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Let p be an odd prime number, k an imaginary abelian field containing a primitive p-th root of unity, and k_∞/k the cyclotomic $\mathbf{Z}_p$-extension. Denote by L/k_∞ the maximal unramified pro–p abelian extension, and by L' the maximal intermediate field of L/k_∞ in which all prime divisors of k_∞ over p split completely. Let N/k_∞ (resp. N'/k_∞) be the pro–p abelian extension generated by all p-power roots of all units (resp. p-units) of k_∞. In the previous paper, we proved that the $\mathbf{Z}_p$-torsion subgroup of the odd part of the Galois group $\mathrm{Gal}(N \cap L/k_\infty)$ is isomorphic, over the group ring $\mathbf{Z}_p[\mathrm{Gal}(k/\mathbf{Q})]$, to a certain standard subquotient of the even part of the ideal class group of k_∞. In this paper, we prove that the same holds also for the Galois group $\mathrm{Gal}(N' \cap L'/k_\infty)$.

1. Introduction.

Let p be a fixed odd prime number, k an imaginary abelian field containing a primitive p-th root ζ_p of unity, and k_∞/k the cyclotomic $\mathbf{Z}_p$-extension. Let L/k_∞ be the maximal unramified pro–p abelian extension, and L' the maximal intermediate field of L/k_∞ in which all prime divisors of k_∞ over p split completely. We put

$$N = k_\infty(\epsilon^{1/p^n} \mid \epsilon \in E_\infty,\ n \geq 1), \quad N' = k_\infty(\epsilon^{1/p^n} \mid \epsilon \in E'_\infty,\ n \geq 1),$$

where E_∞ (resp. E'_∞) is the group of units (resp. p-units) of k_∞. Put

$$\mathcal{X} = \mathrm{Gal}(L/k_\infty), \qquad \mathcal{Y} = \mathrm{Gal}(N \cap L/k_\infty),$$
$$\mathcal{X}' = \mathrm{Gal}(L'/k_\infty), \qquad \mathcal{Y}' = \mathrm{Gal}(N' \cap L'/k_\infty),$$

and let $\mathcal{X}^-$, $\mathcal{Y}^-$, $\mathcal{X}'^-$, $\mathcal{Y}'^-$ be the odd parts of the respective Galois groups. It is well-known that $\mathcal{X}^-$ is (finitely generated and) torsion free over $\mathbf{Z}_p$ (cf. Washington [14, Corollary 13.29]). It is also known (and is shown similarly) that $\mathcal{X}'^-$ is torsion free over $\mathbf{Z}_p$. One naturally asks whether or not the quotients $\mathcal{Y}^-$ of $\mathcal{X}^-$ and $\mathcal{Y}'^-$ of $\mathcal{X}'^-$ are also torsion free over $\mathbf{Z}_p$. This question arised in the previous investigation [5], [6] on a power integral basis problem over cyclotomic $\mathbf{Z}_p$-extensions.

Let A_∞ be the ideal class group of k_∞, and A_∞^+ its even part. It is conjectured by Greenberg [4] that $A_\infty^+ = \{0\}$, which is far from being settled in general. Under this conjecture, it is known that $\mathcal{Y}^- = \mathcal{X}^-$ and $\mathcal{Y}'^- = \mathcal{X}'^-$, and hence $\mathcal{Y}^-$ and $\mathcal{Y}'^-$ are torsion free over $\mathbf{Z}_p$.

In the preceding paper [7], we proved that the $\mathbf{Z}_p$-torsion subgroup $\mathrm{Tor}\mathcal{Y}^-$ of $\mathcal{Y}^-$ is isomorphic, over the group ring $\mathbf{Z}_p[\mathrm{Gal}(k/\mathbf{Q})]$, to a certain standard subquotient of A_∞^+ (under the assumption that p does not divide the degree $[k:\mathbf{Q}]$). Further, we gave some assertions on the vanishing of this subquotient.

