nabla^M(f\circ\phi))H_r \geq 0. \tag{15}$$

Since L_r is selfadjoint and H_r constant, we deduce from (14) and (15) that

$$\int_M \left(H_{r+1}^2 - H_r^2\right) dv_g \geq \int_M (H_{r+1} - FH_r)^2 dv_g + H_r^2 \int_M e^{f\circ\phi} dv_g.$$

Now, if we put $X = \Pi \circ \phi$ and if we denote by X^i its component functions in $\mathbb{R}^{n+2}$, we have

$$\sum_{i \leq n+2} |dX^i|^2 = ne^{f\circ\phi}.$$

Composing Π with a conformal diffeomorphism of $(\mathbb{S}^{n+1}, can)$ if necessary, we can assume that $\int_M X^i dv_g = 0$ ([4]), and thus

$$\begin{aligned} &\int_M \left(H_{r+1}^2 - H_r^2\right) dv_g \\ &\geq \int_M (H_{r+1} - FH_r)^2 dv_g + \frac{H_r^2}{n} \int_M \sum_{i \leq n+2} |dX^i|^2 dv_g \\ &\geq \frac{H_r^2}{n} \lambda_1(M) \int_M \sum_{i \leq n+2} (X^i)^2 dv_g = \frac{H_r^2}{n} \lambda_1(M) V(M). \end{aligned} \tag{16}$$

This proves the inequality in Theorem 2.2.

Equality case. If (M^n, g) is immersed as a geodesic sphere, then $\lambda_1(M) = n(H_1^2 - 1)$. Now, since $H_r = H_1^r$, the inequality in Theorem 2.2 becomes an equality. Conversely, assume that (13) is an equality, then all inequalities in (16) are equalities. Thus, X^i are eigenfunctions of the Laplacian associated to $\lambda_1(M)$ and it follows that

$$ne^{f\circ\phi} = \sum_{i \leq n+2} |dX^i|^2 = -\frac{1}{2} \sum_{i \leq n+2} \Delta (X^i)^2 + \sum_{i \leq n+2} X^i \Delta X^i = \lambda_1(M)$$

and we deduce that $f \circ \phi$ is constant on M. Furthermore, the equality in (16) and Equation (14) imply successively that

$$\frac{H_{r+1}}{H_r} = F \tag{17}$$

and

$$e^{f\circ\phi} = \frac{H_{r+1}^2}{H_r^2} - 1. \tag{18}$$

Now, considering (14) for $r = 0$, we have

$$H_1^2 - 1 = H_1^2 - 2H_1 F + F^2 + e^{f\circ\phi}.$$

Finally, reporting (17) and (18) in this last equality, we get

$$H_r H_1 - H_{r+1} = 0.$$

It is well-known that this implies that M is totally umbilic and thus $\phi(M)$ is a geodesic sphere ([2]). $\square$

In the sequel, since the codimension of the orientable manifold (M^n, g) is 1, we consider r-th mean curvatures as scalar quantities (see (2)) defined on M. As a straightforward consequences of Theorems 1.1, 2.1 and 2.2 we have the following corollaries:

Corollary 2.1. *Let (M^n, g) be a compact, connected orientable n-dimensional Riemannian manifold isometrically immersed by ϕ in $\mathbb{R}^{n+1}$. Let $r \in \{1, \dots, n\}$. If H_r is a positive constant, then we have*

$$\lambda_1(M) \leq n H_r^{2/r}.$$

Moreover, we get equality if and only if ϕ immerses (M^n, g) as a hypersphere in $\mathbb{R}^{n+1}$.

For hypersurfaces of $\mathbb{S}^{n+1}$, we obtain:

Corollary 2.2. *Let (M^n, g) be a compact, connected orientable n-dimensional Riemannian manifold isometrically immersed by ϕ in an open hemisphere of $\mathbb{S}^{n+1}$. Let $r \in \{1, \dots, n-1\}$. If $H_{r+1} > 0$ and if H_r is a positive constant, then we have*

$$\lambda_1(M) \leq n \left(H_r^{2/r} + 1 \right).$$

Moreover, we get equality if and only if ϕ immerses (M^n, g) as a hypersphere in $\mathbb{S}^{n+1}$.

