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RICCI CURVATURE ON THE BLOW-UP OF $\mathbb{C}\mathrm{P}^2$ AT TWO POINTS

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In this note, we compute the Tian's $\alpha_G(M)$-invariant on $\mathbb{C}\mathrm{P}^2\#2\overline{\mathbb{C}\mathrm{P}^2}$. Our result is an improvement of Abdesselem's result in Abdesselem (1997). As a consequence, we obtain a good estimate of Ricci curvature on $\mathbb{C}\mathrm{P}^2\#2\overline{\mathbb{C}\mathrm{P}^2}$ by studying certain complex Monge–Ampère equation.

1. Introduction.

It is well-known that the $\alpha_G(M)$-invariant introduced by Tian plays an important role in the study of the existence of Kähler-Einstein metrics on complex manifolds with positive first Chern class ([**T1**], [**T2**], [**TY**]). Based on the estimate of $\alpha_G(M)$-invariant, Tian in 1990 proved that any complex surface with $c_1(M) > 0$ always admits a Kähler-Einstein metric except in two cases $\mathbb{C}\mathrm{P}^2\#1\overline{\mathbb{C}\mathrm{P}^2}$ and $\mathbb{C}\mathrm{P}^2\#2\overline{\mathbb{C}\mathrm{P}^2}$, i.e., the blow-ups of $\mathbb{C}\mathrm{P}^2$ at one point and two points respectively ([**T2**]). Instead of Kähler-Einstein metric, Koiso constructed a Kähler-Ricci soliton on $\mathbb{C}\mathrm{P}^2\#1\overline{\mathbb{C}\mathrm{P}^2}$ ([**Ko**]). But it is still unknown that there is a Kähler-Ricci soliton on $\mathbb{C}\mathrm{P}^2\#2\overline{\mathbb{C}\mathrm{P}^2}$ or not. Recently, the author studied a sufficient condition for the existence of Kähler-Ricci soliton on a complex manifold with $c_1(M) > 0$ in the sense of Tian's $\alpha_G(M)$-invariant ([**Zh**]). In this note, we compute the Tian's $\alpha_G(M)$-invariant on $\mathbb{C}\mathrm{P}^2\#2\overline{\mathbb{C}\mathrm{P}^2}$ and wish that our estimate was an important step towards finding the Kähler-Ricci soliton on $\mathbb{C}\mathrm{P}^2\#2\overline{\mathbb{C}\mathrm{P}^2}$. Kähler-Ricci soliton can be regarded as a good replacement when a Kähler manifold with $c_1(M) > 0$ doesn't admit a Kähler-Einstein metric ([**Ca**], [**Ha**]). The uniqueness problem of such metrics was solved by Tian and the author recently ([**TZ1**], [**TZ2**], [**TZ3**]). Our result is also an improvement of Abdesselem's result ([**Ab**]). As a consequence, we obtain a good estimate of Ricci curvature on $\mathbb{C}\mathrm{P}^2\#2\overline{\mathbb{C}\mathrm{P}^2}$ by studying certain complex Monge-Ampère equation.

2. Reduction to a local estimate.

Let M be the blow-up of $\mathbb{C}\mathrm{P}^2$ at two points and let π be its natural projection. Without loss of generality, we may assume the two points $p_1 = [0, 0, 1]$ and $p_2 = [0, 1, 0]$. Then $M\setminus(\pi^*p_1 \cup \pi^*p_2)$ is isomorphic to $\mathbb{C}\mathrm{P}^2\setminus(\{p_1\}\cup\{p_2\})$.

