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In this paper we study a family of algebraic deformations of regular coadjoint orbits of compact semisimple Lie groups with the Kirillov Poisson bracket. The deformations are restrictions of deformations on the dual of the Lie algebra. We prove that there are non isomorphic deformations in the family. The star products are not differential, unlike the star products considered in other approaches. We make a comparison with the differential star product canonically defined by Kontsevich's map.

1. Introduction.

Coadjoint orbits of Lie groups are symplectic manifolds that can be used to model physical systems that have a continuous group of symmetries. The Kirillov-Kostant orbit principle allows in many cases to associate canonically a unitary representation to the orbit. The Hilbert space of the representation can then be thought as the Hilbert space of the quantum theory. A quantization map which takes a class of functions on the phase space to operators in such Hilbert space can be constructed. This is the approach of geometric quantization (see reference [**24**] for a review).

On the other hand, the pioneering work by Bayen et al. [**3**] on deformation quantization raised the problem of quantizing the coadjoint orbits with a radically different method. However, being based on the same physical principles, it is natural to expect a relation between the two approaches. In fact, it was thought that deformation quantization, which "forgets" about the Hilbert space on which the quantum algebra is represented, could nevertheless throw light on the Kirillov-Kostant orbit principle [**14**]. The algebra that appears in geometric quantization is defined as the quotient of the enveloping algebra by a prime ideal which is contained in the kernel of the corresponding representation [**24**]. The method of geometric quantization is however more general than the Kirillov-Kostant orbit principle. A comparison with deformation quantization for the case of $\mathbb{R}^{2n}$ with the standard symplectic structure was done in reference [**15**].

In the work of Bayen et al. [**3**] only flat symplectic manifolds were studied. The existence of a deformation quantization of general symplectic manifolds was first established by De Wilde and LeComte [**9**], and using different methods by Omori, Maeda and Yoshioka [**22**] and by Fedosov [**10**]. For a comparison between the methods of De Wilde and Le Comte and Fedosov, see reference [**8**]. In reference [**21**], the existence of tangential deformations for any regular Poisson manifold was proven. In reference [**19**], Kontsevich settled the fundamental question of the existence of deformations for arbitrary (formal) Poisson manifolds. In all these works the deformations are taken to be differential, that is, the product structure in the deformed algebra is defined through bidifferential operators.

Explicit star products for non flat manifolds are not easy to construct. In reference [**16**], Gutt constructed a star product on the cotangent bundle of a Lie group. In reference [**5**], Cahen and Gutt constructed a deformation of the algebra of polynomials on the regular coadjoint orbits of compact semisimple groups, using the fact that the universal enveloping algebra is a deformation of the algebra of polynomials on the dual of the Lie algebra [**17**]. They showed that, although the deformation on the whole space is differential, the one induced on the orbit is not. Moreover, in reference [**6**] they show that for semisimple groups "tangential" deformations (that is, deformations on the ambient space that restrict well to the orbits) that are at the same time differential and that extend over the origin do not exist. Deformations of coadjoint orbits were also studied in [**2**] in terms of a polarization of the orbit (also used in geometric quantization). The resulting star product is covariant. More generally, deformations of Kähler manifolds were studied in [**7**].

In reference [**12**] a family of star products on coadjoint orbits of semisimple Lie groups was constructed as a quotient of the enveloping algebra by a suitable ideal. With a certain choice inside the family of deformations one obtains the same star product as in reference [**5**]. For another choice, in the special case of SU(2), the deformed algebra turns out to be the one of geometric quantization [**12**]. In this case we can associate to the deformation quantization a unitary representation in the spirit of Berezin [**4**].

In the present work we further study the properties of this family of deformations. The organization of the paper is as follows. In Section 2, we review the construction of the algebraic star products on the orbit [**12**] and show that there exist non equivalent products associated with a given algebraic Poisson bracket. We also show that the ideal used to quotient the enveloping algebra is prime. In fact, in geometric quantization the quantum algebra is the enveloping algebra modulo a prime ideal; this ideal is contained in the kernel of the representation. In Section 3, we use Kontsevich's theorem on differential star products to show that the Kirillov Poisson structure on the dual space of the Lie algebra of a semisimple Lie group has only one possible deformation. In Section 4, we study the algebraic star products on the orbit and show that they are not, in general, differential. In Section 5, we show different ways of constructing star products on the orbit.

2. Deformation of the polynomial algebra of a regular orbit.

In this section we review the results of reference [**12**] where a family of deformations of the polynomial algebra of a regular coadjoint orbit of a semisimple Lie group was constructed. We show that the different deformations in the family are not necessarily equivalent by exhibiting a counterexample.

Let G be a complex semisimple Lie group of dimension n and rank m, $\mathcal{G}$ its Lie algebra and U the enveloping algebra of $\mathcal{G}$. Let $T_{\mathbb{C}}(\mathcal{G})$ be the full tensor algebra of $\mathcal{G}$ over $\mathbb{C}$. Consider the algebra $T_{\mathbb{C}}(\mathcal{G})[[h]]$ and its proper two sided ideal

$$(1)\quad \mathcal{L}_{[h]} = \sum_{X,Y\in\mathcal{G}} T_{\mathbb{C}}(\mathcal{G})[[h]] \otimes (X\otimes Y - Y\otimes X - h[X,Y]) \otimes T_{\mathbb{C}}(\mathcal{G})[[h]].$$

We define $U_{[h]} = T_{\mathbb{C}}(\mathcal{G})[[h]]/\mathcal{L}_{[h]}$.