And for hypersurfaces of $\mathbb{H}^{n+1}$, we have:

Corollary 2.3. *Let (M^n, g) be a compact, connected orientable n-dimensional Riemannian manifold isometrically immersed by ϕ in $\mathbb{H}^{n+1}$. For any integer $r \in \{1, \dots, n-1\}$, if H_r is a positive constant and if ϕ is convex (i.e., B is semi definite), then we have*

$$\lambda_1(M) \leq n \left(H_r^{2/r} - 1 \right).$$

Moreover, we get equality if and only if ϕ immerses (M^n, g) as a hypersphere in $\mathbb{H}^{n+1}$.

These corollaries are an immediate consequence of the Maclaurin inequalities which we recall (see for instance [**13**] and [**14**]). Let ϕ be an isometric immersion of a Riemannian manifold (M^n, g) into a simply connected space form $N^{n+1}(\kappa)$ ($\kappa = 0, 1$ or -1 respectively for $\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}$ or $\mathbb{H}^{n+1}$). If for all integer $j \in \{1, \dots, k\}$, we have $H_j > 0$ then

$$H_k^{1/k} \leq H_j^{1/j}$$

with equality at umbilic points. Moreover, we know that if for an integer k, we have:

1. $H_k > 0$ and ϕ is a convex immersion (i.e., B is semi definite), then $H_j > 0$, for any $j \in \{1, \dots, k\}$ ([**20**]).

2. $H_k > 0$ and for $\kappa = 1$, $\phi(M)$ lies in an open hemisphere, then $H_j > 0$, for any $j \in \{1, \ldots, k\}$ ([5]).

Note that the Maclaurin inequalities and Property 1 are still valid for hypersurfaces of any ambiant space.

Another approach allows us to obtain a different upper bounds for $\lambda_1(M)$ of hypersurfaces of $\mathbb{R}^{n+1}$. Indeed, we have:

Theorem 2.3. *Let (M^n, g) be a compact, orientable n-dimensional Riemannian manifold isometrically immersed by ϕ in $\mathbb{R}^{n+1}$. If for $r \in \{0, \ldots, n-2\}$, we have $H_{r+2} > 0$, then*

$$\lambda_1(M) \int_M H_r dv_g \leq nV(M) \sup_M H_{r+2}.$$

Moreover, equality holds if and only if ϕ immerses (M^n, g) as a hypersphere in $\mathbb{R}^{n+1}$.

Proof. From (8), we have

$$\begin{aligned} \frac{1}{2}|\phi| L_r |\phi|^2 &= -\langle \phi, H_{T_r} \rangle |\phi| - \mathrm{tr}(T_r)|\phi| \\ &= -k(r) \left(H_{r+1} \langle \phi, \nu \rangle |\phi| + H_r |\phi| \right) \\ &\leq k(r) \left(|H_{r+1}| |\phi|^2 - H_r |\phi| \right) \end{aligned}$$

hence

$$\int_M |\phi| T_r \left(\nabla^M |\phi|, \nabla^M |\phi| \right) dv_g \leq k(r) \int_M \left(|H_{r+1}| |\phi|^2 - H_r |\phi| \right) dv_g. \tag{19}$$

Now, in [5] (Proposition 3.2), Barbosa and Colares show that if $H_{r+1} > 0$, then T_k is a definite positive $(0,2)$-tensor for any $k \in \{1, \ldots, r\}$. Furthermore, we have in particular that $H_r > 0$. Consequently, we deduce from (19) and the fact that T_r is positive, that

$$\int_M H_r |\phi| dv_g \leq \int_M H_{r+1} |\phi|^2 dv_g$$

and finally from (8) and the above estimate, we obtain

$$\begin{aligned} &\lambda_1(M) k(r) \int_M H_r dv_g \\ &= \lambda_1(M) \int_M \mathrm{tr}(T_r) dv_g \\ &= -\lambda_1(M) \int_M \langle H_{T_r}, \phi \rangle dv_g \end{aligned}$$

$$\leq k(r)\lambda_1(M)\int_M H_{r+1}|\phi|dv_g \leq k(r)\lambda_1(M)\int_M H_{r+2}|\phi|^2 dv_g$$
$$\leq k(r)\lambda_1(M)\sup_M H_{r+2}\int_M |\phi|^2 dv_g \leq k(r)\sup_M H_{r+2}\int_M \sum_i |d\phi_i|^2 dv_g$$
$$= nk(r)V(M)\sup_M H_{r+2}.$$

This completes the proof of Theorem 2.3. Furthermore, it follows from (19) that equality holds if and only if $\phi(M)$ is contained in a geodesic sphere of $\mathbb{R}^{n+1}$. □