If we choose an inhomogeneous coordinates $(z_1, z_2) = [1, z_1, z_2]$ of $\mathbb{C}\mathrm{P}^2$, the Kähler metric

$$\omega_{g_0} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(\log(1+|z_1|^2) + \log(1+|z_2|^2) + \log(1+|z_1|^2+|z_2|^2))$$

can be extended to a Kähler metric g on M which belongs to $c_1(M)$. Clearly, if we take the transformation of inhomogeneous coordinates $\rho_1 : (w_2, w_1) = [w_2, w_1, 1] \to (z_1, z_2) = [1, z_1, z_2]$, i.e., $z_1 = \frac{w_1}{w_2}, z_2 = \frac{1}{w_2}$, then we get a Kähler metric on $\mathbb{C}^2 \setminus \{(0,0)\}$, given by

$$\omega_{g_1} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(\log(1+|w_2|^2) + \log(|w_1|^2+|w_2|^2) + \log(1+|w_1|^2+|w_2|^2)).$$

Similarly, after the transformation of inhomogeneous coordinates $\rho_2 : (w_2, w_1) = [w_2, 1, w_1] \to (z_1, z_2) = [1, z_1, z_2]$, i.e., $z_1 = \frac{1}{w_2}, z_2 = \frac{w_1}{w_2}$, then we also get a Kähler metric on $\mathbb{C}^2 \setminus \{0, 0\}$, given by

$$\omega_{g_2} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(\log(1+|w_2|^2) + \log(|w_1|^2+|w_2|^2) + \log(1+|w_1|^2+|w_2|^2)).$$

Let $\gamma_{j,\theta} (j = 0, 1, 2)$ and σ_0 be automorphisms of $\mathbb{C}\mathrm{P}^2$ given by,

$$\gamma_{j,\theta} : [z_0, z_j, z_2] \to [z_0, e^{i\theta} z_j, z_2],$$
$$\sigma_0 : [z_2, 1, z_1] \to [z_2, z_1, 1].$$

Then $\gamma_{j,\theta}$ and σ_0 generalize a maximal compact subgroups G of automorphisms group of M. Let

$$P_G(M, g) = \left\{\phi \in C^\infty(M) |\ \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\phi > 0, \sup_M \phi = 0, \right.$$
$$\left. \text{and } \phi \text{ is G-invariant}\right\}.$$

In [**T1**], Tian introduced a holomorphic invariant

$$\alpha_G(M) = \sup\left\{\alpha \ \Big| \ \int_M e^{-\alpha\phi} dv_g \le C(\alpha),\ \forall\ \phi \in P_G(M, g)\right\},$$

which is independent of the choice of Kähler form ω_g. In this note, we shall estimate the number of $\alpha_G(M)$.

Let x_i $(i = 1, 2) = |z_i|^2$ (resp. $y_i = |w_i|^2$). Then any G-invariant function is of form $\phi(x_1, x_2)$ and the integral can be divided into three parts,

$$\int_M e^{-\alpha\phi} dv_g = \int_{0\le x_1\le 1, 0\le x_2\le 1} e^{-\alpha\phi} dv_{g_0}$$
$$+ \int_{0<y_1\le 1, 0<y_2\le 1} e^{-\alpha\phi} dv_{g_1} + \int_{0<y_1\le 1, 0<y_2\le 1} e^{-\alpha\phi} dv_{g_2}.$$

So it suffices to estimate each of these three parts of the integral. Note that the computation of part three of the integral is similar to part two.

Let $K_0(x_1,x_2)=\log(1+x_1+x_2)+\log(1+x_1)+\log(1+x_2)$ and $K(x_1,x_2)=K_0(x_1,x_2)+\phi(x_1,x_2)$. Then functions $x_i\frac{\partial K(x_1,x_2)}{\partial x_i}$ $(i=1,2)$ are both strictly increasing for variable $x_i\in[0,+\infty)$. Clearly,

$$x_1\frac{\partial K_0(x_1,x_2)}{\partial x_1}=\frac{x_1}{1+x_1+x_2}+\frac{x_1}{1+x_1},$$
$$x_2\frac{\partial K_0(x_1,x_2)}{\partial x_2}=\frac{x_2}{1+x_1+x_2}+\frac{x_2}{1+x_2}.$$

Since

$$x_1\frac{\partial \phi(x_1,x_2)}{\partial x_1}|_{x_1=+\infty}=0,$$
$$x_2\frac{\partial \phi(x_1,x_2)}{\partial x_2}|_{x_2=+\infty}=0,$$

by using the monotonicity, we get

$$0\le x_1\frac{\partial K(x_1,x_2)}{\partial x_1}\le 2, \tag{2.1}$$
$$0\le x_2\frac{\partial K(x_1,x_2)}{\partial x_2}\le 2.$$