Definition 2.1. An associative algebra $\mathcal{A}_{[h]}$ over $\mathbb{C}[[h]]$ is a formal deformation of a Poisson algebra $(\mathcal{A}, \{\ ,\ \})$ over $\mathbb{C}$ if there exists an isomorphism of $\mathbb{C}[[h]]$-modules $\psi : \mathcal{A}[[h]] \longrightarrow \mathcal{A}_{[h]}$ satisfying the following properties

(a) $\psi(f_1f_2) = \psi(f_1)\psi(f_2) \bmod (h)$.
(b) $\psi(f_1)\psi(f_2) - \psi(f_2)\psi(f_1) = h\psi(\{f_1, f_2\}) \bmod(h^2)$.

Because of its relation with the problem of quantization, $\mathcal{A}_{[h]}$ is sometimes called a *deformation quantization* of $\mathcal{A}$.

Notice that in the above definition we can substitute $\mathbb{C}[[h]]$ by $\mathbb{C}[h]$. The algebra is then a module over $\mathbb{C}[h]$ which will be denoted as $\mathcal{A}_h$. We will say that $\mathcal{A}_h$ is a $\mathbb{C}[h]$-*deformation* of $\mathcal{A}$. Notice that a $\mathbb{C}[h]$-deformation extends to a formal deformation, but the converse is not always true. Also, a $\mathbb{C}[h]$-deformation can be specialized to any value of the parameter h, since the ideal generated by the element $h - h_0$ is proper in $\mathcal{A}_h$. One obtains then a complex algebra, $\mathcal{A}_{h_0} = \mathcal{A}_h/(h - h_0)$.

It is well-known that $U_{[h]}$ is a formal deformation of $\mathbb{C}[\mathcal{G}^*][[h]]$ equipped with the Kirillov Poisson bracket [**17**].

We denote by $p_i,\ i = 1, \dots m$ the algebraically independent homogeneous generators of the subalgebra of invariant polynomials on $\mathcal{G}^*$,

$$(2)\quad I = \{p \in \mathbb{C}[\mathcal{G}^*] \mid p(\mathrm{Ad}^*(g)\xi) = p(\xi) \quad \forall\xi \in \mathcal{G}^*,\, g \in G\} = \mathbb{C}[p_1, \dots p_m],$$

given by Chevalley's theorem. If $S(\mathcal{G})$ is the algebra of symmetric tensors on $\mathcal{G}$, we can identify canonically $\mathrm{Pol}(\mathcal{G}^*) = \mathbb{C}[\mathcal{G}^*] \approx S(\mathcal{G})$.

Let $\{X_1, \dots, X_n\}$ be a basis for $\mathcal{G}$ and let $\{x_1, \dots, x_n\}$ be the corresponding generators of $\mathbb{C}[\mathcal{G}^*]$. Then the symmetrizer map $\mathrm{Sym} : \mathbb{C}[\mathcal{G}^*] \longrightarrow T_{\mathbb{C}}(\mathcal{G})$

is given by

$$\mathrm{Sym}(x_1 \dots x_p) = \frac{1}{p!} \sum_{s \in S_p} X_{s(1)} \otimes \cdots \otimes X_{s(p)} \tag{3}$$

where S_p is the group of permutations of order p. The composition of the symmetrizer with the natural projection $T_{\mathbb{C}}(\mathcal{G}) \longrightarrow U$ is a linear isomorphism that gives the identification $\mathbb{C}[\mathcal{G}^*] \approx U$. Moreover, it sends the invariant polynomials I isomorphically into the center of U (see for example reference [**23**]). We can extend the symmetrizer map as Sym: $\mathbb{C}[\mathcal{G}^*][[h]] \longrightarrow T_{\mathbb{C}}(\mathcal{G})[[h]]$, and the projection $\pi_h : T_{\mathbb{C}}(\mathcal{G})[[h]] \longrightarrow U_{[h]}$. Then $P_i = \pi_h \circ \mathrm{Sym}(p_i)$ are also central elements. Note that $\pi_h \circ \mathrm{Sym}$ can be used as the isomorphism ψ in Definition 2.1, $\psi : \mathbb{C}[\mathcal{G}^*][[h]] \to U_{[h]}$.

We consider now the compact real form of G with $\mathcal{G}^r$ the real Lie algebra ($\mathcal{G}$ still denotes the complex Lie algebra). The coadjoint orbits are algebraic manifolds given by the constrains,

$$p_i(x_1, \dots x_n) = c_i^0, \quad c_i^0 \in \mathbb{R}, \quad i = 1, \dots m.$$

There is a one to one correspondence from the set of orbits with the elements of a Weyl chamber in the Cartan subalgebra. Regular orbits are the orbits of elements in the interior of a Weyl chamber, and they have maximal dimension. Non regular orbits are given by constants c_i^0 satisfying some constrains. This means that if c_i^0 define a regular orbit, there is a neighborhood of the orbit that is foliated with regular orbits. We will use this property in the next section.