Let $\mathcal{O}_\infty$ be the ring of integers of k_∞, and $\mathcal{O}'_\infty = \mathcal{O}_\infty[1/p]$ the ring of p-integers. The pairs (L, N) and (L', N') are objects associated to $\mathcal{O}_\infty$ and $\mathcal{O}'_\infty$, respectively. Since k_∞/k is wildly ramified at p, it is often more natural to use the p-integers $\mathcal{O}'_\infty$ than $\mathcal{O}_\infty$. Therefore, it is desirable to obtain a corresponding result for the pair $(\mathcal{X}', \mathcal{Y}')$. In this paper, we prove that the $\mathbf{Z}_p$-torsion subgroup $\mathrm{Tor}\mathcal{Y}'^-$ of $\mathcal{Y}'^-$ is also isomorphic to the above mentioned subquotient of A_∞^+ as a $\mathbf{Z}_p[\mathrm{Gal}(k/\mathbf{Q})]$-module. Namely, $\mathrm{Tor}\mathcal{Y}^-$ and $\mathrm{Tor}\mathcal{Y}'^-$ are isomorphic to each other over $\mathbf{Z}_p[\mathrm{Gal}(k/\mathbf{Q})]$.

2. Results.

Let k be an imaginary abelian field with $\zeta_p \in k^\times$, and $\Delta = \mathrm{Gal}(k/\mathbf{Q})$, $\Gamma = \mathrm{Gal}(k_\infty/k)$. We assume that

(H) $\quad p$ does not divide the degree $[k:\mathbf{Q}]$.

Then, we have a canonical decomposition

$$\mathrm{Gal}(k_\infty/\mathbf{Q}) = \Delta \times \Gamma.$$

A $\mathbf{Q}_p$-valued character of Δ defined and irreducible over $\mathbf{Q}_p$ is simply called a $\mathbf{Q}_p$-character. For a $\mathbf{Q}_p$-character Φ of Δ and a $\mathbf{Z}_p[\Delta]$-module X, we denote by X^+, X^- and $X(\Phi)$ the even part, the odd part and the Φ-component $e_\Phi X$ of X, respectively. Here, e_Φ is the idempotent of $\mathbf{Q}_p[\Delta]$ defined by

$$e_\Phi = \frac{1}{|\Delta|}\sum_{\sigma\in\Delta}\Phi(\sigma)\sigma^{-1},$$

which is an element of $\mathbf{Z}_p[\Delta]$ by the assumption (H).

Throughout this paper, we *fix* an *even* $\mathbf{Q}_p$-character Ψ of Δ and its irreducible component ψ over the algebraic closure $\overline{\mathbf{Q}}_p$. Denote by Ψ^* and ψ^* the *odd* characters of Δ associated to Ψ and ψ by

$$\Psi^*(\sigma) = \omega(\sigma)\Psi(\sigma^{-1}), \quad \psi^*(\sigma) = \omega(\sigma)\psi(\sigma^{-1}), \quad (\sigma \in \Delta),$$

respectively, where ω is the character of Δ representing the Galois action on ζ_p. We often regard ψ and ψ^* as primitive Dirichlet characters.

Let k_n $(n \geq 0)$ be the n-th layer of k_∞/k with $k_0 = k$, and A_n the Sylow p-subgroup of the ideal class group of k_n. Let

$$A_\infty = \varinjlim A_n$$

be the inductive limit with respect to the inclusion maps $k_n \to k_m$ $(n < m)$. Denote by $\widetilde{A}_0$ the image of A_0 in A_∞. Let A_∞^Γ be the elements of A_∞ fixed by the action of $\Gamma = \mathrm{Gal}(k_\infty/k)$. It is known (cf. [**4**, Proposition 1]) that $(A_\infty^\Gamma)^+$ is a finite abelian group as a consequence of the Leopoldt conjecture for (k,p) proved by Brumer [**1**]. Hence, so is $(A_\infty^\Gamma/\widetilde{A}_0)(\Psi)$. On the other hand, $\mathrm{Tor}\mathcal{Y}(\Psi^*)$ and $\mathrm{Tor}\mathcal{Y}'(\Psi^*)$ are also finite since $\mathcal{X}^-$ is finitely generated over $\mathbf{Z}_p$ by the theorem of Ferrero and Washington [**2**]. For the trivial character Ψ_0, it is known (cf. [**14**, Proposition 6.16]) that $A_\infty(\Psi_0) = \{0\}$ and $\mathcal{X}(\Psi_0^*) = \{0\}$. So, in what follows, we assume that Ψ is nontrivial (and even).

In [**7**], we proved the following:

Theorem 1. *The finite abelian groups* $\mathrm{Tor}\mathcal{Y}(\Psi^*)$ *and* $(A_\infty^\Gamma/\widetilde{A}_0)(\Psi)$ *are isomorphic to each other.*

As for the subquotient $A_\infty^\Gamma/\widetilde{A}_0$ of A_∞, we proved in [**7**, Proposition 1] the following:

Proposition 1. *When* $\psi(p) \neq 1$, *we have* $(A_\infty^\Gamma/\widetilde{A}_0)(\Psi) = \{0\}$.

