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MATRIX OF BOUNDED RANK

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Let $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$ be the spaces of $n \times n$ real matrices and real symmetric matrices respectively. We continue to study $d(n, n - 2, \mathbb{R})$: The minimal number ℓ such that every ℓ -dimensional subspace of $S_n(\mathbb{R})$ contains a nonzero matrix of rank $n - 2$ or less. We show that $d(4, 2, \mathbb{R}) = 5$ and obtain some upper bounds and monotonicity properties of $d(n, n - 2, \mathbb{R})$. We give upper bounds for the dimensions of $n - 1$ subspaces (subspaces where every nonzero matrix has rank $n - 1$) of $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$, which are sharp in many cases. We study the subspaces of $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$ where each nonzero matrix has rank n or $n - 1$. For a fixed integer $q > 1$ we find an infinite sequence of n such that any $\binom{q+1}{2}$ dimensional subspace of $S_n(\mathbb{R})$ contains a nonzero matrix with an eigenvalue of multiplicity at least q .

1. Introduction.

Let \mathbb{F} be a field, $M_{m,n}(\mathbb{F})$ the space of all $m \times n$ matrices over \mathbb{F} and $S_n(\mathbb{F})$ the space of all $n \times n$ symmetric matrices over \mathbb{F} . We write $M_n(\mathbb{F})$ for $M_{n,n}(\mathbb{F})$. Let V be either $M_{m,n}(\mathbb{F})$ or $S_n(\mathbb{F})$, and let k be a positive integer. In the last 80 years there has been interest in the following types of subspaces, all related to the *rank* function:

- (a) A subspace of V where each matrix has rank bounded above by k .
- (b) A subspace of V where each nonzero matrix has rank k . A subspace of this type is said to be a *k-subspace*.
- (c) A subspace of V where each nonzero matrix has rank bounded below by k .

See the works [13], [11], [2], [8], [5], [3], [4], [7] and many others. Roughly speaking these problems are divided into two classes depending on whether \mathbb{F} is algebraically closed or not. The classical case, which goes back to Radon-Hurwitz, discusses the maximal dimension $\rho(n)$ of an n -subspace U of $M_n(\mathbb{R})$ where each $0 \neq A \in U$ is an orthogonal matrix times $r \in \mathbb{R}^*$. Write $n = (2a + 1)2^{c+4d}$, where a and d are nonnegative integers, and $c \in$

$\{0, 1, 2, 3\}$. Then the Radon-Hurwitz number $\rho(n)$ is defined by

$$(1.1) \quad \rho(n) = 2^c + 8d.$$

In his famous work Adams [1] gave a nonlinear version of the Radon-Hurwitz number by showing that $\rho(n) - 1$ is the maximal number of linearly independent vector fields on the $n - 1$ dimensional sphere S^{n-1} . ($S^{n-1} \subset \mathbb{R}^n$ denotes the Euclidian sphere of radius one centered at the origin.) From this result Adams deduced that the maximal dimension of an n -subspace of $M_n(\mathbb{R})$ is exactly $\rho(n)$. Let $\rho(x) = 0$ if x is not a positive integer. Define now

$$(1.2) \quad \rho_s(n) = \rho\left(\frac{n}{2}\right) + 1.$$

Adams, Lax and Phillips [2] showed that the maximal dimension of an n -subspace of $S_n(\mathbb{R})$ is exactly $\rho_s(n)$. Friedland, Robbin and Sylvester [8] and Berger and Friedland [5] gave further nonlinear versions of the above results by considering odd maps ϕ from S^p to matrices of rank n in $M_n(\mathbb{R})$, $S_n(\mathbb{R})$ and $M_{n,n+1}(\mathbb{R})$ respectively. In this paper (§4) we generalize these results to odd maps from S^p to rank $n - 1$ matrices in $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$.