Let Θ^r be a regular orbit. Then, the ideal of polynomials in $\mathbb{R}[\mathcal{G}^{r*}]$ that vanish on Θ^r is generated by the elements $p_i - c_i^0$ [**20**], so we can define

$$\mathcal{I}_0 = (p_i - c_i^0, \ i = 1, \dots, m) \subset \mathbb{R}[\mathcal{G}^{r*}],$$

and the algebra of restrictions of polynomials to the orbit is $\mathbb{R}[\Theta^r] = \mathbb{R}[\mathcal{G}^{r*}]/\mathcal{I}_0$. We take the complexification of this algebra $\mathbb{C}[\mathcal{G}^*]/\mathcal{I}_0$ (we denote still by $\mathcal{I}_0$ the ideal in the complexified algebra), which is the algebra of polynomials on the complex orbit Θ, $\mathbb{C}[\Theta]$. Consider a regular orbit and define the two sided ideal in $U_{[h]}$ generated by the elements $P_i - c_i(h)$ $i = 1, \dots, m$

$$\mathcal{I}_{[h]} = (P_i - c_i(h), \ i = 1, \dots, m) \subset U_{[h]}, \tag{4}$$

where $c_i(h) \in \mathbb{C}[[h]]$ is such that $c_i(0) = c_i^0$ and $P_i = \mathrm{Sym}(p_i)$. In [**12**] it was shown the following:

Theorem 2.1. *The algebra $U_{[h]}/\mathcal{I}_{[h]}$ is a formal deformation of $\mathbb{C}[\Theta] = \mathbb{C}[\mathcal{G}^*]/\mathcal{I}_0$. $U_h/\mathcal{I}_h$ is a $\mathbb{C}[h]$-deformation of $\mathbb{C}[\Theta] = \mathbb{C}[\mathcal{G}^*]/\mathcal{I}_0$.*

Regularity is a technical assumption to show that $U_{[h]}/\mathcal{I}_{[h]}$ is a free module isomorphic to $\mathbb{C}[\Theta][[h]]$.

The ideal $\mathcal{I}_0 \subset \mathbb{C}[\mathcal{G}^*]$ is prime since the corresponding algebraic variety is irreducible. We want to show now that the ideal $\mathcal{I}_h \in U_h$ is prime. We define first a grading in U_h. If $\{X_i, i = 1, \dots, n\}$ is a basis of $\mathcal{G}$ we set $\deg(X_i) = \deg(h) = 1$. This is a set of generators for $T_{\mathbb{C}}(\mathcal{G})[h]$. Notice that the relations in $\mathcal{L}_h$ (1) are homogeneous with respect to this grading, so a grading is defined on U_h. The degree of an inhomogeneous element in U_h is the maximal degree occurring in all of its monomials. Let us restrict to modules over $\mathbb{C}[h]$.

Proposition 2.1. *Assume that* $\deg(c_i(h)) \leq \deg(P_i)$. *Then if* $FG \in \mathcal{I}_h$, *either* $F \in \mathcal{I}_h$ *or* $G \in \mathcal{I}_h$. *Hence* $\mathcal{I}_h$ *is prime.*

Proof. Consider first the projection $\rho : U_h \to U_h/hU_h \approx \mathbb{C}[\mathcal{G}^*]$. One has that for any $F \in U_h$, $\deg(\rho(F)) \leq \deg(F)$.

Since $\rho(P_i) = p_i$, $i = 1, \dots, m$, we have $\rho(\mathcal{I}_h) = \mathcal{I}_0$. Since $\deg(c_i(h)) \leq \deg(P_i)$, if $f \in \mathcal{I}_0$ there exists $F' \in \mathcal{I}_h$ with $\rho(F') = f$ and $\deg(F') = \deg(f)$. If $f = \sum_i f^i(p_i - c_0^i)$ one can take for example $F = \sum_i \pi_h \circ \mathrm{Sym}(f_i)(P_i - c^i(h))$.

Assume that $FG \in \mathcal{I}_h$. Then

$$\rho(FG) = \rho(F)\rho(G) =: fg \in \mathcal{I}_0,$$

where we denote the projections by small case letters. Since $\mathcal{I}_0$ is a prime ideal, either $f \in \mathcal{I}_0$ or $g \in \mathcal{I}_0$.

Assume that $f \in \mathcal{I}_0$. Then there exists $F' \in \mathcal{I}_h$ with $\rho(F') = f$ and $\deg(F') = \deg(f) \leq \deg(F)$. Denote $F - F' = h\Delta F$; it is clear that $\deg(\Delta F) < \deg(F)$. If $\Delta F \in \mathcal{I}_h$ then F itself is in $\mathcal{I}_h$ and we are through; otherwise observe that

$$h\Delta F G = FG - F'G \in \mathcal{I}_h.$$

Since $U_h/\mathcal{I}_h$ is without torsion ([**12**]), we can "divide" by h, and it follows that $\Delta F G \in \mathcal{I}_h$.

We can now proceed to show that either ΔF or G is in $\mathcal{I}_h$. But notice that we have reduced the total degree. We can apply the argument again until we arrive to the situation that one of the factors has degree zero (it is a number). Then it follows that the other factor is in $\mathcal{I}_h$ and eventually that F or G are in $\mathcal{I}_h$, as we wanted to prove. □

We now want to show that there exists two different $\mathbb{C}[h]$-deformations on the same orbit that are not isomorphic. We consider $\mathcal{G} = \mathrm{sl}_2(\mathbb{C})$. Let

$$I_h = (P - \mu^0), \qquad I'_h = (P - \mu^0 - \sqrt{2}h),$$

where P is the quadratic Casimir. Assume that $U_h/I_h \cong U_h/I'_h$. Since any isomorphism will send the ideal $(h-1)$ into the ideal $(h-1)$, the quotient of both algebras by $(h-1)$ must be isomorphic. But the algebra $U/(P-\mu^0)$ has finite dimensional representations only for certain values of μ^0. In particular, for μ^0 irrational, it has no finite dimensional representations [**23**]. It is enough to take μ^0 such that U_h/I_h has finite dimensional representations and we reach a contradiction.

The same is true for formal deformations. In fact, with the same reasoning as in reference [**23**] we have that $U_{[h]}/(P - \mu(h))$ admits finite dimensional representations only for appropriate $\mu(h) = \mu^0$.