For more on this subquotient, see [**7**, Proposition 3] and [**8**].

The main result of this paper is as follows.

Theorem 2. $\mathrm{Tor}\mathcal{Y}'(\Psi^*)$ *is isomorphic to* $(A_\infty^\Gamma/\widetilde{A}_0)(\Psi)$ *as an abelian group.*

We obtain the following corollary from Theorems 1 and 2.

Corollary. *The* $\mathbf{Z}_p[\Delta]$*-modules* $\mathrm{Tor}\mathcal{Y}'^-$ *and* $\mathrm{Tor}\mathcal{Y}^-$ *are isomorphic to each other.*

We put

$$\mathcal{H} = \mathrm{Gal}(N/k_\infty) \quad \text{and} \quad \mathcal{H}' = \mathrm{Gal}(N'/k_\infty).$$

It is known (cf. [**6**, Claim (page 97)]) that, by the restriction map,

$$\mathcal{H}'(\Psi^*) = \mathcal{H}(\Psi^*). \tag{1}$$

This is because the Leopoldt conjecture for (k_n, p) holds for all $n \geq 0$ by [**1**]. It is also known (see Section 4.2 (Proof of Lemma 1)) that, by the restriction map,

$$\mathcal{X}(\Psi^*) = \mathcal{X}'(\Psi^*) \quad \text{when } \psi^*(p) \neq 1. \tag{2}$$

Therefore, when $\psi^*(p) \neq 1$, we have $\mathcal{Y}'(\Psi^*) = \mathcal{Y}(\Psi^*)$. By this and Proposition 1, we see that Theorem 2 follows immediately from Theorem 1 and the following:

Theorem 3. *When* $\psi^*(p) = 1$, $\mathcal{Y}'(\Psi^*)$ *is torsion free over* $\mathbf{Z}_p$.

Remark 1. Let A'_n be the Sylow p-subgroup of the p-ideal-class group of k_n in the sense of Iwasawa [**10**, Section 4.3], and let A'_∞ be the inductive limit of A'_n with respect to the inclusion maps $k_n \to k_m$ $(n < m)$. Denote by $\widetilde{A}'_0$ the image of A'_0 in A'_∞. To talk about the Galois groups $\mathcal{X}'$, $\mathcal{Y}'$, it is more natural to use A'_∞ than A_∞. However, it is known (cf. [**4**, Corollary]) that the natural projections

$$A^+_\infty \longrightarrow A'^+_\infty \quad \text{and} \quad \widetilde{A}^+_0 \longrightarrow \widetilde{A}'^+_0$$

are isomorphisms as a consequence of the Leopoldt conjecture for (k_n, p) $(n \geq 0)$.

Remark 2. It is conjectured that $A^+_\infty = \{0\}$ (cf. [**4**]). We have many numerical examples of (k, p) with $A^+_\infty = \{0\}$, but no counter examples (see Kraft and Schoof [**11**], Kurihara [**12**], Sumida and the author [**9**]). However, the conjecture is not yet proved to be true in general.

3. Proof of Theorem 3.

We recall a standard notation. Let $O = O_\psi$ be the subring of $\overline{\mathbf{Q}}_p$ generated by the values of ψ over $\mathbf{Z}_p$. We identify the subring $e_{\Psi^*}\mathbf{Z}_p[\Delta]$ of $\mathbf{Z}_p[\Delta]$ with O by sending $e_{\Psi^*}\sigma$ to $\psi^*(\sigma)$, $(\sigma \in \Delta)$. Then, for a $\mathbf{Z}_p[\Delta]$-module X, $X(\Psi^*)$ is regarded as an O-module. We fix a topological generator γ of Γ. We identify, as usual, the completed group ring $e_{\Psi^*}\mathbf{Z}_p[\Delta][[\Gamma]]$ with the power series ring $\Lambda = O[[T]]$ by $\gamma = 1 + T$ and the above identification. Thus, for a $\mathbf{Z}_p[\Delta][[\Gamma]]$-module X (such as several Galois groups over k_∞), we can regard $X(\Psi^*)$ as a module over O or Λ. We denote by q the element of $p\mathbf{Z}_p$ such that $\zeta^\gamma = \zeta^{1+q}$ for all $\zeta \in \mu_{p^\infty}$.