The main motivation of this paper is the following quantity studied in [7]. For an integer k , such that $1 \leq k \leq n - 1$, let $d(n, k, \mathbb{F})$ be the smallest integer ℓ such that every ℓ dimensional subspace of $S_n(\mathbb{F})$ contains a nonzero matrix whose rank is at most k . Note that it is clear that the maximal dimension of a subspace of $S_n(\mathbb{F})$ of type (c) above is exactly $d(n, k - 1, \mathbb{F}) - 1$. It is our purpose to continue the study of $d(n, k, \mathbb{R})$ started in [7]. Note that any $d(n, k, \mathbb{R}) - 1$ dimensional subspace of $S_n(\mathbb{R})$ contains a nonzero matrix with an eigenvalue of multiplicity at least $n - k$. (One of the main results of [8] was that any subspace of $S_n(\mathbb{R})$ of dimension $\sigma(n) + 1$, where

$$\begin{aligned} \sigma(n) &= 2 \text{ if } n \not\equiv 0, \pm 1 \pmod{8}, \\ \sigma(n) &= \rho(4b) \text{ if } n = 8b, 8b \pm 1, \end{aligned}$$

contains a nonzero matrix with a multiple eigenvalue.) The equality

$$(1.3) \quad d(n, k, \mathbb{C}) = \binom{n - k + 1}{2} + 1,$$

established in [7] is derived straightforwardly from the following dimension computations. Let $V_{k,n}(\mathbb{C})$ ($V_{k,n}(\mathbb{R})$) be the variety of all matrices in $S_n(\mathbb{C})$ ($S_n(\mathbb{R})$) of rank k or less. Then in the projective space $\mathbb{P}S_n(\mathbb{C})$ the projective variety $\mathbb{P}V_{k,n}(\mathbb{C})$ is an irreducible variety of codimension $d(n, k, \mathbb{C}) - 1$, which yields (1.3). In particular $d(n, n - 1, \mathbb{C}) = 2$. The results of Adams, Lax and Phillips, cited above, yield that $d(n, n - 1, \mathbb{R}) = \rho_s(n) + 1$. This shows that in general the computation of $d(n, k, \mathbb{R})$ is much more difficult than the computation of $d(n, k, \mathbb{C})$. In [7] we gave a simple condition on n when $d(n, k, \mathbb{C}) = d(n, k, \mathbb{R})$ for $k \leq n - 2$, which trivially holds for

$k = n - 1$. It was shown by Harris and Tu [10] that the degree of the variety $\mathbb{P}V_{k,n}(\mathbb{C})$ is given by the formula

$$(1.4) \quad \delta_{k,n} := \deg \mathbb{P}V_{k,n}(\mathbb{C}) = \prod_{j=0}^{n-k-1} \frac{\binom{n+j}{n-k-j}}{\binom{2j+1}{j}}.$$

Then $d(n, k, \mathbb{C}) = d(n, k, \mathbb{R})$ if $\delta_{k,n}$ is odd. (In this case the result that any $d(n, k, \mathbb{R}) - 1$ dimensional subspace of $S_n(\mathbb{R})$ contains a nonzero matrix with an eigenvalue of multiplicity at least $n - k$ is best possible.) We show that $\delta_{n-q,n}$ is odd if $n > q \geq 1$ and

$$(1.5) \quad n \equiv \pm q \pmod{2^{\lceil \log_2 2q \rceil}}.$$

In particular, under the above conditions,

$$(1.6) \quad d(n, n - q, \mathbb{C}) = d(n, n - q, \mathbb{R}) = \binom{q + 1}{2} + 1.$$

If $n \geq q$ and n satisfies (1.5) then any $\binom{q+1}{2}$ subspace of $S_n(\mathbb{R})$ contains a nonzero matrix with an eigenvalue of multiplicity at least q . This statement for $q = 2$ yields the original Lax’s result [12] that any 3 dimensional subspace of $S_n(\mathbb{R})$ contains a nonzero matrix with a multiple eigenvalue for $n \equiv 2 \pmod{4}$. (This result and its generalization in [8] is of importance in the study of singularities of hyperbolic systems.)

An important part of the paper is devoted to the study of the numbers $d(n, n - 2, \mathbb{R})$. Besides the cases given in (1.5-1.6) for $q = 2$ it is easy to see that $d(3, 1, \mathbb{R}) = 6$ [7]. (Hence the inequality $d(n, n - 2, \mathbb{R}) \geq \sigma(n) + 2$ established in [7] is not sharp for some n .) Partial results regarding $d(n, n - 2, \mathbb{R})$ were obtained in [7]. In particular, it was shown that for $m \geq 1$

$$(1.7) \quad d(4m, 4m - 2, \mathbb{R}) \leq 4m + 1.$$

We obtain here additional results on $d(n, n - 2, \mathbb{R})$. In particular, using numerical and symbolic computations we show that $d(4, 2, \mathbb{R}) = 5$, which implies that the upper bound in (1.7) is sharp for $m = 1$.

