Pacific Journal of Mathematics

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Volume 207 No. 1 November 2002

PACIFIC JOURNAL OF MATHEMATICS
Vol. 207, No. 1, 2002

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We prove that the group of isometries of a Finsler space is a Lie transformation group on the original manifold. This generalizes the famous result of Myers and Steenrod on a Riemannian manifold and makes it possible to use Lie theory on the study of Finsler spaces.

Introduction.

Let (M, F) be a Finsler space, where F is positively homogeneous but not necessary absolutely homogeneous. As in the Riemannian case, we have two kinds of definitions of isometry on (M, F). On one hand, we can define an isometry to be a diffeomorphism of M onto itself which preserves the Finsler function. On the other hand, since on M we still have the definition of distance function (although generically it is not a real distance), we can define an ismotry of (M, F) to be a mapping of M onto M which keeps the *distance* of each pair of points of M.

The equivalence of the two definitions of isometry in the Riemannian case is a famous result of Myers and Steenrod. They used this result to prove that the group of isometries of a Riemannian manifold is a Lie transformation groups on the original manifold [**5**]. This result plays a fundamental role on the theory of homogeneous Riemannian manifolds. Since then, many different proofs were provided, cf., e.g., Palais [**6**], S. Kobayashi [**4**].

In this paper we prove that the two definitions of isometry are equivalent for a Finsler space. Then we prove that the group of isometries has a diferentiable structure which turns it into a Lie transformation on the manifold. This result makes it possible to use Lie theory on the study of Finsler spaces.

In this paper, Finsler structure F is only assumed to be positively homogeneous but not necessary absolutely homogeneous. We will not point out this each time. For a mapping ϕ of a manifold M, we use $d\phi$ to denote its differential. If $p \in M$, $d\phi|_p$ will denote the differential of ϕ at p. The notations of forward and backward metric ball in a Finsler spaces comes from the newly published book by D. Bao, S.S. Chern and Z. Shen [**1**].

1. A result on distance function.

Let (M, F) be a Finsler space, d be the distance function of (M, F). We first need to prove a result on the distance function.

Lemma 1.1. *Let $x \in M$. Then for any $\epsilon > 0$, there exists a neighborhood U of the original of $T_x(M)$ such that $\exp_x$ is a C^1-diffeomorphism from U onto its image and for any $A, B \in U$, $A \neq B$, and any C^1 curve $\sigma_0(s), 0 \leq s \leq 1$, connecting A and B which satisfies $\sigma_0(s) \in U$ and $\dot{\sigma}_0(s) \neq 0$, $s \in [0,1]$, we have*

$$\left| \frac{L(\sigma)}{L(\sigma_0)} - 1 \right| \leq \epsilon,$$

where $L(\cdot)$ denotes the arc length of a curve and $\sigma(s) = \exp_x \sigma_0(s)$.

Proof. Let $B_x(r) = \{A \in T_x(M) | \, F(x, A) < r\}$ be a tangent ball in $T_x(M)$ such that $\exp = \exp_x$ is a C^1-diffeomorphism from $B_x(r)$ onto $\mathcal{B}_x^+(x) = \{w \in M | d(x, w) \leq r\}$ (cf. [**1**]). Assume $A, B \in B_x(r)$, $A \neq B$. Let $\sigma_0(s)$, $0 \leq s \leq 1$ be a C^1 curve connecting A and B and $\forall s, \sigma_0(s) \in B_x(r)$ and $\dot{\sigma}_0(s) \neq 0$. Then we can write the velocity vector of $\sigma_0(s)$ as $\dot{\sigma}_0(s) = t(s)X(s)$, where $X(s)$ satisfies $F(x, X(s)) = \frac{r}{2}, \forall s$, and $t(s) \geq 0$ is a continuous function on $[0,1]$. Therefore the arc length of σ_0 is

$$L(\sigma_0) = \int_0^1 t(s) F(x, X(s)) ds.$$

Denote $X_1(s) = d(\exp_x)|_{\sigma_0(s)} X(s)$. Then the velocity vector of the curve $\sigma(s) = \exp_x(\sigma_0(s)), 0 \leq s \leq 1$ is

$$\dot{\sigma}(s) = d(\exp_x)|_{\sigma_0(s)}(t(s)X(s)) = t(s) d(\exp_x)|_{\sigma_0(s)}(X(s)) = t(s) X_1(s).$$