3. Upper bounds of $\lambda_1(M)$ in terms of scalar curvature.

First, we deduce from the previous corollaries an unified estimate of $\lambda_1(M)$ in terms of the scalar curvature S for hypersurfaces immersed in a space form $N^{n+1}(\kappa)$ ($\kappa = 0, 1$ or -1 respectively for $\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$). Indeed, we have:

Corollary 3.1. *Let (M^n, g) be a compact, orientable n-dimensional Riemannian manifold isometrically immersed in a simply connected space form $N^{n+1}(\kappa)$. Assume that:*

1. *If $\kappa = 0$, $r \in \{2, \ldots, n\}$ and H_r is a positive constant;*
2. *if $\kappa = 1$, $r \in \{2, \ldots, n-1\}$, $\phi(M)$ is contained in an open hemisphere of $\mathbb{S}^{n+1}$, $H_{r+1} > 0$ and H_r is a constant;*
3. *if $\kappa = -1$, $r \in \{2, \ldots, n-2\}$, ϕ is convex and H_r is a positive constant.*

Then $S > 0$, and we have

$$\lambda_1(M) \leq \frac{\inf_M S}{n-1}.$$

Moreover, equality holds if and only if ϕ immerses (M^n, g) as a geodesic sphere.

Remark 3.1. If (M^n, g) is an Einstein manifold ($n \geq 3$) with positive scalar curvature, then the Lichnerowicz-Obata ([**12**]) estimate for $\lambda_1(M)$ gives us: $\lambda_1(M) \geq S/(n-1)$, equality holding only for the spheres. Now, if (M^n, g) is an Einstein manifold of positive scalar curvature isometrically immersed in $\mathbb{R}^{n+1}$, H_2 is a positive constant and we deduce from Corollary 3.1, that $\phi(M)$ is a geodesic sphere. This is another way to prove that the spheres are the only hypersurfaces of $\mathbb{R}^{n+1}$ which are endowed with an Einstein structure of positive scalar curvature (see for instance Theorem 5.3 p. 36 of [**11**]). We can obtain similar results for the other space forms. Recall that, more generally, Fialkow in [**8**] proved that geodesic spheres are the only compact Einstein hypersurfaces of positive scalar curvature immersed in a space form $N^{n+1}(\kappa)$. Recall also that A. Montiel and A. Ros in [**14**] have shown that geodesic spheres are the only compact hypersurfaces of constant scalar curvature **embedded** in $N^{n+1}(\kappa)$ (with the additionaly hypothesis "$\phi(M)$ contained in a hemisphere" for the spherical case $\kappa = 1$).

Another consequence concerns the Yamabe problem. Indeed, note that T. Aubin ([**4**]) shows that if g is a Yamabe metric of positive scalar curvature on a compact manifold (M^n, g) $(n \geq 3)$, then $\lambda_1(M) \geq S/(n-1)$. Then from our Corollary 3.1, we deduce the following:

Corollary 3.2. *If (M^n, g) is a compact hypersurface of positive scalar curvature immersed in $\mathbb{R}^{n+1}$ and if g is a Yamabe metric (i.e., minimizes the Yamabe functional in its conformal class) then (M^n, g) is a standard sphere.*

Proof of Corollary 3.1*.* This corollary follows from Corollaries 2.1, 2.2 and 2.3, in the case $r = 2$. Under the assumptions of these corollaries and by using the Maclaurin inequalities about r-th mean curvatures, we obtain

$$\lambda_1(M) \leq n\left(H_r^{2/r} + \kappa\right) \leq n(H_2 + \kappa) \tag{20}$$

and equality holds if and only if ϕ immerses (M^n, g) as a geodesic sphere. Now, let $(e_i)_{1 \leq i \leq n}$ be a g-orthonormal basis which diagonalizes the second fundamental form b (i.e., $b(e_i, e_j) = \langle B(e_i, e_j), \nu\rangle = \mu_i \delta_{ij}$). From the Gauss equation, we have

$$S = \kappa n(n-1) + \sum_{i \neq j} \mu_i \mu_j = n(n-1)(\kappa + H_2) \tag{21}$$

and reporting this relation in (20), we obtain the desired inequality. $\square$

As an immediate consequence of Theorem 2.3, we have $\lambda_1(M) \leq \sup_M S/(n-1)$, by applying the inequality for $r = 0$. The techniques used in this theorem don't allow us to extend it to hypersurfaces of $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$. But, by a different method inspired by Heintze's work ([**10**]), we can prove:

Theorem 3.1. *Let (M^n, g) be a compact, orientable n-dimensional Riemannian manifold isometrically immersed in a simply connected space form $N^{n+1}(\kappa)$ ($\kappa = 0, 1$ or -1 respectively for $\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}$ or $\mathbb{H}^{n+1}$) and assume in addition that for $\kappa = 1$, $\phi(M)$ lies in a geodesic ball of radius $\pi/4$. If $S > n(n-1)\kappa$ then we have*

$$\lambda_1(M) \leq \frac{\sup_M S}{n-1}$$

and equality holds if and only if ϕ immerses M as a geodesic sphere.

