Pacific Journal of Mathematics

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Volume 207 No. 1 November 2002

PACIFIC JOURNAL OF MATHEMATICS
Vol. 207, No. 1, 2002

TODA LATTICE AND TORIC VARIETIES FOR REAL SPLIT SEMISIMPLE LIE ALGEBRAS

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The paper concerns the topology of an isospectral real smooth manifold for certain Jacobi element associated with real split semisimple Lie algebra. The manifold is identified as a compact, connected completion of the disconnected Cartan subgroup of the corresponding Lie group $\widetilde{G}$ which is a disjoint union of the split Cartan subgroups associated to semisimple portions of Levi factors of all standard parabolic subgroups of $\widetilde{G}$. The manifold is also related to the compactified level sets of a generalized Toda lattice equation defined on the semisimple Lie algebra, which is diffeomorphic to a toric variety in the flag manifold $\widetilde{G}/B$ with Borel subgroup B of $\widetilde{G}$. We then give a cellular decomposition and the associated chain complex of the manifold by introducing colored-signed Dynkin diagrams which parametrize the cells in the decomposition.

1. Introduction.

In this paper, we study the topological structure of certain manifolds that are interesting in two different ways. First they are isospectral manifolds for a signed Toda lattice flow [14]; an integrable system that arises in several physical contexts and has been studied extensively. Secondly they are shown in §8 to be the closures of generic orbits of a split Cartan subgroup on a real flag manifold. These are certain smooth toric varieties that glue together the disconnected pieces of a Cartan subgroup of a semisimple Lie group of the form $\widetilde{G}$. In this paper we start from the Toda lattice aspect of this object and end the paper inside a real flag manifold. We thus start by motivating and describing our main constructions from the point of view of the Toda lattice; then we trace a path that starts with this Toda lattice and that naturally leads to the (disconnected) Cartan subgroup of a split semisimple Lie group and a toric orbit. Another path that also leads to a Cartan subgroup starts with Kostant's paper [16]; this approach is described in §8.

From the Toda lattice end of this story, these manifolds are related to the compactified level set of a generalized (nonperiodic) Toda lattice equation defined on the semisimple Lie algebra (see for example [16]) and, although they share some features with the Tomei manifolds in [21], they are different from those (e.g., nonorientable). As background information, we start with a definition of the generalized Toda lattice equation which led us to our present study of the manifolds.

Let $\mathfrak{g}$ denote a real split semisimple Lie algebra of rank l. We fix a split Cartan subalgebra $\mathfrak{h}$ with root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, real root vectors e_{α_i} associated with simple roots $\{\alpha_i : i = 1, \ldots, l\} = \Pi$. We also denote $\{h_{\alpha_i}, e_{\pm\alpha_i}\}$ the Cartan-Chevalley basis of $\mathfrak{g}$ which satisfies the relations,

$$[h_{\alpha_i}, h_{\alpha_j}] = 0, \quad [h_{\alpha_i}, e_{\pm\alpha_j}] = \pm C_{j,i} e_{\pm\alpha_j}, \quad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{i,j} h_{\alpha_j}$$

where the $l \times l$ matrix $(C_{i,j})$ is the Cartan matrix corresponding to $\mathfrak{g}$, and $C_{i,j} = \alpha_i(h_{\alpha_j})$.

Then the generalized Toda lattice equation related to real split semisimple Lie algebra is defined by the following system of 2nd order differential equations for the real variables $f_i(t)$ for $i = 1, \ldots, l$,

$$\frac{d^2 f_i}{dt^2} = \epsilon_i \exp\left(-\sum_{j=1}^{l} C_{i,j} f_j\right) \tag{1}$$

where $\epsilon_i \in \{\pm 1\}$ which correspond to the signs in the indefinite Toda lattices introduced in [**5**, **14**]. The main feature of the indefinite Toda equation having at least one of ϵ_i being -1 is that the solution blows up to infinity in finite time [**14**, **10**]. Having introduced the signs, the group corresponding to the Toda lattice is a real split Lie group $\widetilde{G}$ with Lie algebra $\mathfrak{g}$ which is defined in §3. For example, in the case of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, if n is odd, $\widetilde{G} = SL(n, \mathbb{R})$, and if n is even, $\widetilde{G} = \mathrm{Ad}(SL(n, \mathbb{R})^{\pm})$.

The original Toda lattice equation in [**20**] describing a system of l particles on a line interacting pairwise with exponential forces corresponds to the case with $\mathfrak{g} = \mathfrak{sl}(l + 1, \mathbb{R})$ and $\epsilon_i = 1$ for all i, and it is given by

$$\frac{d^2 q_i}{dt^2} = \exp(q_{i-1} - q_i) - \exp(q_i - q_{i+1}), \quad i = 1, \ldots, l,$$

where the physical variable q_i, the position of the i-th particle, is given by

$$q_i = f_i - f_{i+1}, \quad i = 1, \ldots, l,$$

with $f_{l+1} = 0$ and $f_0 = f_{l+2} = -\infty$ indicating $q_0 = -\infty$ and $q_{l+1} = \infty$.

The system (1) can be written in the so-called Lax equation which describes an iso-spectral deformation of a Jacobi element of the algebra $\mathfrak{g}$. This is formulated by defining the set of real functions $\{(a_i(t), b_i(t)) : i = 1, \ldots, l\}$ with

$$a_i(t) := \frac{df_i(t)}{dt}, \quad b_i(t) := \epsilon_i \exp\left(-\sum_{j=1}^{l} C_{i,j} f_j(t)\right) \tag{2}$$

from which the system (1) reads

$$\frac{da_i}{dt} = b_i, \qquad \frac{db_i}{dt} = -b_i\left(\sum_{j=1}^{l} C_{i,j}a_j\right). \tag{3}$$

This is then equivalent to the Lax equation defined on $\mathfrak{g}$ (see [7] for a nice review of the Toda equation).

$$\frac{dX(t)}{dt} = [P(t), X(t)], \tag{4}$$

where the Lax pair $(X(t), P(t))$ in $\mathfrak{g}$ is defined by

$$\begin{cases} X(t) &= \sum_{i=1}^{l} a_i(t)h_{\alpha_i} + \sum_{i=1}^{l}(b_i(t)e_{-\alpha_i} + e_{\alpha_i}) \\ P(t) &= -\sum_{i=1}^{l} b_i(t)e_{-\alpha_i}. \end{cases} \tag{5}$$

Although a case with some $b_i = 0$ is not defined in (2) the corresponding system (3) is well-defined and is reduced to several noninteracting subsystems separated by $b_i = 0$. The constant solution a_i for $b_i = 0$ corresponds to an eigenvalue of the Jacobi (tridiagonal) matrix for $X(t)$ in the adjoint representaion of $\mathfrak{g}$. We denote by $S(F)$ the ad diagonalizable elements in $\mathfrak{g}\otimes_{\mathbb{R}}F$ with eigenvalues in F, where F is $\mathbb{R}$ or $\mathbb{C}$.

Then the purpose of this paper is to study the disconnected manifold $Z_{\mathbb{R}}$ of the set of Jacobi elements in $\mathfrak{g}$ associated to the generalized Toda lattices,

$$Z_{\mathbb{R}} = \left\{X = x + \sum\nolimits_{i=1}^{l}(b_ie_{-\alpha_i} + e_{\alpha_i}) \ : \ x \in \mathfrak{h}, b_i \in \mathbb{R}\setminus\{0\}, X \in S(\mathbb{R})\right\},$$

its iso-spectral leaves $Z(\gamma)_{\mathbb{R}}$, $\gamma \in \mathbb{R}^l$ and the construction of a smooth *connected* compactification, $\hat{Z}(\gamma)_{\mathbb{R}}$ of each $Z(\gamma)_{\mathbb{R}}$. The construction of $\hat{Z}(\gamma)_{\mathbb{R}}$ generalizes the construction of such a smooth compact manifold which was carried out in [15] in the important case of $\mathfrak{g} = \mathfrak{sl}(l{+}1, \mathbb{R})$. The construction there is based on the explicit solution structure in terms of the so-called τ-functions, which provide a local coordinate system for the blow-up points. Then by tracing the solution orbit of the indefinite Toda equation, the disconnected components in $Z(\gamma)_{\mathbb{R}}$ are all glued together to make a smooth compact manifold. The result is maybe well explained in Figure 1 for the case of $\mathfrak{sl}(3, \mathbb{R})$. In the figure, the Toda orbits are shown as the dotted lines, and each region labeled by the same signs in (ϵ_1, ϵ_2) with $\epsilon_i \in \{\pm\}$ are glued together through the boundary (the wavy-lines) of the hexagon. At a point of the boundary the Toda orbit blows up in finite time, but the orbit can be uniquely traced to the one in the next region (marked by the same letter

A, B or C). Then the compact smooth manifold $\hat{Z}(\gamma)_{\mathbb{R}}$ in this case is shown to be isomorphic to the connected sum of two Klein bottles. In the case of $\mathfrak{sl}(l+1, \mathbb{R})$ for $l \geq 2$, $\hat{Z}(\gamma)_{\mathbb{R}}$ is shown to be nonorientable and the symmetry group is the semi-direct product of $(\mathbb{Z}_2)^l$ and the Weyl group $W = S_{l+1}$, the permutation group. One should also compare this with the result in [21] where the compact manifolds are associated with the definite (original) Toda lattice equation and the compactificationis done by adding only the subsystems. (Also see [4] for some topological aspects of the manifolds.)

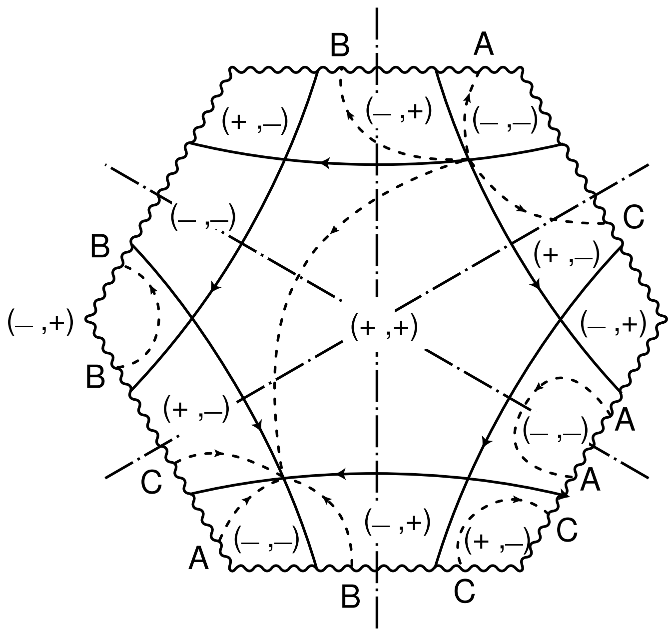

Figure 1. The manifold $Z(\gamma)_{\mathbb{R}}$ and the Toda orbits for $\mathfrak{sl}(3, \mathbb{R})$.

The study of $Z(\gamma)_{\mathbb{R}}$ and of the compact manifolds $\hat{Z}(\gamma)_{\mathbb{R}}$, can be physically motivated by the appearance of the indefinite Toda lattices in the context of symmetry reduction of the Wess-Zumino-Novikov-Witten (WZNW) model. For example, the reduced system is shown in [6] to contain the indefinite Toda lattices. The compactification $\hat{Z}(\gamma)_{\mathbb{R}}$ can then be viewed as a concrete description of (an expected) regularization of the integral manifolds of these indefinite Toda lattices, where infinities (i.e., blow up points) of the solutions of these Toda systems glue everything into a smooth compact manifold.

In addition our work is mathematically motivated by:

a) The work of Kostant in [16] where he considered the real case with all $b_i > 0$,

b) the construction of the Toda lattice in [**17**].

In [**17**], the solution $\{b_i(t) : i = 1, \ldots, l\}$ in (2) of the generalized Toda lattice equation is shown to be expressed as an orbit on a connected component H_ϵ labeled by $\epsilon = (\epsilon_1, \ldots, \epsilon_l)$ with $\epsilon_i = \pm 1$, of the Cartan subgroup $H_{\mathbb{R}}$ defined in §3,

$$H_{\mathbb{R}} = \bigcup_{\epsilon \in \{\pm 1\}^l} H_\epsilon.$$

This can be seen as follows: Let g_ϵ be an element of H_ϵ given by

$$g_\epsilon = h_\epsilon \exp\left(\sum_{i=1}^{l} f_i h_{\alpha_i}\right), \tag{6}$$

which gives a map from $Z_{\mathbb{R}}$ into $H_{\mathbb{R}}$. Here the element $h_\epsilon \in H_\epsilon$ satisfies $\chi_{\alpha_i}(h_\epsilon) = \epsilon_i$ for the group character χ_ϕ determined by $\phi \in \Delta$. The solution $\{b_i(t) : i = 1, \ldots, l\}$ in (2) is then directly connected to the group character $\chi_{-\alpha_i}$ evaluated at g_ϵ, i.e.,

$$b_i(t) = \chi_{-\alpha_i}(g_\epsilon).$$

The Toda lattice equation is now written as an evolution of g_ϵ,

$$\frac{d}{dt} g_\epsilon^{-1} \frac{d}{dt} g_\epsilon = \left[g_\epsilon^{-1} e_+ g_\epsilon,\ e_-\right],$$

where $e_\pm$ are fixed elements in the simple root spaces $\mathfrak{g}_{\pm\Pi}$ so that all the elements in $\mathfrak{g}_{\pm\Pi}$ can be generated by $e_\pm$, i.e., $\mathfrak{g}_{\pm\Pi} = \{\mathrm{Ad}_h(e_\pm) : h \in H\}$. In particular, we take

$$e_\pm = \sum_{i=1}^{l} e_{\pm\alpha_i}.$$

Thus the Cartan subgroup $H_{\mathbb{R}}$ can be identified as the position space (e.g., $f_i = q_i + \cdots + q_l$ for $\mathfrak{sl}(l+1, \mathbb{R})$-Toda lattice) of the generalized Toda lattice equation, whose phase space is given by the tangent bundle of $H_{\mathbb{R}}$.

One should also note that the boundary of each connected component H_ϵ is given by either $b_i = 0$ (corresponding to a subsystem) or $|b_i| = \infty$ (to a blow-up). We are then led to our main construction of the compact smooth manifold of $H_{\mathbb{R}}$ in attempting to generalize [**16**] Theorem 2.4 to our indefinite Toda case including $b_i < 0$.

We define a set $\hat{H}_{\mathbb{R}}$ containing the Cartan subgroup $H_{\mathbb{R}}$ by adding pieces corresponding to the blow-up points and the subsystems (Definition 6.1). Thus the set $\hat{H}_{\mathbb{R}}$ is defined as a disjoint union of split Cartan subgroups $H_{\mathbb{R}}^A$ associated to semisimple portions of Levi factors of all standard parabolic subgroups determined by $A \subset \Pi$, (Definition 3.10),

$$\hat{H}_{\mathbb{R}} = \bigcup_{A \subset \Pi} \bigcup_{w \in W/W_{\Pi \setminus A}} w\left(H_{\mathbb{R}}^A\right) \times \{[w]^A\}.$$

The space $\hat{H}_{\mathbb{R}}$ then constitutes a kind of compact, connected completion of the disconnected Cartan subgroup $H_{\mathbb{R}}$. Figure 1 also describes how to connect the connected components of $H_{\mathbb{R}}$ to produce the connected manifold $\hat{H}_{\mathbb{R}}$ in the case of $\mathfrak{sl}(3,\mathbb{R})$. The signs must now be interpreted as the signs of simple root characters on the various connected components. The pair of signs $(+,+)$ corresponds to the connected component of the identity H and regions with a particular sign represent one *single* connected component of $H_{\mathbb{R}}$. Boundaries between regions with a fixed sign correspond to connected components of Cartan subgroups arising from Levi factors of parabolic subgroups. In addition the Weyl group W acts on $\hat{H}_{\mathbb{R}}$.

Most of this paper is then devoted to describing the detailed structure of the manifold $\hat{H}_{\mathbb{R}}$, and in §8 we conclude with a diffeomorphism defined between $\hat{H}_{\mathbb{R}}$ and the isospectral manifold $\hat{Z}(\gamma)_{\mathbb{R}}$ as identified with a toric variety $\overline{(H_{\mathbb{R}}gB)}$ in the flag manifold $\widetilde{G}/B$.

1.1. Main Theorems. In connection with the construction $\hat{H}_{\mathbb{R}}$, we introduce in Definition 4.1 the set of colored Dynkin diagrams. The colored Dynkin diagrams are simply Dynkin diagrams D where some of the vertices have been colored red or blue. For example in the case $\mathfrak{g}=\mathfrak{sl}(3,\mathbb{R})$:$\circ_R-\circ$ (the sub-index R indicates that $\circ$ is colored red). The full set of colored Dynkin diagrams consists of pairs: $(D,[w]^{\Pi\setminus S})$ where D is a colored Dynkin diagram, $S\subset\Pi$ denotes the set of vertices that are colored in D and $[w]^{\Pi\setminus S}$ is the coset of w in W/W_S. To each pair $(D,[w]^{\Pi\setminus S})$ one can associate a *set* which is actually a *cell*. First Notation 6.2 associates a subset of $\hat{H}_{\mathbb{R}}$ also denoted $(D,[w]^{\Pi\setminus S})$.

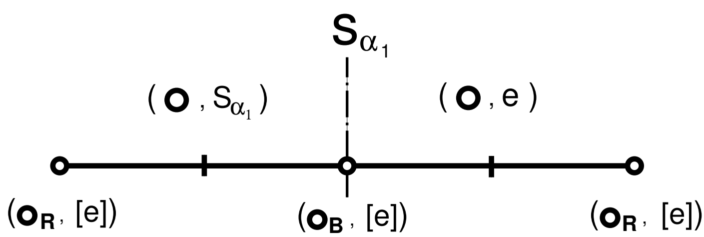

Figure 2. The manifold $\hat{H}_{\mathbb{R}}$ parametrized by colored Dynkin diagrams for $\mathfrak{sl}(2,\mathbb{R})$.

This turns out to be a cell of codimension k with $k=|S|$. We illustrate the example of $\mathfrak{sl}(2,\mathbb{R})$ in Figure 2.

We consider the set $\mathbb{D}^k$ of colored Dynkin diagrams $(D,[w]^{\Pi\setminus S})$ with $|S|=k$. In Figure 2, we have $\mathbb{D}^0=\{(\circ,e),(\circ,s_{\alpha_1})\},\mathbb{D}^1=\{(\circ_B,[e]),(\circ_R,[e])\}$. We then obtain the following theorem:

Theorem 1.1. *The collection of the sets* $\{\mathbb{D}^k : k = 0, 1, \ldots, l\}$ *gives a cell decomposition of* $\hat{H}_{\mathbb{R}}$.

The chain complex $\mathcal{M}_*$ is introduced in §4. The construction, in the case of $\mathfrak{sl}(2;\mathbb{R})$, is as follows:

$$\mathcal{M}_1 = \mathbb{Z}\left[\mathbb{D}^0\right] \xrightarrow{\partial_1} \mathcal{M}_0 = \mathbb{Z}\left[\mathbb{D}^1\right].$$

Here the boundary map ∂_1 is given by

$$\begin{cases} \partial_1(\circ, e) & = (\circ_B, [e]) - (\circ_R, [e]), \\ \partial_1(\circ, s_{\alpha_1}) & = (\circ_B, [s_{\alpha_1}]) - (\circ_R, [s_{\alpha_1}]). \end{cases}$$

In particular, $(\circ) := \sum_{w \in W}(-1)^{\ell(w)}(\circ, w) = (\circ, e) - (\circ, s_{\alpha_1})$ is a cycle, where $\ell(w)$ is the length of w, and $(\circ)$ represents the $\hat{H}_{\mathbb{R}}$. The following theorem gives a topological description of $\hat{H}_{\mathbb{R}}$:

Theorem 1.2. *The manifold* $\hat{H}_{\mathbb{R}}$ *is compact, nonorientable* (*except if* $\mathfrak{g}$ *is of type* A_1), *and it has an action of the Weyl group* W. *The integral homology of* $\hat{H}_{\mathbb{R}}$ *can be computed as a* $\mathbb{Z}[W]$*-module as the homology of the chain complex* $\mathcal{M}_*$ *in* (13).

Theorem 1.2 is completed in Proposition 7.7. The W-action is (abstractly) introduced in Definition 4.4 in terms of a representation-theoretic induction process from smaller parabolic subgroups of W. The proof that the W-action is well-defined is given in Proposition 4.5 and Proposition 5.3. The chain complex $\mathcal{M}_*^{CW}$ in Definition 7.6 is defined so that it computes integral homology. Since each $X_r \setminus X_{r-1}$ in Definition 7.6 is a union of cells $(D, [w]^{\Pi\setminus S}) \in \mathbb{D}^{l-r}$, we obtain an identification between $\mathcal{M}_*^{CW}$ and the chain complex $\mathcal{M}*$.

Then using the Kostant map which can be described as a map from $\hat{H}_{\mathbb{R}}$ into the flag manifold $\widetilde{G}/B$ (in Definition 8.8), that is, a torus imbedding, we obtain the following theorem:

Theorem 1.3. *The toric variety* $\hat{Z}(\gamma)_{\mathbb{R}}$ *is a smooth compact manifold which is diffeomorphic to* $\hat{H}_{\mathbb{R}}$.

The complex version of this theorem has been proven in [**9**], and our proof is essentially given in the same manner.

1.2. Outline of the paper. The paper is organized as follows:

In §2, we begin with two fundamental examples, $\mathfrak{g} = \mathfrak{sl}(l+1, \mathbb{R})$ for $l = 1, 2$, which summarize the main results in the paper.