Therefore, the arc length of σ is

$$L(\sigma) = \int_0^1 t(s) F(\sigma(s), X_1(s)) ds.$$

Now we select a neighborhood V_1 of x in M with compact closure which is contained in $\mathcal{B}_x^+(r)$ and fix a coordinate system $(x_1, x_2, \ldots, x_n)$ in V_1. Let $U_1 = \exp^{-1} V_1$. Suppose $\sigma_0 \subset U_1$. Denote by $M(s)$ the matrix of $d(\exp_x)|_{\sigma_0(s)}$ under the base $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}$. Given any positive number $\delta < \frac{r}{2}$. Since $d(\exp_x)|_0 = I_n$ and exp is C^1 smooth, there exists a neighborhood $U_2 \subset U_1$ of the original of $T_x(M)$ such that for any C^1 curve σ_0 satisfying $\sigma_0(s) \in U_2, \forall s$, we have

$$\|M(s) - I\| < \frac{\delta}{n}, \quad 0 \leq s \leq 1,$$

where $\|\cdot\|$ denote the maximum of the absolute value of the entries of a matrix. Write $X(s)$ and $X_1(s)$ as:

$$X(s) = \sum_{j=1}^{n} y_j(s) \frac{\partial}{\partial x_j}\Big|_x;$$

$$X_1(s) = \sum_{j=1}^{n} y_j'(s) \frac{\partial}{\partial x_j}\Big|_{\sigma(s)}.$$

Then we have

$$|y_j'(s) - y_j(s)| < \delta, \quad 1 \leq j \leq n.$$

Consider the set

$$C_0 = \Big\{(w,\, (d(\exp_x))|_W)y \mid w \in V_1,\ W = \exp_x^{-1} w,$$
$$y \in T_W(Tx(M)) = Tx(M), F(x,y) = \frac{r}{2}\Big\}.$$

Since exp is C^1 smooth, the closure of C_0 is compact. Hence the Finsler function F is bounded on C_0. Suppose $F < r_1$ on C_0, $r_1 > 0$. Now write the Finsler function $F(w,y)$ as $F(w, y_1, y_2, \ldots, y_n)$ for $y = \sum\limits_{j=1}^{n} y_j \frac{\partial}{\partial x_j}\big|_w$. Consider the closure D_1 of the set $D_0 = \{(w,y) \in TM | w \in V_1, F(x,y) \leq \frac{r}{2} + r_1\}$. Since F is continuous and D_1 is compact, F is uniformly continuous on D_1. Therefore for the given $\epsilon > 0$, there exists $\delta_1 > 0$ and a neighborhood $V_2 \subset V_1$ of x such that for any $w \in V_2$, $|y_j - y_j'| < \delta_1, j = 1, 2, \ldots, n$, $F(x, y_1, y_2, \ldots, y_n) < \frac{r}{2} + r_1, F(w, y_1', \ldots, y_n') < \frac{r}{2} + r_1$, we have

$$|F(x, y_1, y_2, \ldots, y_n) - F(w, y_1', y'2, \ldots, y_n')| < \frac{r}{2}\epsilon.$$

Therefore if we select the above δ such that $\delta < \delta_1$. Then for the corresponding U_2 and any C^1 curve σ_0, $\sigma_0 \subset U_2 \cap (\exp)^{-1} V_2$, we have

$$\begin{aligned}\left|\frac{L(\sigma)}{L(\sigma_0)} - 1\right| &= \frac{\left|\int_0^1 t(s)(F(x, X(s)) - F(\sigma(s), X_1(s)))ds\right|}{\left|\int_0^1 t(s) F(x, X(s)) ds\right|} \\ &\leq \frac{\int_0^1 t(s)|F(x, X(s)) - F(\sigma(s), X_1(s))|ds}{r\int_0^1 t(s)ds} \\ &\leq \frac{\frac{r}{2}\epsilon}{\frac{r}{2}} \frac{\int_0^1 t(s)ds}{\int_0^1 t(s)ds} = \epsilon.\end{aligned}$$

□

Theorem 1.2. *Let $x \in M$ and $B_x(r)$ be a tangent ball of $T_x(M)$ such that $\exp_x$ is a C^1 diffeomorphism from $B_x(r)$ onto $\mathcal{B}_x^+(r)$. For $A, B \in B_x(r)$, $A \neq B$, let $a = \exp_x A$, $b = \exp_x B$. Then we have*

$$\frac{F(x, A - B)}{d(a, b)} \to 1$$

as $(A, B) \to (0, 0)$.