3. Star products and equivalence.

Definition 3.1. Given $\mathcal{A}_{[h]}$, a formal deformation of a Poisson algebra $\mathcal{A}$ and a $\mathbb{C}[[h]]$-module isomorphism $\psi : \mathcal{A}[[h]] \longrightarrow \mathcal{A}_{[h]}$ as in Definition 2.1, we say that the associative product in $\mathcal{A}[[h]]$ defined by

$$a \star b = \psi^{-1}(\psi(a) \cdot \psi(b)), \qquad a, b \in \mathcal{A}[[h]]$$

is a star product on $\mathcal{A}[[h]]$.

It follows from property **a** in Definition 2.1 that a star product can always be written as

$$a \star b = ab + \sum_{n>0} h^n B_n(a, b) \tag{5}$$

where B_n are bilinear operators and by juxtaposition ab we denote the commutative product in $\mathcal{A}[[h]]$. Property **b** in Definition 2.1 implies that $\{a, b\} = B_1^-(a, b) := B_1(a, b) - B_1(b, a)$.

For a given $\mathcal{A}_{[h]}$ there are many choices of the isomorphism ψ (it is not canonical). Once ψ is given, the star product $\star$ is defined and one regards $\mathcal{A}[[h]]$ as an associative non commutative $\mathbb{C}[[h]]$-algebra. Let $\star$ and $\star'$ be different star products corresponding to the same deformation, defined by the maps

$$\begin{aligned} \psi &: \mathcal{A}[[h]] \longrightarrow \mathcal{A}_{[h]}, \qquad & a \star b &= \psi^{-1}(\psi(a) \cdot \psi(b)), \\ \psi' &: \mathcal{A}[[h]] \longrightarrow \mathcal{A}_{[h]}, \qquad & a \star' b &= \psi'^{-1}(\psi'(a) \cdot \psi'(b)). \end{aligned}$$

They define isomorphic algebras. The isomorphism $T : \mathcal{A}[[h]] \longrightarrow \mathcal{A}[[h]]$ is given by

$$T = \psi'^{-1} \circ \psi, \qquad T(a \star b) = T(a) \star' T(b).$$

T can also be expressed as a power series

$$T = \sum_{n \geq 0} h^n T_n \tag{6}$$

in terms of the linear operators T_n. It is easy to show that T_0 is an automorphism of the commutative algebra $\mathcal{A}[[h]]$

$$T_0(ab) = T_0(a) T_0(b), \qquad a, b \in \mathcal{A}[[h]]$$

and of the Poisson algebra $\mathcal{A}$

$$T_0\{a,b\} = \{T_0(a), T_0(b)\}, \qquad a, b \in \mathcal{A}.$$

Definition 3.2. If $\star$ and $\star'$ are two isomorphic star products on $\mathcal{A}[[h]]$, the isomorphism being $T : \mathcal{A}[[h]] \longrightarrow \mathcal{A}[[h]]$ as in (6), we say that they are gauge equivalent if $T_0 = \mathrm{Id}$.

A star product is *differential* if $\mathcal{A} = C^\infty(M)$ for a smooth manifold M, and the operators B_n in (5) are bidifferential operators. An example of differential star product is the one induced on $\mathcal{G}^*$ by the map (3). It is in principle defined on polynomials, but it can be extended to $C^\infty(M)$ through operators B_i that are bidifferential. It was shown in reference [**8**] that with a gauge transformation any differential star product can be brought to a form under which the bilinear operators B_n are null on the constants (that is, the zero degree doesn't appear).

One can consider gauge equivalence inside the class of differential star products by considering only differential maps T. For this case, it was shown by Kontsevich in [**19**] the following important theorem:

Theorem 3.1. *The set of gauge equivalence classes of differential star products on a smooth manifold M can be naturally identified with the set of equivalence classes of Poisson structures depending formally on h,*

$$\alpha = h\alpha_1 + h^2\alpha_2 + \dots$$

modulo the action of the group of formal paths in the diffeomorphism group of M, starting at the identity isomorphism.

In particular, for a given Poisson structure α_1, we have the equivalence class of differential star products canonically associated to $h\alpha_1$.

We explain briefly the concept of formal paths in the diffeomorphism group of M. For further details we refer to reference [**19**, **1**]. Let $\mathbf{m}$ be the maximal ideal in $\mathbb{R}[[t]]$. Consider $\mathcal{L}$ the algebra of polyvector fields with the Schouten-Nijenhuis bracket. It is a differential graded Lie algebra with zero differential. We recall that a Poisson structure is a bivector field such that its Schouten-Nijenhuis bracket with itself is zero. Let $\mathcal{L}_0$ be the algebra of vector fields on M. They are the 0-cochains of the complex. Consider $\mathcal{L}_0 \otimes \mathbf{m}$. The exponential of this algebra is the group of formal paths in the diffeomorphism group starting with the identity. $\mathcal{L}_1$ is the set of (skew-symmetric) bivector fields. $\mathcal{L}_0$ acts on $\mathcal{L}_1$ with the Schouten-Nijenhuis bracket,

$$Z(B)(f_1, f_2) = Z(B(f_1, f_2)) - B(Z(f_1), f_2) - B(f_1, Z(f_2)),$$
$$Z \in \mathcal{L}_0, \;\; B \in \mathcal{L}_1, \;\; f_1, f_2 \in C^\infty(M)$$

and this action can be exponentiated to the group.