The paper is organized as follows: In Section 2 we show that $d(4, 2, \mathbb{R}) = 5$. In Section 3 we obtain some upper bounds and some monotonicity properties of $d(n, n - 2, \mathbb{R})$. In Section 4 we consider the existence of continuous and smooth odd maps from the sphere S^p to matrices of rank $n - 1$ in $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$. As a consequence we obtain new upper bounds for the dimension of $(n - 1)$ -subspaces of $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$. These upper bounds are shown to be sharp in many cases. In Section 5 we discuss the existence of subspaces $U \subset S_{2n-1}(\mathbb{R})$ of dimension $2n$, which do not contain a nonzero matrix of rank less than $2n - 2$. The last section is devoted to remarks and conjectures.

2. Computation of $d(4, 2, \mathbb{R})$.

As indicated in the introduction, the first unknown number in the sequence $\{d(n, n-2, \mathbb{R})\}_{n=3}^{\infty}$ is $d(4, 2, \mathbb{R})$. By (1.7) $d(4, 2, \mathbb{R}) \leq 5$. It is our purpose to prove:

Theorem 2.1. $d(4, 2, \mathbb{R}) = 5$.

Proof. It suffices to exhibit a 4-dimensional subspace L of $S_4(\mathbb{R})$ with the property that every $0 \neq A \in L$ has rank 3 or 4.

Let L be the subspace of $S_4(\mathbb{R})$ spanned by the matrices

$$B_1 = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

It is straightforward to check that $\dim L = 4$. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be indeterminates and let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, and $B(\alpha) = \sum_{i=1}^4 \alpha_i B_i$.

We show that if $0 \neq \alpha \in \mathbb{R}^4$ then $\text{rank } B(\alpha) \geq 3$. So let $d(\alpha) = \det B(\alpha)$, and for $1 \leq j \leq 4$ let $d_j(\alpha)$ denote the determinant of the principal submatrix of $B(\alpha)$ obtained by deleting row and column j . We let Maple compute the common zeros of the 5 polynomials $d(\alpha), d_1(\alpha), d_2(\alpha), d_3(\alpha), d_4(\alpha)$ in \mathbb{C}^4 .

It turns out that there are no common zeros with $\alpha_1 = 0$ except the trivial solution $\alpha = 0$. Now we can assume $\alpha_1 = 1$. We list the ten solutions found by Maple. They are:

- (i) The two solutions given by

$$\alpha = (1, 0, 1 + x, x),$$

where x is a zero of the polynomial $f(z) = z^2 + z + 1$.

- (ii) The two solutions given by

$$\alpha = (1, x, 2x - 1, 3x - 2),$$

where x is a zero of the polynomial $f(z) = 7z^2 - 8z + 3$.

- (iii) The two solutions given by

$$\alpha = (1, x, -x, -5x - 1),$$

where x is a zero of the polynomial $f(z) = 14z^2 + 4z + 1$.

(iv) The four solutions given by

$$\alpha = \left(1, x, \frac{1}{2}x^2 + x + \frac{1}{2}, -\frac{1}{2}x^2 - x - \frac{1}{2} \right),$$

where x is a zero of the polynomial

$$f(z) = z^4 + 4z^3 + 8z^2 + 4z + 3 = (z + 1)^4 + 2z^2 + 2.$$

It follows that for any $0 \neq \alpha \in \mathbb{R}^4$ either $d(\alpha) \neq 0$ or $\exists 1 \leq j \leq 4$ such that $d_j(\alpha) \neq 0$. Thus, the rank of every nonzero matrix in L is at least 3.

We explain now why Maple did indeed give us *all* the common zeros of $d(\alpha), d_1(\alpha), d_2(\alpha), d_3(\alpha), d_4(\alpha)$. Let L_1 be the subspace of $S_4(\mathbb{C})$ spanned by B_1, B_2, B_3, B_4 . We have by (1.3)

$$d(4, 2, \mathbb{C}) = 4.$$

Moreover, by (1.4) we get

$$(2.1) \quad \deg \mathbb{P}(V_{2,4}(\mathbb{C})) = \frac{\binom{4}{2} \binom{5}{1}}{3} = 10.$$

We also use the known fact that a rank r symmetric matrix with entries in a field has a nonsingular $r \times r$ principal submatrix. Thus, the ten distinct solutions found by Maple yield ten matrices in L_1 whose rank is at most 2, and no two of those matrices lie on the same line. *In fact, a computation showed that each of those ten matrices has rank 2.*