Let M/k_∞ be the maximal pro–p abelian extension unramified outside p. The fields N, L, N' and L' are intermediate fields of M/k_∞. We put

$$\mathcal{G} = \mathrm{Gal}(M/k_\infty), \qquad \mathcal{Z}' = \mathrm{Gal}(M/N')$$
$$\mathcal{I} = \mathrm{Gal}(M/L), \qquad \mathcal{I}' = \mathrm{Gal}(M/L').$$

For a $\mathbf{Q}_p$-character Φ of Δ, denote by $M(\Phi)$ the intermediate field of M/k_∞ corresponding to $\bigoplus'_{\Phi'} \mathcal{G}(\Phi')$ by Galois theory where Φ' runs over the $\mathbf{Q}_p$-characters of Δ with $\Phi' \neq \Phi$. Then, $\mathrm{Gal}(M(\Phi)/k_\infty) = \mathcal{G}(\Phi)$. We define $N(\Phi)$, $L(\Phi)$, etc, in a similar way.

As we have mentioned in Section 2, $\mathcal{H}'(\Psi^*) = \mathcal{H}(\Psi^*)$. Therefore, by the assertion [**6**, Lemma 1] on $\mathcal{H}(\Psi^*)$, there exists an injective Λ-homomorphism

$$\iota : \mathcal{H}'(\Psi^*) \hookrightarrow \begin{cases} \Lambda, & \text{when } \psi(p) \neq 1, \\ \Lambda \oplus \Lambda/(T - q), & \text{when } \psi(p) = 1, \end{cases}$$

with a finite cokernel. This is the Δ-decomposed version of [**10**, Theorem 15]. In the next section, we prove the following two lemmas.

Lemma 1. *There exists a* Λ*-isomorphism:*

$$\mathcal{I}'(\Psi^*) \cong \begin{cases} \Lambda, & \text{when } \psi(p) \neq 1, \\ \Lambda \oplus \Lambda/(T-q), & \text{when } \psi(p) = 1. \end{cases}$$

Lemma 2. *We have* $M(\Psi^*) = N'(\Psi^*)L'(\Psi^*)$.

Proof of Theorem 3. Assume that $\psi^*(p) = 1$. We put

$$\overline{\mathcal{I}'}(\Psi^*) = \mathcal{I}'(\Psi^*)\mathcal{Z}'(\Psi^*)/\mathcal{Z}'(\Psi^*).$$

Then, we have $\overline{\mathcal{I}'}(\Psi^*) \subseteq \mathcal{H}'(\Psi^*)$, and

$$\mathcal{Y}'(\Psi^*) \cong \mathcal{H}'(\Psi^*)/\overline{\mathcal{I}'}(\Psi^*).$$

As $\psi^*(p) = 1$, we see from Lemmas 1 and 2 that

$$\overline{\mathcal{I}'}(\Psi^*) \cong \mathcal{I}'(\Psi^*) \cong \Lambda.$$

Let ι be an embedding of $\mathcal{H}'(\Psi^*)$ into Λ with a finite cokernel. By the above, the image $\iota(\overline{\mathcal{I}'}(\Psi^*))$ of $\overline{\mathcal{I}'}(\Psi^*)$ equals a principal ideal (f) of Λ for some $f \in \Lambda$. Therefore, we obtain an injective Λ-homomorphism

$$\mathcal{Y}'(\Psi^*) \hookrightarrow \Lambda/(f)$$

with a finite cokernel. On the other hand, f is relatively prime to p by [**2**]. Hence, $\mathcal{Y}'(\Psi^*)$ is torsion free over $\mathbf{Z}_p$. □

4. Proof of lemmas.

4.1. Preliminaries. In this subsection, we give and recall some assertions on some groups of local universal norms of k_∞/k and the Galois groups $\mathcal{I} = \mathrm{Gal}(M/L)$, $\mathcal{I}' = \mathrm{Gal}(M/L')$. For a while, we fix a prime ideal $\mathfrak{p}$ of k over p. We denote the unique prime ideal of k_n over $\mathfrak{p}$ simply by $\mathfrak{p}$. Let $k_{n,\mathfrak{p}}$ be the completion of k_n at $\mathfrak{p}$, and $\mathcal{U}_{n,\mathfrak{p}}$ the group of principal units of $k_{n,\mathfrak{p}}$. Let