3.1. Uniqueness of the deformation of the Kirillov Poisson structure. We want to determine whether the equivalence class of the Kirillov Poisson bracket in $\mathcal{G}^*$ is the only class of formal Poisson structures whose first order term is the Kirillov Poisson bracket (as it happens, for example, in any flat symplectic manifold [**3**]). This is actually the case, at least for algebraic Poisson structures (we say that a Poisson structure β is algebraic if $\beta(p,q)$ is a polynomial whenever p and q are polynomials) and $\mathcal{G}$ semisimple.

Proposition 3.1. *Let $\mathcal{G}$ be a real semisimple algebra. Let β be an algebraic differential formal Poisson structure*

$$\beta = \sum_{i=0}^{\infty} h^{i+1}\beta_i,$$

such that β_0 is the Kirillov Poisson structure in $\mathcal{G}^*$. Then β is equivalent to β_0.

Proof. The Jacobi identity at first order is satisfied since β_0 itself is a Poisson structure and at second order it implies that β_1 is a two-cocycle in the Chevalley cohomology of β_0. If β_1 is a coboundary, then

$$\beta_1(f_1,f_2) = \delta Z(f_1,f_2) = Z(\beta_0(f_1,f_2)) - \beta_0(Z(f_1),f_2) - \beta_0(f_1,Z(f_2)),$$

with Z a 0-cochain. Then a gauge transformation (formal path in the diffeomorphism group) of the form

$$\varphi = \mathrm{Id} + hZ + \dots$$

shows that β is equivalent to a formal Poisson structure without term of order h^2, i.e., we can assume that $\beta_1 = 0$. But then β_2 is a cocycle and we can proceed recursively. Hence, to prove that $\beta \sim \beta_0$ it is actually sufficient to show that the Chevalley cohomology of β_0 is zero. Since β is a bivector field and it is algebraic, it is sufficient to check that there is no non trivial algebraic two cocycle with order of differentiability (1,1). We will show that this is the case.

Such a cocycle is an antisymmetric bidifferential map, null on the constants and with polynomial coefficients,

$$C^2 : \mathrm{Sym}(\mathcal{G}) \otimes \mathrm{Sym}(\mathcal{G}) \longrightarrow M$$

where $M = \mathrm{Sym}(\mathcal{G})$ is a left (Lie algebra) $\mathrm{Sym}(\mathcal{G})$-module, with the action given by the Poisson bracket β_0. If C^2 has order of differentiability (1,1), we can restrict C^2 non trivially to first order polynomials. We denote that restriction by

$$\hat{C}^2 : \mathcal{G} \otimes \mathcal{G} \longrightarrow \mathrm{Sym}(\mathcal{G}).$$

Then $\hat{C}^2$ is a cocycle in the Lie algebra cohomology of order two of $\mathcal{G}$ with values in $\mathrm{Sym}(\mathcal{G})$. Since $\mathcal{G}$ is semisimple, as a consequence of Whitehead's lemma, this cohomology is zero (see for example reference [**18**]). Hence $\hat{C}^2$ is trivial, i.e., there exists a 1-cochain $\hat{C}^1 : \mathcal{G} \longrightarrow \mathrm{Sym}(\mathcal{G})$ such that $\hat{C}^2 = \delta\hat{C}^1$. If $\hat{C}^1$ is given on a basis of $\mathcal{G}$ by $\hat{C}^1(X_i) = \hat{C}^1_i$, this means that

$$\hat{C}^2(X_i, X_j) = \delta\hat{C}^1 = \hat{C}_1([X_i, X_j]) - \beta_0(X_i, \hat{C}^1_j) - \beta_0(\hat{C}^1_i, X_j),$$

and $\hat{C}^1$ can be extended to a 1-cochain in the Chevalley complex by

$$C^1(f) = \hat{C}^1_k \frac{\partial f}{\partial x^k}, \qquad f \in \mathrm{Sym}(\mathcal{G}).$$

We then have $C^2 = \delta C^1$, showing that C^2 is trivial. Hence β is equivalent to β_0, as we wanted to show. $\square$

Using Theorem 3.1, we conclude that there is only one equivalence class of star products whose first order term is the Kirillov Poisson bracket. All these star products give algebra structures on the polynomials on $\mathcal{G}^*$ isomorphic to $U_{[h]}$.

4. Star products on the orbit.

A star product on the orbit Θ associated to the deformation of Theorem 2.1 is given by a linear isomorphism

$$\widetilde{\psi} : \mathbb{C}[\Theta][[h]] \longrightarrow U_{[h]}/\mathcal{I}_{[h]}.$$

In particular, if $\{x_{i_1} \dots x_{i_k}, (i_1, \dots i_k) \in S\}$ is a basis of $\mathbb{C}[\Theta]$ for some set of multiindices S, then $\{X_{i_1} \dots X_{i_k}, (i_1, \dots i_k) \in S\}$ is a basis of $U_{[h]}/\mathcal{I}_{[h]}$ [**12**]. This defines a particular isomorphism $\widetilde{\psi}(x_{i_1} \dots x_{i_k}) = X_{i_1} \dots X_{i_k}$ and the corresponding star product.

This star product can be seen as the restriction to the orbit of a star product on $\mathbb{C}[\mathcal{G}^*]$. We have only to extend the map $\widetilde{\psi}$ to an isomorphism $\psi : \mathbb{C}[\mathcal{G}^*][[h]] \longrightarrow U_{[h]}$. This is guaranteed since $\mathbb{C}[\mathcal{G}^*] = \mathbb{C}[\Theta] \oplus \mathcal{I}_0$, $U_{[h]} = U_{[h]}/\mathcal{I}_{[h]} \oplus \mathcal{I}_{[h]}$, and $\mathcal{I}_0$ and $\mathcal{I}_{[h]}$ are also isomorphic as $\mathbb{C}[[h]]$-modules (we denote with the same symbol $\mathcal{I}_0$ the ideal generated by $(p_i - c_i)$ both in $\mathbb{C}[\mathcal{G}^*]$ and in $\mathbb{C}[\mathcal{G}^*][[h]]$).