Proof. Let $\mathcal{B}_x^-(r) = \{w \in M \mid d(w, x) < r\}$. Suppose r is so small that each pair of points in $\mathcal{B}_x^+(\frac{r}{2}) \cap \mathcal{B}_x^-(\frac{r}{2})$ can be joint by a unique minimal geodesic contained in $\mathcal{B}_x^+(r)$(cf. [**1**]). Let $\Gamma_0(s), 0 \leq s \leq 1$ be the line segment connecting A and B, and $\Gamma(s) = \exp_x \Gamma_0(s)$. By Lemma 1.1, we have

$$\frac{L(\Gamma_0)}{L(\Gamma)} = \frac{F(x, A - B)}{L(\Gamma)} \to 1$$

as $(A, B) \to (0, 0)$. Now let $a = \exp_x A$, $b = \exp_x B$. Suppose $a, b \in \mathcal{B}_x^+(\frac{r}{2}) \cap \mathcal{B}_x^-(\frac{r}{2})$. Let $\gamma_{ab}(s), 0 \leq s \leq 1$ be the unique minimal geodesic of constant speed connecting a and b. Let $\gamma_0(s), 0 \leq s \leq 1$ be the unique curve in $B_x(r)$ which satisfies $\gamma_{ab}(s) = \exp_x \gamma_0(s)$. Then by Lemma 1.1, we also have

$$\frac{L(\gamma_0)}{L(\gamma_{ab})} \to 1$$

as $(A, B) \to (0, 0)$. Since

$$d(a, b) \leq L(\Gamma), \ L(\gamma_0) \geq F(x, A - B),$$

we have

$$\frac{F(x, A - B)}{L(\Gamma)} \leq \frac{F(x, A - B)}{d(a, b)} \leq \frac{L(\gamma_0)}{L(\gamma_{ab})}.$$

Theorem 1.2 follows. □

2. Differentiability of isometries.

First we have:

Proposition 2.1. *Let $\| \cdot \|_1$, $\| \cdot \|_2$ be two Minkowski norms on $\mathbb{R}^n$. Let ϕ be a mapping of $\mathbb{R}^n$ into itself such that $\|\phi(A) - \phi(B)\|_2 = \|A - B\|_1$, $\forall A, B \in \mathbb{R}^n$. Then ϕ is a diffeomorphism.*

Proof. Consider $\mathbb{R}^n$ endowed with $\| \cdot \|_j$, $j = 1, 2$ as two Finsler spaces, denoted by (M_1, F_1), (M_2, F_2). Then geodesics in M_j, $j = 1, 2$ are straight lines (cf. [**1**]). And the distance function of M_j are $d_j(A, B) = \|A - B\|_j$, $j = 1, 2$. Consider ϕ as a mapping from the Finsler space (M_1, F_1) to (M_2, F_2). Then ϕ preserves the distance function. Since in a Finsler space short geodesics minimize distance between its start and end points (cf. [**1**]), we can prove (similarly as in the Riemannian case) that ϕ transforms geodesics to geodesics. First suppose $\phi(0) = 0$. For $A \in \mathbb{R}^n$, $A \neq 0$, the curve $\phi(tA)$,