Furthermore, we have:

Lemma 2.1.

$$\frac{\partial K}{\partial x_1}\le\frac{3}{2x_1},\quad x_1\le x_2;$$
$$\frac{\partial K}{\partial x_2}\le\frac{3}{2x_2},\quad x_2\le x_1.$$

Proof. Since ϕ is G-invariant, by the transformatiom, $w_1=\frac{z_1}{z_2}$, $w_2=\frac{1}{z_2}$, i.e.,

$$y_1=\frac{x_1}{x_2},y_2=\frac{1}{x_2},$$

we have $\phi(y_1,y_2)=\phi\left(\frac{1}{y_1},\frac{y_2}{y_1}\right)$ (for simplicity, we still use $\phi(y_1,y_2)$ to mean $\phi(x_1(y_1,y_2),x_2(y_1,y_2))$ here; similarly, $K(y_1,y_2)$ and $K_0(y_1,y_2)$ will denote $K(x_1(y_1,y_2),x_2(y_1,y_2))$ and $K_0(x_1(y_1,y_2),x_2(y_1,y_2))$, respectively). It follows

$$2\partial_1\phi(1,y_2)+y_2\partial_2\phi(1,y_2)=0, \tag{2.2}$$

and

$$2\partial_1K(1,y_2)+y_2\partial_2K(1,y_2)=2\partial_1K_0(1,y_2)+y_2\partial_2K_0(1,y_2)=3.$$

On the other hand, by using the convexity of K, one can check the function with variable u,

$$u\frac{d}{du}K(u^2y_1,uy_2)=2u^2y_2\partial_1K(u^2y_1,uy_2)+uy_2\partial_2K(u^2y_1,uy_2)$$

is strictly increasing (cf. [**Re**]). Hence we obtain that for any $y_1 \le 1$,

$$2y_1\partial_1 K(y_1,y_2) + y_2\partial_2 K(y_1,y_2) \le 2\partial_1 K(1,y_2) + y_2\partial_2 K(1,y_2) = 3.$$

In particular, for any $0 < y_1 \le 1$, we have

$$\partial_1 K(y_1,y_2) \le \frac{3}{2y_1}. \tag{2.3}$$

Since

$$\frac{\partial K(x_1,x_2)}{\partial x_1} = \frac{1}{x_2}\frac{\partial K(y_1,y_2)}{\partial y_1},$$

by (2.3), we get

$$\frac{\partial K(x_1,x_2)}{\partial x_1} \le \frac{3}{2x_1}, \quad x_1 \le x_2.$$

On the other hand, by using the symmetry of $K(x_1,x_2)$ for variables x_1 and x_2, we have

$$\frac{\partial K(x_1,x_2)}{\partial x_2} = \frac{\partial K(x_2,x_1)}{\partial x_2} = \frac{1}{x_1}\frac{\partial K(y_1,y_2)}{\partial y_1}.$$

Again by (2.3), we get

$$\frac{\partial K}{\partial x_2} \le \frac{3}{2x_2}, \quad x_2 \le x_1.$$

□

Lemma 2.2. *Let $C_1 = \{[z_0,1,0]\}$,$C_2 = \{[0,1,z_2]\}$, $C_3 = \{[z_0,0,1]\}$ be three lines of $\mathbb{C}\mathrm{P}^2$. Then $\phi \in P_G(M,g)$ are uniformly locally bounded away from the set of five curves $\cup_{i=1}^3 \pi^* C_i \cup_{i=0}^2 \pi^* p_i$.*

Proof. Since ϕ are almost subharmonic functions, by the normalization condition $\sup_M \phi = 0$, one sees that there is a subset $K \subset [0,2]\times[0,2]$ with Lebseque measure bigger than 1 such that ϕ are uniform bounded on K. Then by (2.1), it is easy to see that $\phi(x_1,x_2)$ are uniform locally bounded on $[0,2]\times[0,2] \setminus ((x_1,0)\cup(0,x_2))$. On the other hand, similar to (2.1), we have

$$0 \le y_1\frac{\partial K(y_1,y_2)}{\partial y_1} \le 2,$$

$$0 \le y_2\frac{\partial K(y_1,y_2)}{\partial y_2} \le 3.$$

Hence we can also prove that $\phi(y_1,y_2)$ are uniform locally bounded on $[0,2]\times[0,2] \setminus ((y_1,0)\cup(0,y_2))$. This completes the proof of lemma. □