We have then that the following diagram

$$\begin{array}{ccc} \mathbb{C}[\mathcal{G}^*][[h]] & \xrightarrow{\psi} & U_{[h]} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi_h} \\ \mathbb{C}[\Theta][[h]] & \xrightarrow{\widetilde{\psi}} & U_{[h]}/I_{[h]} \end{array} \tag{7}$$

commutes. In general, we say that a star product on $\mathcal{G}^*$ is *tangential* to the orbit Θ if it defines a star product on Θ by restriction. So the star product in (7) is tangential.

Example 4.1. Star product on an orbit of SU(2).

Consider the Lie algebra of SU(2),

$$[X,Y]=Z, \qquad [Y,Z]=X, \qquad [Z,X]=Y.$$

The subalgebra of invariant polynomials on $\mathcal{G}^*$ is generated by $p=x^2+y^2+z^2$, so the corresponding Casimir is $P=X^2+Y^2+Z^2$. We consider the orbit $p=c^2$, $c\in\mathbb{R}$, $c\neq 0$. A basis of $\mathcal{I}_0$ is $B_1=\{x^ry^sz^t(p-c^2),\ r,s,t=0,1,2,\dots\}$ and one can complete it to a basis in $\mathcal{G}^*$ by adding $B_2=\{x^ry^sz^\nu,\ \nu=0,1,\ r,s=0,1,2,\dots\}$. The equivalence classes of the elements in B_2 are a basis of $\mathbb{C}[\mathcal{G}^*]/\mathcal{I}_0$.

Let $\mathcal{I}_{[h]}$ be the ideal in $U_{[h]}$ generated by $P-c^2$. We can define the isomorphism $\psi:\mathbb{C}[\mathcal{G}^*][[h]]\longrightarrow U_{[h]}$ as

$$\begin{aligned} \psi(x^ry^sz^t(p-c^2)) &= X^rY^sZ^t(P-c^2), \qquad r,s,t=0,1,2,\dots \\ \psi(x^ry^sz^\nu) &= X^rY^sZ^\nu, \qquad \nu=0,1,\ r,s=0,1,2,\dots. \end{aligned} \tag{8}$$

Clearly $\psi(\mathcal{I}_0)=\mathcal{I}_{[h]}$, so the star product defined by ψ is tangential to the orbit. It is easy to check that if we move to a neighboring orbit, $p={c'}^2$, then ψ, as defined in (8) doesn't preserve the new ideal, that is, $\psi(\mathcal{I}'_0)\neq\mathcal{I}'_{[h]}$.

One can construct a star product that is tangential to all the orbits in a neighborhood of the regular orbit (this is in fact the definition of "tangential star product" given in [**6**]). If $p_i=c_i^0$, $i=1,\dots m$ define the regular orbit $\Theta_{(c_1^0,\dots c_m^0)}$ with ideal $\mathcal{I}_{(c_1^0,\dots c_m^0)}$ one can construct a map ψ such that

$$\psi(\mathcal{I}_{(c_1,\dots c_m)})=\mathcal{I}_{(c_1,\dots c_m),h} \tag{9}$$

for $(c_1,\dots c_m)$ in a neighborhood of $(c_1^0,\dots c_m^0)$ and $\mathcal{I}_{(c_1,\dots c_m),h}$ an ideal in $U_{[h]}$ of the type required in Theorem 2.1. The construction follows similar lines to the one in [**5**]. We consider the decomposition $\mathbb{C}[\mathcal{G}^*]=I\otimes H$ where I is the subalgebra of invariant polynomials as in (2) and H is the set of harmonic polynomials (this result is due to Kostant [**20**]). Harmonic polynomials are in one to one correspondence with the polynomials on the orbit, so we have in fact

$$\mathbb{C}[\mathcal{G}^*]\approx I\otimes\mathbb{C}[\Theta_{(c_1^0,\dots c_m^0)}].$$

Consider now the basis in I $\{(p_{i_1}-c_{i_1})\dots(p_{i_k}-c_{i_k}), i_1\leq\dots\leq i_k\}$ and the basis in $\mathbb{C}[\Theta_{(c_1^0,\dots c_n^0)}]$ as before, $\{x_{j_1}\dots x_{j_l},(j_1,\dots j_l)\in S\}$. We define the $\mathbb{C}[[h]]$-module isomorphism

$$\begin{aligned} &\psi\big((p_{i_1}-c_{i_1}^0)\dots(p_{i_k}-c_{i_k}^0)\otimes x_{j_1}\dots x_{j_l}\big) \\ &\quad=(P_{i_1}-c_{i_1}(h))\dots(P_{i_k}-c_{i_k}(h))\otimes(X_{j_1}\dots X_{j_l}). \end{aligned} \tag{10}$$

It is obvious that it preserves the ideal, $\psi(\mathcal{I}_{(c_1^0,\dots c_n^0)})=\mathcal{I}_{(c_1^0,\dots c_m^0),h}$. A closer look reveals that, in fact $\psi(\mathcal{I}_{(c_1,\dots c_m)})=\mathcal{I}_{(c_1,\dots c_m),h}$, and then $\widetilde{\psi}$ in (7) is well-defined for any $(c_1,\dots c_m)$ in a neighborhood of $(c_1^0,\dots c_m^0)$. Consequently we have a star product that is tangential to all the orbits in a neighborhood of the regular orbit.