It remains to explain why Maple did not omit other solutions, assuring us it did not miss any real solutions in particular. It follows from (2.1) that if L' is a four dimensional generic subspace of $S_4(\mathbb{C})$ then $\mathbb{P}(V_{2,4}(\mathbb{C}))$ meets $\mathbb{P}(L')$ in at most ten distinct points. Hence, we have to check that L_1 is indeed generic. It suffices to show that $\mathbb{P}(V_{2,4}(\mathbb{C}))$ and $\mathbb{P}(L_1)$ intersect transversally. So let A denote any of the ten matrices in L_1 of rank 2 found by Maple. Let $x, y \in \mathbb{C}^4$ be linearly independent vectors such that $Ax = Ay = 0$. A computation showed that the rank of

$$\begin{bmatrix} x^t B_2 x & x^t B_3 x & x^t B_4 x \\ y^t B_2 y & y^t B_3 y & y^t B_4 y \\ x^t B_2 y & x^t B_3 y & x^t B_4 y \end{bmatrix} \in M_3(\mathbb{C})$$

is 3. This implies that the linear system of three equations

$$\begin{aligned} x^t B x &= 0 \\ y^t B y &= 0 \\ x^t B y &= 0, \end{aligned}$$

has no solution B in L_1 such that A, B are linearly independent. This shows that L_1 is generic.

In addition to the computation using Maple, a numerical procedure (using Matlab) was performed to check that there are no nonzero matrices of rank 1 or 2 in L .

3. Upper bounds and monotonicity properties of $d(n, n - 2, \mathbb{R})$.

It follows from (1.5) and (1.6) that if $n \equiv 2 \pmod{4}$ then $d(n, n - 2, \mathbb{R}) = d(n, n - 2, \mathbb{C}) = 4$. It was shown in [7] that

$$(3.1) \quad d(n, n - 2, \mathbb{R}) \leq 7 \quad \text{for } n \equiv 4 \pmod{8}.$$

It is our purpose here to get some additional upper bounds and some monotonicity properties of $d(n, n - 2, \mathbb{R})$.

Proposition 3.1. *Let $n \equiv 3, 5 \pmod{8}$. Then $d(n, n - 2, \mathbb{R}) \leq 7$.*

Proof. It follows from (1.4) that for every $n \geq 3$

$$\deg \mathbb{P}(V_{n-3,n}(\mathbb{C})) = \frac{\binom{n}{3} \binom{n+1}{2} \binom{n+2}{1}}{1 \cdot 3 \cdot 10} = \frac{(n+2)(n+1)n^2(n-1)(n-2)}{2^3 \cdot 3^2 \cdot 5},$$

and this is odd for $n \equiv 3, 5 \pmod{8}$. So, as indicated in the introduction, and using (1.3) we have for $n \equiv 3, 5 \pmod{8}$

$$d(n, n - 3, \mathbb{R}) = d(n, n - 3, \mathbb{C}) = \binom{4}{2} + 1 = 7,$$

and since $d(n, n - 2, \mathbb{R}) \leq d(n, n - 3, \mathbb{R})$, the proposition follows. \square

It follows from [7] that $d(n, n - 2, \mathbb{R}) \geq 4$ for $n \equiv 3, 4, 5 \pmod{8}$. Thus, we have the exact value for $d(n, n - 2, \mathbb{R})$ whenever $n \equiv 2, 6 \pmod{8}$, and good upper and lower bounds whenever $n \equiv 3, 4, 5 \pmod{8}$.

Proposition 3.2. *Let k be a fixed integer such that $k \in \{3, 4, 5, 11, 12, 13\}$. Then the sequence $\{d(16m + k, 16m + k - 2, \mathbb{R})\}_{m=0}^{\infty}$ is a (weakly) monotone increasing sequence, bounded above by 7.*

There exists $M = M(k)$ such that

$$d(16m + k, 16m + k - 2, \mathbb{R}) = d(16M + k, 16M + k - 2, \mathbb{R}) \quad \text{for all } m \geq M.$$

Proof. It suffices to prove the monotonicity of the given sequence, because the required boundedness follows from (3.1) and Proposition 3.1.

Let L be any 8-dimensional 8-subspace of $M_8(\mathbb{R})$. Such a subspace exists because $\rho(8) = 8$ by (1.1). Let

$$L_1 = \left\{ \left[\begin{array}{cc} 0 & A \\ A^t & 0 \end{array} \right], A \in L \right\}.$$

Then L_1 is an 8-dimensional subspace of $S_{16}(\mathbb{R})$. Note that every nonzero matrix in L_1 is nonsingular. Let $\{B_i\}_{i=1}^8$ be a basis of L_1 .