Before giving the proof of Theorem 3.1, we need to give some preliminary results. Let $p_0 \in N^{n+1}(\kappa)$ and $\exp_{p_0}$ the exponential map at this point. We denote $(x_i)_{1 \leq i \leq n+1}$ the normal coordinates of $N^{n+1}(\kappa)$ centered at p_0 and for all $x \in N^{n+1}(\kappa)$, we set $r(x) = d(p_0, x)$, the geodesic distance between

p_0 and x on $N^{n+1}(\kappa)$. Assume in the case $\kappa = 1$, that $\phi(M)$ lies in an open hemisphere.

Let s_κ and c_κ be the functions defined by

$$s_\kappa(r) = \begin{cases} \sin r & \text{if } \kappa = 1 \\ r & \text{if } \kappa = 0 \\ \sinh r & \text{if } \kappa = -1 \end{cases} \quad \text{and} \quad c_\kappa(r) = \begin{cases} \cos r & \text{if } \kappa = 1 \\ 1 & \text{if } \kappa = 0 \\ \cosh r & \text{if } \kappa = -1. \end{cases}$$

Note that $c_\kappa^2 + \kappa s_\kappa^2 = 1$ and $s'_\kappa = c_\kappa$ and $c'_\kappa = -s_\kappa$.

In the sequel, we denote respectively by ∇^M and ∇^N the gradient associated to g and to the canonical metric of $N^{n+1}(\kappa)$ denoted by h. Then, if we put $X = s_\kappa(r)\nabla^N r$, it is easy to see that the normal coordinates of X are $\left(\frac{s_\kappa(r)}{r}x_i\right)_{1\leq i\leq n+1}$. Furthermore, the tangential and the normal projection of a vector field Y respectively on the tangent bundle and the normal bundle to $\phi(M)$ will be denoted by Y^T and $Y^\perp$.

We recall two lemmas shown by Heintze ([**10**]):

Lemma 3.1. *At any $x \in M$, we have*

$$\sum_{1\leq i\leq n+1} g_x\left(\nabla^M\left(\frac{s_\kappa(r)}{r}x_i\right), \nabla^M\left(\frac{s_\kappa k(r)}{r}x_i\right)\right) = n - \kappa g_x(X^T, X^T). \tag{22}$$

Lemma 3.2. *The vector field $X = s_\kappa(r)\nabla^N r$ satisfies*

$$\kappa\int_M |X^T|^2 dv_g = n\int_M c_\kappa^2 dv_g - n\int_M |H| s_\kappa c_\kappa dv_g.$$

Now, we need the following inequality for the proof of Theorem 3.1:

Lemma 3.3. *For all symmetric free divergence definite positive $(0,2)$-tensor T, we have*

$$\mathrm{tr}\,(T)c_\kappa \leq s_\kappa|H_T| + \mathrm{div}_M(T^\sharp X^T)$$

and if T is the identity, then equality holds.

Proof of Lemma 3.3. Since $T^\sharp$ is a positive symmetric $(1,1)$-tensor, we can define a natural positive symmetric $(1,1)$-tensor $\sqrt{T^\sharp}$ such that $\sqrt{T^\sharp}\circ\sqrt{T^\sharp} = T^\sharp$.

Now let $(e_i)_{1\leq i\leq n}$ be an orthonormal frame at x, such that $\sqrt{T^\sharp}e_n$ lies in the direction of $\nabla^M r$ and let e_n^* be a unit vector orthogonal to $\nabla^N r$ in order to have: $\sqrt{T^\sharp}e_n = \lambda\nabla^N r + \mu e_n^*$. Then at x, we have

$$\operatorname{div}_M(T^\sharp X^T) = \sum_{1\le i\le n} g_x(\nabla^M_{e_i}(T^\sharp X^T), e_i) = \sum_{1\le i\le n} h_x(\nabla^N_{e_i} X^T, T^\sharp e_i) \tag{23}$$
$$= \sum_{1\le i\le n} h_x(\nabla^N_{e_i} X, T^\sharp e_i) - \sum_{1\le i\le n} h_x(\nabla^N_{e_i} X^\perp, T^\sharp e_i)$$
$$= \sum_{1\le i\le n} h_x(\nabla^N_{e_i} X, T^\sharp e_i) + h_x(X, H_T).$$