Proposition 2.1. *For any $\alpha < \frac{4}{7}$, there is a uniform C such that*

$$\int_{0<x_1\le 1, 0<x_2\le 1} e^{-\alpha\phi} dv_{g_0} \le C.$$

Proof. Let $(x_1, x_2) \in S = \{0 \le x_1 \le 1, 0 \le x_2 \le 1, x_1 \le x_2\}$. Then by Lemma 2.1, we have

$$\begin{aligned} -K(x_1,x_2) &= \int_{x_2}^{1} \partial_2 K(x_1,y)dy + \int_{x_1}^{1} \partial_1 K(x_1,1)dx - K(1,1) \\ &\le -\frac{3}{2}\ln x_1 - 2\ln x_2 - K(1,1). \end{aligned}$$

Similarly, if $(x_1, x_2) \in S' = \{0 \le x_1 \le 1, 0 \le x_2 \le 1, x_2 \le x_1\}$, we have

$$-K(x_1,x_2) \le -\frac{3}{2}\ln x_2 - 2\ln x_1 - K(1,1).$$

Since $dv_{g_0} \le C_1 dx_1 \wedge dx_2 \wedge d\Theta$ (where $d\Theta = d\theta_1 \wedge d\theta_2$, $0 \le \theta_1 \le 2\pi, 0 \le \theta_2 \le 2\pi$), we have

$$\begin{aligned} &\int_{0\le x_1\le 1, 0\le x_2\le 1} e^{-\alpha\phi} dv_{g_0} \\ &\le C_2 \left(\int_S + \int_{S'}\right) e^{-\alpha K(x_1,x_2)} dx_1 dx_2 \\ &\le 2C_2 \int_0^1 \int_0^{x_2} x_1^{\frac{-3\alpha}{2}} x_2^{-2\alpha} dx_1 dx_2 \\ &= \frac{4}{2-3\alpha} C_2 \int_0^1 x_2^{1-\frac{3\alpha}{2}-2\alpha} dx_2. \end{aligned}$$

Clearly, if $\alpha < \frac{4}{7}$, we get

$$\int_{0\le x_1\le 1, 0\le x_2\le 1} e^{-\alpha\phi} dv_{g_0} \le C.$$

□

3. Blow-up transformation.

Lemma 3.1. *Let $k < 1$ be a positive number and $\Delta_k = \{0 < y_1 \le k, 0 < y_2 \le 1,$ and $y_1 \le k y_2\}$. Then for any $(y_1, y_2) \in \Delta_k$, it holds*

$$y_2 \partial_2 K(y_1,y_2) \le \frac{3}{2} + \frac{2+3k}{4+2k} + \frac{k}{2+2k}.$$

Proof. Make transformation $y_1' = y_2 \le 1, y_2' = \frac{y_1}{y_2} \le k$. Then $y_1 = y_1' y_2', y_2 = y_1'$. Moreover, one can check

$$\phi(y_1', y_2') = \phi\left(\frac{1}{y_1'}, \frac{y_2'}{y_1'}\right).$$

Hence

$$2\partial_1\phi(1,y_2') + y_2'\partial_2\phi(1,y_2') = 0. \tag{3.1}$$

Since

$$\begin{aligned}\widetilde{K}_0(y_1', y_2') &= \log y_1' + \log(1+y_1') + \log(1+y_2') + \log(1+y_1'y_2'+y_1') \\ &= \log y_1' + K_0(y_1', y_2'),\end{aligned}$$

then

$$\begin{aligned}y_2\partial_2 K(y_1,y_2) &= y_1'\frac{\partial y_1'}{\partial y_2}\partial_1\widetilde{K}(y_1',y_2') + y_2\frac{\partial y_2'}{\partial y_2}\partial_2\widetilde{K}(y_1',y_2') \\ &= y_1'\partial_1 K(y_1',y_2') - y_2'\partial_2 K(y_1',y_2') + 1 \\ &\leq y_1'\partial_1 K(y_1',y_2') + 1 \\ &\leq \partial_1 K(1,y_2') + 1 \\ &\leq \partial_1 K(1,y_2') + \frac{y_2'}{2}\partial_2 K(1,y_2') + 1.\end{aligned} \tag{3.2}$$