In §3, we present the basic notations necessary for our discussions. We then define a real group $\widetilde{G}$ of rank l whose split Cartan subgroup $H_{\mathbb{R}}$ contains 2^l connected components. We also define the Lie subgroups of $\widetilde{G}$ corresponding to the subsystems and blow-ups of the generalized Toda lattice equations. The reason for the introduction of $\widetilde{G}$ and the Cartan subgroup $H_{\mathbb{R}}$ can be appreciated in Remark 8.10 and Corollary 8.11.

In §4, we introduce colored Dynkin diagrams which will be shown to parametrize the cells in a cellular decomposition of the manifold $\hat{H}_{\mathbb{R}}$. We then construct a chain complex $\mathcal{M}_*$ of the $\mathbb{Z}[W]$-modules $\mathcal{M}_{l-k}$ (Definition 4.9). The parameters involved in the statement of Theorem 1.1 are given here.

In §5 we show that Weyl group representations introduced in §4 are well-defined. In addition we define $H^{\circ}_{\mathbb{R}}$ in §5 by adding some Cartan subgroups associated to semisimple Levi factors of parabolic subgroups to $H_{\mathbb{R}}$.

In §6, we define $\hat{H}_{\mathbb{R}}$ as a union of the Cartan subgroup associated to semisimple Levi factors of certain parabolic subgroups (Definition 6.1) using translation by Weyl group elements. We also associate subsets of $\hat{H}_{\mathbb{R}}$ to colored Dynkin diagrams.

In §7, we discuss the toplogical structure of $\hat{H}_{\mathbb{R}}$ expressing $\hat{H}_{\mathbb{R}}$ as the union of the subsets determined by the colored Dynkin diagrams. We then show that $\hat{H}_{\mathbb{R}}$ is a smooth compact manifold and those subsets naturally determine a cell decomposition (Theorem 1.2).

In §8, we consider a Kostant map between the isotropy subgroup $G^z_{\mathbb{C}}$ of $G_{\mathbb{C}}$ with $\mathrm{Ad}(g)z = z$ and the isospectral manifold $J(\gamma)_{\mathbb{C}}$ for some $\gamma \in \mathbb{R}^l$, which can be also described as a map into the flag manifold $\widetilde{G}/B$. Then we show that the toric variety $\overline{(H_{\mathbb{R}} nB)}$ is a smooth manifold and obtain Theorem 1.3.

2. Examples of $\mathfrak{sl}(l+1, \mathbb{R})$, $l = 1, 2$.

This section contains two examples which are the source of insight for the main theorems in this paper. Most of the notation and constructions used later on in the paper can be anticipated through these examples.

Our main object of study in this paper, $\hat{H}_{\mathbb{R}}$, has nothing to do, in its construction, with the moment map and the Convexity Theorem of [**1**]. However, because of [**1**] it is expected to contain a polytope with vertices given by the action of the Weyl group which, gives rise to it through some gluings along its faces. We will identify a convenient polytope of this kind in $\mathfrak{h}'$ (dual of Cartan subalgebra) and show how all the pieces of $\hat{H}_{\mathbb{R}}$ would fit inside it. The polytope in the example of $\mathfrak{sl}(3, \mathbb{R})$ is a hexagon. Since we are not dealing with the moment map in this paper, this is just done for the purposes of motivation and illustration. The polytope of the Convexity Theorem in [**1**], strictly speaking, is the convex hull of the orbit of ρ. This sits inside the convex hull of the orbit of 2ρ which is all a part of $\hat{H}_{\mathbb{R}}$.

The terms *dominant* and *antidominant* being relative to a choice of a Borel subalgebra, we refer to the chamber in $\mathfrak{h}'$ containing ρ as *antidominant* for no other reason than the fact that we will make it correspond to what we call the antidominant chamber of the Cartan subgroup $H_{\mathbb{R}}$. This odd convention is applicable throughout this section always in connection with the 2ρ polytope and our figures.

At this stage it is useful to keep in mind that:

1) The bulk — interior — of $\hat{H}_{\mathbb{R}}$ is made up of a split Cartan subgroup $H_{\mathbb{R}}$ which has 2^l connected components H_ϵ parametrized by signs $\epsilon = (\pm, \ldots, \pm)$ for $\mathfrak{g}$ with rank l, i.e., $H_{\mathbb{R}} = \bigcup_{\epsilon \in \{\pm 1\}^l} H_\epsilon$. These disconnected pieces are glued together into a connected manifold by using Cartan subgroups $H_{\mathbb{R}}^A$ associated to Levi factors of parabolic subgroups, which are determined by the set of simple roots $\Pi \setminus A$ for each $A \subset \Pi$.
2) The language of *colored Dynkin* diagrams and *signed colored* Dynkin diagrams is introduced in this paper in order to parametrize the pieces of $\hat{H}_{\mathbb{R}}$. However the motivation for their introduction is that these diagrams parametrize the pieces of a convex polytope (hexagon) which lives in $\mathfrak{h}'$, including its external faces and internal walls. The parameters for the pieces are colored and signed-colored Dynkin diagrams.
3) If one just looks at the antidominant chamber intersected with the 2ρ polytope, then it is easy to see that it forms a *box*. This box is the closure of the *antidominant chamber* $H_{\mathbb{R}}^<$ of the Cartan subgroup $H_{\mathbb{R}}$ (Definition 3.10) inside $\hat{H}_{\mathbb{R}}$, i.e., $\hat{H}_{\mathbb{R}} = \bigcup_{w \in W} w\left(\overline{H_{\mathbb{R}}^<}\right)$. The box $\overline{H_{\mathbb{R}}^<}$ has *internal* chamber walls corresponding to simple roots (thus a Dynkin diagram appears) and it has *external* faces which are also parametrized by simple roots. Hence one needs not just a Dynkin diagram but also two *colors* to indicate internal walls and external faces. The color *blue* indicates internal chamber walls and the color *red* indicates external faces. *The S is usually reserved for the set of colored vertices of a colored Dynkin diagram.*
4) To have a correspondence with the 2^l connected components of the Cartan subgroup $H_{\mathbb{R}}$ of diagonal matrices, the box $\overline{H_{\mathbb{R}}^<}$ must be *further subdivided* into 2^l boxes $H_\epsilon^<$ with signs ϵ, i.e., $H_{\mathbb{R}}^< = \bigcup_{\epsilon \in \{\pm 1\}^l} H_\epsilon^<$. These boxes are parametrized by a Dynkin diagram where each simple root has a sign attached to it. The boundaries of these boxes are portions of the internal and external walls. Hence we use Dynkin diagrams with both signs and colors. The boundaries between two signs require to be labeled as 0. *The A is usually reserved for the set of vertices of a signed-colored Dynkin diagram assigned* 0 (*subsystems*).
5) To *translate* the notation of colored and signed-colored Dynkin diagrams to other chamber one needs to consider pairs (D, w) where w is a Weyl group element. However since the Weyl group action has nontrivial isotropy groups in portions of the polytope, it is necessary to

consider Weyl group cosets $[w]^{\Pi\backslash S}$ for $S \subset \Pi$ the set of colored simple roots (giving reflections generating an isotropy group).

6) To *translate* the Levi subgroups around, note that the Weyl group of the Levi factor stabilizes the Cartan subgroup of the Levi factor corresponding to the simple roots in $\Pi\backslash A$. Because of that we consider products $w(H_{\mathbb{R}}^A) \times \{[w]^A\}$ so that $[w]^A$ is a coset in $W/W_{\Pi\backslash A}$,
$$\hat{H}_{\mathbb{R}} = \bigcup_{A\subset\Pi} \bigcup_{w\in W/W_{\Pi\backslash A}} w\left(H_{\mathbb{R}}^A\right) \times \{[w]^A\}.$$

2.1. The example of $\mathfrak{sl}(2,\mathbb{R})$. The corresponding group in this example is given by $\widetilde{G} = \mathrm{Ad}(SL(2,\mathbb{R})^{\pm})$. The geometric picture that corresponds to $\hat{H}_{\mathbb{R}}$ is a circle. Consider the interval $[-2,2]$ where -2 and 2 are identified. Here 2 represents 2ρ. Inside this interval $(-2,2)$ we consider the subset $[-1,1]$. The points $-1,0,1$ divide $[-2,2]$ into four open intervals. These open intervals will correspond to the connected components of a Cartan subgroup $H_{\mathbb{R}}$ of $\mathrm{Ad}(SL(2,\mathbb{R})^{\pm})$ when the walls corresponding to the points 0 and 2 are deleted. Below we list each cell $w(H_\epsilon^{A,<}) \times \{[w]^A\}$ in $\hat{H}_{\mathbb{R}}$, where $H_\epsilon^{A,<}$ is the intersection of H_ϵ^A with the strictly antidominant chamber (the superscript $\le$ means that the walls are included):

Let us take $h_{\alpha_1} = \mathrm{diag}(1,-1) \in \mathfrak{h}$ and $h_\epsilon = \mathrm{Ad}(\mathrm{diag}(\epsilon,1)) \in H_\epsilon$. Then any element in $H_{\mathbb{R}}$ can be expressed as $h_\epsilon \exp(th_{\alpha_1})$ with some parameter $t \in \mathbb{R}$. We denote $\exp(th_{\alpha_1}) = \mathrm{diag}(a, a^{-1})$.

We first consider $A = \emptyset$. Then the cell $H_+^{\le} \times \{[e]\}$ is given by
$$\left\{\mathrm{Ad}(\mathrm{diag}(a,a^{-1}))\ :\ 0<a<1\right\} \times \{[e]\} = (\circ_+, e) \leftrightarrow (0,1).$$
Here $\chi_{\alpha_1}(h) = a^2$ for $h \in H_+$. Also we have the set $H_-^{\le} \times \{[e]\}$ as
$$\left\{\mathrm{Ad}(\mathrm{diag}(-a,a^{-1}))\ :\ 0<a<1\right\} \times \{[e]\} = (\circ_-, e) \leftrightarrow (1,2)$$
with $\chi_{\alpha_1}(h) = -a^2$ for $h \in H_-$.

We now consider the case of $A = \{\alpha_1\} = \Pi$, which corresponds to a subsystem of Toda lattice with $b_1 = 0$ in (5). Then we have $H_{\mathbb{R}}^{\Pi,\le} = \{e\}$. This is the degenerate case of $A = \Pi$ which gives rise to the Levi factor of a Borel subgroup. Since the Levi factor does not contain a semisimple Lie subgroup, $H_{\mathbb{R}}^A$ is defined to be $\{e\}$ (Definition 3.9). Here $[w]^A$ is just the element w. We have
$$\{\mathrm{Ad}(h_+)\} \times \{e\} = (\circ_0, e) \leftrightarrow \{1\}.$$
We now describe the box containing the strictly antidominant chamber $H_{\mathbb{R}}^<$ of $H_{\mathbb{R}}$. Since $H_{\mathbb{R}}^<$ is disconnected, $(\circ_0, e)$ has been used to glue the pieces together. We then have a box given by
$$(\circ, e) = (\circ_+, e) \cup (\circ_-, e) \cup (\circ_0, e) \leftrightarrow (0,2). \tag{7}$$

The bijection which gives rise to local coordinates ϕ_e in Subsection 7.2 is given by either $\pm a^2$ or 0. The set $(\circ, e)$ is sent by ϕ_e to the interval $(-1, 1)$.

We now apply s_{α_1} on $H_\epsilon^< \times \{[e]\}$ to obtain the cell in the s_{α_1}-chamber which corresponds to the negative intervals:

$$\left\{\mathrm{Ad}(\mathrm{diag}(a^{-1}, a)) \ : \ 0 < a < 1\right\} \times \{[e]\} = (\circ_+, s_{\alpha_1}) \leftrightarrow (-1, 0)$$

with $\chi_{\alpha_1}(h) = a^{-2}$. However the local coordinate $\phi_{s_{\alpha_1}}$ is $\chi_{-\alpha_1}$ which equals a^2. We also have

$$\left\{\mathrm{Ad}(\mathrm{diag}(a^{-1}, -a)) \ : \ 0 < a < 1\right\} \times \{[e]\} = (\circ_-, s_{\alpha_i}) \leftrightarrow (-2, -1).$$

Also for the case $A = \{\alpha_1\}$, and the set $s_{\alpha_1}(H_{\mathbb{R}}^{A,\leq}) \times \{s_{\alpha_1}\}$ is given by

$$\{\mathrm{Ad}(h_+)\} \times \{s_{\alpha_1}\} = (\circ_0, s_{\alpha_1}) \leftrightarrow \{-1\}.$$

Once again $\phi_{s_{\alpha_1}}$ is given as 0, and we have the set $s_{\alpha_1}(H_{\mathbb{R}}^<)$, giving rise to an open box by gluing two disconnected pieces as before,

$$(\circ, s_{\alpha_1}) = (\circ_+, s_{\alpha_1}) \cup (\circ_-, s_{\alpha_1}) \cup (\circ_0, s_{\alpha_1}) \leftrightarrow (-2, 0).$$

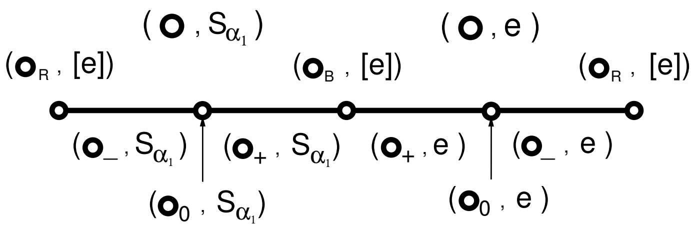

Figure 3. The manifold $\hat{H}_{\mathbb{R}}$ parametrized by signed-colored Dynkin diagrams for $\mathfrak{sl}(2, \mathbb{R})$. The endpoints in the interval are identified giving rise to a circle.

The image of $(\circ, s_{\alpha_1})$ under $\phi_{s_{\alpha_1}}$ is thus$(-1, 1)$. We also have the internal and external walls of the Cartan subgroup,respectively:

$$\{\mathrm{Ad}(h_+)\} \times \{[e]\} = (\circ_B, [e]) \leftrightarrow \{0\}$$
$$\{\mathrm{Ad}(h_-)\} \times \{[e]\} = (\circ_R, [e]) \leftrightarrow \{2\}.$$

These are already associated to colored Dynkin diagrams $\circ_B$ and $\circ_R$.

We can write down $\{-2\}$ by applying s_{α_1} as we did above. However noting $\mathrm{Ad}(\mathrm{diag}(-1, 1)) = \mathrm{Ad}(\mathrm{diag}(1, -1))$, we obtain the same set that defines $\{2\}$. Thus $\{2\}$ and $\{-2\}$ are identified. We then obtain the interval $[-2, 2]$ with -2 identified with 2, which is $\hat{H}_{\mathbb{R}}$ diffeomorphic to a circle. We illustrate this example in Figure 3.

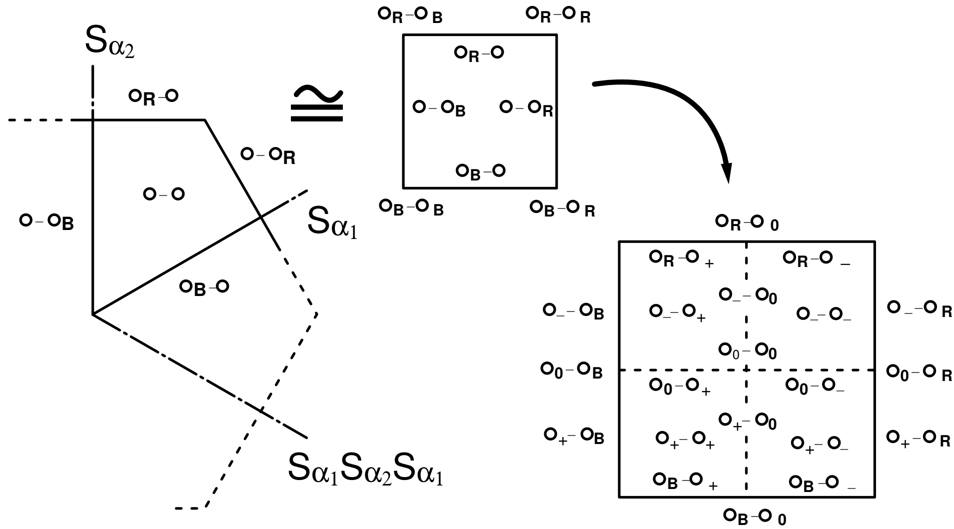

Figure 4. The *square* formed by the intersection of the antidominant chamber with the 2ρ hexagon for $\mathfrak{sl}(3,\mathbb{R})$.

Maps can be easily found between the intervals on the left and the sets on the right above. With a suitable topology associated to the $(D,[w])$, topology defined in terms of the coordinate functions ϕ_e and $\phi_{s_{\alpha_1}}$ in Section 7.3, each interval or point on the left side is homeomorphic to the interval on the right. This is what is indicated with $\leftrightarrow$.

2.2. The example of $\mathfrak{sl}(3,\mathbb{R})$. We here consider the dual of its Cartan subalgebra $\mathfrak{h}'$ and, inside it, a convex region bounded by a hexagon which is determined by the W orbit of 2ρ. We will later describe how to identify some of the faces of this hexagon. In Figure 4, we illustrate the parametrization of the faces. The two walls of the antidominant chamber intersected with this convex region are denoted by the colored Dynkin diagrams $\circ_B - \circ$ (the s_{α_1}-wall) and $\circ - \circ_B$ (the s_{α_2}-wall). The intersection of two of the sides or faces of the 2ρ hexagon with the antidominant chamber are each denoted by a colored Dynkin diagram $\circ_R - \circ$ or $\circ - \circ_R$. The four colored Dynkin diagrams $\circ_B - \circ$, $\circ - \circ_B$, $\circ_R - \circ$, $\circ - \circ_R$ form a *square*. This square is the intersection of the antidominant chamber and the 2ρ hexagon.

We now further subdivide this *square* into four *subsquares* (see Figure 1 and Figure 4). Inside the 2ρ hexagon is the orbit of ρ which gives rise to a new smaller hexagon. Consider the two faces in the smaller hexagon intersecting the antidominant chamber. Denote these (intersected with the antidominant chamber) by signed-colored diagrams. The $\circ_0 - \circ_+$ corresponds to the unique face intersecting the wall $\circ - \circ_B$, and $\circ_+ - \circ_0$ corresponds to the unique face intersecting the wall $\circ_B - \circ$. Denote by $\circ_0 - \circ_0$ the vertex of the ρ hexagon which is the intersection of these two faces. We now add a segment joining the vertex $\circ_0 - \circ_0$, to a point in the interior of $\circ_R - \circ$, say the midpoint. Denote this segment by the signed-colored diagram $\circ_- - \circ_0$. We add another segment joining $\circ_0 - \circ_0$ to a point in the interior of $\circ - \circ_R$ and denote this second segment by the signed-colored diagram $\circ_0 - \circ_-$. Now the square is divided into four "square" regions denoted by $\circ_\pm - \circ_\pm$.

Note that both $\circ_R - \circ$ and $\circ - \circ_R$ segments parametrized by colored Dynkin diagrams, are now subdivided into two segments parametrized with signed-colored Dynkin diagrams. For instance $\circ - \circ_R$ contains $\circ_+ - \circ_R$ and $\circ_- - \circ_R$ and the intersection of these two is the point $\circ_0 - \circ_R$. Now the square $\circ_- - \circ_-$, for example, has a boundary which consists of the segments parametrized by $\circ_R - \circ_-$, $\circ_- - \circ_R$, $\circ_0 - \circ_-$ and $\circ_- - \circ_0$. The square $\circ_- - \circ_+$ has boundary $\circ_R - \circ_+$, $\circ_- - \circ_B$, $\circ_0 - \circ_+$, $\circ_- - \circ_0$. Here the α_2-wall $\circ - \circ_B$ has also been subdivided into two pieces $\circ_+ - \circ_B$ and $\circ_- - \circ_B$ by the intersection with the ρ hexagon.

If we now consider the full set of signed-colored Dynkin diagrams by translating with W, we can fill the interior of the 2ρ hexagon with a total of 12 regions. The four squares $(\circ_\pm - \circ_\pm, e)$ form the intersection of the antidominant chamber with the inside of the 2ρ hexagon.

2.2.1. The sets in $\hat{H}_{\mathbb{R}}$ parametrized by the colored and signed-colored Dynkin diagrams. We now proceed to describe explicitly some of the pieces of $\hat{H}_{\mathbb{R}}$ corresponding to the signed-colored Dynkin diagrams $(D, [w]^{\Pi\setminus S})$ with $+, -$ or 0 on the vertices in $\Pi \setminus S$ of the diagram D and $[w]^{\Pi\setminus S} \in W/W_S$.

When $A = \emptyset$ implying no 0's in the vertices, we have $H_{\mathbb{R}}^A = H_{\mathbb{R}}$ which has 4 connected components,

$$H_{\mathbb{R}} = \bigcup_{\epsilon\in\{\pm 1\}^2} \{h_\epsilon \mathrm{diag}(a,b,c) \ : \ a>0,\ b>0,\ abc=1\}, \tag{8}$$

where $h_\epsilon = h_{(\epsilon_1,\epsilon_2)} = \mathrm{diag}(\epsilon_2, \epsilon_1\epsilon_2, \epsilon_1)$ satisfying $\chi_{\alpha_i}(h_\epsilon) = \epsilon_i$. Since $A = \emptyset$, $W_{\Pi\setminus A} = W$, and there is only one coset $[e]^A = [e]$ in the $\hat{H}_{\mathbb{R}}$ construction. We now consider the signed-colored Dynkin diagrams setting $S = \emptyset$, that is, all the vertices are uncolored and they give a subdivision of the antidominant chamber inside the 2ρ hexagon. In order to move around this chamber and its subdivisions using the Weyl group we must also consider six elements $[w]^\Pi = w \in W$. We then have, for $S = \emptyset$ and $A = \emptyset$, and $(\epsilon_1, \epsilon_2) = (\pm 1, \pm 1)$,

$$(\circ_{\epsilon_1} - \circ_{\epsilon_2}, e) = \left\{ h_\epsilon \mathrm{diag}(a, b, (ab)^{-1}) \ : \ ab^{-1} < 1,\ ab^2 < 1 \right\} \times \{[e]\}$$

Note here that the inequalities $|\chi_{\alpha_i}| < 1$ guarantee that the set in question is contained in the chamber associated to e. Here the local coordinate ϕ_e is given by $(\chi_{\alpha_1}, \chi_{\alpha_2})$ which equals (ab^{-1}, ab^2).