$t \geq 0$ is a ray which coincides with the ray $t\phi(A)$ for $t = 0$ and $t = 1$. Therefore they coincide as point sets. Thus $\phi(tA) = \mu(t)\phi(A)$ for some nonnegative function $\mu(t)$. Since

$$\begin{aligned}\|\phi(tA) - 0\|_2 &= \|tA - 0\|_1 = t\|A\|_1 \\ &= \|\mu(t)\phi(A) - 0\|_2 = \mu(t)\|\phi(A)\|_2 = \mu(t)\|A\|_1, t \geq 0,\end{aligned}$$

we have $\mu(t) = t$. Thus $\phi(tA) = t\phi(A)$, for $t \geq 0$. Suppose $A \neq B$, a similar argument as the above shows that there exists a nonnegative function $\lambda(t)$ such that $\phi(tA + (1-t)B) = \lambda(t)\phi(A) + (1 - \lambda(t))\phi(B)$, $t \geq 0$. And we can similarly show that $\lambda(t) = t$. In particular, for $t = \frac{1}{2}$ we have,

$$\frac{1}{2}\phi(A + B) = \phi\left(\frac{1}{2}(A + B)\right) = \frac{1}{2}\phi(A) + \frac{1}{2}\phi(B).$$

Thus $\phi(A+B) = \phi(A)+\phi(B)$. Taking $A = -B$ in the above equality we have $\phi(-A) = -\phi(A)$. Therefore ϕ is a linear transformation. Since $\mathrm{Ker}(\phi) = \{0\}$, it is a diffeomorphism. If $A_1 = \phi(0) \neq 0$, consider the composition mapping $\phi_1 = \pi_{A_1} \circ \phi$, where $\pi_{A_1}(A) = A - A_1$ is the parallel translation, which is a diffeomorphism. Since $\phi_1(0) = 0$ and $\|\phi_1(A)-\phi(B)\|_2 = \|A-B\|_1$, ϕ_1 is a diffeomorphism. Hence ϕ is a diffeomorphism. □

Remark. The proposition is an interesting application of Finsler geometry to Functional Analysis.

Now we can prove the main result of this paper.

Theorem 2.2. *Let (M, F) be a Finsler space and ϕ be a distance-preserving mapping of M onto itself. Then ϕ is a diffeomorphism.*

Proof. Let $p \in M$ and put $q = \phi(p)$. Let $r > 0, \epsilon > 0$ be so small that both $\exp_p$ and $\exp_q$ are C^1 diffeomorphisms on the tangent ball $B_p(r+\epsilon)$, $B_q(r+\epsilon)$ of $T_p(M)$ and $T_q(M)$, respectively. For any nonzero $X \in T_p(M)$, consider the radial geodesic $\exp_p(tX)$, $0 \leq t \leq \frac{r}{2F(p,X)}$. The image $\gamma(t) = \phi(\exp_p(tX))$ is a geodesic since ϕ is distance-preserving. Let X' denote the tangent vector of γ at the point q. We have obtained a mapping $X \to X'$ of $T_p(M)$ into $T_q(M)$. Denoting this mapping by ϕ' we have $\phi'(\lambda X) = \lambda\phi'(X)$, for $X \in T_p(M)$ and $\lambda \geq 0$. Let $A, B \in T_p(M)$, $A \neq B$ and t is so small that both tA and tB lie in $B_p(r)$. Let $a_t = \exp_p(tA)$, $b_t = \exp_p(tB)$. Then by Theorem 1.2 we have

$$\lim_{t \to 0^+} \frac{F(p, tA - tB)}{d(a_t, b_t)} = 1.$$

On the other hand, by the definition of ϕ' we have

$$\exp_q(\phi'(tX)) = \phi(\exp \mathrm{tX}),$$

for any X and t small enough. Thus by Theorem 1.2 we also have

$$\lim_{t\to 0^+} \frac{F(q, \phi'(tA) - \phi'(tB))}{d(\phi(a_t), \phi(b_t))} = 1.$$

Since $d(\phi(a_t), \phi(b_t)) = d(a_t, b_t)$, we get

$$\begin{aligned} 1 &= \lim_{t\to 0^+} \frac{F(p, tA - tB)}{F(q, \phi'(tA) - \phi'(tB))} \\ &= \lim_{t\to 0^+} \frac{tF(p, A - B)}{tF(q, \phi'(A) - \phi'(B))} = \frac{F(p, A - B)}{F(q, \phi'(A) - \phi'(B))}. \end{aligned}$$

Therefore $F(q, \phi'(A) - \phi'(B)) = F(p, A - B)$. By Proposition 2.1, ϕ' is a diffeomorphism of $T_p(M)$ onto $T_q(M)$.