On the other hand,

$$\begin{aligned}&2\partial_1 K_0(1,y_2') + y_2'\partial_2 K_0(1,y_2') \\ &= 1 + \frac{2+3y_2'}{2+y_2'} + \frac{y_2'}{1+y_2'}.\end{aligned} \tag{3.3}$$

Hence combining (3.1), (3.2), (3.3), the lemma is proved. □

Lemma 3.2. *Let* $0 < \delta < \frac{3}{2}$. *Then for any* $(y_1, y_2) \in \Delta_k$, *we have*

$$-K(y_1,y_2) \leq \begin{cases} -\frac{3}{2}\log y_1 - (\frac{3}{2}-\delta)\log y_2 - K(k,1), & or \\ -\frac{1}{2}(\frac{3}{2}+\delta)\log y_1 - c_k\log y_2 - K(k,1), \end{cases}$$

where $c_k = \frac{3}{2} + \frac{2+3k}{4+2k} + \frac{k}{2+2k}$.

Proof. First we assume that $\partial_2 K(k,1) \geq \frac{3}{2} - \delta$. Then by the fact

$$2k\partial_1 K(k,1) + \partial_2 K(k,1) \leq 3,$$

we have

$$k\partial_1 K(k,1) \leq \frac{1}{2}\left(\frac{3}{2}+\delta\right).$$

By using the monotonicity, we get

$$x\partial_1 K(x,1) \leq \frac{1}{2}\left(\frac{3}{2}+\delta\right), \quad \forall\ 0 < x \leq k. \tag{3.4}$$

On the other hand, by Lemma 3.1, for any $0 < y \leq 1$, we have

$$\partial_2 K(y_1,y) \leq \frac{1}{y}\left(\frac{3}{2} + \frac{2+3k}{4+k} + \frac{k}{2+2k}\right). \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$\begin{aligned} & -K(y_1,y_2) \\ & = \int_{y_2}^{1} \partial_2 K(y_1,y)dy + \int_{y_1}^{k} \partial_1 K(x,1)dx - K(k,1) \\ & \leq -\frac{1}{2}\left(\frac{3}{2}+\delta\right)\log y_1 - \left(\frac{3}{2}+\frac{2+3k}{4+2k}+\frac{k}{2+k}\right)\log y_2 - K(k,1). \end{aligned}$$

In the other case of $\partial_2 K(k,1) < \frac{3}{2}-\delta$, by the monotonicity, we have

$$\partial_2 K(k,y) < \left(\frac{3}{2}-\delta\right)\frac{1}{y}, \quad \forall\ 0<y\leq 1.$$

Combining $\partial_1 K(y_1,y_2) \leq \frac{3}{2y_1}$, we get

$$\begin{aligned} & -K(y_1,y_2) \\ & = \int_{y_2}^{1} \partial_2 K(y_1,y)dy + \int_{y_1}^{k} \partial_1 K(x,1)dx - K(k,1) \\ & \leq -\frac{3}{2}\log y_1 - \left(\frac{3}{2}-\delta\right)\log y_2 - K(k,1). \end{aligned}$$