For $A = \{\alpha_1\}$, we have $h_\epsilon = \mathrm{diag}(\epsilon_2, \epsilon_2, 1) \in H_\epsilon^A$ with $\epsilon = (1, \epsilon_2)$; and we multiply this element with $\mathrm{diag}(1, a, a^{-1})$. This is a typical element in the connected component of the Cartan associated to the Levi factor, in accordance with the definition of $H^A_{\mathbb{R}}$ in Definition 3.10. This gives:

$$(\circ_0 - \circ_{\epsilon_2}, e) = \left\{ \mathrm{diag}(\epsilon_2, \epsilon_2 a, a^{-1}) \ : \ 0 < a < 1 \right\} \times \{[e]^A\}$$

where the local coordinate ϕ_e is given by $(0, \epsilon_2 a^2)$.

For $A = \{\alpha_2\}$, we have in a similar way with $h_{(\epsilon_1,1)} = \mathrm{diag}(1, \epsilon_1, \epsilon_1)$:

$$(\circ_{\epsilon_1} - \circ_0, e) = \left\{ \mathrm{diag}(a, \epsilon_1 a^{-1}, \epsilon_1) \ : \ 0 < a < 1 \right\} \times \{[e]^A\}$$

where ϕ_e now equals $(\epsilon_1 a^2, 0)$.

We have an open *square* associated with the interior of the antidominant chamber,

$$(\circ - \circ, e) = \bigcup_{(\nu_1,\nu_2)\in\{\pm 1,0\}^2} (\circ_{\nu_1} - \circ_{\nu_2}, e). \tag{9}$$

The image of this set under the map ϕ_e is an open square $(-1,1)\times(-1,1)$.

We now write down the boundary of this square: We here give an explicit form of $(\circ_R - \circ, [e]^{\Pi\setminus S})$, and the others can be obtained in the similar way. We first have, for $S = \{\alpha_1\}$, $A = \emptyset$, so that $[e]^A = [e] = [s_{\alpha_i}]$ for $i = 1, 2$,

$$\left(\circ_R - \circ_{\epsilon_2}, [e]^{\{\alpha_2\}} \right) = \left\{ \mathrm{diag}\left(\epsilon_2 a, -\epsilon_2 a, -a^{-2} \right) \ : \ 0 < a < 1 \right\} \times \{[e]\}.$$

Here ϕ_e equals $(\chi_{\alpha_1}, \chi_{\alpha_2})$, and is given by $(-1, \epsilon_2 a^3)$.

With $A = \{\alpha_2\}$, we have:

$$\left(\circ_R - \circ_0, [e]^{\{\alpha_2\}} \right) = \{\mathrm{diag}(1,-1,-1)\} \times \left\{ [e]^{\{\alpha_2\}} \right\}.$$

The map ϕ_e is $(\chi^{\Delta^A}_{\alpha_1}, 0)$ and equals $(-1, 0)$. We then have

$$\left(\circ_R - \circ, [e]^{\{\alpha_2\}} \right) = \bigcup_{\nu\in\{\pm 1,0\}} \left(\circ_R - \circ_\nu, [e]^{\{\alpha_2\}} \right).$$

The image of this set under ϕ_e is thus $\{-1\} \times (-1, 1)$.

We now consider the parts of the Cartan subgroup of Levi factors corresponding to other chambers inside the hexagon. Note that if we apply $s_{\alpha_1} = w$ to $(\circ_{\epsilon_1} - \circ_{\epsilon_2}, e)$, we obtain

$$(\circ_{\epsilon_1} - \circ_{\epsilon_1\epsilon_2}, s_{\alpha_1}) = \left\{ \mathrm{diag}\left(\epsilon_1\epsilon_2 b, \epsilon_2 a, \epsilon_1 (ab)^{-1} \right) : ab^{-1} < 1, ab^2 < 1 \right\} \times \{[e]\}.$$

Since $\mathrm{sign}(\chi_{\alpha_1}) = \epsilon_1$ and $\mathrm{sign}(\chi_{\alpha_2}) = \epsilon_1\epsilon_2$, this set is no longer contained in $H_{(\epsilon_1,\epsilon_2)}$ but rather in $H_{(\epsilon_1,\epsilon_1\epsilon_2)}$. This justifies the notation $(\circ_{\epsilon_1} - \circ_{\epsilon_1\epsilon_2})$. One should note that ϵ for the component H_ϵ in $H_{\mathbb{R}}$ did not change when one uses the simple roots associated to the *new* positive system $s_{\alpha_1}\Delta_+$.

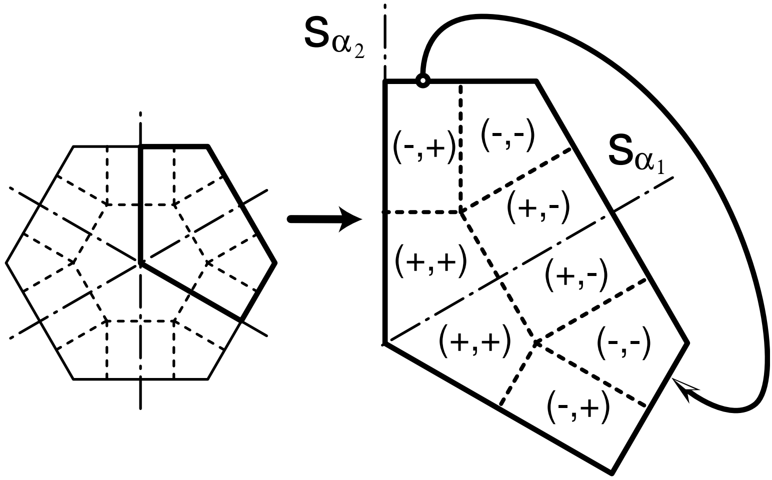

Figure 5. Gluing creates a Mobius band out of two contiguous chambers.

Also notice that for $S = \{\alpha_1\}$ we have

$$\left(\circ_R - \circ_+, [s_{\alpha_1}]^{\Pi\backslash S}\right) = \left\{\mathrm{diag}\left(-a, a, -a^{-2}\right) \ : \ a > 0\right\} \times \{[e]\}$$

and we have an identification of $(\circ_R - \circ_+, [s_{\alpha_1}]^{\Pi\backslash S})$ and $(\circ_R - \circ_-, [e]^{\Pi\backslash S})$. They are the same set. In fact, now $S = \{\alpha_1\}$ and e and s_{α_1} give the same coset in W/W_S so the corresponding signed-colored Dynkin diagrams agree too. Similarly $(\circ_R - \circ_-, [s_{\alpha_1}]^{\Pi\backslash S})$ is the same as $(\circ_R - \circ_+, [e]^{\Pi\backslash S})$. In our hexagon this means that *two of the outer walls must be glued.* The fact that a segment with a sign $+$ is glued to one with $-$ corresponds to the fact that the two contiguous chambers will form a *Mobius band* after the gluing (see Figure 5).

What this means is that in our geometric picture consisting of the inside of the 2ρ hexagon, some portions of the boundary need to be identified. Such identifications take place on all the chambers. This identification provides the gluing rule given in Lemma 4.2 in [**15**] for the case of $\mathfrak{sl}(n, \mathbb{R})$.

2.3. The chain complex $\mathcal{M}_*$. We describe $\mathcal{M}_*$ in terms of colored Dynkin diagrams. The $\mathbb{Z}$ modules of chains $\mathcal{M}_k$ are then given by (see Figure 6):

- $\mathcal{M}_2 = \mathbb{Z}\left[(\circ - \circ, w) : w \in W\right]$. The cells are the *chambers of the* 2ρ *hexagon*, and $\dim\mathcal{M}_2 = 6$.
- $\mathcal{M}_1$ consists of the cells parametrized by the colored Dynkin diagrams $(\circ_R - \circ, [w]^{\{\alpha_2\}})$, $(\circ_B - \circ, [w]^{\{\alpha_2\}})$ with $w \in \{e, s_{\alpha_2}, s_{\alpha_1}s_{\alpha_2}\}$ and $(\circ - \circ_R, [w]^{\{\alpha_1\}})$, $(\circ - \circ_B, [w]^{\{\alpha_1\}})$ with $w \in \{e, s_{\alpha_1}, s_{\alpha_2}s_{\alpha_1}\}$. These are the *sides of the different chambers of the hexagon.* The *blue* (B) stands for *internal chamber wall* and the *red* (R) for *external face of the hexagon.* The dimension is then given by $\dim \mathcal{M}_1 = 12$.

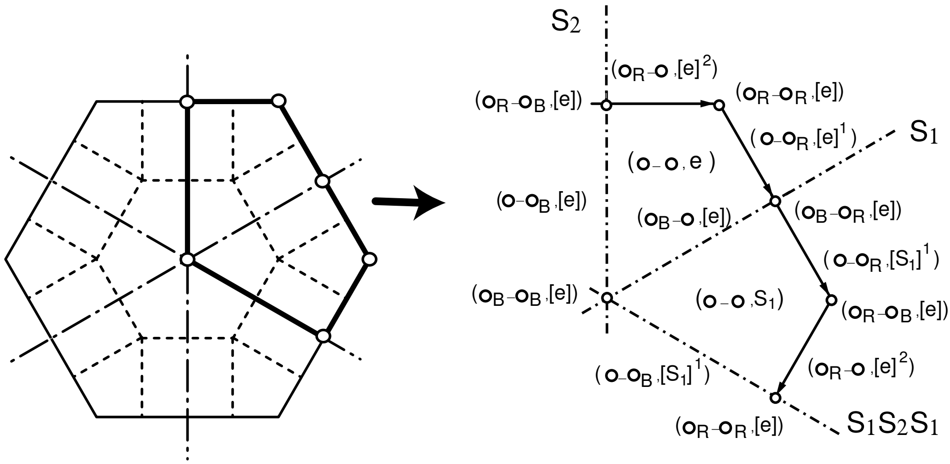

Figure 6. The manifold $\hat{H}_{\mathbb{R}}$ parametrized by colored Dynkin diagrams for $\mathfrak{sl}(3,\mathbb{R})$.

- $\mathcal{M}_0 = \mathbb{Z}\left[(\circ_s - \circ_t, [e]) : s,t \in \{R,B\}\right]$. These are the four vertices of a chamber. Because of identifications, there are only four different points coming from all the six chambers, that is, $\dim\mathcal{M}_0 = 4$.

The chain complex $\mathcal{M}_* : \mathcal{M}_2 \xrightarrow{\partial_2} \mathcal{M}_1 \xrightarrow{\partial_1} \mathcal{M}_0$ leads to the following integral homology: $H_2 = 0$, $H_1 = \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z}$, $H_0 = \mathbb{Z}$. This implies that $\hat{H}_{\mathbb{R}}$ is nonorientable and is equivalent to the connected sum of two Klein bottles. Also note that the Euler character is $6 - 12 + 4 = -2$. According to Proposition 7.7 this computes the homology $H_*(\hat{H}_{\mathbb{R}}, \mathbb{Z})$ of the compact smooth manifold $\hat{H}_{\mathbb{R}}$. The torsion $\mathbb{Z}/2\mathbb{Z}$ in H_1 has the following representative:

$$c_1 = \sum_{w \in W/W_{\{s_{\alpha_2}\}}} (-1)^{\ell(w)} \left(\circ_R - \circ, [w]^{\{s_{\alpha_2}\}}\right) - \sum_{w \in W/W_{\{s_{\alpha_1}\}}} (-1)^{\ell(w)} \left(\circ_R - \circ, [w]^{\{s_{\alpha_1}\}}\right).$$

Here $\ell(w)$ denotes the length of w. If we let

$$c_2 = \sum_{w \in W} (-1)^{\ell(w)} (\circ - \circ, w),$$

then $\partial_2(c_2) = 2c_1$.

From the chain complex one can compute the three dimensional vector space $H_1(\hat{H}_{\mathbb{R}}, \mathbb{Q})$ as a W-module. This is a direct sum of a one dimensional nontrivial (sign) representation and the two dimensional reflection representation. The representation $H_0(\hat{H}_{\mathbb{R}}, \mathbb{Q})$ is trivial.

3. Notation, the group $\widetilde{G}$.

3.1. Basic Notation. The important Lie group for the purposes of this paper is a group $\widetilde{G}$ that will be technically introduced in Subsection 3.2. This group is $\mathbb{R}$ split and has a split Cartan subgroup with 2^l components with $l = \operatorname{rank}(G)$. For example $SL(3,\mathbb{R})$ is of this kind but $SL(4,\mathbb{R})$ is not. We need to introduce the following standard objects before it is possible to define and study $\widetilde{G}$.

Notation 3.1 (Standard Lie theoretic notation)**.** We adhere to standard notation and to the following conventions: $\ldots_{\mathbb{C}}$ denotes a complexification; for any set $S \subset \Pi$, $\ldots_S$ is an object related to a parabolic subgroup or subalgebra determined by S; for any subset $A \subset \Pi$, $\ldots^A$ is associated to the parabolic determined by $\Pi \setminus A$. However we will not simultaneously employ $\ldots^A$ and $\ldots_S$ for the same object $\ldots$. As in the introduction $\mathfrak{g}$ denotes a real split semisimple Lie algebra of rank l with complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \underset{\mathbb{R}}{\otimes} \mathbb{C}$. We also have $G_{\mathbb{C}}$ the connected adjoint Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ ($G_{\mathbb{C}} = \operatorname{Ad}(G^s_{\mathbb{C}})$) and G the connected real semisimple Lie subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{g}$. Denote $G^s_{\mathbb{C}}$ the simply connected complex Lie group associated to $\mathfrak{g}_{\mathbb{C}}$. We list some additional very standard Lie theoretic notation:

- $\mathfrak{g}' = \operatorname{Hom}_{\mathbb{R}}(\mathfrak{g},\mathbb{R})$ and $\mathfrak{g}'_{\mathbb{C}} = \operatorname{Hom}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}},\mathbb{C})$.
- Given $\lambda \in \mathfrak{g}'$ and $x \in \mathfrak{g}$, $\langle \lambda, x\rangle$ is λ evaluated in x, $\langle \lambda, x\rangle = \lambda(x)$.
- $(,)$ the bilinear form on $\mathfrak{g}$ or $\mathfrak{g}_{\mathbb{C}}$ given by the Killing form (the same notation applies to the Killing form on $\mathfrak{g}'$ and $\mathfrak{g}'_{\mathbb{C}}$).
- θ a Cartan involution on $\mathfrak{g}$.
- $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$ associated to θ, where $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup K of the adjoint group G and $\mathfrak{p}$ is the orthogonal complement to $\mathfrak{k}$ with respect to the Killing form.
- $e_\phi \in \mathfrak{g}$, $\mathfrak{h}$ root vectors chosen so that $(e_\phi, e_{-\phi}) = 1$.
- $\Delta_+ \subset \Delta$ be a fixed system of positive roots.
- $\mathfrak{b} = \mathfrak{h} + \sum_{\phi\in\Delta_+} \mathbb{R}e_\phi$ (Borel subalgebra).
- $\mathfrak{n} = [\mathfrak{b},\mathfrak{b}] = \sum_{\phi\in\Delta_+} \mathbb{R}e_\phi$ and $\bar{\mathfrak{n}} = \sum_{\phi\in\Delta_+} \mathbb{R}e_{-\phi}$.
- $\mathfrak{n}_{\mathbb{C}} = \mathfrak{n} \underset{\mathbb{R}}{\otimes} \mathbb{C}$, $\bar{\mathfrak{n}}_{\mathbb{C}} = \bar{\mathfrak{n}} \underset{\mathbb{R}}{\otimes} \mathbb{C}$.
- $H_{\mathbb{C}}$ Cartan subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{h}_{\mathbb{C}}$.
- $H = H(\Delta)$ the connected Lie subgroup of G with Lie algebra $\mathfrak{h}$, $H = \exp(\mathfrak{h})$.
- $H^1_{\mathbb{R}}$ Cartan subgroup of G with Lie algebra $\mathfrak{h}$.
- W the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$ or the Weyl group of $H^1_{\mathbb{R}}$.

- $W_S \subset W$, the group generated by the simple reflections corresponding to the elements in $S \subset \Pi$.

Remark 3.2. The group $H^1_{\mathbb{R}}$ [22] p. 59, 2.3.6 consists of all $g \in G$ such that $\mathrm{Ad}(g)$ restricted to $\mathfrak{h}$ is the identity. This Cartan subgroup will be usually disconnected and H is the connected component of the identity e. The Weyl group W of the Cartan subgroup $H^1_{\mathbb{R}}$ is isomorphic to the group which is generated by the simple reflections s_{α_i} with $\alpha_i \in \Pi$ and which agrees with the Weyl group associated to the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. This is because our group is assumed to be $\mathbb{R}$ split.

Example 3.3. The reader may wish to read the whole paper with the following well-known example in mind. Let $G^s_{\mathbb{C}} = SL(n, \mathbb{C})$ and $G_{\mathbb{C}} = \mathrm{Ad}(SL(n, \mathbb{C}))$. The second group is obtained by dividing $SL(n, \mathbb{C})$ by the finite abelian group consisting of the n roots of unity times the identity matrix. We can set $G = \mathrm{Ad}(SL(n, \mathbb{R}))$. Note that if n is odd then $SL(n, \mathbb{R}) = \mathrm{Ad}(SL(n, \mathbb{R})) = G$. If n is even then $\mathrm{Ad}(SL(n, \mathbb{R}))$ is obtained by dividing by $\{\pm I\}$ (I the identity matrix). In this example, $\mathfrak{g}$ consists of traceless $n \times n$ real matrices. The Cartan subalgebra $\mathfrak{h}$ in the $\mathrm{Ad}(SL(n, \mathbb{R}))$ case can be taken to be the space of traceless $n \times n$ real diagonal matrices. The root vectors e_ϕ are the various matrices with all entries $a_{i,j} = 0$ if $(i, j) \neq (i_o, j_o)$ and $a_{i_o,j_o} = 1$ where $i_o \neq j_o$ are fixed integers. In this case of $G = \mathrm{Ad}(SL(n, \mathbb{R}))$, with $\mathfrak{h}$ chosen as above, the Cartan subgroup $H^1_{\mathbb{R}}$ of G consists of Ad applied to the group of real diagonal matrices of determinant one. The group H consists of Ad applied to all diagonal matrices $\mathrm{diag}(r_1, \dots, r_n)$ with $r_i > 0$ the connected component of the identity of $H^1_{\mathbb{R}}$.

Notation 3.4. The following elements in $\mathfrak{h}$ and its dual $\mathfrak{h}'$ will appear often in this paper:

- $\check{\alpha}_i = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$ the coroots.
- m_{α_i} $i = 1, \dots, l$, the fundamental weights, $(m_{\alpha_i}, \check{\alpha}_j) = \delta_{i,j}$.
- $m^\circ_{\alpha_i}$ the unique element in $\mathfrak{h}$ defined by $\langle m_{\alpha_i}, x\rangle = (m^\circ_{\alpha_i}, x)$.
- h_{α_i} is the unique element in $\mathfrak{h}$ such that $(h_{\alpha_i}, x) = \langle \check{\alpha}_i, x\rangle$.
- $y_i = \frac{2\pi m^\circ_{\alpha_i}}{(\alpha_i, \alpha_i)}$.
- $\mathfrak{h}^< = \{x \in \mathfrak{h} : \langle \alpha_i, x\rangle < 0 \text{ for all } \alpha_i \in \Pi\}$.

Example 3.5. Consider $G_{\mathbb{C}} = \mathrm{Ad}(SL(n, \mathbb{C})) \subset \mathrm{Ad}(GL(n, \mathbb{C}))$ and n even. We view $\mathfrak{g}_{\mathbb{C}}$ inside the Lie algebra of $GL(n, \mathbb{C})$ (as $n \times n$ complex traceless matrices). Then $m^\circ_{\alpha_i} = \mathrm{diag}(t_1, \dots, t_n) + z$ where $t_j = 1$ for $j \le i$ and $t_j = 0$ for $j > i$. The element z is in the center of the Lie algebra of $GL(n, \mathbb{C})$ and thus $\mathrm{ad}(z) = 0$. For instance if $n = 2$ then $m^\circ_{\alpha_1} = \mathrm{diag}(\frac{1}{2}, -\frac{1}{2}) = \mathrm{diag}(1, 0) - \mathrm{diag}(\frac{1}{2}, \frac{1}{2})$.