We need to estimate $\sum_{1\le i\le n} h_x(\nabla^N_{e_i} X, T^\sharp e_i)$. We first have

$$\sum_{1\le i\le n} h_x(\nabla^N_{e_i} X, T^\sharp e_i) \tag{24}$$
$$= \sum_{1\le i\le n} h_x(\nabla^N_{e_i}(s_\kappa \nabla^N r), T^\sharp e_i)$$
$$= c_\kappa h_x(\nabla^N r, T^\sharp(\nabla^N r)^T) + s_\kappa \sum_{1\le i\le n} h_x(\nabla^N_{e_i}\nabla^N r, T^\sharp e_i)$$
$$= c_\kappa h_x(T^\sharp(\nabla^N r)^T, (\nabla^N r)^T) + s_\kappa \sum_{1\le i\le n} h_x(\nabla^N_{\sqrt{T^\sharp}e_i}\nabla^N r, \sqrt{T^\sharp}e_i).$$

Now, we compute the last term of (24). Using the Jacobi fields of $N^{n+1}(\kappa)$, one can prove that $D^2 r = (c_\kappa/s_\kappa)(h - dr\otimes dr)$ (see for instance [**18**]). Then, for all orthogonal vector ξ to $\nabla^N r$ at x, we have the equality

$$h_x(\nabla^N_\xi \nabla^N r, \xi) = \frac{c_\kappa}{s_\kappa}|\xi|_x^2.$$

Thus

$$\sum_{1\le i\le n} h_x(\nabla^N_{\sqrt{T^\sharp}e_i}\nabla^N r, \sqrt{T^\sharp}e_i)$$
$$= \sum_{1\le i\le n-1} h_x(\nabla^N_{\sqrt{T^\sharp}e_i}\nabla^N r, \sqrt{T^\sharp}e_i) + h_x(\nabla^N_{\sqrt{T^\sharp}e_n}\nabla^N r, \sqrt{T^\sharp}e_n)$$
$$= \frac{c_\kappa}{s_\kappa}\sum_{1\le i\le n-1} |\sqrt{T^\sharp}e_i|_x^2 + \mu^2 h_x(\nabla^N_{e_n^*}\nabla^N r, e_n^*)$$
$$= \frac{c_\kappa}{s_\kappa}\sum_{1\le i\le n-1} |\sqrt{T^\sharp}e_i|_x^2 + \mu^2\frac{c_\kappa}{s_\kappa}$$

and reporting this inequality in (24), we obtain

$$\sum_{1\le i\le n} h_x(\nabla^N_{e_i} X, T^\sharp e_i) \tag{25}$$
$$= c_\kappa|\sqrt{T^\sharp}(\nabla^N r)^T|_x^2 + c_\kappa \sum_{1\le i\le n-1} |\sqrt{T^\sharp}e_i|_x^2 + \mu^2 c_\kappa$$

now

$$\lambda^2 = h_x(\sqrt{T^\sharp}e_n, \nabla^N r)^2 = h_x(e_n, \sqrt{T^\sharp}(\nabla^N r)^T)^2 \leq |\sqrt{T^\sharp}(\nabla^N r)^T|_x^2$$

and if $T^\sharp$ is the identity, this last inequality is in fact an equality. Furthermore, it is easy to verify that

$$\lambda^2 + \mu^2 = |\sqrt{T^\sharp}e_n|_x^2.$$

Thus, from (25) and these two last facts, we have

$$\begin{aligned}\sum_{1\leq i\leq n} h_x(\nabla^N_{e_i}X, T^\sharp e_i) &\geq c_\kappa \left(\lambda^2 + \mu^2 + \sum_{1\leq i\leq n-1} |\sqrt{T^\sharp}e_i|_x^2\right)\\ &= \mathrm{tr}\,(T)c_\kappa.\end{aligned}$$

Now, we report this last inequality in (23) and we complete the Proof of Lemma 3.3 by noting that $h_x(X, H_T) \geq -|X||H_T| = -s_\kappa|H_T|$. □

Now, we can give the Proof of Theorem 3.1:

Proof of Theorem 3.1. Let $p_0 \in N$ and $r(x) = d(p_0, x)$. We will use $\frac{s_\kappa(r)}{r}x_i$ as test functions in the variational characterization of $\lambda_1(M)$ but the mean of these functions must be zero. For this purpose, we use a standard argument used by Chavel and Heintze before ([**10**] and [**6**]). Indeed, let Y be the vector field defined by

$$Y_q = \int_M \frac{s_\kappa(d(q,p))}{d(q,p)} \exp_q^{-1}(p) dv_g(p) \in T_qN.$$

From the theorem of fixed point of Brouwer, there exists a point $p_0 \in N$ such that $Y_{p_0} = 0$ and consequently, for a such p_0, the mean of $\frac{s_\kappa(r)}{r}x_i$ will be zero. But for $\kappa = 1$, we must assume $\phi(M)$ contained in a ball of radius $\pi/4$. This guarantees the inclusion of $\phi(M)$ in a ball of center p_0 (the point p_0 such that $Y_{p_0} = 0$) with a radius less or equal to $\pi/2$ (this hypothesis is necessary in the proof of the preceding lemmas). It follows from the variational characterization of $\lambda_1(M)$, that

$$\begin{aligned}&\lambda_1(M)\int_M s_\kappa^2(r)dv_g\\ &= \lambda_1(M)\int_M |X|^2 dv_g = \lambda_1(M)\int_M \sum_{1\leq i\leq n+1}\left(\frac{s_\kappa(r)}{r}x_i\right)^2 dv_g\\ &\leq \int_M \sum_{1\leq i\leq n+1} g\left(\nabla^M\left(\frac{s_\kappa(r)}{r}x_i\right), \nabla^M\left(\frac{s_\kappa(r)}{r}x_i\right)\right)dv_g\end{aligned}$$

and using Lemmas 3.1 and 3.2, we deduce that

$$\begin{aligned}(26)\qquad \lambda_1(M)\int_M s_\kappa^2(r)dv_g &\leq nV(M)-\kappa\int_M |X^T|^2 dv_g\\ &\leq n\kappa\int_M s_\kappa^2 dv_g + n\int_M |H| s_\kappa c_\kappa dv_g\\ &= n\kappa\int_M s_\kappa^2 dv_g + \frac{1}{n-1}\int_M \mathrm{tr}\,(T_1) s_\kappa c_\kappa dv_g\end{aligned}$$

now, from Lemma 3.3, we have

$$\mathrm{tr}\,(T_1)s_\kappa c_\kappa \leq s_\kappa \mathrm{div}_M(T_1^\sharp X^T) - h(X, H_{T_1})s_\kappa$$

and reporting this inequality in (26), we obtain

$$\begin{aligned}&\lambda_1(M)\int_M s_\kappa^2 dv_g\\ &\leq n\kappa\int_M s_\kappa^2 dv_g - \frac{1}{n-1}\int_M h(X,H_{T_1})s_\kappa dv_g + \frac{1}{n-1}\int_M s_\kappa \mathrm{div}_M(T_1^\sharp X^T)dv_g\\ &\leq n\kappa\int_M s_\kappa^2 dv_g + \frac{1}{n-1}\int_M |H_{T_1}|s_\kappa^2 dv_g - \int_M g(\nabla^M s_\kappa, T_1^\sharp X^T)dv_g\\ &= n\kappa\int_M s_\kappa^2 dv_g + n\int_M H_2 s_\kappa^2 dv_g - \int_M s_\kappa c_\kappa T_1(\nabla^M r, \nabla^M r)dv_g.\end{aligned}$$

Since we assume that $S > n(n-1)\kappa$, it follows from (21), that $H_2 > 0$, and from the same argument used in the proof of Theorem 2.3, T_1 is a definite positive $(0,2)$-tensor ([**5**]). Furthermore c_κ and s_κ are positive functions and thus

$$\lambda_1(M)\int_M s_\kappa^2 dv_g \leq n\int_M (H_2+\kappa)s_\kappa^2 dv_g = \frac{1}{n-1}\int_M S s_\kappa^2 dv_g$$

which gives the inequality of Theorem 3.1. Now, equality in this inequality holds if and only if $T_1(\nabla^M r, \nabla^M r) = 0$. Since T_1 is definite positive, this is the case if and only if $\phi(M)$ is a geodesic sphere. This concludes the Proof of Theorem 3.1. □

Acknowledgements. The author would like to express his gratefulness to his advisor Professor S. Ilias.

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Received December 28, 2000.

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