The lemma is proved. □

Lemma 3.3. *Let $k>1$ and $\overline{\Delta}_k = \{0<y_1\leq \frac{1}{2}, 0<y_2\leq \frac{1}{2k},$ and $y_1\leq ky_2\}$. Then for any $(y_1,y_2)\in\overline{\Delta}_k$, we have*

$$\begin{aligned} & -K(y_1,y_2) \\ & < \begin{cases} -\frac{3}{2}\log y_1 - b_1\log y_2 - K(\frac{1}{2},\frac{1}{2k}), & \text{if } \frac{1}{2k}\partial_2 K(\frac{1}{2},\frac{1}{2k}) < b_1 \\ -\frac{1}{2}(3-b_j)\log y_1 - b_{j+1}\log y_2 & \\ \quad -K(\frac{1}{2},\frac{1}{2k}), & \text{if } b_j \leq \frac{1}{2k}\partial_2 K(\frac{1}{2},\frac{1}{2k}) < b_{j+1}, \end{cases} \end{aligned}$$

where $b_1 = \frac{2}{3}-\delta$, and $b_{j+1} = 3-\left(\frac{1}{2}\right)^j(3-b_1)$, $j=1,2,\dots$.

Proof. The proof is similar to that of Lemma 3.2. We omit it. □

Lemma 3.4. *There are a positive number $\alpha>\frac{1}{2}$ and a uniform constant C such that*

$$\int_{\Delta_{\frac{1}{4}}} e^{-\alpha\phi} dv_{g_1} \leq C,$$

where $\Delta_{\frac{1}{4}} = \{0<y_1\leq \frac{1}{4}, 0<y_2\leq 1,$ and $y_1\leq \frac{1}{4}y_2\}$.

Proof. Let $c_0 = 2+\frac{19}{90} < \frac{9}{4}$. Then it is clear that there are two positive numbers $\alpha_0>\frac{1}{2}$ and δ_0 such that $c_0-\frac{1}{4} < \frac{1}{\alpha_0}-\frac{1}{2}\delta_0$. We first suppose that for all $(y_1,y_2)\in(0,\frac{1}{4}]\times(0,1]$,

$$-K(y_1,y_2) \leq \frac{3}{2}\log y_1 - \left(\frac{3}{2}-\delta_0\right)\log y_2 - K\left(\frac{1}{4},1\right).$$

Since

$$dv_{g_1} \le C_1 e^{-K_0} dy_1 \wedge dy_2 \wedge d\Theta \le C_1'(y_1+y_2)^{-1} dy_1 \wedge dy_2 \wedge d\Theta,$$

we have

$$\begin{aligned}
&\int_{0<y_1\le\frac{1}{4},0<y_2\le 1} e^{-\alpha\phi} dv_{g_1} \\
&\int_{0<y_1\le\frac{1}{4},0<y_2\le 1} e^{-\alpha K} e^{-(1-\alpha)K_0} dy_1 \wedge dy_2 \wedge d\Theta \\
&\le C_2 \int_0^{\frac{1}{4}} \int_0^1 (y_1+y_2)^{-(1-\alpha)} y_1^{-\frac{3\alpha}{2}} y_2^{-(\frac{3}{2}-\delta_0)} dy_1 dy_2 \\
&\le C_3 \int_0^{\frac{1}{4}} \int_0^1 y_1^{-\frac{3\alpha}{2}-\frac{1-\alpha}{s}} y_2^{-(\frac{3}{2}-\delta_0)-\frac{1-\alpha}{t}} dy_1 dy_2,
\end{aligned}$$

where s and t are two positive numbers satisfying $\frac{1}{s}+\frac{1}{t}=1$. By choosing $t<2$ sufficiently closely to 2, we see that there are positive numbers s,t and $\alpha>\frac{1}{2}$ such that

$$\frac{3\alpha}{2}+\frac{1-\alpha}{s}<1, \text{ and } \alpha\left(\frac{3}{2}-\delta_0\right)+\frac{1-\alpha}{t}<1.$$

Hence we obtain a uniform constant such that

$$\int_{0<y_1\le k,0<y_2\le 1} e^{-\alpha\phi} dv_{g_1} \le C. \tag{3.6}$$

By (3.6) and Lemma 3.2, we may assume that for any $(y_1,y_2)\in\Delta_{\frac{1}{4}}$,

$$-K(y_1,y_2) < -\left(\frac{3}{2}+\delta_0\right)\log y_1 - c_0 \log y_2 - K\left(\frac{1}{4},1\right).$$