3.2. The group $\widetilde{G}$. We now define, following [16], 3.4.4 in p. 241 an enlargement $\widetilde{G}$ of the split group G. The purpose of this is to "complete" the Cartan subgroup $H^1_{\mathbb{R}}$ forcing it to have 2^l connected components where l is the rank of G. In the case $G = SL(l+1,\mathbb{R})$ where l is even, $\widetilde{G} = G$ already. We thus consider another real split group, slightly bigger than G. Let $x, y \in \mathfrak{h}$ so that $x + \sqrt{-1}y \in \mathfrak{h}_{\mathbb{C}}$. We recall the conjugate linear automorphism of $\mathfrak{g}_{\mathbb{C}}$ given by $z^c = x - \sqrt{-1}y$ where $z = x + \sqrt{-1}y$ and $x, y \in \mathfrak{g}$. We recall that this automorphism induces an automorphism $G_{\mathbb{C}} \to G_{\mathbb{C}}$, $g \mapsto g^c$. We thus let

$$\widetilde{G} = \{g \in G_{\mathbb{C}} : g^c = g\}.$$

By [**16**] Proposition 3.4, we also have:

$$\widetilde{G} = \{g \in G_{\mathbb{C}} : \mathrm{Ad}(g)\mathfrak{g} \subset \mathfrak{g}\}.$$

Then we have the following Proposition whose proof will be given later (right after Proposition 3.15):

Proposition 3.6. *Let $G = \mathrm{Ad}(SL(n,\mathbb{R}))$, then $\widetilde{G} \cong SL(n,\mathbb{R})$ for n odd and $\widetilde{G} \cong \mathrm{Ad}(SL(n,\mathbb{R})^{\pm})$ if n is even. Thus $\widetilde{G}$ is disconnected whenever n is even.*

We now describe an element h_i in the Cartan subgroup of $\widetilde{G}$. First we have:

Lemma 3.7. *Let $x, y \in \mathfrak{h}$ and $\alpha_i \in \Pi$. Then the numbers $\exp(\langle \alpha_i, x + \sqrt{-1}y\rangle)$ are real if and only if y has the form: $y = \sum_{i=1}^{l} k_i y_i$ where k_i is an integer and y_i is as in Notation 3.4. The elements $h_i = \exp(\sqrt{-1}k_i y_i)$ with k_i odd are in $\widetilde{G}$ and satisfy $h_i^2 = e$ with $h_i \neq e$.*

Proof. It is enough to consider the case when $x = 0$. Suppose first that $y = \sum_{i=1}^{l} k_i y_i$ where each k_i is an integer. Then $e^{\sqrt{-1}\langle \alpha_i, y\rangle} = e^{\sqrt{-1}\pi k_i}$ takes either $+1$ or -1. Conversely, suppose that all the $e^{\langle \alpha_i, \sqrt{-1}y\rangle}$ with $i = 1, \ldots, l$ are real. Then $e^{\langle \alpha_i, \sqrt{-1}y\rangle}$ equals ± 1 and $\langle \alpha_i, y\rangle = k_i \pi$ for all $i = 1, \ldots, l$. This implies that, for each i, $\langle \check{\alpha}_i, y\rangle = \frac{2k_i\pi}{(\alpha_i,\alpha_i)}$. Since $\mathfrak{g}$ is semisimple and the $\{m^{\circ}_{\alpha_i} : i = 1, \ldots, l\}$ forms a basis of $\mathfrak{h}$, necessarily $y = \sum_{i=1}^{l} c_i m^{\circ}_{\alpha_i}$ with $\langle \check{\alpha}_i, y\rangle = c_i$ and $\langle \check{\alpha}_i, y\rangle = \frac{2k_i\pi}{(\alpha_i,\alpha_i)}$. This proves the first part of the statement in Lemma 3.7. We note that since all the $e^{\langle \alpha_j, y_i\rangle}$ are real then $e^{\langle \phi, \sqrt{-1}y_i\rangle}$ is also real for any root ϕ which is not necessarily simple. Therefore each $\mathrm{Ad}(h_i)$ stabilizes all the root spaces of $\mathfrak{h}$ in $\mathfrak{g}$. Since $\mathfrak{h}$ is also stabilized, we obtain that $\mathrm{Ad}(h_i)(\mathfrak{g}) \subset \mathfrak{g}$ and thus $h_i \in \widetilde{G}$. Clearly $h_i = \exp(\sqrt{-1}y_i)$ satisfies $h_i^2 = 1$ where $h_i \neq e$. This is because h_i is representable by a diagonal matrix with entries of the form ± 1. Moreover, at least one diagonal entry must be equal to -1. $\square$

Example 3.8. In the case of $\mathrm{Ad}(SL(n,\mathbb{R})^{\pm})$ with n even, $h_i = \exp(\sqrt{-1}y_i)$ is just $\mathrm{Ad}(\mathrm{diag}(r_1, \ldots, r_n))$ where $r_j = 1$ if $j \leq i$ and $r_j = -1$ if $j > i$. When

n is odd then we have $h_i = (-1)^{n-i}\mathrm{diag}(r_1, \ldots, r_n)$ with the same notation as above for the r_i. In this case $SL(n, \mathbb{R}) = \mathrm{Ad}(SL(n, \mathbb{R}))$.

We now describe a split Cartan subgroup $H_{\mathbb{R}}$ of $\widetilde{G}$ and other items related to it. The $H_{\mathbb{R}}$ is the real part of $H_{\mathbb{C}}$ on $\widetilde{G}$,

$$H_{\mathbb{R}} = H_{\mathbb{C}} \cap \widetilde{G},$$

which has 2^l components (see Proposition 3.15 below). We denote by B a Borel subgroup with Lie algebra $\mathfrak{h} + \mathfrak{n}$ contained in $\widetilde{G}$ or G as will be clear from the context. Thus in $\widetilde{G}$ this is $H_{\mathbb{R}}N$, $N = \exp(\mathfrak{n})$. From the Bruhat decomposition applied to $\widetilde{G}$, we have

$$\widetilde{G} = \bigcup_{w \in W} \overline{N}\hat{w}H_{\mathbb{R}}N,$$

with $\overline{N} = \exp(\bar{\mathfrak{n}})$. Here $\hat{w}$ stands for any representative of the Weyl group element $w \in W$ in the normalizer of the Cartan subgroup. Keeping this in mind we will harmlessly drop the $\hat{}$ from the notation.

In addition let χ_ϕ denote the group character determined by $\phi \in \Delta$; on $H_{\mathbb{R}}$ each χ_ϕ is real and cannot take the value zero. Thus a group character has a fixed sign on each connected component. We denote by $\mathrm{sign}(\chi_\phi(g))$ the sign of this character on a specified element g of $H_{\mathbb{R}}$. We let $\mathcal{E} = \mathcal{E}(\Delta) = \{\pm 1\}^l = \{(\epsilon_1, \ldots, \epsilon_l) : \epsilon_i \in \{\pm 1\}, i = 1, \ldots, l\}$. Then the set $\mathcal{E}$ of all 2^l elements ϵ (or functions $\epsilon : \Pi \to \{\pm 1\}$) will parametrize connected components of the Cartan subgroup $H_{\mathbb{R}}$ of $\widetilde{G}$ and corresponds to the signs $\epsilon_k = s_k s_{k+1}$ in the indefinite Toda lattice in p. 323 of [**14**]. In connection with the connected components let $h_\epsilon = \prod_{\epsilon_i \neq 1} h_i$ for $\epsilon \in \mathcal{E}$ and $h_i = \exp(\sqrt{-1}y_i)$ in Lemma 3.7. Now $H_\epsilon = h_\epsilon H = \{h \in H_{\mathbb{R}} : \mathrm{sign}(\chi_{\alpha_i}(h)) = \epsilon_i \text{ for all } \alpha_i \in \Pi\}$, and we have:

$$H_{\mathbb{R}} = \bigcup_{\epsilon \in \mathcal{E}} H_\epsilon.$$

Notation 3.9. We need notation to parametrize connected the components of a split Cartan subgroup, roots and root characters. Unfortunately we need such notation for *all* the parabolic subgroups associated to arbitrary subsets of Π. Recall (Notation 3.1) that notation associated to a parabolic subgroup determined by a subset $\Pi \setminus A$, $A \subset \Pi$ is usually indicated by changing the standard notation with the use of a superscript $\ldots^A$. Thus we have: $\Delta^A \subset \Delta$, root system giving rise to a *semisimple* Lie algebra $\mathfrak{l}^A \subset \mathfrak{g}$. Also there are corresponding connected semisimple Lie subgroups $L^A_{\mathbb{C}} \subset G_{\mathbb{C}}$, $L^A \subset L^A_{\mathbb{C}}$. The adjoint group is denoted by $L_{\mathbb{C}}(\Delta^A) = \mathrm{Ad}(L^A_{\mathbb{C}})$ and it has a real connected Lie subgroup $L(\Delta^A)$. Let $\widetilde{L}(\Delta^A)$ be defined in the same way as $\widetilde{G}$ but relative to the root system Δ^A. We let $\mathfrak{h}^A$ be the real span of the h_{α_i} with $\alpha_i \notin A$. This is a (split) Cartan subalgebra of $\mathfrak{l}^A$ denoted as $H^A_{\mathbb{R}}$. The corresponding connected Lie subgroup is $H^A = \exp(\mathfrak{h}^A)$ (exponentiation taking place inside $G_{\mathbb{C}}$).

We also consider Lie subgroups of $\widetilde{G}$ corresponding to the subsystems of the Toda lattice. In accordance to our convention for Levi subgroups associated with $A \in \Pi$, we have

$$\mathcal{E}^A = \{\epsilon = (\epsilon_1, \dots, \epsilon_l) : \epsilon_i = 1 \text{ if } \alpha_i \in A\}.$$

Thus H^A is by definition a subgroup of H. This Lie subgroups of H is isomorphic to the connected component of the identity of a Cartan subgroup of a real semisimple Lie group that corresponds to $\mathfrak{l}^A$. There is a bijection

$$\mathcal{E}^A \cong \mathcal{E}(\Delta^A) = \{(\epsilon_{j_1}, \dots, \epsilon_{j_m}) : \alpha_{j_i} \notin A \text{ for } i = 1, \dots, m = |\Pi \setminus A|\} \tag{10}$$

which is given in the obvious way by restricting a function $\epsilon : \Pi \to \{\pm 1\}$ such that $\epsilon_i = \epsilon(\alpha_i) = 1$ for $\alpha_i \in A$ to a new function $\epsilon(\Delta^A)$ with domain $\Pi \setminus A$.

We now consider the Cartan subalgebras and the Cartan subgroups for Levi factors of parabolic subalgebras and subgroups dtermined by $A \subset \Pi$:

Definition 3.10. For $A \subset \Pi$, we denote:

- $H^A_{\mathbb{R}} = \bigcup_{\epsilon \in \mathcal{E}^A} h_\epsilon H^A$ (if $\Delta^A = \emptyset$ (i.e., $A = \Pi$) then $H^A_{\mathbb{R}} = \{e\}$).
- $H_{\mathbb{R}}(\Delta^A)$ a Cartan subgroup of $\widetilde{L}(\Delta^A)$, defined in the same way as $H_{\mathbb{R}}$ and having Lie algebra $\mathfrak{h}^A$ ($H_{\mathbb{R}}(\Delta^A) = \{e\}$ if $\Delta^A = \emptyset$).
- $H^{A,\leq}_\epsilon = \{h \in h_\epsilon H^A :$ for all $\alpha_i \in \Pi \setminus A : |\chi_{\alpha_i}(h)| \leq 1\}$ the antidominant chamber of $H^A_\epsilon = h_\epsilon H^A$. Similarly we consider $H^{A,<}_\epsilon$ using *strict* inequalities.
- $H(\Delta^A)^{\leq}_\epsilon = \{h \in H(\Delta^A)_\epsilon :$ for all $\alpha_i \in \Pi \setminus A : |\chi^{\Delta^A}_{\alpha_i}(h)| \leq 1\}$. Similarly we consider a version with strict inequalities.
- $\chi^{\Delta^A}_{\alpha_i}$ the root character associated to α_i on the Cartan $H_{\mathbb{R}}(\Delta^A)$.

We use notation $\dots^{\leq}$ to indicate the antidominant chamber on Cartan subalgebras and subgroups and $\dots^{<}$ for strictly antidominant chambers.

Example 3.11. In the case of $\widetilde{G} = SL(3, \mathbb{R})$, $H_{\mathbb{R}}$ is the group

$$H_{\mathbb{R}} = \{\mathrm{diag}\,(a, b, c) \ : \ a \neq 0, \ b \neq 0, \ abc = 1\}.$$

For $A = \{\alpha_1\}$, H^A is the group $\{\mathrm{diag}(1, a, a^{-1}) : a > 0\}$. The set $\mathcal{E}^A$ consists of $(1, 1)$ and $(1, -1)$. The element $h_{(1,-1)} = \mathrm{diag}(-1, -1, 1)$ and thus $h_{(1,-1)} H^A = \{\mathrm{diag}(-1, -a, a^{-1}) : a > 0\}$. These two components form $H^A_{\mathbb{R}}$. Note that $H_{\mathbb{R}} L^A$ is the Lie subgroup of $SL(3, \mathbb{R})$ consisting of all real matrices of the form,

$$H_{\mathbb{R}} L^{\{\alpha_1\}} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix}$$

having determinant one. The group L^A is obtained by setting $a = 1$ and the determinant equal to one. The group $L(\Delta^A)$ is isomorphic to $\mathrm{Ad}(SL(2,\mathbb{R}))$ the adjoint group obtained from L^A and $\widetilde{L}(\Delta^A)$ is isomorphic to $\mathrm{Ad}(SL(2,\mathbb{R})^{\pm})$. Thus these three groups L^A, $L(\Delta^A)$ and $\widetilde{L}(\Delta^A)$ are all different in this case.

Definition 3.12. Recall that the S is a subset of Π indicating *colored* vertices in a Dynkin diagram. Let $\eta : S \to \{\pm 1\}$ be any function. We let $\epsilon_\eta \in \mathcal{E}^{\Pi\setminus S}$ defined by $\epsilon_\eta(\alpha_i) = \eta(\alpha_i)$ if $\alpha_i \in S$, $\epsilon_\eta(\alpha_i) = 1$ if $\alpha_i \notin S$. Thus there is a bijective correspondence between the set of all functions η and the set $\mathcal{E}^{\Pi\setminus S}$ and (by Equation (10)) a second bijection with the set $\mathcal{E}(\Delta^{\Pi\setminus S})$:

$$\eta \mapsto \epsilon_\eta \in \mathcal{E}^{\Pi\setminus S},$$

and

$$\eta \mapsto \epsilon_\eta(\Delta^{\Pi\setminus S}) \in \mathcal{E}(\Delta^{\Pi\setminus S}).$$

The exponential map $h_\epsilon \exp : \mathfrak{h} \to h_\epsilon H$ allows us to define chamber walls in H and therefore in any of the connected components of $H_{\mathbb{R}}$. For any root $\phi \in \Delta$ the set $\{h \in h_\epsilon H : |\chi_\phi(h)| = 1\}$ defines the ϕ-wall of $H_\epsilon = h_\epsilon H$. The intersection of all the α_i-walls of H_ϵ is the set $\{h_\epsilon\}$. This is also the intersection of all the ϕ-walls of H_ϵ with $\phi \in \Delta$.

We also define the following subsets of H_ϵ:

Definition 3.13. We denote:

- $\mathcal{D} = \mathcal{D}(\Delta) = \{h_\epsilon :\ \epsilon \in \mathcal{E}\}$.
- $\mathcal{D}(\Delta^{\Pi\setminus S}) = \{h_\epsilon :\ \epsilon \in \mathcal{E}(\Delta^{\Pi\setminus S})\}$.

The set $\mathcal{D}$ has two structures: It is a finite group and also a set with an action of W. In Proposition 3.15 it is the first structure that is emphasized but in the Proof of Proposition 4.5 it is the second structure which is relevant. We now look at the W-action.

Since W acts on $H_{\mathbb{R}}$ and $w \in W$ sends a ϕ-wall of H_ϵ to the $w(\phi)$-wall of some other $H_{\epsilon'}$, this set $\mathcal{D}$ is preserved by W and thus acquires a W-action. Given $S \subset \Pi$ we similarly obtain that $\mathcal{D}(\Delta^{\Pi\setminus S})$ has a W_S-action.

The map

$$\mathcal{D}(\Delta^{\Pi\setminus S}) \to \mathcal{E}(\Delta^{\Pi\setminus S}) \tag{11}$$

sending $h_\epsilon \mapsto \epsilon$ also defines an action of W_S on the set of signs $\mathcal{E}(\Delta^{\Pi\setminus S})$. Recall (Notation 3.9) that $\mathcal{E}^{\Pi\setminus S} \subset \mathcal{E}$ denotes those ϵ for which $\epsilon_i = 1$ whenever $\alpha_i \notin S$. We have a bijection (by (11) together with(10))

$$\mathcal{E}^{\Pi\setminus S} \cong \mathcal{D}(\Delta^{\Pi\setminus S}).$$

The W_S-action on the set $\mathcal{D}(\Delta^{\Pi\setminus S})$ thus gives a W_S-action on $\mathcal{E}^{\Pi\setminus S}$ such that for any $\epsilon \in \mathcal{E}^{\Pi\setminus S}$ only the ϵ_j with $\alpha_j \in S$ may change in sign under the action. Note that this construction requires looking at the h_ϵ with $\epsilon \in \mathcal{E}(\Delta^{\Pi\setminus S})$ in the adjoint representation of $\mathfrak{l}^A$.

Remark 3.14. The root characters can be expressed as a product of the simple root characters raised to certain integral powers $\phi = \sum_{i=1}^{l} c_i\alpha_i$ with $c_i \in \mathbb{Z}$, $e^\phi = \prod_{i=1}^{l} \chi_{\alpha_i}^{c_i}$. Therefore if $h \in H_\mathbb{C}$ the scalars $\chi_{\alpha_i}(h)$ determine all the scalars $e^\phi(h) = \chi_\phi(h)$ and thus h is uniquely determined. Moreover $e^{\langle \phi, x+\sqrt{-1}y\rangle}$ is real for all $\phi \in \Delta$ if and only if $e^{\langle \alpha_i, x+\sqrt{-1}y\rangle}$ is real for all $\alpha_i \in \Pi$.

Proposition 3.15. *The Cartan subgroup $H_\mathbb{R}$ of $\widetilde{G}$ has 2^l components. We have $H_\mathbb{R} = \mathcal{D}H$ where $\mathcal{D}$ (Definition 3.13) is the finite group of all the h_ϵ, $\epsilon \in \mathcal{E}$.*

Proof. It was shown in Lemma 3.7 that the elements $\exp(x + \sqrt{-1}y)$ with $x, y \in \mathfrak{h}$ such that $e^{\langle \alpha_i; x+\sqrt{-1}y\rangle}$ is real for all $\alpha_i \in \Pi$ are those for which y has the form $y = \sum_{i=1}^{l} k_i y_i$ with y_i as in Notation 3.4, and k_i are integers. Moreover as in Remark 3.14 we also have that $e^{\langle \phi, x+\sqrt{-1}y\rangle}$ is real for any $\phi \in \Delta$ exactly when $y = \sum_{i=1}^{l} k_i y_i$ for some integers k_i . Therefore all the root spaces of $\mathfrak{g}$ are stabilized under the adjoint action of $\exp(x + \sqrt{-1}y)$. Since clearly $\mathfrak{h}$ is also stabilized, then $\exp(x + \sqrt{-1}y)$ stabilizes all of $\mathfrak{g}$ and this implies that $\exp(x+\sqrt{-1}y) \in \widetilde{G}$. In fact this shows that $\exp(x+\sqrt{-1}y) \in \widetilde{G}$ if and only if y has the form $y = \sum_{i=1}^{l} k_i y_i$ for certain k_i integers. Thus Lemma 3.7 and these remarks compute the intersection $\widetilde{G} \cap H_\mathbb{C}$. From here it is easy to conclude. ☐

Note that Proposition 3.15 implies that $\mathcal{D}(\Delta)$ in Definition 3.13 is isomorphic to $H_\mathbb{R}/H$ as a set with a W-action. That the W-actions agree is verified in Corollary 4.6.

We now give the Proof of Proposition 3.6:

Proof. Let Ad denote the representation of $GL(n, \mathbb{C})$ on $\mathfrak{sl}(n, \mathbb{C})$. Then we have $\mathrm{Ad}(GL(n, \mathbb{C})) = \mathrm{Ad}(SL(n, \mathbb{C}))$. If n is odd, $\mathrm{Ad}(SL(n, \mathbb{C}))$ is isomorphic to $SL(n, \mathbb{C})$. Denote $D_i = \mathrm{diag}(r_1, \ldots, r_n)$ with r_i as in Example 3.8. We set $\bar{h}_i = D_i$ when n is even and $\bar{h}_i = (-1)^{n-i}D_i$ when n is odd. Let $h_i = \mathrm{Ad}(\bar{h}_i)$. We have that for each $h_i \in \widetilde{G}$, $\chi_{\alpha_i}(h_i) = e^{\pi\sqrt{-1}\delta_{i,j}}$ for $\alpha_i \in \Pi$. The h_i generate the group $\mathcal{D}$ with 2^l elements where $l = n - 1$ and now the group $G_1 = \langle \mathrm{Ad}(SL(n, \mathbb{R})), h_i, i = 1, \ldots, l\rangle$ is a subgroup of $\widetilde{G}$ and it is isomorphic to $\mathrm{Ad}(SL(n, \mathbb{R})^\pm)$ for n even and to $\mathrm{Ad}(SL(n, \mathbb{R})) \cong SL(n, \mathbb{R})$ for n odd. What remains is to verify that $\widetilde{G} \subset G_1$.