In [**5**] it is shown that for SU(2) a star product of this type (with $c_{i_l}(h) = c^0_{i_l}$) is not differential. More generally, it was shown in [**6**] the following theorem:

Theorem 4.1. *If $\mathcal{G}$ is a semisimple Lie algebra there is no differential star product on any neighborhood of the origin in $\mathcal{G}^*$ which is tangential to the coadjoint orbits.*

The only property of tangential star products that is used in the proof of this theorem is that if f is a function that is constant on the orbits (in particular, the quadratic Casimir p_1), then, $g \star f = gf$. It is easy to show that the tangential star products defined by (10) satisfy this property on all $\mathcal{G}^*$ (in particular in a neighborhood of 0), so they are not differential.

On any regular Poisson manifold there exists a star product that is tangential and differential [**21**]. But on all of $\mathcal{G}^*$, which is not regular, Theorem 4.1 states that a star product with both properties does not exist. To induce a star product on a particular orbit, it is enough to assume that the star product on $\mathcal{G}^*$ is tangent to only such orbit. One can find star products on $\mathrm{Pol}(\mathcal{G}^*)$ isomorphic to U_h that restrict well to only one orbit (in the sense of (7)). Example 4.2 shows one of such star products for $\mathcal{G}$=su(2). We prove that it is not differential, so at least in this case, the relaxation of the tangentiality condition does not allow in general for differentiability. In Section 5 we will investigate how these deformations are related to differential deformations on the orbit.

Example 4.2. Non differential star product on $\mathcal{G}^* = \mathrm{su}(2)^*$.

Consider again the Lie algebra of SU(2), with the same notation, and the orbit Θ^r given by $p = 1$. It is a 2-sphere in $\mathbb{R}^3$. Fix the star product $\star$ on Θ by choosing the $\mathbb{C}[[h]]$-isomorphism

$$\begin{aligned} \widetilde{\psi} : \mathbb{C}[\Theta][[h]] \quad &\longrightarrow \quad U_{[h]}/\mathcal{I}_{[h]} \\ x^n y^m z^\nu \quad &\mapsto \quad X^n Y^m Z^\nu, \qquad \nu = 0,1,\ m,n = 0,1,2,\dots. \end{aligned}$$

We regard the Cartesian coordinates x and y as functions on the sphere and let V be an open set in Θ where (x, y) are coordinates. On this open set V the 1-forms dx and dy form a basis for the module of 1-forms. Let ∂_x and ∂_y be the elements of the dual basis, that is, ∂_x and ∂_y are vector fields on V such that

$$\langle \partial_x, dx\rangle = \langle \partial_y, dy\rangle = 1, \qquad \langle \partial_y, dx\rangle = \langle \partial_x, dy\rangle = 0.$$

Any differential operator on V is an element of the algebra generated by functions and by ∂_x and ∂_y. The advantage of ∂_x and ∂_y is that they behave well on polynomials in x and y. We have

$$0 = \partial_x(1) = \partial_x(x^2 + y^2 + z^2) = 2x + 2z\partial_x(z),$$

hence $\partial_x(z) = -\frac{x}{z}$ and $\partial_y(z) = -\frac{y}{z}$. Observe that ∂_x and ∂_y commute.

Assume that $\star$ is differential,

$$f \star g = \sum_{i \geq 0} h^i B_i(f, g)$$

where B_i are bidifferential operators. To determine B_i it is enough to compute them on the monomials x and y. With the following lemma we compute B_1.

Lemma 4.1. *Let p_1, p_2 be two polynomials in x and y, then we have:*

$$p_1 \star p_2 = p_1 p_2 - hz\partial_y(p_1)\partial_x(p_2) \mod(h^2).$$

Proof. It is enough to show it for p_1, p_2 monomials. Let $p_1 = x^n y^m$, $p_2 = x^r y^s$. We use induction on $N = m + r$. For $N = 0$ it is clear. Let $N > 0$. By the definition of $\star$,

$$p_1 \star p_2 = x^n y^m \star x^r y^s = x^n (y^m \star x^r) y^s = x^n [(y^m \star x^{r-1}) \star x] y^s.$$

By induction we have:

$$p_1 \star p_2 = x^n [(x^{r-1} y^m - hzm(r-1) y^{m-1} x^{r-2}) \star x] y^s \mod(h^2),$$

and by induction again we have:

$$\begin{aligned} p_1 \star p_2 &= x^n [x^r y^m - hzm y^{m-1} x^{r-1} - hzm(r-1) y^{m-1} x^{r-1}] y^s \mod(h^2) \\ &= x^n [x^r y^m - hzmr y^{m-1} x^{r-1}] y^s \mod(h^2) \\ &= x^{n+r} y^{m+s} - hzmr y^{m-1+s} x^{r-1+n} \mod(h^2), \end{aligned}$$

which is what we wanted to prove. □

According to the previous lemma

$$z \star z = z^2 - h\frac{xy}{z} \mod(h^2) = 1 - x^2 - y^2 - h\frac{xy}{z} \mod(h^2),$$

on the other hand, by definition,

$$z \star z = \widetilde{\psi}^{-1}(\widetilde{\psi}(z)\widetilde{\psi}(z)) = \widetilde{\psi}^{-1}(Z^2) = 1 - x^2 - y^2,$$

a contradiction that shows that $\star$ cannot be differential.