Let $d_{m,k} = d(16m + k, 16m + k - 2, \mathbb{R})$ and $d_0 = d_{m,k} - 1$. Let $W_{m,k}$ be any d_0 -dimensional subspace of $S_{16m+k}(\mathbb{R})$ such that the rank of any

nonzero matrix in $W_{m,k}$ is at least $16m + k - 1$. Note that $d_0 \leq 7 - 1 = 6$. Let $\{C_i\}_{i=1}^{d_0}$ be a basis of $W_{m,k}$. Let $\widetilde{W}_{m,k} = \text{span}\{C_i \oplus B_i : 1 \leq i \leq d_0\}$. Then $\widetilde{W}_{m,k}$ is a d_0 -dimensional subspace of $S_{16(m+1)+k}(\mathbb{R})$, and clearly every nonzero matrix in it has rank $\geq 16 + 16m + k - 1 = 16(m + 1) + k - 1$. This shows that

$$\begin{aligned} d(16(m + 1) + k, 16(m + 1) + k - 2, \mathbb{R}) &\geq d_0 + 1 \\ &= d_{m,k} = d(16m + k, 16m + k - 2, \mathbb{R}). \end{aligned}$$

□

The following proposition justifies some claims we made in the introduction.

Proposition 3.3. *Let $1 \leq k \leq n - 2$. Then any $d(n, k, \mathbb{R}) - 1$ dimensional subspace of $S_n(\mathbb{R})$ contains a nonzero matrix which has an eigenvalue of multiplicity at least $n - k$. If $d(n, k, \mathbb{R}) = d(n, k, \mathbb{C})$ then this result is sharp.*

Proof. Let V be a $d(n, k, \mathbb{R}) - 1$ dimensional subspace of $S_n(\mathbb{R})$. If $I_n \in V$ then 1 is the eigenvalue of I_n of multiplicity $n \geq n - k$. Suppose that $I_n \notin V$. Let $\hat{V} = \text{span}\{V, I_n\}$. Then \hat{V} contains $0 \neq \hat{A}$ such that $\text{rank } \hat{A} \leq k$. $\hat{A} = A + aI, 0 \neq A \in V$, so $-a$ is an eigenvalue of A of multiplicity at least $n - k$.

Let $\Sigma_{n-k} \subset S_n(\mathbb{R})$ be the variety of all $A \in S_n(\mathbb{R})$ such that A has an eigenvalue of multiplicity at least $n - k$. Clearly

$$\Sigma_{n-k} = \{B \in S_n(\mathbb{R}) : B = A + aI_n \text{ for some } A \in V_{k,n}(\mathbb{R}) \text{ and } a \in \mathbb{R}\}.$$

Since $V_{k,n}(\mathbb{R})$ is an irreducible variety of codimension $d(n, k, \mathbb{C}) - 1$ it follows that Σ_{n-k} is an irreducible variety of codimension $d(n, k, \mathbb{C}) - 2$. Hence there exists an $d(n, k, \mathbb{C}) - 2$ dimensional subspace $W \subset S_n(\mathbb{R})$ such that $\Sigma_{n-k} \cap W = \{0\}$.

4. Upper bounds for the dimension of $(n - 1)$ -subspaces of $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$.

In this section we obtain upper bounds for the dimension of $(n - 1)$ -subspaces of $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$. This is done by considering certain odd, smooth maps from S^p into $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$. We also need several known results.

Let $M_n^{(k)}(\mathbb{R}) \left(S_n^{(k)}(\mathbb{R}) \right)$ denote the set of all rank k matrices in $M_n(\mathbb{R})(S_n(\mathbb{R}))$. Given any field \mathbb{F} and positive integers m, n , let $M_{m,n}^0(\mathbb{F})$ denote the set of all matrices in $M_{m,n}(\mathbb{F})$ whose rank is equal to $\min\{m, n\}$.

Lemma 4.1. *Let $C \in M_{n,n+1}^0(\mathbb{F})$. For $j = 1, 2, \dots, n + 1$, let $C^{(j)}$ denote the $n \times n$ matrix obtained from C by deleting its j -th column. Then the*

solution of the homogeneous linear system $Cy = 0$ is a line spanned by

$$(4.1) \quad y = \left(-\det C^{(1)}, \det C^{(2)}, -\det C^{(3)}, \dots, (-1)^{n+1} \det C^{(n+1)} \right)^t.$$

Proof. This is an easy consequence of well-known properties of the determinant. □

The next theorem appears as a part of Theorems A and F in [8]. See there how it is related to classical results due to Radon, Hurwitz, Adams and Adams-Lax-Phillips.