Let $\widetilde{K} = \{\mathrm{Ad}(g) : g \in U(n)\} \cap \widetilde{G}$. By the Iwasawa decomposition of $\widetilde{G}$ and G_1, it suffices to show that $\widetilde{K} = K\mathcal{D}$, $K = SO(n)$. The right side, $K\mathcal{D}$ is either $O(n)$ or $SO(n)$ according to the parity of n. Recall that the maximal compact Lie subgroup $\widetilde{K}$ of $\widetilde{G}$ acts transitively on the set X of all maximal abelian Lie subalgebras $\mathfrak{a}$ which are contained in the vector space $\mathfrak{p}$. The action of $K = \mathrm{Ad}(SO(n))$ is also transitive on X (by (2.1.9) of [**22**]) and thus for any $g \in \widetilde{K}$ there is $k \in K$ such that $g = kD$ where D is the isotropy group (in $\widetilde{K}$) of an element in X, for instance of the element $\mathfrak{h} \in X$. However this isotropy group D has been computed implicitly in the Proof of Proposition 3.15 and $D = \mathcal{D}$. Therefore $\widetilde{K} = K\mathcal{D}$. □

Proposition 3.16. *Let $\alpha_i \in \Pi$ and assume that $\epsilon = (\epsilon_1, \ldots, \epsilon_l) \in \mathcal{E}$. Then $s_{\alpha_i} h_\epsilon = h_{\epsilon'}$ where $\epsilon'_j = \epsilon_j \epsilon_i^{-C_{j,i}}$ with $(C_{i,j})$ the Cartan matrix.*

Proof. This follows from the expression of the Weyl group action on elements in $\mathfrak{h}'_{\mathbb{C}}$ given by: $s_{\alpha_i} x = x - (\check{\alpha}_i, x)\alpha_i$. If this expression is applied to $x = \alpha_j$ it gives $s_{\alpha_i}\alpha_j = \alpha_j - C_{j,i}\alpha_i$. On the level of root characters this becomes, by exponentiation of the previous identity,

$$s_{\alpha_i} \chi_{\alpha_j} = \chi_{\alpha_j} \chi_{\alpha_i}^{-C_{j,i}}.$$

Now recall that ϵ_j is just χ_{α_j} evaluated at h_ϵ. Also ϵ'_j will be χ_{α_j} evaluated at $s_{\alpha_i} h_\epsilon$. When we evaluate χ_{α_j} on $s_{\alpha_i} h_\epsilon$ in order to compute the corresponding j-th sign, we obtain $\chi_{s_{\alpha_i}\alpha_j}(h_\epsilon)$. Therefore the sign of χ_{α_j} on the $s_{\alpha_i} h_\epsilon$ is given by the product $\epsilon_j \epsilon_i^{-C_{j,i}}$. Finally we use the fact that the set of all scalars $\chi_{\alpha_i}(h)$ determines h. Thus ϵ' determines the element $h_{\epsilon'}$ giving rise to the equation $s_{\alpha_i} h_\epsilon = h_{\epsilon'}$. □

The sign change $\epsilon_j \to \epsilon'_j$ in Proposition 3.16 is precisely the gluing rule in Lemma 4.2 for the indefinite Toda lattice in [**15**]. Then the gluing pattern using the Toda dynamics is just to identify each piece of the connected component H_ϵ (see Figure 1). The sign change on subsystem corresponding to $H^A_{\mathbb{R}}$ with $A \subset \Pi$ can be also formulated as:

Proposition 3.17. *Let $\alpha_i \in \Pi \setminus A$ and assume that $\epsilon = (\epsilon_1, \ldots, \epsilon_l) \in \mathcal{E}^A$. Then:*

a) *If $\epsilon_i = 1$, $s_{\alpha_i} h_\epsilon = h_\epsilon$. If $\epsilon_i = -1$ then $s_{\alpha_i} h_\epsilon = h_{\epsilon'}$ where $\epsilon'_j = \epsilon_j \epsilon_i^{-C_{j,i}}$. In addition $h_{\epsilon'}$ factors as a product $(\prod_{\alpha_j \in A, C_{j,i} \text{ is odd}} h_j) h_{\epsilon_A}$ where $\epsilon_A \in \mathcal{E}^A$.*

b) *If $\epsilon_i = 1$, $s_{\alpha_i} H^A_\epsilon \subset H^A_\epsilon$. If $\epsilon_i = -1$ then $s_{\alpha_i} H^A_\epsilon \subset h_{\epsilon'} H^A$ where $\epsilon'_j = \epsilon_j \epsilon_i^{-C_{j,i}}$. In addition $h_{\epsilon'}$ factors as a product $(\prod_{\alpha_j \in A, C_{j,i} \text{ is odd}} h_j) h_{\epsilon_A}$ where $\epsilon_A \in \mathcal{E}^A$.*

c) *The sign $\epsilon_k = \mathrm{sign}(\chi_{\alpha_k}(h))$ for any $h \in H_\epsilon$ agrees with $\mathrm{sign}(\chi_{s_{\alpha_i}\alpha_k}(h'))$ for any $h' \in s_{\alpha_i}(H_\epsilon)$.*

Proof. Part a) follows from Proposition 3.16 but with the observation that in this case, ϵ' may fail to be in $\mathcal{E}^A$ even if $\epsilon \in \mathcal{E}^A$. This happens exactly when $\epsilon_i = -1$ and $\alpha_j \in A$ with $C_{j,i}$ odd (either -3 or -1). Under these circumstances $\epsilon'_j = -1$ (rather than one as required in the definition of $\mathcal{E}^A$). We can fix this problem by factoring $h_{\epsilon'}$ as a product $(\prod_{\alpha_j \in A, C_{j,i} \text{ is odd}} h_j) h_{\epsilon_A}$ where $\epsilon_A \in \mathcal{E}^A$.

Part b) follows from Part a) and the fact that each $h_\epsilon H$ is a connected component of $H_{\mathbb{R}}$ in Proposition 3.15.

Part c) follows easily from $\chi_{s_{\alpha_i}\alpha_k}(s_{\alpha_i}h) = \chi_{\alpha_k}(s_{\alpha_i}(s_{\alpha_i}h)) = \chi_{\alpha_k}(h)$. Also by the fact that the sign of a root character is constant along a connected component and that we have $h \in H_\epsilon$ and $s_{\alpha_i}h$ is in the new component $s_{\alpha_i}H_\epsilon$. The two desired signs have thus been computed in the two connected components and they agree. □

Remark 3.18. From Proposition 3.17 it follows that any connected component H_ϵ^A, is the union of chambers of the form $w\left(H^{A,\leq}_{\epsilon(w)}\right)$, with $w \in W_{\Pi\setminus A}$, $\epsilon(w) \in \mathcal{E}$,

$$H_\epsilon^A = \bigcup_{w \in W_{\Pi\setminus A}} w\left(H^{A,\leq}_{\epsilon(w)}\right).$$

4. Colored Dynkin diagrams.

We now introduce some notation that will ultimately parametrize the cells in a cellular decomposition of the smooth compact manifold $\hat{H}_{\mathbb{R}}$ to be defined in §6.

4.1. Colored Dynkin Diagrams. Let us first define:

Definition 4.1 (Colored Dynkin diagrams $\mathbb{D}(S)$)**.** A colored Dynkin diagram is a Dynkin diagram where all the vertices in a set $S \subset \Pi$ have been colored either red R or blue B. For example, in $\mathfrak{sl}(4,\mathbb{R})$, $\circ_R - \circ - \circ_B$ is a colored Dynkin diagram with $S = \{\alpha_1,\ \alpha_3\}$. Thus a colored Dynkin diagram where $S \neq \emptyset$, corresponds to a pair(S, ϵ_η) with $S \subset \Pi$ and $\eta : S \to \{\pm 1\}$ any function. Here $\eta(\alpha_i) = -1$ if α_i is colored R and $\eta(\alpha_i) = 1$ if α_i is colored B. If $S = \emptyset$ then ϵ_o with $\epsilon_o(\alpha_i) = 1$ for all $\alpha_i \in \Pi$ replaces ϵ_η. We denote:

- $D = (S, \epsilon_\eta)$ or $(S, \epsilon_\eta(\Delta^{\Pi\setminus S}))$ with $\epsilon_\eta \in \mathcal{E}^{\Pi\setminus S}$;
- $\mathbb{D}(S) = \{D = (S, \epsilon_\eta)\ :\ \text{the vertices in } S \text{ are colored}\}$.

We also introduce *an oriented colored Dynkin diagram* which is defined as a pair (D, o) with $o \in \{\pm 1\}$ and D a colored Dynkin diagram.

4.2. Boundary of a colored Dynkin diagram. We now define the boundaries of a cell parametrized by a colored Dynkin diagram D.

Definition 4.2 (The boundary $\partial_{j,c}D$)**.** For each (j, c) with $c = 1, 2$ and $j = 1, \ldots, m$ we define a new colored Dynkin diagram $\partial_{j,c}D$, the (j, c)-boundary of the D by considering $\{\alpha_{i_j} : 1 \leq i_1 < \cdots < i_m \leq l\}$ the set

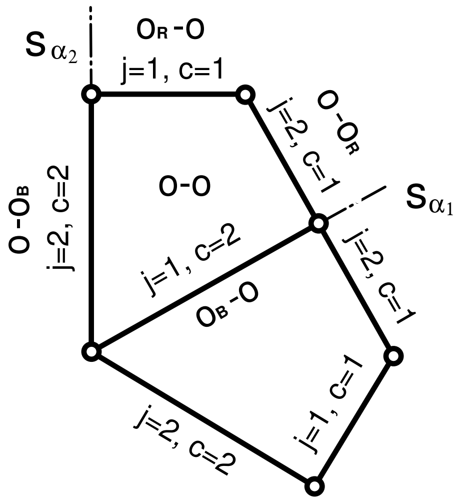

Figure 7. The boundary $\partial_{j,c}$ in the case of $\mathfrak{sl}(3,\mathbb{R})$.

$\Pi\setminus S$ of uncolored vertices and $m = |\Pi\setminus S|$. The $\partial_{j,c}D$ is then a new colored Dynkin diagram obtained by coloring the i_j-th vertex with R if $c = 1$ and with B if $c = 2$. The boundary of an oriented colored Dynkin diagram (D, o), $o \in \{\pm 1\}$ is, in addition, given an orientation defined to be the sign $(-1)^{j+c+1}o$. Recall that a colored Dynkin diagram D corresponds to a pair (S, ϵ_η) with $S \subset \Pi$ and $\eta : S \to \{\pm 1\}$ (Definition 3.12). Thus the boundary $\partial_{j,c}$ determines a new pair $(S \cup \{\alpha_{i_j}\}, \epsilon_{\eta'})$ associated to $\partial_{j,c}D$. We can then define the following boundary maps,

$$(-1)^{j+c+1}\partial_{j,c} : \mathbb{Z}\left[\mathbb{D}(S)\right] \to \mathbb{Z}\left[\mathbb{D}(S\cup\{\alpha_{i_j}\})\right]. \tag{12}$$

Example 4.3. The boundary of $\circ - \circ$ (which we can picture as a box) consists of segments (one dimensional boxes) given by $\partial_{1,1}(\circ - \circ) = \circ_R - \circ$, $\partial_{2,1}(\circ - \circ) = \circ - \circ_R$, $\partial_{1,2}(\circ - \circ) = \circ_B - \circ$ and $\partial_{2,2}(\circ - \circ) = \circ - \circ_B$. The orientation sign associated to $\circ_R - \circ$ is the following: With $c = 1$ and $j = 1$ one has $(-1)^{j+c+1} = (-1)^3 = -1$. We illustrate the example in Figure 7.

4.3. W_S-action on colored Dynkin diagrams. We now move these colored Dynkin diagrams around with elements in W. A W_S-action on the diagram $D \in \mathbb{D}(S)$, $W_S : \mathbb{D}(S) \to \mathbb{D}(S)$, is defined as follows:

Definition 4.4. For any $\alpha_i \in S$ (the α_i vertex is colored), $s_{\alpha_i}D = D'$ is a new colored Dynkin diagram having the colors according to the sign change $\epsilon'_j = \epsilon_j\epsilon_i^{-C_{j,i}}$ in Proposition 3.17 with the identification that R if the sign is -1, and B if it is $+1$. For example, in the case of $\mathfrak{sl}(3,\mathbb{R})$,$s_{\alpha_1}(\circ_R - \circ_B) = \circ_R - \circ_R$, $s_{\alpha_1}(\circ_B - \circ_R) = \circ_B - \circ_R$.

We also define a W_S-action on the set $\mathbb{D}(S) \times \{\pm 1\}$ of oriented colored Dynkin diagrams. If $\alpha_i \in S$, the action of s_{α_i} on the pair (D, o) is given by $s_{\alpha_i}(D, o) = (s_{\alpha_i} D, (\epsilon_i)^{r_{\alpha_i}} o)$ where r_{α_i} is the number of elements in the set $\{\alpha_j \in \Pi \setminus S : C_{j,i} \text{ is odd}\}$ and $\epsilon_i = \pm 1$ depending on the color of α_i.

We confirm the Definition:

Proposition 4.5. *Definition 4.4 above gives a well-defined action of W_S on the set $\mathbb{D}(S)$ of colored Dynkin diagrams with set of colored vertices S.*

Proof. This follows from Proposition 3.16 and the correspondence $(S, \epsilon_\eta) \to h_{\epsilon_\eta}$ giving a bijection between $\mathbb{D}(S)$ and $\mathcal{D}(\Delta^{\Pi \setminus S})$. □

Corollary 4.6. *There is a bijection of $\mathcal{D}(\Delta) \to H_{\mathbb{R}}/H$ intertwining the W-actions on both sets.*

Proof. The two sets $\mathcal{D}(\Delta)$ and $H_{\mathbb{R}}/H$ are clearly in bijective correspondence and an element h_ϵ corresponds to the coset $h_\epsilon H$. By Proposition 3.17, Notation 4.4 and Proposition 4.5, the actions on these two sets agree. They are both given by Part a) in Proposition 3.17. □

Notation 4.7. Motivated by Example 2.2 we consider the set $W \underset{W_S}{\times} \mathbb{D}(S)$ the W translations of the set $\mathbb{D}(S)$. We write the elements in this set as pairs (w, D) and introduce an equivalence relation $\sim$ on these pairs, where for any $x \in W_S$, $(wx, D) \sim (w, xD)$. Denote the equivalence classes $[(w, D)]$. This set of equivalence classes is then in bijective correspondence with the set $\mathbb{D}(S) \times W/W_S$ of pairs $(D, [w])$ with $[w] = [w]^{\Pi \setminus S} \in W/W_S$ and D in $\mathbb{D}(S)$. The correspondence is such that $[(w^\bullet, D)]$ corresponds to $(D, [w^\bullet])$ with $w^\bullet \in [w]^{\Pi \setminus S}$ a minimal length representative of a coset in W/W_S.

We denote

$$\mathbb{D}^k = \{(D, [w]^{\Pi \setminus S}) : S \subset \Pi, |S| = k, w \in W\},$$

which also parametrizes all the connected components of the Cartan subgroups of the form $H_{\mathbb{R}}(w(\Delta^{\Pi \setminus S}))$.

Remark 4.8. For the reader who is not interested in the W-action or torsion, the object defined in Definition 4.9 becomes over $\mathbb{Q}$ the vector space with basis given by the full set of colored Dynkin diagrams having a fixed number of uncolored vertices. The boundary maps are obtained by *translating* around the $\partial_{j,c}$ defined for the antidominant chamber (see (14) below). Perhaps this boundary construction can be appreciated in Example 2.2 in Section 2 where portions of the boundary of a chamber are translated and some identifications take place. The tensor product notation below accomplishes the required boundary identifications algebraically.

Definition 4.9 (The $\mathbb{Z}[W]$-modules $\mathcal{M}(S)$)**.** The full set of colored Dynkin diagrams is the set $\mathbb{D}(S) \times W/W_S$ of all pairs $(D, [w]^{\Pi \setminus S})$.

We can also define the full set of *oriented* colored Dynkin diagrams by considering $\mathbb{D}(S) \times \{\pm 1\} \times W/W_S$ for different subsets $S \subset \Pi$. As W sets these correspond to $W \underset{W_S}{\times} \mathbb{D}(S) \times \{\pm 1\}$.

If we consider $\mathbb{D}(S) \times \{\pm 1\}$ as imbedded in $\mathbb{Z}[\mathbb{D}(S)]$ by sending (D, o) to oD, $o \in \{\pm 1\}$ we may consider a W_S-action on $\pm\mathbb{D}(S)$, namely the action on oriented colored Dynkin diagrams. This produces a $\mathbb{Z}[W]$-module $\mathbb{Z}[W] \underset{\mathbb{Z}[W_S]}{\otimes} \mathbb{Z}[\mathbb{D}(S)]$, and we denote this module by

$$\mathcal{M}(S) = \mathbb{Z}[W] \underset{\mathbb{Z}[W_S]}{\otimes} \mathbb{Z}[\mathbb{D}(S)].$$

Also denote by $\mathcal{M}_{l-k}$ the direct sum of all these modules over all sets S with exactly k elements,

$$\mathcal{M}_{l-k} = \bigoplus_{|S|=k} \mathcal{M}(S).$$

Remark 4.10. Assume that A is a $\mathbb{Z}[W_S]$-module and B is a $\mathbb{Z}[W_{S'}]$-module with $S \subset S'$. In particular B can be regarded as a $\mathbb{Z}[W_S]$-module by restriction. Let $f : A \to B$ be a map intertwining these two $\mathbb{Z}[W_S]$-module structures involved. Then, tensoring with $\mathbb{Z}[W]$ we obtain a map of $\mathbb{Z}[W]$-modules:

$$F(f) : \mathbb{Z}[W] \underset{\mathbb{Z}[W_S]}{\otimes} A \to \mathbb{Z}[W] \underset{\mathbb{Z}[W_S]}{\otimes} B.$$

Since, in addition, B is a $\mathbb{Z}[W_{S'}]$-module where $S \subset S'$ then there is a second map:

$$g : \mathbb{Z}[W] \underset{\mathbb{Z}[W_S]}{\otimes} B \to \mathbb{Z}[W] \underset{\mathbb{Z}[W_{S'}]}{\otimes} B.$$

Let $g \circ F(f) = T(f)$. This is then a map of $\mathbb{Z}[W]$ modules

$$T(f) : \mathbb{Z}[W] \underset{\mathbb{Z}[W_S]}{\otimes} A \to \mathbb{Z}[W] \underset{\mathbb{Z}[W_{S'}]}{\otimes} B.$$

We now give the boundary maps of $\mathcal{M}_{l-k}$. We regard $\mathbb{D}(S) \times \{\pm 1\}$ as a $\mathbb{Z}[W_S]$-module using Definition 4.4 the oriented case. We can view a pair (D, o) instead as $\pm D \in \mathbb{Z}[\mathbb{D}(S)]$. This gives a $\mathbb{Z}[W_S]$-module structure to each of the $\mathbb{Z}$ modules involved on the domain of the map in (12) and a $W_{S \cup \{\alpha_{i_j}\}}$ to those on the co-domain. The map in (12) now has the form described in Remark 4.10. We apply the construction $T(\partial_{j,c})$ in Remark 4.10 to (12) and add over all possible subsets $S \subset \Pi$ with $|S| = k$ on the left side and with $|S| = k + 1$ on the right side. We then obtain the boundary maps,

$$\partial_{l-k} : \mathcal{M}_{l-k} \to \mathcal{M}_{l-(k+1)}, \tag{13}$$

which are all given by

$$\partial_{l-k}(w^\bullet \otimes X) = \sum_{j=1}^{l-k}\sum_{c=1}^{2}(-1)^{j+c+1} w^\bullet \otimes T(\partial_{j,c})X, \tag{14}$$

where $X \in \mathbb{D}(S)$ and $w^\bullet \in [w]^{\Pi\backslash S}$ is the minimum length representative in W/W_S (see Notation 4.7). We thus have:

Proposition 4.11. *The maps ∂_{l-k} of (14) define a chain complex $\mathcal{M}_*$ of $\mathbb{Z}[W]$-modules.*

5. Cartan subgroups and Weyl group actions.

We here discuss relations between some Cartan subgroups of Levi factors, and verify the W_S-action on oriented colored Dynkin diagrams.

5.1. Cartan subgroups of Levi factors. Let Ad^{Δ^A} denote the adjoint representation of the Lie subgroup $H_{\mathbb{R}}L^A$ of G on the Lie algebra $\mathfrak{l}^A$. Note that we are deviating slightly from the standard convention and denoting by Ad^{Δ^A} the representation of $H_{\mathbb{R}}L^A$ acting on the semisimple part of the Levi factor $\mathfrak{l}^A$, seen as a *quotient* of the group action on $\mathfrak{h}^A + \mathfrak{l}^A$ (thus dividing by the center of this Lie algebra which corresponds to a trivial representation summand). We will use this same notation Ad^{Δ^A} when restricting to various Lie subgroups of $H_{\mathbb{R}}L^A$ containing L^A. The notation $\mathrm{Ad}^{w(\Delta^A)}$ with $w \in W$ refers to the similar construction with respect to $w(\Delta^A)$. When $A = \Pi$ and $\Delta^A = \emptyset$ then $\mathrm{Ad}^{w(\Delta^A)}$ will refer to the (trivial) one dimensional representation of $\{e\}$.

Definition 5.1. Let H^A_{fund} be defined by:

$$\exp\left(\left\{\sum_{\alpha_i\notin A} c_i m^\circ_{\alpha_i} \;:\; c_i \in \mathbb{R}\right\}\right) = H^A_{\mathrm{fund}}. \tag{15}$$

Note that usually $H^A \neq H^A_{\mathrm{fund}}$.

Then we have:

Proposition 5.2. *The images of Ad^{Δ^A} on subsets of Cartan subgroups satisfy*

1) $\mathrm{Ad}^{\Delta^A}(\mathcal{D}(\Delta)) = \mathcal{D}(\Delta^A)$,
2) $\mathrm{Ad}^{\Delta^A}(H^A_{\mathbb{R}}) = H_{\mathbb{R}}(\Delta^A)$,
3) $\mathrm{Ad}^{\Delta^A}(H^A_{\mathrm{fund}}) = \mathrm{Ad}^{\Delta^A}(H^A)$.

Proof. We first point out that the exponential map in the Lie groups $L^A_{\mathbb{R}}$, Ad^{Δ^A} gives a diffeomorphism between $\mathfrak{h}^A$ and the corresponding connected

Lie group. Thus the Lie groups H^A, $\mathrm{Ad}^{\Delta^A}(H^A)$ are isomorphic. This takes care of Part 2) on the level of the connected component of the identity.