Let $p=1-\frac{\alpha_0}{2}(\frac{3}{2}+\delta_0)>0$. Then

$$\begin{aligned}
&\int_{\Delta_{\frac{1}{4}}} e^{-\alpha_0\phi} dv_{g_1} \\
&\le C_4 \int_0^1 dy_2 \int_0^{\frac{1}{4}y_2} (y_1+y_2)^{-(1-\alpha_0)} y_1^{-\frac{\alpha_0}{2}(\frac{3}{2}+\delta_0)} y_2^{-c_0\alpha_0} dy_1 \\
&= \frac{C_4}{p} \int_0^1 dy_2 \int_0^{\frac{1}{4^p}y_2^p} (y_1^{\frac{1}{p}}+y_2)^{-(1-\alpha_0)} y_2^{-c_0\alpha_0} dy_1 \\
&\le \frac{C_5}{p} \int_0^1 dy_2 \int_0^{\frac{1}{4^p}y_2^p} y_1^{\frac{-(1-\alpha_0)}{ps}} y_2^{-\frac{1-\alpha_0}{t}} y_2^{-c_0\alpha_0} dy_1 \\
&\le C_6 \int_0^1 y_2^{p-\alpha_0 c_0-(1-\alpha_0)} dy_2,
\end{aligned} \tag{3.7}$$

where s and t are two positive numbers satisfying $\frac{1}{s}+\frac{1}{t}=1$. By the choice of numbers δ_0 and α_0, it is clear $p-\alpha_0 c_0-(1-\alpha_0)>-1$. Hence

$$\int_{\Delta_{\frac{1}{4}}} e^{-\alpha_0\phi}dv_{g_1}\le C. \tag{3.8}$$

By combining (3.6) and (3.8), the lemma is proved. □

Lemma 3.5. *For any positive number ϵ, there is a uniform constant C depending only on ϵ such that*

$$\int_{\overline{\Delta}_5} e^{-(\frac{1}{2}-\epsilon)\phi}dv_{g_1}\le C,$$

where $\overline{\Delta}_5=\{0<y_1\le\frac{1}{2}, 0<y_2\le\frac{1}{10},$ *and* $y_1\le 5y_2\}$.

Proof. From the proof of Lemma 3.4, we may assume that

$$b_j\le\frac{1}{10}\partial_2 K\left(\frac{1}{2},\frac{1}{10}\right)<b_{j+1},$$

for some integer j, and

$$-K(y_1,y_2)<-\frac{1}{2}(3-b_j)\log y_1-b_{j+1}\log y_2-K\left(\frac{1}{2},\frac{1}{10}\right),$$

where $b_{j+1}=3-\left(\frac{1}{2}\right)^j(3-b_1)$, and $b_1=\frac{3}{2}-\delta$.

Let $\alpha_0=\frac{1}{2}-\epsilon$ and $p=1-\frac{\alpha_0}{2}(3-b_j)>0$. Then one can check $p-\alpha_0 b_{j+1}-(1-\alpha_0)\ge -1+\epsilon'$, for some positive number ϵ' depending only on ϵ. Hence similar to (3.7), we get,

$$\int_{\overline{\Delta}_5} e^{-\alpha_0\phi}dv_{g_1}\le C\int_0^{\frac{1}{10}} y_2^{p-\alpha_0 b_{j+1}-(1-\alpha_0)}dy_2\le C'.$$

The lemma is proved. □

Lemma 3.6. *There is a positive number $\alpha>\frac{1}{2}$ and a uniform constant C such that*

$$\int_{\Delta'_5} e^{-\alpha\phi}dv_{g_1}\le C,$$

where $\Delta'_5=\{0<y_1\le 1, 0<y_2\le\frac{1}{5},$ *and* $y_1\ge 5y_2\}$.