$$\mathcal{V}_{n,\mathfrak{p}} = \bigcap_{m \geq n} N_{m/n}\mathcal{U}_{m,\mathfrak{p}} \quad \text{and} \quad \mathcal{W}_{n,\mathfrak{p}} = \bigcap_{m \geq n} N_{m/n}((k_{m,\mathfrak{p}}^\times)^{(p)})$$

be the groups of universal norms. Here, $N_{m/n}$ denotes the norm map from $k_m^\times$ to $k_n^\times$, and for an abelian group X, $X^{(p)}$ denotes the maximal pro–p quotient. We put

$$\mathcal{U}_n = \prod_{\mathfrak{p}|p} \mathcal{U}_{n,\mathfrak{p}}, \quad \mathcal{V}_n = \prod_{\mathfrak{p}|p} \mathcal{V}_{n,\mathfrak{p}}, \quad \mathcal{W}_n = \prod_{\mathfrak{p}|p} \mathcal{W}_{n,\mathfrak{p}},$$

where $\mathfrak{p}$ runs over the primes of k over p. These are closed subgroups of the maximal pro–p quotient $\widehat{k_n^\times} = (\prod_{\mathfrak{p}|p} k_{n,\mathfrak{p}}^\times)^{(p)}$. Denote by φ_n the natural embedding of $k_n^\times$ into $\widehat{k_n^\times}$. Let E_n (resp. E_n') be the group of units (resp. p-units) of k_n, and let $\mathcal{E}_n$ (resp. $\mathcal{E}_n'$) be the closure of $\varphi_n(E_n)$ (resp. $\varphi_n(E_n')$)

in $\widehat{k_n^\times}$. Let $\mathcal{U}_\infty$, $\mathcal{E}_\infty$, $\mathcal{W}_\infty$, $\mathcal{E}'_\infty$ be the projective limits of $\mathcal{U}_n$, $\mathcal{E}_n$, $\mathcal{W}_n$, $\mathcal{E}'_n$ with respect to the relative norms, respectively:

$$\mathcal{U}_\infty = \varprojlim \mathcal{U}_n \ (= \varprojlim \mathcal{V}_n), \quad \mathcal{E}'_\infty = \varprojlim \mathcal{E}'_n \ (= \varprojlim (\mathcal{W}_n \cap \mathcal{E}'_n)), \quad \text{etc.}$$

These groups are naturally regarded as modules over $\mathbf{Z}_p[\Delta][[\Gamma]]$.

Lemma 3. *The projection $P : \mathcal{W}_\infty \to \mathcal{W}_0$ induces an isomorphism*

$$\mathcal{W}_\infty / \mathcal{W}_\infty^T \cong \mathcal{W}_0.$$

Proof. It is clear that the projection P is surjective and that $\mathcal{W}_\infty^T \subseteq \ker P$. So, it suffices to show that $\ker P \subseteq \mathcal{W}_\infty^T$. Let $u = (u_n)_{n\geq 0}$ be an element of $\ker P$ with $u_n \in \mathcal{W}_n$. As $u_0 = 1$, we see that u_n is contained in $\mathcal{U}_n$. We can write $u_n = w_n^T$ for some $w_n \in \prod_{\mathfrak{p}|p} k_{n,\mathfrak{p}}^\times$ by Hilbert Satz 90. Hence, $u_n = \overline{w}_n^T$, $\overline{w}_n$ being the projection of w_n in $\widehat{k_n^\times}$. Denote by $x^{(n)}$ the element of the product $X = \prod_\ell \widehat{k_\ell^\times}$ whose ℓ-th component is $N_{n/\ell}(\overline{w}_n)$ (resp. 1) for $\ell \leq n$ (resp. $\ell > n$). Since X is compact, $\{x^{(n)}\}$ has an accumulation point x in X. We easily see that $x \in \mathcal{W}_\infty$ and $x^T = u$. Therefore, $\ker P \subseteq \mathcal{W}_\infty^T$. □