Also the image under ad^{Δ^A} of the set $\{m^\circ_{\alpha_i} : \alpha_i \notin A\}$ gives rise to a basis of the Cartan subalgebra of $\mathrm{ad}^{\Delta^A}(\mathfrak{l}^A)$. Thus exponentiating we obtain Part 3).

Recall that $\chi_{\alpha_i}(h_j) = \exp(\pi\sqrt{-1}\delta_{i,j})$ and $\mathrm{Ad}^{\Delta^A}(h) = I$ if and only if $\chi_{\alpha_i}(h) = 1$ for all $\alpha_i \in \Pi \setminus A$. We have thus $\mathrm{Ad}^{\Delta^A}(h_i) = I$ for $\alpha_i \in A$ and $\{\mathrm{Ad}^{\Delta^A}(h_\epsilon) : \epsilon \in \mathcal{E}\} = \{\mathrm{Ad}^{\Delta^A}(h_\epsilon) : \epsilon \in \mathcal{E}^A\} = \mathcal{D}(\Delta^A)$. The elements h_i with $\alpha_i \in A$ are in the center of $H_{\mathbb{R}}L^A$. This proves Part 1).

For Part 2), we proceed by noting that by definition of $H_{\mathbb{R}}(\Delta^A)$ it must be generated by the Lie group $\mathrm{Ad}^{\Delta^A}(H^A)$ with Lie algebra $\mathfrak{h}^A$ and the elements $\mathrm{Ad}(h_i)$ with $\alpha_i \in \Pi \setminus A$ (playing the role that the h_i play in the definition of $H_{\mathbb{R}}$). This is the same as $\mathrm{Ad}^{\Delta^A}(H^A_{\mathbb{R}})$. □

5.2. Action of the Weyl group on oriented colored Dynkin diagrams.

Proposition 5.3. *The action of W_S on $\mathbb{D}(S) \times \{\pm 1\}$ in Definition 4.4 is well-defined.*

Proof. Recall that W is a Coxeter group, (Proposition 3.13 [**12**]) and it thus has defining relations $s^2_{\alpha_i} = e$ and $(s_{\alpha_i}s_{\alpha_j})^{m_{ij}} = e$ where m_{ij} is $2, 3, 4, 6$ depending on the number of lines joining α_i and α_j in the Dynkin diagram. The case $m_{ij} = 2$ occurring when α_i and α_j are not connected in the Dynkin diagram. The only relevant cases then are when α_i, α_j are both *colored* and connected in the Dynkin diagram. The only relevant vertices in the Dynkin diagram are those connected with these two and which are uncolored. We are thus reduced to very few nontrivial possibilities: D_5, A_4, F_4, B_4, C_4; smaller rank cases being very easy cases. We verify only one of these cases, the others being almost identical with no additional difficulties. Consider the case of D_5 where $\alpha_i = \alpha$, $\alpha_j = \beta$ are the two simple roots which are not "endpoints" in the Dynkin diagram and all the others are uncolored. Then $r_\alpha = 1$ but $r_\beta = 2$. For book-keeping purposes it is convenient to temporarily represent *red* as -1 and *blue* as 1 and simply follow what happens to these two roots and an orientation $o = 1$. Hence all the information can be encoded in a triple $(\epsilon_1, \epsilon_2, o)$ representing the colors of these two roots and the orientation o. Now we apply $(s_\alpha s_\beta)^3$ to $(\epsilon_1, \epsilon_2, o)$:

$$
\begin{gathered}
(\epsilon_1, \epsilon_2, o = 1) \xrightarrow{s_\beta} (\epsilon_1\epsilon_2, \epsilon_2, (\epsilon_2)^2 o = 1) \xrightarrow{s_\alpha} \\
(\epsilon_1\epsilon_2, \epsilon_1, (\epsilon_1\epsilon_2)^1 o) \xrightarrow{s_\beta} (\epsilon_2, \epsilon_1, (\epsilon_1)^2\epsilon_1\epsilon_2 o = \epsilon_1\epsilon_2 o) \xrightarrow{s_\alpha} \\
(\epsilon_2, \epsilon_1\epsilon_2, (\epsilon_2)^1\epsilon_1\epsilon_2 o = \epsilon_1 o) \xrightarrow{s_\beta} (\epsilon_1, \epsilon_1\epsilon_2, (\epsilon_1\epsilon_2)^2\epsilon_1 o = \epsilon_1 o) \xrightarrow{s_\alpha} \\
(\epsilon_1, \epsilon_2, o).
\end{gathered}
$$

□

5.3. The set $H_{\mathbb{R}}^{\circ}$. By Proposition 5.2, if $\chi_{\alpha_i}^{\Delta^A}$, $\alpha_i \in \Pi \setminus A$ is a root character of $H_{\mathbb{R}}(\Delta^A)$ acting on $\mathfrak{l}^A$, then it is related to χ_{α_i} by $\chi_{\alpha_i}^{\Delta^A} \circ \mathrm{Ad}^{\Delta^A} = \chi_{\alpha_i}$. The map $\chi_{\alpha_i}^{\Delta^A}$, $\alpha_i \in \Pi \setminus A$ provides the local coordinates on $H_{\mathbb{R}}^{\circ}$ which consists of the Cartan subgroups $H_{\mathbb{R}}(\Delta^A)$.

Definition 5.4. Let $H_{\mathbb{R}}^{\circ}$ be defined as

$$H_{\mathbb{R}}^{\circ} = \bigcup_{A \subset \Pi} H_{\mathbb{R}}(\Delta^A) \times \{[e]^A\}.$$

We then define a map $\phi_e : H_{\mathbb{R}}^{\circ} \to \mathbb{R}^l$ as follows:

$$\phi_e(h, [e]^A) = (\phi_{e,1}(h), \ldots, \phi_{e,l}(h)),$$

where $\phi_{e,i}(h) = \chi_{\alpha_i}^{\Delta^A}(h)$ whenever $\alpha_i \notin A$ and $\phi_{e,i}(h) = 0$ if $\alpha_i \in A$. Denote ϕ_e^A the restriction of ϕ_e to $H_{\mathbb{R}}(\Delta^A) \times \{[e]^A\}$.

By Proposition 5.2 Part 2) we can compose with $\mathrm{Ad}^{\Delta^A} \times 1$ and re-write the domain of $\phi_e \circ (\mathrm{Ad}^{\Delta^A} \times 1)$ as $\bigcup_{A \subset \Pi} H_{\mathbb{R}}^A \times \{[e]^A\}$. We also define

$$\phi_w = (\phi_{w,1}, \ldots, \phi_{w,l}) \ : \ w(H_{\mathbb{R}}^{\circ}) \longrightarrow \mathbb{R}^l,$$

with

$$w(H_{\mathbb{R}}^{\circ}) := \bigcup_{A \subset \Pi} H_{\mathbb{R}}(w(\Delta^A)) \times \{[w]^A\},$$

by setting

$$\phi_{w,i}(x, [w]^A) = \chi_{w\alpha_i}^{w(\Delta^A)}(\mathrm{Ad}^{w(\Delta^A)}(wh)) = \chi_{w\alpha_i}(wh) = \chi_{\alpha_i}(h),$$

where $x = \mathrm{Ad}^{w(\Delta^A)}(wh) \in H_{\mathbb{R}}(w(\Delta^A))$ with $h \in H_{\mathbb{R}}$, if $\alpha_i \notin A$. For the case when $\alpha_i \in A$, we set

$$\phi_{w,i}(x, [w]^A) = 0.$$

5.4. The sets $H_{\mathbb{R}}^{A}$. We here note an isomorphism between several presentations of a split Cartan subgroup of a Levi factor.

Proposition 5.5. *All the following are isomorphic as Lie groups:*

1) $H_{\mathbb{R}}^A$,
2) $\mathrm{Ad}^{\Delta^A}(H_{\mathbb{R}})$,
3) $\mathrm{Ad}^{\Delta^A}(H_{\mathbb{R}}^A)$,
4) $H_{\mathbb{R}}(\Delta^A)$.

Proof. The groups in 3) and 4) are isomorphic by Proposition 5.2. We can write $\mathfrak{h}_{\mathbb{C}} = \mathfrak{z} + \mathfrak{h}_{\mathbb{C}}^A$ where $\mathfrak{z}$ is defined by $z \in \mathfrak{z}$ if and only if $\langle \alpha_i, z \rangle = 0$ for all $\alpha_i \in \Pi \setminus A$.

Now the group $\exp(\mathfrak{z})$ is the center of $H_{\mathbb{C}}L^A_{\mathbb{C}}$. Intersecting with $\widetilde{G}$ we obtain the center of $H_{\mathbb{R}}L^A$. To find this intersection with $\widetilde{G}$, we must find all $x+\sqrt{-1}y \in \mathfrak{z}$ such that $e^{\langle \alpha_j, x+\sqrt{-1}y\rangle}$ is real for all $\alpha_i \in \Pi$ (see the Proof of Lemma 3.7). We find that $x \in \mathfrak{z}\cap\mathfrak{h}$ and that y is an integral linear combination of the y_i. As in the Proof of Proposition 5.2 the elements $\exp(\sqrt{-1}y_i)$ which are in the center of $H_{\mathbb{R}}L^A$ are those for which $\alpha_i \in A$. The center of $H_{\mathbb{R}}L^A$ is then the group generated by h_i with $\alpha_i \in A$ and $\exp(\mathfrak{z}\cap\mathfrak{h})$. Since $\exp(\mathfrak{z}\cap\mathfrak{h})\cap H^A = \{e\}$, we have that $H_{\mathbb{R}}$ divided by $\exp(\mathfrak{z}\cap\mathfrak{h})$ is isomorphic to $\mathcal{D}H^A_{\mathbb{R}}$. The groups $H^A_{\mathbb{R}}$, $\mathcal{D}H^A_{\mathbb{R}}$ and those involved in 2), 3) or 4) all differ by a subgroup of the $\mathcal{D} \subset \exp(\mathfrak{z})$ which is annihilated by Ad^{Δ^A}. From here and Proposition 5.2 Part 1), the isomorphism between 2), 3) and 4) follows. □

6. The set $\hat{H}_{\mathbb{R}}$.

We here define our main object $\hat{H}_{\mathbb{R}}$ as a union of Cartan subgroups of Levi factors and their W-translations.

Let $\hat{W}$ be the disjoint union of all the quotients $W/W_{\Pi\setminus A}$ over $A \subset \Pi$. Each of the elements $[w]^A \in W/W_{\Pi\setminus A}$ parametrizes a parabolic subgroup. First $[e]^A$ corresponds to a standard parabolic subgroup of $G_{\mathbb{C}}$ (Proposition 7.76 or Proposition 5.90 of [**13**]). This is just the parabolic subgroup determined by the subset $\Pi\setminus A$ of Π. Then we translate such a parabolic subgroup with $w \in W$. The resulting parabolic subgroup corresponds to $[w]^A$. Thus $\hat{W}$ is the set of all the W-translations of standard parabolic subgroups.

Definition 6.1. We define

$$\hat{H}_{\mathbb{R}} = \bigcup_{A\subset\Pi}\bigcup_{w\in W} H_{\mathbb{R}}(w(\Delta^A)) \times \{[w]^A\}. \tag{16}$$

From Proposition 5.5, $H^A_{\mathbb{R}}$ can be replaced by $H_{\mathbb{R}}(\Delta^A)$ or $\mathrm{Ad}^{w(\Delta^A)}(H_{\mathbb{R}})$. Thus we can alternatively write $\hat{H}_{\mathbb{R}}$ as a subset of $H_{\mathbb{R}} \times \hat{W}$. We recall $w^{\bullet,A}$ or just $w^{\bullet}$ as in Notation 4.7. We then have:

$$\hat{H}_{\mathbb{R}} = \bigcup_{A\subset\Pi}\ \bigcup_{w^{\bullet,A}\in W/W_{\Pi\setminus A}} w^{\bullet,A}(H^A_{\mathbb{R}}) \times \{[w]^A\}. \tag{17}$$

Also fixing an isomorphism $\xi_w : H_{\mathbb{R}}(\Delta^A) \to H_{\mathbb{R}}(w(\Delta^A))$ inducing a set bijection (by composition) $\xi^*_w : w(\Pi) \to \Pi$, we have

$$\hat{H}_{\mathbb{R}} \cong \bigcup_{A\subset\Pi} W \underset{W_{\Pi\setminus A}}{\times} H_{\mathbb{R}}(\Delta^A),$$

where an element $(w^{\bullet,A}, h)$ on the right-hand side is sent to $(\xi_w(h), [w]^A)$. This endows $\hat{H}_{\mathbb{R}}$ with a W-action.

In what follows it is useful to think of a colored Dynkin diagram (e.g., $\circ_R - \circ$) as parametrizing a "box" (e.g., $[-1, 1]$). We need to further subdivide this box into 2^l smaller boxes by dividing it into regions according to the sign of each of the coordinates (e.g., $[-1, 0]$ $[0, 1]$). We also need to consider the boundary between these 2^l regions (e.g., $\{0\}$). We will introduce an additional sign or a zero to keep track of such subdivisions (e.g., $\circ_R - \circ_+$, $\circ_R - \circ_-$, $\circ_R - \circ_0$ for $[-1, 0], [0, 1], \{0\}$) (see also Example in Section 2). We will then do the same thing with a colored Dynkin diagram of the form $(D, [w]^{\Pi\setminus S})$ by assigning labels in $\{\pm 1, 0\}$ to the vertices in $\Pi \setminus S$.

The main purpose of the following is to associate certain sets in $\hat{H}_{\mathbb{R}}$ to the colored Dynkin diagrams, the signed-colored Dynkin diagrams and the sets associated to them only play an auxiliary role.

We introduce the following notation. This notation is illustrated in Example in Section 2 and Figure 3.

Notation 6.2. A *signed-colored Dynkin diagram* $\check{D}$ is a Dynkin diagram with some vertices colored (R or B) and the remaining vertices labled $+, -$ or 0. The followings are auxiliary objects to keep track of signs, zeros and colors.

- $\check{\eta} : \Pi \to \{\pm 1, 0\}$ function which agrees with η on S and determines the sign labels in $\check{D}$.
- $A = A(\check{D}) = A(\check{\eta}) = \{\alpha_i \in \Pi : \check{\eta}(\alpha_i) = 0\}$.
- $K(\eta)$ the set of all $\check{\eta} : \Pi \to \{\pm 1, 0\}$ which agree with η in S.
- $\epsilon_{\check{\eta}} \in \mathcal{E}^A$ the element which agrees with $\check{\eta}$ on $\Pi \setminus A$ ($A = A(\check{D})$).
- $\check{D}(S, A, \epsilon_{\check{\eta}}, [w]^{\Pi\setminus S})$ the unique signed-colored Dynkin diagram attached to a colored Dynkin diagram $(D, [w]^{\Pi\setminus S})$ or to $(S, \eta, [w]^{\Pi\setminus S})$, where the vertices in $\Pi \setminus S$ are given a label in $\{\pm 1, 0\}$.

We now associate a subset of $\hat{H}_{\mathbb{R}}$ to a signed-colored Dynkin diagram with $S = \emptyset$. Notice that in this case $\epsilon_{\check{\eta}}$ can be any element of $\mathcal{E}^A$. Recall that $\epsilon_{\check{\eta}}(\Delta^A)$ is then the element of $\mathcal{E}(\Delta^A)$ that corresponds.

We associate to a signed colored Dynkin diagram $(\emptyset, A, \epsilon_{\check{\eta}}, [e]^{\Pi\setminus S})$ two sets, one includes walls and the other doesn't (in order to avoid duplicate notation we denote the signed-colored Dynkin diagram and the set with the same notation):

$$(\emptyset, A, \epsilon_{\check{\eta}}, [e]^{\Pi\setminus S})^{\leq} = H(\Delta^A)^{\leq}_{\epsilon_{\check{\eta}}(\Delta^A)} \times \{[e]^A\}.$$

When $A = \emptyset$ (no zeros), these are the 2^l boxes in the antidominant chamber. We define the second related set as:

$$(\emptyset, A, \epsilon_{\check{\eta}}, [e]^{\Pi\setminus S}) = H(\Delta^A)^{<}_{\epsilon_{\check{\eta}}(\Delta^A)} \times \{[e]^A\}.$$

The chamber walls of the antidominant chamber of the Cartan subgroup are defined as:

- $D(\alpha_i, A, \epsilon)^{\leq} = \{h \in H(\Delta^A)^{\leq}_{\epsilon(\Delta^A)} : |\chi_{\alpha_i}(h)| = 1\}$ (the α_i-wall),
- $D(\alpha_i, A, \epsilon)^{<} = D(\alpha_i, A, \epsilon)^{\leq} \cap \{h \in H(\Delta^A)^{\leq}_{\epsilon(\Delta^A)} : |\chi_{\alpha_j}(h)| < 1 \text{ if } j \neq i, \alpha_j \in \Pi \setminus A\}$.

We next consider the case of $S = \{\alpha_i\} \notin A$ and then the general case of any $S \subset \Pi$ with $A \subset \Pi \setminus S$ and any $\epsilon_{\check{\eta}} \in \mathcal{E}^A$. This defines the walls for Levi factor pieces corresponding to subsystems of the Toda lattice (we here list open walls):

- $(\{\alpha_i\}, A, \epsilon_{\check{\eta}}) = D(\alpha_i, A, \epsilon_{\check{\eta}})^{<} \times \{[e]^A\}$.
- $(S, A, \epsilon_{\check{\eta}}) = \bigcap_{\alpha_i \in S} (\{\ \alpha_i\}, A, \epsilon_{\check{\eta}})$.

We now associate a set in $\hat{H}_{\mathbb{R}}$ to a colored Dynkin diagram. We define a set denoted $D = (S, \epsilon_\eta)$ as follows: For $S \neq \emptyset$ (so that there is an $\eta : S \to \{\pm 1\}$),

- $(S, \epsilon_\eta) = \bigcup_{\check{\eta} \in K(\eta)} (S, A(\check{\eta}), \epsilon_{\check{\eta}})$, and if $S = \emptyset$, $(\emptyset, \epsilon_o) = \bigcup_{\check{\eta} \in \{\pm 1, 0\}^l} (\emptyset, \epsilon_{\check{\eta}})$.

Here $\epsilon_o = (1, \ldots, 1)$, and see (7) and (9) in Section 2.

We consider the w-translations of the colored Dynkin diagrams: For this write w uniquely as $w = w^\bullet w_\bullet$ with $w_\bullet \in W_S$.

- $(w_\bullet D, [w]^{\Pi \setminus S}) = w((D, [e]^{\Pi \setminus S}))$.

In order to define the W-translations of sets associated to signed-colored Dynkin diagrams we have to extend the definition of the W_S-action to the signed-colored Dynkin diagrams. The definition is exactly the same if we treat the label $-$ as if it were an R and the label $+$ as if it were a B as in Definition 4.4. Then consider $(w_\bullet \check{D}, [w]^{\Pi \setminus S})$ and let:

- $(w_\bullet \check{D}, [w]^{\Pi \setminus S}) = w((\check{D}, [e]^{\Pi \setminus S}))$.

Remark 6.3. We refer to the set $\overline{(\emptyset, \epsilon_o)} \subset \hat{H}_{\mathbb{R}}$ as the *antidominant chamber* of $\hat{H}_{\mathbb{R}}$. Recall Notation 4.7 $w^{\bullet, A} \in [w]^A$. The following justifies our definition of the "antidominant chamber" of $\hat{H}_{\mathbb{R}}$ using (17) in the definition of $\hat{H}_{\mathbb{R}}$.

Proposition 6.4. *We have*

$$\hat{H}_{\mathbb{R}} = \bigcup_{A \subset \Pi} \bigcup_{\substack{w^{\bullet, A} \in W \\ \sigma \in W_{\Pi \setminus A}}} w^{\bullet, A} \left(\sigma \left(H^{A, \leq}_{\epsilon(\sigma)} \right) \right) \times \{[w]^A\}. \tag{18}$$

Proof. We set $w^\bullet = w^{\bullet, A}$ for simplicity. By Proposition 3.17 Part b), we have that each $H^A_{\mathbb{R}}$ can be written as a union over $\sigma \in W_{\Pi \setminus A}$ of sets of the form $\sigma \left(H^{A, \leq}_{\epsilon(\sigma)} \right)$ and thus by the definition (17) of $\hat{H}_{\mathbb{R}}$, we conclude the statement. $\square$

7. Colored Dynkin diagrams and the corresponding cells.

Here we consider the manifold structure and the topology of $\hat{H}_{\mathbb{R}}$ as the union of cells parametrized by colored Dynkin diagrams.

7.1. Action of the Weyl group on the sets (S, ϵ_η). Let $\mathcal{M}^{\text{geo}}$ be the complex with the sets $(D, [w]^{\Pi\setminus S})$. We then consider the action of W_S on the union of all the sets $D = (S, \epsilon_\eta)$ having a fixed nonempty set S of colored vertices and endowed with an orientation o. Similarly we can endow each of the terms in the chain complex $\mathcal{M}^{\text{geo}}$ with a action of W by translating the sets corresponding to colored Dynkin diagrams with the W-action and taking into account changes of orientation induced on the oriented boxes. This new action of W on $\mathcal{M}^{\text{geo}}$ could in principle be different from the W-action on the chain complex $\mathcal{M}_*$ (see Proposition 4.11).