Theorem 4.1. *Let $\varphi : S^p \rightarrow M_n(\mathbb{R})$ be an odd continuous map, i.e., $\varphi(-\alpha) = -\varphi(\alpha) \quad \forall \alpha \in S^p$.*

- (i) *Suppose that $\varphi(S^p) \subset M_n^{(n)}(\mathbb{R}) = GL(n, \mathbb{R})$. Then, $p \leq \rho(n) - 1$.*
- (ii) *Suppose that $\varphi(S^p) \subset S_n^{(n)}(\mathbb{R})$. Then, we have $p = 0$ if n is odd, and $p \leq \rho(\frac{n}{2})$ if n is even.*

All the inequalities in (i) and (ii) are sharp.

The next theorem appears in [5].

Theorem 4.2. *Let $\varphi : S^p \rightarrow M_{n,n+1}^0(\mathbb{R})$ be an odd continuous map. Then $p \leq \max\{\rho(n) - 1, \rho(n + 1) - 1\}$, and this inequality is sharp.*

Theorem 4.3. *Let $n \geq 2$. Let $\varphi : S^p \rightarrow M_n^{(n-1)}(\mathbb{R})$ be n odd smooth map. Then, if $n \neq 3, 5, 9$*

$$(4.2) \quad p \leq \max\{\rho(n - 1) - 1, \rho(n) - 1, \rho(n + 1) - 1, 2\}.$$

Furthermore, if $\varphi(S^p) \subset S_n^{(n-1)}(\mathbb{R})$ then $p = 0$ if n is even, and

$$(4.3) \quad p \leq \max \left\{ \rho \left(\frac{n - 1}{2} \right), \rho(n + 1) - 1 \right\} \quad \text{if } n \text{ is odd.}$$

Proof. We prove first (4.2). If $p \leq 1$ (4.2) certainly holds, so we may assume $p \geq 2$. Hence S^p is simply connected. Therefore, we can choose $x(\alpha) \in S^{n-1}$ in a well-defined and continuous way such that

$$(4.4) \quad x^t(\alpha)\varphi(\alpha) = 0 \quad \forall \alpha \in S^p.$$

For $\alpha \in S^p$ let $B(\alpha)$ be the matrix in $M_{n,n+1}(\mathbb{R})$ obtained from $\varphi(\alpha)$ by augmenting it with the column vector $x(\alpha)$, i.e.,

$$B(\alpha) = (\varphi(\alpha), x(\alpha)) \in M_{n,n+1}(\mathbb{R}).$$

Then $\text{rank } \varphi(\alpha) = n - 1$ and (4.4) imply that $B(\alpha) \in M_{n,n+1}^0(\mathbb{R})$ for all $\alpha \in S^p$.

Note that the smoothness of φ implies that $x(\alpha)$ is also smooth. It is also clear that for each $\alpha \in S^p$ $x(-\alpha) = \pm x(\alpha)$. Hence, we must have $x(-\alpha) = -x(\alpha)$ for all $\alpha \in S^p$, or $x(-\alpha) = x(\alpha)$ for all $\alpha \in S^p$.

Case 1. Suppose that $x(-\alpha) = -x(\alpha)$ for all $\alpha \in S^p$. Then the map from S^p to $M_{n,n+1}^0(\mathbb{R})$ defined by $\alpha \rightarrow B(\alpha)$ is an odd continuous map, so by Theorem 4.2 we have

$$(4.5) \quad p \leq \max\{\rho(n) - 1, \rho(n + 1) - 1\},$$

and (4.2) holds.

Case 2. We now assume $x(-\alpha) = x(\alpha)$ for all $\alpha \in S^p$. We apply Lemma 4.1 to $B(\alpha) \in M_{n,n+1}^0(\mathbb{R})$, for each $\alpha \in S^p$, and denote the normalized solution given by (4.1) (that is, $\frac{1}{\|y\|}y$) by $\psi(\alpha)$. Let

$$\psi(\alpha) = (\psi_1(\alpha), \psi_2(\alpha), \dots, \psi_n(\alpha), \psi_{n+1}(\alpha)).$$

There are two subcases now.