Although on $\mathcal{B}_p^+(r) = \exp_p B_r(p)$ we have $\phi = \exp_q \circ \phi' \circ (\exp_p)^{-1}$, we still cannot conclude that ϕ is smooth on $\mathcal{B}_p^+(r)$, since in a Finsler space the exponential mapping is only C^1 at the zero section. That is, we can only conclude that ϕ is smooth in $\mathcal{B}_p(r) - \{p\}$. To finish the proof, we proceed to take r so small so that every pair of points in $\mathcal{B}_p^+(r) \cap \mathcal{B}_p^-(r)$ can be joint by a unique minimizing geodesic. Select $p_1 \in \mathcal{B}_p^+(\frac{r}{2}) \cap \mathcal{B}_p^-(\frac{r}{2})$, $p_1 \neq p$. Consider the tangent ball $B_{p_1}(\frac{r}{2})$ of $T_{p_1}(M)$. The exponential mapping is a C^1 diffeomorphism from $B_{p_1}(\frac{r}{2})$ onto $\mathcal{B}_{p_1}^+(\frac{r}{2})$. The above argument shows that ϕ is smooth in $\mathcal{B}_{p_1}^+(\frac{r}{2}) - \{p_1\}$, which is a neighborhood of p. This completes the proof. □

3. Group of isometries.

Theorem 2.2 justifies the following definition of isometry for a Finsler space.

Definition 3.1. Let (M, F) be a Finsler space. A mapping ϕ of M onto itself is called an isometry if ϕ is a diffeomorphism and for any $x \in M, X \in T_x(M)$, $F(\phi(x), d\phi_x(X)) = F(x, X)$.

In the following we denote the group of isometries of (M, F) by $I(M)$.

Let N be a connected, locally compact metric space and $\mathcal{I}(N)$ be the group of isometries of N, for each point x of N, let $\mathcal{I}_x(N)$ denote the isotropy subgroup of $\mathfrak{I}(N)$ at x. Van Danzig and van der Waerden [**7**] proved that $\mathcal{I}(N)$ is a locally compact topological transformation group on N with respect to the compact-open topology and $\mathcal{I}_x(N)$ is compact.

Now on M we have a distance function d defined by the Finsler function F. By Theorem 2.2, the group $I(M)$ coincides with the group of isometries $\mathcal{I}(M)$ of (M, d). Although generically d is not a distance (d is not symmetric unless F is absolutely homogeneous), we still have:

Theorem 3.2. *Let (M, F) be a connected Finsler space. The compact-open topology turns $I(M)$ into a locally compact transformation group of M. Let*

$x \in M$ and $I_x(M)$ denote the subgroup of $I(M)$ which leaves x fixed. Then $I_x(M)$ is compact.

Proof. A proof of this result for the Riemannian case was given in Helgason [**3**] (cf. Helgason [**3**], pp. 201-204), which is valid in general cases after some minor changes. Just note that on a Finsler manifold the topology generated by the forward metric balls $\mathcal{B}_p^+(r) = \{x \in M | d(p,x) < r\}, p \in M, r > 0$ is precisely the underlying manifold topology and this is true for the topology generated by the backward metric balls $\mathcal{B}_p^-(r) = \{x \in M | d(x,p) < r\}, p \in M, r > 0$ (cf. [**1**]). □

Bochner-Montgomery [**2**] proved that a locally compact group of differentiable transformations of a manifold is a Lie transformation group. Therefore we have the following theorem.

Theorem 3.3. *Let (M, F) be a Finsler space. Then the group of isometries $I(M)$ of M is a Lie transformation group of M. Let $x \in M$ and $I_x(M)$ be the isotropy subgroup of $I(M)$ at x. Then $I_x(M)$ is compact.*

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Received March 15, 2001. This work is Project 19901015 and 19731004 supported by NSFC.

Department of Mathematics
Nankai University
Tianjin, P. R. China
E-mail address: matdsq@263.net

Department of Mathematics
Nankai University
Tianjin, P. R. China
E-mail address: houzx@nankai.edu.cn