We now become more explicit about the W-action that was just introduced on $\mathcal{M}_*^{\text{geo}}$: Note that since $\chi_{\alpha_{j_i}}(h) = \pm 1$ for any $\alpha_{j_i} \in S$ and $h \in \overline{(S, \epsilon_\eta)}$ (Notation 6.2), if $h \in (S, \epsilon_\eta)$ then $s_{\alpha_{j_i}} \chi_{\alpha_j}(h) = \chi_{\alpha_j}(h) \chi_{\alpha_{j_i}}(h)^{-C_{j,j_i}} = \pm \chi_{\alpha_j}(h)$. Hence, in terms of the coordinates given by ϕ_e, the action of $s_{\alpha_{j_i}}$ is given by a diagonal matrix whose nonzero entries are ± 1. This matrix has some entries corresponding to the set S and other entries corresponding to $\Pi \setminus S$. The entries corresponding to the set S change by a sign as described in Proposition 3.16. The statement in Proposition 3.16 just means that the set (S, ϵ_η) is sent to $(S, \epsilon_{\eta'})$ with $\eta' = (\eta_{j_1}, \ldots, \eta_{j_s})$, and $\eta'_{j_i} = \eta_{j_i}(-1)^{C_{j,j_i}}$. The determinant of the diagonal submatrix corresponding to elements in $\Pi \setminus S$ is the sign $(\eta_i)^r$ with (see Definition 4.4)

$$r = \Big| \{\alpha_j \in \Pi \setminus S \, : \, C_{j,i} \text{ is odd}\} \Big|.$$

Consider the *set* (S, ϵ_η) endowed with a fixed orientation ω corresponding to $o = 1$. Then s_{α_i} with $\alpha_i \in S$ sends (S, ϵ_η) to $(S, \epsilon_{\eta'})$ endowed with the orientation $(\eta_i)^r \omega$. We now consider the $\mathbb{Z}$ module of formal integral combinations of the sets (S, ϵ_η),

$$\mathbb{Z}\left[(S, \epsilon_\eta) \, : \, S \subset \Pi, \; |S| = k, \; \epsilon_\eta \in \mathcal{E}^{\Pi\setminus S}\right].$$

We keep track of the orientation by putting a sign $\pm$ in front of (S, ϵ_η). This $\mathbb{Z}$ module acquires a $\mathbb{Z}[W_S]$-action that corresponds to the abstract construction given in Definition 4.9 with colored Dynkin diagrams. By considering all the W-translations of the (S, ϵ_η) we generate the module denoted above by $\mathcal{M}(S)$. The direct sum of all these $\mathcal{M}(S)$ over $|S| = k$ is denoted $\mathcal{M}_{l-k}$ (see Definition 4.9). Hence there is no difference as W-modules between $\mathcal{M}_*^{\text{geo}}$ and $\mathcal{M}_*$.

We can now summarize this discussion in the following:

Proposition 7.1. *The action of W on $\mathcal{M}_*$ (Subsection 4.3) and the action of W on $\mathcal{M}^{\text{geo}}$ are isomorphic.*

We will then drop the superscript $\dots^{\mathrm{geo}}$ from the notation in view of Proposition 7.1.

7.2. Manifold structure on $\hat{\boldsymbol{H}}_{\mathbb{R}}$. Recall the map ϕ_e in Definition 5.4 whose domain is $\overset{\circ}{H}_{\mathbb{R}} = \bigcup_{A\subset\Pi} H_{\mathbb{R}}(\Delta^A) \times \{[e]^A\}$ and co-domain is $\mathbb{R}^l$. We also have defined ϕ_w, with domain $\bigcup_{A\subset\Pi} H_{\mathbb{R}}(w(\Delta^A) \times \{[w]^A\})$. We will use these maps to give $\hat{H}_{\mathbb{R}}$ coordinate charts leading to a manifold structure. We then have the following three Propositions:

Proposition 7.2. *The image $\phi_e(H_{\mathbb{R}}(\Delta^A)) \times \{[e]^A\}$ consists of all $(t_1, ..., t_l) \in \mathbb{R}^l$ such that $t_i \neq 0$ if and only if $\alpha_i \notin A$. The map ϕ_e is a bijection between $\overset{\circ}{H}_{\mathbb{R}} = \bigcup_{A\subset\Pi} (H_{\mathbb{R}}(\Delta^A)) \times \{[e]^A\}$ and $\mathbb{R}^l$.*

Proof. We start with the last statement, that ϕ_e is a bijection; we have that ϕ_e is injective because the scalars $\chi^{\Delta^A}_{\alpha_{j_1}}(h), \dots, \chi^{\Delta^A}_{\alpha_{j_m}}(h)$ determine all the root characters $\chi^{\Delta^A}_{\phi}(h)$, $\phi \in \Delta^A$ for $h \in H_{\mathbb{R}}(\Delta^A)$ and these scalars determine h in the adjoint group (Remark 3.14). From Proposition 5.2 Part 2) it follows that we can regard $H_{\mathbb{R}}(\Delta^A)$ as $\mathrm{Ad}^{\Delta^A}(H_{\mathbb{R}})$ (see (15)). We prove surjectivity by proving first the statement concerning the image $\phi_e \circ (\mathrm{Ad}^{\Delta^A} \times 1)(H_A \times \{[e]^A\})$. When all the sets A are considered then all of $\mathbb{R}^l$ will be seen to be in the image of ϕ_e. First consider $h_\epsilon h$ with $h = \exp\left(\sum_{\alpha_i\notin A} c_i m^\circ_{\alpha_i}\right)$ and $\epsilon \in \mathcal{E}^A$. We now apply $\phi_e \circ (\mathrm{Ad}^{\Delta^A} \times 1)$. Since $\left\langle \alpha_i, \sum_{\alpha_j\notin A} c_j m^\circ_{\alpha_j} \right\rangle = c_i\frac{(\alpha_i,\alpha_i)}{2}$ we obtain, by exponentiating, $\chi_{\alpha_i}(h_\epsilon h) = \epsilon_i e^{c_i\frac{(\alpha_i,\alpha_i)}{2}}$. The set $\phi_e(H_{\mathbb{R}}(\Delta^A) \times \{[e]^A\})$ becomes the image of the map: $\mathbb{R}^l \to \mathbb{R}^l$ given by first defining a map that sends $(\epsilon_1 t_1, \dots, \epsilon_l t_l) \to (f_1, \dots, f_l)$ with $f_i = \epsilon_i t_i^{\frac{(\alpha_i,\alpha_i)}{2}}$ for $t_i > 0$. This map is modified so that whenever $\alpha_i \in A$ then the i-th coordinate is replaced with 0. We denote this modified map by F^A. The domain and the image of F^A therefore consists of the set $\{(s_1, \dots, s_l) : s_i = 0 \text{ if } \alpha_i \in A\}$.

Together all these F^A give rise to one single map $F : \mathbb{R}^l \to \mathbb{R}^l$ which is surjective. □

Proposition 7.3. *The image $\phi_w(H_{\mathbb{R}}(w(\Delta^A)) \times \{[w]^A\})$ consists of all $(t_1, \dots, t_l) \in \mathbb{R}^l$ such that $t_i \neq 0$ if and only if $\alpha_i \notin A$.*

Proof. This follows from Proposition 6.4 and the fact that $\phi_{e,i}(w(h), A) = \chi_{w\alpha_i}(w(h)) = \chi_{\alpha_i}(h)$ for $\alpha_i \notin A$. □

Proposition 7.4. *The image $\phi_w((\emptyset, \epsilon_o))$ consists of all $(t_1, \dots, t_l) \in \mathbb{R}^l$ such that $-1 < t_i < 1$. The sets $\phi_w((S, \epsilon_\eta, [w]^{\Pi\setminus S}))$ as $S \subset \Pi$, $S \neq \emptyset$ varies,* *give a cell decomposition of the boundary of the box $[-1,1]^l$. In particular, the sets $(S, \epsilon_\eta, [w]^{\Pi\setminus S})$ give a cell decomposition of the smooth manifold $\hat{H}_{\mathbb{R}}$.*

Proof. This follows from Proposition 7.3 but is better understood in Example 2.2. We omit details. ☐

Remark 7.5. There is a more convenient cell decomposition of $\hat{H}_{\mathbb{R}}$ for the purpose of calculating homology explicitly. The only change is that the l dimensional cell becomes the union of all the l-cells together with all the (internal) boundaries corresponding to colored Dynkin diagrams where all the colored vertices are colored B. This is the set:

$$\hat{H}_{\mathbb{R}} \setminus \bigcup_{\substack{S\subset\Pi,\ w\in W \\ \eta \text{ such that } \eta(\alpha_i)=-1 \text{ for some } \alpha_i\in S}} (S, \epsilon_\eta, [w]^{\Pi\setminus S}).$$

This set can be seen to be homeomorphic to $\mathbb{R}^l$. With this cell decomposition there is exactly one l cell; and the other lower dimensional cells correspond to colored Dynkin diagrams which are parametrized by pairs $(D, [w]^{\Pi\setminus S})$, such that, at least one vertex of D has been colored R. In terms of the $(S, \epsilon_\eta, [w]^{\Pi\setminus S})$, the top cell would instead be defined to consist of $H^\circ_{\mathbb{R}}$. (See Figure 6 for this remark.)

The big cell in this decomposition with a fixed orientation corresponds to the element $c_l = \sum_{w\in W}(-1)^{l(w)}(D, \epsilon_o, w)$ $(S=\emptyset)$. This element satisfies $\partial_l(c_l) = 2(c_{l-1})$ for some $c_{l-1} \neq 0$ (except in the case of type A_1). Note that (D, w) and (D, ws_{α_i}), with $S=\emptyset$ and $l(ws_{\alpha_i}) = l(w)+1$, appear with opposite signs in c_l. When ∂_l is applied and the α_i is colored R the sign $(-1)^{r_{\alpha_i}}$ in Definition 4.4 makes these two terms contribute as $2(D', [w]^{\Pi\setminus\{\alpha_i\}})$ with D' the new colored Dynkin diagram obtained. The terms from the boundary ∂_l obtained by coloring α_i with B will cancel since the action of s_{α_i} on colored Dynkin diagrams is trivial when α_i is colored B. The case of A_1 is an exception because, in that case, once α_1 is colored R no more uncolored vertices remain. The set $\{\alpha_j \in \Pi \setminus \alpha_i : C_{j,i} \text{ is odd}\}$ is empty and $r_{\alpha_i} = 0$. Thus there is cancellation in this case.

7.3. Topology on $\hat{H}_{\mathbb{R}}$, coordinate charts, integral homology. We define a topology on $\hat{H}_{\mathbb{R}}$ in which $U \subset \bigcup_{A\subset\Pi} H(w(\Delta^A)) \times \{[w]^A\}$ is open if and only if $\phi_w(U)$ is open in the usual topology of $\mathbb{R}^l$. The maps ϕ_w become coordinate charts and since the compositions $\phi_w \circ \phi_\sigma^{-1}$ are C^∞ on their domain, then $\hat{H}_{\mathbb{R}}$ acquires the structure of a smooth manifold. The W-action becomes a smooth action.

Definition 7.6 (Filtration of $\hat{H}_{\mathbb{R}}$ and the chain complex $\mathcal{M}_*^{CW}$)**.** We construct a filtration of the topological space $\hat{H}_{\mathbb{R}}$ in the sense of [**18**] p. 222.

Let X_{l-k} denote the union of all the sets of the form $(D, [w]^{\Pi\setminus S})$ over all $w \in W$ and $S \in P(\Pi)$ such that $|S| \geq k$. This is a closed set and $X_r \setminus X_{r-1}$ is a union of sets of the form $(D, [w]^{\Pi\setminus S})$ with $|S| = l - r$. The filtration X_r $r = 0, 1, \ldots, l$ satisfies the conditions of Theorem 39.4 in [**18**]. We define a chain complex $\mathcal{M}_*^{CW}$ with boundary operators as in [**18**]

$$\partial_r : H_r(X_r, X_{r-1}, \mathbb{Z}) \to H_{r-1}(X_{r-1}, X_{r-2}, \mathbb{Z}).$$

Proposition 7.7. *The smooth manifold $\hat{H}_{\mathbb{R}}$ is compact, nonorientable (except if $\mathfrak{g}$ is of tupe A_1). The homology of the chain complex $\mathcal{M}_*$, $H^k(\mathcal{M}_*)$ $= \operatorname{Ker} \partial_k / \operatorname{image}(\partial_{k-1})$ is isomorphic as a $\mathbb{Z}[W]$ module to $H_k(\hat{H}_{\mathbb{R}}, \mathbb{Z})$.*

Proof. The manifold $\hat{H}_{\mathbb{R}}$ is the finite union of the chambers as in Proposition 6.4. Since the W-action on $\hat{H}_{\mathbb{R}}$ is by continuous transformations, it then suffices to observe that the antidominant chamber is compact. The antidominant chamber $\overline{(\emptyset, \epsilon_o)}$ of Definition 6.2 can be seen to be compact by describing explicitly its image under ϕ_e. This image is a "box" inside $\mathbb{R}^l$, as can be seen in Propositions 7.2, 7.4 namely the set $\{(t_1, \ldots, t_l) : -1 \leq t_i \leq 1\}$. The space $\hat{H}_{\mathbb{R}}$ is now the finite union of the W-translates of this compact set. That the boundary operators of $\mathcal{M}^{CW}$ agree with the boundary operators of $\mathcal{M}_*$ will follow from the fact that in the ϕ_w coordinates the $(D, [w])$ is a "box" which is itself part of the boundary of a bigger "box" (Proposition 7.4). We start with the set $\{(t_1, \ldots, t_l) : -1 \leq t_i \leq 1\}$ and note that its boundary is combinatorially described by (12) or (13). Note that (D, w) (with $S = \emptyset, [w]^{\Pi} = w$) represents the open box. The faces are parametrized by coloring each of the l vertices R or B which then represent *opposite* faces in the boundary. The signs are just chosen so that $\partial_{k-1} \circ \partial_k = 0$ for $k = 1, \ldots, l$. This description may be best understood by working out Example 2.2.

All the cells thus appear by taking the faces of a box $[-1, 1]^l$ and then faces of faces etc. By the same process of coloring uncolored vertices R or B which give rise, each time, to a pair of opposite faces in a box. In each case (12) or (13) correctly describe the process of taking the boundary of a box. Note that it is enough to study what happens when $w = e$ and then consider the W-translates.

We now use Theorem 39.4 of [**18**] to conclude that $\mathcal{M}_*^{CW}$ computes integral homology. However each $\mathbb{Z}[W]$-module appearing in $\mathcal{M}_*^{CW}$ in a fixed degree, can easily be seen to be identical with the corresponding term in $\mathcal{M}_*$. By Proposition 7.1 and the agreement of the boundary operators, we obtain that $\mathcal{M}_*$ computes integral homology. The nonorientability follows if we use the second cell decomposition described in Remark 7.5. The unique top cell then has a nonzero boundary (except in the case of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$). $\square$

8. Toda lattice and the manifold $\hat{H}_{\mathbb{R}}$.

We now associate the manifold $\hat{H}_{\mathbb{R}}$ with the Toda lattice by extending the results of Kostant in [16]. We start with the definition of the variety $Z_{\mathbb{R}}$ of Jacobi elements on $\mathfrak{g}$ where the Toda lattice is defined.

8.1. The variety $Z_{\mathbb{R}}$ and isotropy group $\widetilde{G}^{z}$.

Definition 8.1 (Varieties of Jacobi elements)**.** Let $S(\mathfrak{g})$ be the symmetric algebra of $\mathfrak{g}$. We may regard $S(\mathfrak{g})$ as the algebra of polynomial functions on the dual $\mathfrak{g}'$.

If we consider the algebra of G-invariants of $S(\mathfrak{g})$, then by Chevalley's theorem there are homogeneous polynomials $I_1, \dots, I_l$ in $S(\mathfrak{g})^G$ which are algebraically independent and which generate $S(\mathfrak{g})^G$. Thus $S(\mathfrak{g})^G$ can be expressed as $\mathbb{R}[I_1, \dots, I_n]$.

For $F = \mathbb{C}$ or $F = \mathbb{R}$, we consider the variety Z_F of normalized Jacobi elements of $\mathfrak{g}_F$. Our notation, however, is slightly different from the notation of [16] in the roles of e_{α_i} and $e_{-\alpha_i}$. Thus we let

$$J_F = \left\{ X = x + \sum_{i=1}^{l} (b_i e_{-\alpha_i} + e_{\alpha_i}) \,:\, x \in \mathfrak{h}, b_i \in F \setminus \{0\} \right\},$$
$$Z_F = \left\{ X = x + \sum_{i=1}^{l} (b_i e_{-\alpha_i} + e_{\alpha_i}) \,:\, x \in \mathfrak{h}, b_i \in F \setminus \{0\}, X \in S(F) \right\}.$$

We also allow *subsystems* which correspond to the cases having some $b_i = 0$:

$$\mathring{J}_F = \left\{ X = x + \sum_{i=1}^{l} (b_i e_{-\alpha_i} + e_{\alpha_i}) \,:\, x \in \mathfrak{h}, b_i \in F \right\},$$
$$\mathring{Z}_F = \left\{ X = x + \sum_{i=1}^{l} (b_i e_{-\alpha_i} + e_{\alpha_i}) \,:\, x \in \mathfrak{h}, b_i \in F, X \in S(F) \right\}.$$

Kostant defines in [16] p. 218 a real manifold Z by considering all elements $x + \sum_{i=1}^{l} (b_i e_{-\alpha_i} + e_{\alpha_i})$ in $Z_{\mathbb{R}}$ which in addition satisfy $b_i > 0$. We are departing in a crucial way from [16] by allowing the b_i to be negative or even zero when $A \neq \emptyset$. This extension gives the indefinite Toda lattices introduced in [14]. We let for any $\epsilon \in \mathcal{E}$,

$$Z_\epsilon = \left\{ X = x + \sum_{i=1}^{l} (b_i e_{-\alpha_i} + e_{\alpha_i}) \,:\, \epsilon_i b_i > 0, X \in S(\mathbb{R}) \right\}.$$

This is a set of real normalized Jacobi elements, that is, the elements of $Z_{\mathbb{R}}$. Thus the union of all the Z_ϵ is all of $Z_{\mathbb{R}}$,

$$Z_{\mathbb{R}} = \bigcup_{\epsilon \in \mathcal{E}} Z_\epsilon.$$

The elements in $Z_{\mathbb{R}}$ are thus the signed normalized Jacobi elements (in $S(\mathbb{R})$ and Z simply denotes Z_{ϵ_o} where $\epsilon_o = (1, \ldots, 1)$).

Definition 8.2 (Chevalley invariants and isospectral manifold)**.** If $x \in \mathfrak{g}$ and $g_x \in \mathfrak{g}'$ is defined by $\langle g_x, y\rangle = (x, y)$ for any $y \in \mathfrak{g}$ then the map $\mathfrak{g} \to \mathfrak{g}'$ sending x to g_x defines an isomorphism. We can then regard $S(\mathfrak{g})$ as the algebra of polynomial functions on $\mathfrak{g}$ itself by setting for $f \in S(\mathfrak{g})$ and $x \in \mathfrak{g}$, $f(x) = f(g_x)$.

The functions $I_1, \ldots, I_l$ now on $\mathfrak{g}$ and then restricted to $J_{\mathbb{R}}$, $Z_{\mathbb{R}}$ or to $Z_{\mathbb{C}}$ are called the *Chevalley invariants* which are the polynomial functions of $\{a_1, \ldots, a_l, b_1, \ldots, b_l\}$ for $X = \sum_{i=1}^{l} (a_i h_{\alpha_i} + b_i e_{-\alpha_i} + e_{\alpha_i})$. The map $\mathbb{I} = (I_1, \ldots, I_l)$ then defines by restriction a map

$$\mathcal{I} = \mathcal{I}_F : Z_F \to F^l.$$

Fix $\gamma \in F^l$ in the image of the map $\mathbb{I}$, and denote

$$Z(\gamma)_F = \mathcal{I}_F^{-1}(\gamma) = \mathbb{I}^{-1}(\gamma) \bigcap Z_F,$$

which defines the isospectral manifold of Jacobi elements of $\mathfrak{g}$. Note that in the real (isospectral) manifold $Z(\gamma)$ studied in [**16**] will just be one connected component of $Z(\gamma)_{\mathbb{R}}$.

Definition 8.3 (The isotropy subgroup $\widetilde{G}^y$ on $\widetilde{G}$)**.** Let $G_{\mathbb{C}}^y$ be the isotropy subgroup of $G_{\mathbb{C}}$ for an element $y \in \mathfrak{g}_{\mathbb{C}}$. The group $G_{\mathbb{C}}^y$ is an abelian connected algebraic group of complex dimension l (Proposition 2.4 of [**16**]). If $y \in \mathfrak{g}$ we denote by $\widetilde{G}^y$ the intersection

$$\widetilde{G}^y = G_{\mathbb{C}}^y \cap \widetilde{G}.$$

If $x \in \mathfrak{g}$, the centralizer of x is denoted $\mathfrak{g}^x$ and $\dim \mathfrak{g}^x \geq l$. We say that x is *regular* if $\dim \mathfrak{g}^x = l$.

We consider an open subset of $G_{\mathbb{C}}$ given by the biggest piece in the Bruhat decomposition:

$$(G_{\mathbb{C}})_* = \overline{N}_{\mathbb{C}} H_{\mathbb{C}} N_{\mathbb{C}} = \overline{N}_{\mathbb{C}} B_{\mathbb{C}}, \tag{19}$$

where $N_{\mathbb{C}} = \exp(\mathfrak{n}_{\mathbb{C}}), \overline{N}_{\mathbb{C}} = \exp(\bar{\mathfrak{n}}_{\mathbb{C}})$ and $B_{\mathbb{C}} = H_{\mathbb{C}} N_{\mathbb{C}}$. We let $\widetilde{G}_* = \widetilde{G} \cap (G_{\mathbb{C}})_*$ and, as in Notation 3.9, $B = H_{\mathbb{R}} \exp(\mathfrak{n})$. We then have a map

$$\overline{N}_{\mathbb{C}} \times B_{\mathbb{C}} \to (G_{\mathbb{C}})_*,$$

given by $(n, b) \mapsto nb$ which is an isomorphism of algebraic varieties. Given $d \in (G_{\mathbb{C}})_*$, d has a unique decomposition as $d = n_d b_d$ as in (2.4.6) of [**16**].