Case 2a. Suppose that n is even. It follows from (4.1) that

$$\psi(-\alpha) = (-\psi_1(\alpha), -\psi_2(\alpha), \dots, -\psi_n(\alpha), \psi_{n+1}(\alpha)).$$

We define

$$C(\alpha) = \begin{bmatrix} B(\alpha) \\ \psi(\alpha) \end{bmatrix} \in M_{n+1}(\mathbb{R}).$$

This matrix is nonsingular because $\psi(\alpha)$ is orthogonal to all the rows of $B(\alpha)$. Let $\eta(\alpha) = (\psi_1(\alpha), \psi_2(\alpha), \dots, \psi_n(\alpha))$ and let

$$D(\alpha) = \begin{bmatrix} \varphi(\alpha) \\ \eta(\alpha) \end{bmatrix} \in M_{n+1,n}(\mathbb{R}),$$

that is, $D(\alpha)$ is the matrix obtained from $C(\alpha)$ by deleting its last column. Its rank is n for each $\alpha \in S^p$. Since $D(-\alpha) = -D(\alpha)$ for all $\alpha \in S^p$ we apply Theorem 4.2 again and conclude that (4.5) holds, so (4.2) holds.

Case 2b. Suppose that n is odd. So $n \geq 3$. Suppose first that $p < n - 1$. Since $x(\alpha)$ is smooth, it follows from Sard's theorem that $\{x(\alpha) : \alpha \in S^p\}$ has measure zero in S^{n-1} . In particular, $\exists \xi \in S^{n-1}$ such that $x(\alpha) \neq \pm \xi$ for all $\alpha \in S^p$.

For every $\alpha \in S^p$ let $L_\alpha = \text{span}\{x(\alpha), \xi\}$. Let $Q(\alpha)$ be the unique $n \times n$ orthogonal matrix satisfying: $Q(\alpha)$ is the identity on L_α^\perp and its restriction to L_α is the rotation in that plane by an angle $< \pi$ that sends $x(\alpha)$ to ξ . Since $x(-\alpha) = x(\alpha)$ we have $Q(-\alpha) = Q(\alpha)$. Also, $Q(\alpha)$ is continuous in α (cf. the Proposition in [8]). It follows from (4.4) that

$$(4.6) \quad \xi^t(Q^t(\alpha)\varphi(\alpha)Q(\alpha)) = 0, \quad \text{for all } \alpha \in S^p.$$

Let $\varphi_1(\alpha) = Q^t(\alpha)\varphi(\alpha)Q(\alpha)$. Then $\varphi_1(\alpha)$ is an odd continuous function. Without loss of generality we may assume $\xi^t = (0, 0, \dots, 0, 1)$. It follows from (4.6) that for each $\alpha \in S^p$ the last row of $\varphi_1(\alpha)$ is 0, so deleting it

results in an $(n - 1) \times n$ matrix of rank $n - 1$. Applying Theorem 4.2 again we conclude that

$$p \leq \max\{\rho(n - 1) - 1, \rho(n) - 1\}$$

so (4.2) holds.

Now suppose that $p \geq n - 1$. We may consider S^{n-2} as contained in S^p , and then consider the restriction of the given $\varphi(\alpha)$ and $x(\alpha)$ to S^{n-2} . We repeat the proof given for $p < n - 1$ and conclude in the same way that

$$n - 2 \leq \max\{\rho(n - 1) - 1, \rho(n) - 1\} = \rho(n - 1) - 1.$$

Since this cannot happen by (1.1) when $n \neq 3, 5, 9$ we have completed the proof of (4.2).

Suppose now that $\varphi(S^p) \subset S_n^{(n-1)}(\mathbb{R})$. Recall that the *inertia* of $A \in S_n(\mathbb{R})$ is the triple (π, ν, δ) , where π is the number of positive eigenvalues of A , ν is the number of negative eigenvalues of A and δ is the number of zero eigenvalues of A .

Let n be even and suppose that $p > 0$. Let $\alpha \in S^p$, and consider a path in S^p from α to $-\alpha$. Let \mathcal{J} denote the image of this path under φ . Since every matrix in \mathcal{J} has rank $n - 1$, it follows that the inertia of each matrix on this path is equal to inertia $(\varphi(\alpha))$. In particular, inertia $(\varphi(-\alpha)) = \text{inertia}(\varphi(\alpha))$, but this is impossible since $\varphi(-\alpha) = -\varphi(\alpha)$. Hence $p = 0$.