By class field theory, it is known (cf. [**14**, Corollary 13.6]) that the inertia group $\mathcal{I}$ is canonically isomorphic to $\mathcal{U}_\infty/\mathcal{E}_\infty$ over $\mathbf{Z}_p[\Delta][[\Gamma]]$. As Ψ^* is odd and $\Psi^* \neq \omega$, it follows that $\mathcal{E}_\infty(\Psi^*) = \{0\}$ by a theorem on units of CM-fields (cf. [**14**, Theorem 4.12]). Therefore, we obtain a Λ-isomorphism

$$\mathcal{I}(\Psi^*) \cong \mathcal{U}_\infty(\Psi^*). \tag{3}$$

On the Λ-structure of $\mathcal{U}_\infty$, it is known (cf. Gillard [**3**, Proposition 1]) that

$$\mathcal{U}_\infty(\Psi^*) \cong \begin{cases} \Lambda, & \text{when } \psi(p) \neq 1, \\ \Lambda \oplus \Lambda/(T-q), & \text{when } \psi(p) = 1. \end{cases} \tag{4}$$

It is also known (cf. [**3**, Proposition 2]) that

$$\mathcal{V}_0(\Psi^*) \cong \begin{cases} O, & \text{when } \psi(p) \neq 1 \text{ and } \psi^*(p) \neq 1, \\ O \oplus O/q, & \text{when } \psi(p) = 1, \\ \{0\}, & \text{when } \psi^*(p) = 1. \end{cases} \tag{5}$$

As for the decomposition group $\mathcal{I}'$, we need to prove the following:

Proposition 2. *The reciprocity law map induces a canonical isomorphism*

$$\mathcal{I}' \cong \mathcal{W}_\infty / \mathcal{E}'_\infty$$

over $\mathbf{Z}_p[\Delta][[\Gamma]]$.

Proof. Let M_n (resp. L'_n) be the maximal abelian extension of k_n contained in M (resp. L'). It suffices to prove that

$$\mathrm{Gal}(M_n/L'_n) \cong \mathcal{W}_n/(\mathcal{W}_n \cap \mathcal{E}'_n) \tag{6}$$

since $\mathcal{I}'$ is the projective limit of $\mathrm{Gal}(M_n/L'_n)$ with respect to the restriction maps. It suffices to show the assertion (6) only when $n=0$ by considering k_n as the base field.

For an integer m (≥ 0), we put

$$W^{(m)} = \prod_{\mathfrak{p}|p} N_{m/0} k^{\times}_{m,\mathfrak{p}} \ (\supseteq \mathcal{U}_0^{p^m}).$$

For a prime divisor $\mathfrak{q}$ of k relatively prime to p, let $U_{\mathfrak{q}}$ be the group of local units (resp. the multiplicative group) of the completion $k_{\mathfrak{q}}$ of k at $\mathfrak{q}$ when $\mathfrak{q}$ is finite (resp. infinite). Let J_k be the group of idèles of k. We define its subgroups A, B, C as follows:

$$A = W^{(m)} \times \prod_{\mathfrak{q}\nmid p}\{1\}, \quad B = \mathcal{U}_0^{p^m} \times \prod_{\mathfrak{q}\nmid p}\{1\}, \quad C = \prod_{\mathfrak{p}|p}\{1\} \times \prod_{\mathfrak{q}\nmid p} U_{\mathfrak{q}},$$

where $\mathfrak{p}$ (resp. $\mathfrak{q}$) runs over the primes of k dividing p (resp. relatively prime to p).

Denote by H the Hilbert p-class field of k. Let $M_{0,m}$ be the maximal intermediate field of M_0/H whose Galois group over H is of exponent p^m. Clearly, $M_{0,m}$ contains k_m. Let $L'_{0,m}$ be the maximal intermediate field of $M_{0,m}/k_m$ in which all prime divisors of k_m over p split completely. We have a natural isomorphism

$$\mathrm{Gal}(M_0/L'_0) \cong \varprojlim \mathrm{Gal}(M_{0,m}/L'_{0,m}), \tag{7}$$

the projective limit being taken with respect to the restriction maps.

It is known that the reciprocity law map induces isomorphisms

$$\mathrm{Gal}(M_{0,m}/k) \cong (J_k/k^{\times}BC)^{(p)} \quad \text{and} \quad \mathrm{Gal}(L'_{0,m}/k) \cong (J_k/k^{\times}AC)^{(p)}.$$

For this, see Sumida [**13**, pp. 692-693]. Therefore, we obtain a canonical isomorphism

$$\mathrm{Gal}(M_{0,m}/L'_{0,m}) \cong (k^{\times}AC/k^{\times}BC)^{(p)} \cong (A/(A\cap(k^{\times}BC)))^{(p)}.$$