We caution the reader that $\widetilde{G}^y_*$ is not an intersection with $\exp(\bar{\mathfrak{n}})H\exp(\mathfrak{n})$ but rather with $\exp(\bar{\mathfrak{n}})H_{\mathbb{R}}\exp(\mathfrak{n})$. This is the object that appears, for example, in (3.4.10) of [**16**] and properly contains (3.2.9) in Lemma 3.2 of [**16**].

The following Proposition gives the relation between $\widetilde{G}^y$ and $H_{\mathbb{R}}$:

Proposition 8.4. *Let $y \in Z_{\epsilon_o}$. Then $\widetilde{G}^y$ is $\widetilde{G}$ conjugate to the Cartan subgroup $H_{\mathbb{R}}$ of $\widetilde{G}$.*

Proof. By Lemma 2.1.1 in [**16**], y is regular. Then as in Lemma 3.2 of [**16**], y must be conjugate, under an element in H, to an element x in $\mathfrak{p}$. Using Proposition 2.4 of [**16**] $G^x_{\mathbb{C}}$ is connected. Since x is also a regular element it follows that $\mathfrak{g}^x_{\mathbb{C}}$ is a Cartan subalgebra and $G^x_{\mathbb{C}}$ is a Cartan subgroup of $G_{\mathbb{C}}$. Using conjugation by an element in K we may conjugate this Cartan subgroup if necessary to $H_{\mathbb{C}}$ (Proposition 6.61 or Lemma 6.62 of [**13**]). We can thus assume that $G^x_{\mathbb{C}} = H_{\mathbb{C}}$. We obtain that $\widetilde{G}^y = \widetilde{G} \cap G^y_{\mathbb{C}}$ is $\widetilde{G}$ conjugate to $\widetilde{G} \cap H_{\mathbb{C}} = H_{\mathbb{R}}$ (see Notation 3.9). ☐

8.2. Kostant's map $\beta^y_{\mathbb{C}}$. Fix $y \in J(\gamma)_{\mathbb{R}}$. Kostant defines a map

$$\begin{array}{cccl} \beta^y_{\mathbb{C}} & : & (G^y_{\mathbb{C}})_* & \longrightarrow J(\gamma)_{\mathbb{C}} \\ & & d & \mapsto \mathrm{Ad}(n_d^{-1})(y) \end{array} \tag{20}$$

with $d = n_d b_d, n_d \in \overline{N}$ and $b_d \in B$. Note that we have deviated from the convention in [**16**] by exchanging the roles of $\overline{N}_{\mathbb{C}}$ and $N_{\mathbb{C}}$. We did not exchange the roles of these two groups in (19) but this is compensated by our use of an inverse in the definition of the Kostant map. Theorem 2.4 of [**16**] then implies that $\beta^y_{\mathbb{C}}$ is an isomorphism of algebraic varieties. Denote β^y the restriction of $\beta^y_{\mathbb{C}}$ to the intersection with $\widetilde{G}$. Thus we have $\beta^y : \widetilde{G}^y_* \to Z(\gamma)_{\mathbb{R}}$ and $\widetilde{G}^y_* = (G^y_{\mathbb{C}})_* \cap \widetilde{G}$.

Proposition 8.5. *Let $y \in J(\gamma)_{\mathbb{R}}$. The map β^y is an isomorphism of smooth manifolds $\widetilde{G}^y_* \to J(\gamma)_{\mathbb{R}}$.*

Proof. The map β^y is the restriction to the Lie group $\widetilde{G}^y$ of the diffeomorphism of complex analytic manifolds $\beta^y_{\mathbb{C}}$. We obtain that β^y must be an injective map. We show surjectivity. If $z \in J(\gamma)_{\mathbb{R}}$ then by surjectivity of $\beta^y_{\mathbb{C}}$ there is $g_{\mathbb{C}} \in (G^y_{\mathbb{C}})_*$ such that $\beta^y_{\mathbb{C}}(g_{\mathbb{C}}) = z$ and $g_{\mathbb{C}} = n_{\mathbb{C}} b_{\mathbb{C}}$. Thus $g^c_{\mathbb{C}} = n^c_{\mathbb{C}} b^c_{\mathbb{C}}$ with $n^c_{\mathbb{C}} \in \overline{N}_{\mathbb{C}}$ and $b^c_{\mathbb{C}} \in B_{\mathbb{C}}$. Therefore $\beta^y(g^c_{\mathbb{C}}) = \mathrm{Ad}(n^c_{\mathbb{C}})^{-1} y$. Since $y^c = y$ we obtain that $(\mathrm{Ad}(n_{\mathbb{C}})^{-1} y)^c = z^c$. But our assumption is that $z^c = z$. Hence we have obtained that $\beta^y_{\mathbb{C}}(g^c_{\mathbb{C}}) = z$. By the *injectivity* of $\beta^y_{\mathbb{C}}$ we obtain that $g^c_{\mathbb{C}} = g_{\mathbb{C}}$. Therefore $g_{\mathbb{C}} \in \widetilde{G}$ and thus $g_{\mathbb{C}} = g \in \widetilde{G}^y$. This proves β^y is a bijection.

By Proposition 2.3.1 of [**16**], $J(\gamma)_{\mathbb{R}}$ is a submanifold of real dimension l of $J(\gamma)_{\mathbb{C}}$. The diffeomorphism $\beta^y_{\mathbb{C}}$ restricts to the smooth nonsingular map β^y. Since we have shown that β^y is a bijection, then it is a diffeomorphism. ☐

Remark 8.6. If $y \in Z_{\epsilon_o}$ then $\widetilde{G}^y$ is a Cartan subgroup conjugate to $H_{\mathbb{R}}$ (see Proposition 8.4). However, in general it may happen that $y \in J(\gamma)_{\mathbb{R}}$ is not semisimple or that y is semisimple but $\widetilde{G}^y$ is a Cartan subgroup not conjugate to $H_{\mathbb{R}}$. For example, take $y = a_1 h_{\alpha_1} + b_1 e_{-\alpha_1} + e_{\alpha_1}$ in the case of $\mathfrak{sl}(2,\mathbb{R})$ ($\widetilde{G} = \mathrm{Ad}(SL(2,\mathbb{R})^{\pm})$). We get a nilpotent matrix if $a_1^2 + b_1 = 0$. When $a_1^2 + b_1 < 0$ then y is semisimple but $\widetilde{G}^y$ is a compact Cartan subgroup and thus it is not conjugate to $H_{\mathbb{R}}$. In the cases $a_1^2 + b > 0$ one obtains that $\widetilde{G}^y$ is conjugate to $H_{\mathbb{R}}$.

Assume that we are in the case when $\widetilde{G}^y$ is conjugate to $H_{\mathbb{R}}$ (for example $y \in Z_{\epsilon_o}$). Combining Proposition 8.4 and Proposition 8.5, Kostant's map gives an imbedding of $Z(\gamma)_{\mathbb{R}}$, where $\gamma \in \mathbb{R}^l$ is in the image of $\mathbb{I}$, into $\hat{H}_{\mathbb{R}}$ as an open dense subset.

8.3. Kostant's map and toric varieties. We first remark that for a fixed $y \in Z(\gamma)_{\epsilon_o}$ Kostant's map β^y in (20) is just the map $d \to n_d^{-1}$ with $d = n_d b_d \in \widetilde{G}^y_*$ and $n_d \in \overline{N}$, so that it can be described as a map into the flag manifold: $d \to gB$ in $\widetilde{G}/B$, restricted to $\widetilde{G}^y_*$.

By Proposition 8.5 the map into the flag manifold is a diffeomorphism onto its image when restricted to $\widetilde{G}^y_*$. Hence, since the map is given by the action of a Cartan subgroup on the flag manifold; this action has a trivial isotropy group and the map to the flag manifold sends $\widetilde{G}^y$ diffeomorphically to its image.

The Cartan subgroup $\widetilde{G}^y$ is as good as its conjugate $H_{\mathbb{R}}$ but, for convenience, we prefer to deal with $H_{\mathbb{R}}$ for which we have established notation. We let x be an element that conjugates $H_{\mathbb{R}}$ to $\widetilde{G}^y$, $x^{-1}H_{\mathbb{R}}x = \widetilde{G}^y$. Then the $\widetilde{G}^y$-orbit of B in $\widetilde{G}/B$ is $x^{-1}H_{\mathbb{R}}xB$. Since we can translate this set using multiplication by the fixed element x, we can just study the $H_{\mathbb{R}}$-orbit of xB in $\widetilde{G}/B$.

We denote:

- The map $q : \widetilde{G}^y \to \widetilde{G}/B$.
- $\hat{Z}(\gamma)_{\mathbb{R}} = \overline{(q \circ (\beta^y)^{-1} Z(\gamma)_{\mathbb{R}})} = x^{-1}(\overline{(H_{\mathbb{R}}xB)})$.

To study $\hat{Z}(\gamma)$, it is enough to describe in detail the toric variety $\overline{(H_{\mathbb{R}}xB)}$. Thus we focus our attention on objects that have this general form:

Definition 8.7. For any $n \in \widetilde{G}$, such that $nB \cap H_{\mathbb{R}} = \{e\}$, the toric variety $\overline{(H_{\mathbb{R}}nB)}$ is called *generic* in the sense of [8], if $n \in \bigcap_{w \in W} w(\overline{N}B)w^{-1}$.

We then assume $n \in \bigcap_{w \in W} w(\overline{N}B)w^{-1}$ and $nB \cap H_{\mathbb{R}} = \{e\}$. With these hypotheses we have:

$$nB = \prod_{\phi_j \in -\Delta_+} \exp(t_j^e e_{\phi_j})B$$

for $t_j^e \in \mathbb{R}$, and

$$nB = n(w(\Delta))wB = \prod_{\phi_j \in -\Delta_+} \exp(t_j^w e_{w(\phi_j)})wB$$

for $w \in W$ and $t_j^w \in \mathbb{R}$.

We now define:

Definition 8.8. Assume $n \in \bigcap_{w \in W} w(\overline{N}B)w^{-1}$ and $nB \cap H_{\mathbb{R}} = \{e\}$. Denote for any $A \subset \Pi$,

$$n(w(\Delta^A))wB = \prod_{\phi_j \in -\Delta_+^A} \exp(t_j^w e_{w(\phi_j)})wB.$$

We define a map,

$$\begin{array}{cccc} \check{B} & : & \hat{H}_{\mathbb{R}} & \longrightarrow & \overline{(H_{\mathbb{R}} nB)} \\ & & (\mathrm{Ad}^{w(\Delta^A)} g, [w]^A) & \mapsto & gn(w(\Delta^A))wB \end{array}$$

where $g \in H_{\mathbb{R}}$ (see Definition 6.1 for $\hat{H}_{\mathbb{R}}$). Note here that

$$gn(w(\Delta^A))wB = \prod_{\phi_j \in -\Delta_+^A} \exp\left(t_j^w \chi_{w(\phi_j)}(g) e_{w(\phi_j)}\right) wB. \tag{21}$$

The map $\check{B}$ can be interpreted as a version of Kostant's map β^y for the subsystem determined by the set $A \in \Pi$ and its w-translation. A detailed correspondence with the set $\mathring{Z}_{\mathbb{R}}(\gamma)$ could be made but it requires additional notation. Note also that we have $\chi_{w(\phi_j)}(g) = \chi_{w(\phi_j)}^{w(\Delta^A)} \circ \mathrm{Ad}^{w(\Delta^A)}(g)$ for $\phi_j \in \Delta^A$. Then we obtain:

Theorem 8.9. *Assume $n \in \bigcap_{w \in W} w(\overline{N}B)w^{-1}$ and $nB \cap H_{\mathbb{R}} = \{e\}$. The function $\check{B}$ is a homeomorphism of topological spaces. The toric variety $\overline{(H_{\mathbb{R}} nB)}$ is a smooth manifold and the map $\check{B}$ is a diffeomorphism.*

Proof. First we point out that we already have a smooth manifold $\hat{H}_{\mathbb{R}}$ and the assumption $n \in \bigcap_{w \in W} w(\overline{N}B)w^{-1}$ will simply ensure that we can define the map to the flag manifold.

We first show the continuity of $\check{B}$: We use the local coordinates, $\{\phi_w : w \in W\}$ for $\hat{H}_{\mathbb{R}}$ given in Definition 5.4. In these local coordinates, we assume that for each $i = 1, \dots, l$, $\phi_{ww_o,i}(\mathrm{Ad}^{ww_o(\Delta^{w_o(A)})}(g), [ww_o]^{w_o(A)})$ (which equals $\chi_{w(-\alpha_i)}^{w(\Delta^A)}(g)$ or zero) converges to a scalar $\chi_{w(-\alpha_i)}^o$. We note that if $\phi_j =$

$-\sum_{i=1}^{l} c_{ij}\alpha_i$ with each c_{ij} a nonnegative integer, then

$$\chi_{w(\phi_j)} = \prod_{i=1}^{l} \chi_{-w(\alpha_i)}^{c_{ij}} = \prod_{i=1}^{l} \phi_{ww_o,i}^{c_{ij}}(\mathrm{Ad}^{ww_o(\Delta^{w_o(A)})}, [ww_o]^{w_o(A)}).$$

We thus let

$$\chi^o_{w(\phi_j)} = \prod_{i=1}^{l} \left(\chi^o_{-w(\alpha_i)}\right)^{c_{ij}}.$$

Let

$$A' = \{\alpha_i \in \Pi : \chi^o_{-w(\alpha_i)} = 0\}.$$

Then note that $\chi^o_{w(\phi_j)} = 0$ if and only if $c_{ij} \neq 0$ for some $\alpha_i \in A'$. Thus the only $\chi^o_{w(\phi_j)}$ which are nonzero correspond to roots in $\Delta_+^{A'}$.

By the assumption made, we have

$$gn(w(\Delta^A))wB = \prod_{\phi_j \in -\Delta_+} \exp\left(t_j^w \chi_{w(\phi_j)}(g) e_{w(\phi_j)}\right) wB,$$

which can be written in terms of the coordinate functions ϕ_{ww_o} as

$$\prod_{\phi_j \in -\Delta_+} \exp\left(t_j^w \prod_{i=1}^{l} \phi_{ww_o,i}^{c_{ij}}(\mathrm{Ad}^{ww_o(\Delta^{w_o(A)})}, [ww_o]^{w_o(A)}) e_{w(\phi_j)}\right) wB.$$

This then converges (by continuity of ϕ_{ww_o}) to

$$\prod_{\phi_j \in -\Delta_+} \exp\left(t_j^w \chi^o_{w(\phi_j)}(g) e_{w(\phi_j)}\right) wB,$$

which only involves roots in $\Delta_+^{A'}$, and can be written as $gn(w(\Delta^{A'}))wB$. Since any $(\mathrm{Ad}^{w(\Delta^A)}(g), [w]^A) \in \hat{H}_{\mathbb{R}}$ is completely determined by the coordinates $\phi_w(\mathrm{Ad}^{w(\Delta^A)}(g), [w]^A)$ and some $(\mathrm{Ad}^{w(\Delta^{A'})}(g), [w]^{A'})$ uniquely corresponds to the coordinates $(\chi^o_{w(\alpha_1)}(g), \ldots, \chi^o_{w(\alpha_l)}(g))$, then we can conclude that $\check{B}(\mathrm{Ad}^{w(\Delta^A)}(g), [w]^A)$ converges to $\check{B}(\mathrm{Ad}^{w(\Delta^{A'})}(g), [w]^{A'})$ whenever the argument $(\mathrm{Ad}^{w(\Delta^A)}(g), [w]^A)$ approaches to $(\mathrm{Ad}^{w(\Delta^{A'})}(g), [w]^{A'})$ in $\hat{H}_{\mathbb{R}}$. This proves the continuity of the map $\check{B}$.

Since $\hat{H}_{\mathbb{R}}$ is compact (Proposition 7.7) and $\check{B}(\hat{H}_{\mathbb{R}})$ contains the orbit $H_{\mathbb{R}}nB$, then $\overline{(H_{\mathbb{R}}nB)} \subset \check{B}(\hat{H}_{\mathbb{R}})$. From the construction of the map $\check{B}$ it is easy to see that its image is contained in $\overline{(H_{\mathbb{R}}nB)}$ and thus $\check{B}(\hat{H}_{\mathbb{R}}) = \overline{(H_{\mathbb{R}}nB)}$.

The smoothness of the map and that it gives a diffeomorphism follows from the fact that $(s_1,\ldots,s_{|\Delta_+|}) \to \prod_{j=1}^{|\Delta_+|} \exp(s_j e_{w(\phi_j)})wB$ constitutes a coordinate system in the flag manifold. $\square$

Remark 8.10. We caution the reader that if one replaces the Cartan subgroup $H^1_{\mathbb{R}}$ of G instead of $H_{\mathbb{R}}$ in the statement of Theorem 8.9 then the closure of the orbit $H^1_{\mathbb{R}}nB$ may not be smooth. In fact the structure of $\overline{H^1_{\mathbb{R}}nB}$ can be explicitly described too. Consider only the connected components of each $H_{\mathbb{R}}(w(\Delta^A))$ associated to ϵ such that $\mathrm{Ad}(h_\epsilon) \in \mathrm{Ad}(H^1_{\mathbb{R}})$ in Definition 5.4. This gives a subspace of $\hat{H}_{\mathbb{R}}$ that will correspond to $\overline{H^1_{\mathbb{R}}nB}$. In terms of the coordinate charts ϕ_w one gets locally $\mathbb{R}^l$ but now some of its 2^l quadrants may be missing (since $\mathrm{Ad}(H^1_{\mathbb{R}})$ may have fewer than 2^l connected components). Thus smoothness is obtained exactly when $\mathrm{Ad}(H^1_{\mathbb{R}})$ contains 2^l connected components. Examples that lead to nonsmooth closures if one uses the Cartan subgroup $H^1_{\mathbb{R}}$ are all the $G = SL(n,\mathbb{R})$ with n even. In terms of the Toda lattice this corresponds to considering the indefinite Toda lattice in (1) in the Introduction but leaving out some of the signs ϵ_i. When $n = 2$, for example, one obtains a closed interval inside G/B (which is a circle). The disconnected Lie group $\widetilde{G}$ which leads to the Cartan subgroup $H_{\mathbb{R}}$ is then a requirement in all our constructions and main results. We thus have:

Corollary 8.11. *Assume $n \in \bigcap_{w\in W} w(\overline{N}B)w^{-1}$ and $nB \cap H^1_{\mathbb{R}} = \{e\}$. Then $\overline{H^1_{\mathbb{R}}nB}$ is smooth if and only if $\mathrm{Ad}(H^1_{\mathbb{R}})$ has 2^l connected components.*

We also have:

Corollary 8.12. *If $n \in \bigcap_{w\in W} w(\overline{N}B)w^{-1}$ and $nB \cap H_{\mathbb{R}} = \{e\}$, then the toric variety $X = \overline{(H_{\mathbb{R}}nB)}$ satisfies: $X^{H_{\mathbb{R}}} = (G/B)^{H_{\mathbb{R}}}$, the $H_{\mathbb{R}}$-fixed points.*

Proof. We use Theorem 8.9. The manifold $\hat{H}_{\mathbb{R}}$ has an $H_{\mathbb{R}}$-action and the only fixed points are the (e,w) with $w \in W$. These get mapped to the fixed points of the $H_{\mathbb{R}}$-action in G/B, the cosets wB, $w \in W$. Thus $X^{H_{\mathbb{R}}} = (G/B)^{H_{\mathbb{R}}}$. This also follows directly if we consider (21) with $A = \emptyset$ and we just let each $\chi_{w(\phi_j)}$ go to zero and obtain wB. $\square$

Remark 8.13.

1) By Theorem 3.6 in [**9**] and Remark 3 in p. 257 of [**8**] we may assume that the map into the flag manifold which appears as a consequence of Kostant's map in Subsection 8.3 is such that one in fact obtains the $H_{\mathbb{R}}$ orbit of an element $n \in \bigcap_{w \in W} w(\overline{N}B)w^{-1}$.
2) In §7 of [**2**] and in [**3**] a more restrictive definition of the notion of *genericity* is given. This discussion is only relevant for the case of toric varieties in G/P where P is a parabolic subgroup rather than just a Borel subgroup as is the case in the present work. In these more general situations sometimes generic varieties as defined in [**8**] are not normal. In any case, note that if $P = B$ our Corollary 8.12 implies that if $n \in \bigcap_{w \in W} w(\overline{N}B)w^{-1}$ and $nB \cap H_{\mathbb{R}} = \{e\}$ then the corresponding toric variety is also generic in the sense discussed in [**2**] or [**3**]. Finally we observe that normality is not used in any of our results. Here we rely instead on explicit coordinate charts to obtain the smoothness of our toric varieties and describe their topological structure. (We would like to thank H. Flaschka for sending the papers [**2**, **3**] to us.)

We conclude:

Theorem 8.14. *Let $\gamma \in \mathbb{R}^l$, then $\hat{Z}(\gamma)_{\mathbb{R}}$ is a smooth compact manifold diffeomorphic to $\hat{H}_{\mathbb{R}}$.*

Proof. This is just Theorem 8.9 and the definition of $\hat{Z}(\gamma)$. The two conditions in Theorem 8.9 are satisfied by Proposition 8.5 (Subsection 8.3) and by Theorem 3.6 in [**9**] and Remark 3 in p. 257 of [**8**] as noted above in Remark 8.13. □

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Received January 19, 2000 and revised April 13, 2001. This research was supported by NSF grant DMS-0071523.

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