Proof. As in the proof of Lemma 3.1, we make a transformation, $y'_1=y_1, y'_2=\frac{y_2}{y_1}\le\frac{1}{5}$. Then $y_1=y'_1, y_2=y'_1y'_2$. Moreover, one can check

$$\phi(y'_1,y'_2)=\phi\left(\frac{1}{y'_1},\frac{y'_2}{y'_1}\right).$$

Hence

$$2\partial_1\phi(1,y'_2)+y'_2\partial_2\phi(1,y'_2)=0.$$

Since

$$K_0(y_1', y_2') = \log(1 + y_2') + \log(1 + y_1'y_2') + \log(1 + y_1'y_2' + y_1'),$$

then, for any $y_2' \leq \frac{1}{5}$,

$$2\partial_1 K_0(1, y_2') + y_2'\partial_2 K_0(1, y_2') = 1 + \frac{2y_2'}{2 + y_2'} + \frac{4y_2'}{1 + y_2'} < 2.$$

It follows that for any $y_1' \leq 1$, and $y_2' \leq \frac{1}{5}$,

$$2y_1'\partial_1 K(y_1', y_2') + y_2'\partial_2 K(y_1', y_2') < 2.$$

In particular, there is a positive number δ such that

$$\partial_1 K(y_1', y_2') < \frac{1 - \delta}{y_1'} \text{ and } \partial_2 K(y_1', y_2') < \frac{2(1 - \delta)}{y_2'}.$$

Hence one can choose a positive number $\alpha > \frac{1}{2}$ such that

$$\int_{\Delta_5'} e^{-\alpha\phi} dv_{g_1} \leq C_1 \int_0^{\frac{1}{5}} dy_2' \int_0^1 e^{-\alpha K(y_1', y_2')} dy_1' \leq C.$$

□

Combining Lemma 3.4, Lemma 3.5 and Lemma 3.6, we obtain:

Proposition 3.1. *For any positive number ϵ, there is a uniform constant C depending only on ϵ such that*

$$\int_{0<y_1\leq 1, 0<y_2\leq 1} e^{-(\frac{1}{2}-\epsilon)\phi} dv_{g_1} \leq C.$$

Proof. By Lemma 3.4, Lemma 3.5 and Lemma 3.6, it suffices to prove that for any $(y_1, y_2) \in (0, 1] \times (0, 1] \setminus (\cup\Delta_{\frac{1}{4}} \cup \overline{\Delta}_5 \cup \Delta_5')$, $\phi(y_1, y_2)$ is uniformly bounded. This follows from Lemma 2.2 immediately. □

Combining Propositions 2.1 and 3.1, we prove:

Theorem 3.1. *Let M be the blow-up of CP^2 at two points. Then $\alpha_G(M) \geq \frac{1}{2}$.*

Remark 3.1. In [**Ab**], Abdesselem proved $\alpha(M) \geq \frac{1}{4}$ on $\mathbb{C}\mathrm{P}^2\#2\overline{\mathbb{C}\mathrm{P}^2}$. Theorem 3.1 is an improvement of Abdesselem's result. we guess that $\alpha_G(M) = \frac{1}{2}$ on $\mathbb{C}\mathrm{P}^2\#2\overline{\mathbb{C}\mathrm{P}^2}$.

4. Estimate of Ricci curvature.

In this section, we prove:

Theorem 4.1. *Let* $M = \mathbb{C}\mathrm{P}^2\#2\overline{\mathbb{C}\mathrm{P}^2}$. *Then there exists a Kähler metric with its Kähler form* $\omega \in c_1(M)$ *such that Ricci curvature of* ω *is not less than* $\frac{3}{4}$.

Proof. Choose a G-invariant Kähler form $\omega_g \in c_1(M)$ of M. Then there is a smooth function h such that

$$\begin{cases} \mathrm{Ric}(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}h, \\ \int_M e^h\omega_g^n = \int_M \omega_g^n. \end{cases}$$

We consider the following complex Monge-Ampère equations with one parameter $t \in [0, 1]$,

$$\begin{cases} \det(g_{i\bar{j}} + \phi_{i\bar{j}}) = \det(g_{i\bar{j}})e^{h-t\phi}, \\ \det(g_{i\bar{j}} + \phi_{i\bar{j}}) > 0. \end{cases}$$

Then by a result in [**T1**] together with Theorem 3.1, we conclude that for any $t < \frac{3}{4}$, there is a smooth function ϕ solves the above equation on t. It follows

$$\mathrm{Ric}(\omega_\phi) = t\omega_\phi + (1 - t)\omega_g > t\omega_\phi.$$

The theorem is proved. □

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