We easily see that

$$A\cap(k^{\times}BC) = (W^{(m)} \cap (E'_0\mathcal{U}_0^{p^m})) \times \prod_{\mathfrak{q}\nmid p}\{1\}.$$

Here, we are regarding E'_0 as a subgroup of $\prod_{\mathfrak{p}|p} k^{\times}_{0,\mathfrak{p}}$ in the natural way. Hence, we have

$$\mathrm{Gal}(M_{0,m}/L'_{0,m}) \cong (W^{(m)}/(W^{(m)} \cap (E'_0\mathcal{U}_0^{p^m})))^{(p)}.$$

From this and (7), we obtain

$$\mathrm{Gal}(M_0/L'_0) \cong \mathcal{W}_0/(\mathcal{W}_0 \cap \mathcal{E}'_0)$$

by an elementary but tedious argument on the topology of $\widehat{k_0^{\times}} = (\prod_{\mathfrak{p}|p} k^{\times}_{0,\mathfrak{p}})^{(p)}$, which we leave to the reader. □

4.2. Proof of Lemmas 1 and 2.

Proof of Lemma 1 (*and the formula* (2)). Let B_n be the subgroup of A_n consisting of classes which contain a product of prime ideals of k_n over p, and let B_∞ be the projective limit of B_n with respect to the relative norms.

From class field theory, we see that $\mathcal{I}'/\mathcal{I}$ is canonically isomorphic to B_∞. Let D ($\subseteq \Delta$) be the decomposition group of p at k. Then, we have a natural surjection

$$\mathbf{Z}_p[\Delta/D] \longrightarrow B_\infty \cong \mathcal{I}'/\mathcal{I}$$

over $\mathbf{Z}_p[\Delta]$. We see that $\mathbf{Z}_p[\Delta/D](\Psi^*) = \{0\}$ or O according as $\psi^*(p) \neq 1$ or $\psi^*(p) = 1$. Let $\psi^*(p) \neq 1$. Then, from the above surjection, we see that $\mathcal{I}'(\Psi^*) = \mathcal{I}(\Psi^*)$ (from which (2) follows). Hence, the assertion of Lemma 1 follows from (3) and (4) in this case.

Let $\psi^*(p) = 1$. We have the following exact sequence of $\mathbf{Z}_p[\Delta]$-modules.

$$\{0\} \longrightarrow \mathcal{U}_0 \longrightarrow \left(\prod_{\mathfrak{p}|p} k_{0,\mathfrak{p}}^\times\right)^{(p)} \longrightarrow \mathbf{Z}_p[\Delta/D] \longrightarrow \{0\}.$$

As $\psi^*(p) = 1$, we see from (5) that

$$(\mathcal{W}_0 \cap \mathcal{U}_0)(\Psi^*) = \mathcal{V}_0(\Psi^*) = \{0\}.$$

Therefore, by the above exact sequence, we see that the O-module $\mathcal{W}_0(\Psi^*)$ is free of rank one (or $\mathcal{W}_0(\Psi^*) = \{0\}$). Hence, $\mathcal{W}_\infty(\Psi^*)$ is cyclic over Λ by Lemma 3 and Nakayama's lemma (cf. [**14**, Lemma 13.16]). By this and Proposition 2, $\mathcal{I}'(\Psi^*)$ is cyclic over Λ. Then, we obtain $\mathcal{I}'(\Psi^*) \cong \Lambda$ since $\mathcal{I} \subseteq \mathcal{I}'$ and $\mathcal{I}(\Psi^*) \cong \Lambda$ by (3) and (4). □

Proof of Lemma 2. It is known (cf. [**6**, Proposition 3]) that

$$M(\Psi^*) = N(\Psi^*)L(\Psi^*).$$

Let $\psi^*(p) \neq 1$. Then, $N'(\Psi^*) = N(\Psi^*)$ and $L'(\Psi^*) = L(\Psi^*)$ by (1) and (2). Hence, the assertion follows from the above in this case. Let $\psi^*(p) = 1$. Then, by Lemma 1, $\mathcal{I}'(\Psi^*) \cong \Lambda$. On the other hand, $\mathcal{Z}'(\Psi^*)$ is finitely generated and torsion over Λ by [**10**, Theorems 5, 14]. Therefore, we obtain $\mathcal{Z}'(\Psi^*) \cap \mathcal{I}'(\Psi^*) = \{0\}$, and hence $M(\Psi^*) = N'(\Psi^*)L'(\Psi^*)$. □

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