5. Algebraic and differential star products on the regular orbit.

Let us consider the regular orbit Θ^r as a symplectic manifold. By Theorem 3.1 we can associate to the Poisson structure a (equivalence class of) differential star product. It was already known [**9**, **10**] that a star product exists for any symplectic manifold. In fact, differential star products are not in general unique. The space of equivalence classes of differential star products such that

$$f \star g - g \star f = h\{f, g\},$$

being $\{\cdot, \cdot\}$ a symplectic Poisson bracket, are classified by the sequences $\{\omega_n\}_{n\geq 1}$ of de Rham cohomology classes in $H^2(M)$ such that ω_1 is the symplectic form associated to the Poisson bracket. In fact, the space of equivalence classes of star products is a principal homogeneous space under the group $H^2(M)[[h]]$ [**9**, **8**].

The symplectic two form is not defined in arbitrary Poisson manifolds, so the natural structure to consider is the Poisson bivector. We want to describe the space of equivalence classes of star products for symplectic manifolds in terms of the Poisson bivector, being this approach closer to the one of Kontsevich's theorem for arbitrary manifolds. Let M be a symplectic manifold and consider $\omega_h = \sum_{j\geq 0} h^j \omega_j \in H^2(M)[[h]]$, where ω_0 is the original symplectic two form and ω_j are closed two forms. Since ω_0 is non degenerate, ω_h defines an invertible map between tangent and cotangent vector fields in the usual way,

$$\mu_h : \Gamma(TM)[[h]] \longrightarrow \Gamma(T^*M)[[h]],$$

which can be extended to tensors. In fact, by closedness of ω_h, the map

$$(f, g) \mapsto \{f, g\}_h = h\omega_h(\mu_h^{-1}(df), \mu_h^{-1}(dg))$$

is a formal Poisson structure in the sense of Kontsevich and this formal Poisson structure is gauge equivalent to zero (the gauge group is the group of formal paths in the diffeomorphism group starting with the identity) if and only if all the ω_j are exact. We have then that the set of equivalence classes in $H^2(M)[[h]]$ is in one to one correspondence with the set of formal Poisson structures modulo the action of the gauge group. So for symplectic manifolds both descriptions, in terms of the symplectic form or in terms of the Poisson bivector are equivalent.

Coadjoint orbits of compact groups are an example of manifolds that admit inequivalent quantizations. In fact they have a non trivial de Rham cohomology $H^2(\Theta^r)$. In particular, the symplectic form is a closed, non exact form, so we have many inequivalent deformations.

Let Θ^r be the orbit defined by

$$p_i(x_1, \ldots x_n) = c_i^0, \quad c_i^0 \in \mathbb{R}, \quad i = 1, \ldots m,$$

Θ^r is regular if and only if the differentials dp_i are independent. One can consider all the regular orbits given by the constraints $p_i = c_i$, $i = 1, \dots m$ with $(c_1, \dots c_m)$ in a neighborhood of $(c_1^0, \dots c_m^0)$ where the differentials are still independent. The set of these points is a neighborhood $\mathcal{N}_{\Theta^r} \approx \Theta^r \times \mathbb{R}^m$ of the regular orbit. $\mathcal{N}_{\Theta^r}$ is a regular Poisson manifold. The Poisson structure on $\mathcal{N}_{\Theta^r}$ can be seen as a symplectic structure on Θ^r which depends on certain parameters, the invariant polynomials p_i, which determine the leaf of the foliation in $\mathcal{N}_{\Theta^r}$.

We now want to examine various star products that can be defined on the open set $\mathcal{N}_{\Theta^r}$. We can consider the star product $\star_S$ induced by U_h by means of the map (3). It is differential, but not tangential. It was shown in reference [**19**] that the canonical deformation of the Kirillov Poisson structure on $\mathcal{G}^*$ is isomorphic to U_h.

We can also consider the quantization of the Kirillov symplectic structure on the orbit given by Kontsevich's theorem. From the local expression of Kontsevich's quantization, one can see that it is a smooth with respect to the parameters p_i. Interpreting the parameters as transverse coordinates, Kontsevich's theorem applied on Θ^r gives indeed a star product on $\mathcal{N}_{\Theta^r}$ that is tangential and differential. We denote it by $\star_T$.

Finally we can consider the star product $\star_P$ on $\mathcal{N}_{\Theta^r}$, induced by a map ψ as in formula (10). $\star_P$ is tangential to the orbit, but, in general, not differential. To sum up we get Table 1.

$*_S$	Isomorphic to U_h (on the polynomials), induced by Sym.	Defined on all $\mathcal{G}^*$, differential, not tangential.
$*_P$	Isomorphic to U_h, induced by a map ψ like (10).	Defined on all $\mathcal{G}^*$, not differential (only given on polynomials), tangential to the orbits.
$*_T$	Gluing Kontsevich construction on the leaves.	Defined on $\mathcal{N}_{\Theta^r}$, differential, tangential.

Table 1. Star products on $\mathcal{N}_{\Theta^r}$.

The relation among these star products on $\mathcal{N}_{\Theta^r}$ and the corresponding star products induced on the orbit Θ^r will be studied in [**13**].

6. Summary.

In this paper we consider different methods of quantization for regular orbits of compact semisimple Lie groups. From the algebraic point of view, one can obtain non isomorphic deformations of the same Poisson structure. These deformations can be compared with geometric quantization since the formulation is in terms of a certain prime ideal in the enveloping algebra. The comparison with differential deformations becomes more difficult since the polynomials are "global" objects, very different from the "local" C^∞ functions, and in fact we see that the star products obtained are not differential in general. At the end we define three star products on a regularly foliated neighborhood of the orbit.

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