Let n be odd. Suppose first that $n = 3$. We show that $p \leq 1$ in this case, so (4.3) holds. Suppose to the contrary that $p \geq 2$. It is clear that $\varphi(\alpha)$ has the same inertia for each $\alpha \in S^p$. In particular, given any $\alpha \in S^p$, $\varphi(\alpha)$ and $\varphi(-\alpha) = -\varphi(\alpha)$ have the same inertia. Hence each $\varphi(\alpha)$ has one positive, one negative and one zero eigenvalue. So for each $\alpha \in S^p$, the eigenvalues of $\varphi(\alpha)$ are pairwise distinct, contradicting Theorem B of [8].

Note that while we have shown that there is no odd continuous map from S^2 to $S_3^{(2)}(\mathbb{R})$, there is an odd continuous map from S^2 to $M_3^{(2)}(\mathbb{R})$. For example, consider the map that sends $(x, y, z) \in S^2$ to

$$\begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}.$$

We assume now $n \geq 5$. Since $\max\{\rho(\frac{n-1}{2}), \rho(n + 1) - 1\} \geq 2$, (4.3) holds if $p \leq 1$. Hence we may assume that $p \geq 2$. We go along the proof of the nonsymmetric part of the theorem. If $x(\alpha)$ is odd, then (4.5) holds, and since $\rho(n) = 1$ for odd n , we get

$$p \leq \rho(n + 1) - 1 \leq \max\left\{\rho\left(\frac{n - 1}{2}\right), \rho(n + 1) - 1\right\},$$

and (4.3) holds.

So we may assume that $x(\alpha)$ is even. Suppose also that $p < n - 1$. Then we get (4.6) again, where $\varphi_1(\alpha) = Q^t(\alpha)\varphi(\alpha)Q(\alpha)$ is an odd continuous function, and we may assume $\xi^t = (0, 0, \dots, 0, 1)$. It follows that for each $\alpha \in S^p$ the matrix obtained from $\varphi_1(\alpha)$ by deleting its last row and column is in $S_{n-1}^{(n-1)}(\mathbb{R})$. Hence, by part (ii) of Theorem 4.1 we get

$$p \leq \rho\left(\frac{n-1}{2}\right),$$

so (4.3) holds.

If $p \geq n - 1$ we consider S^{n-2} as contained in S^p , and then consider the restriction of $\varphi(\alpha), x(\alpha)$ to S^{n-2} . A repetition of the argument for $p < n - 1$ yields $n - 2 \leq \rho(\frac{n-1}{2})$, which is impossible for n odd, $n \geq 5$. This completes the proof. \square

Corollary 4.1. *Let $n \geq 2$ and let L be an $(n - 1)$ -subspace of $M_n(\mathbb{R})$.*

(a) *If $n \neq 3, 5, 9$, then*

$$(4.7) \quad \dim L \leq \max\{\rho(n - 1), \rho(n), \rho(n + 1), 3\}.$$

(b) *If $L \subset S_n(\mathbb{R})$ (that is, L is a subspace of $S_n(\mathbb{R})$), then*

$$(4.8) \quad \dim L \leq \begin{cases} 1 & \text{if } n \text{ is even,} \\ \max\{\rho(\frac{n-1}{2}) + 1, \rho(n + 1)\} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. (a) Let $d = \dim L$ and let A_1, A_2, \dots, A_d be a basis of V . For $A \in V$ we write $A = \sum_{i=1}^d \alpha_i A_i$ and let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$. Then we may view the map $\alpha \rightarrow A$, restricted to α such that $\|\alpha\| = 1$, as an odd smooth map from S^{d-1} to $M_n^{(n-1)}(\mathbb{R})$. Then $d - 1$ is bounded by the right-hand side of (4.2), so (4.7) follows.

(b) The proof is similar to the previous case, using now (4.3) instead of (4.2). \square

Consider next the sharpness of the inequalities (4.2), (4.3), (4.7) and (4.8). For that purpose we have the following lemma.

Lemma 4.2.

- (i) *There exists a $\rho(n - 1)$ dimensional $(n - 1)$ -subspace V of $M_n(\mathbb{R})$.*
- (ii) *There exists an $(n - 1)$ -subspace V of $S_n(\mathbb{R})$ such that*

$$\dim V = \begin{cases} 1 & \text{if } n \text{ is even,} \\ \rho(\frac{n-1}{2}) + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. (i) As indicated in the introduction, there exists an $\rho(n - 1)$ dimensional $(n - 1)$ -subspace V_1 of $M_{n-1}(\mathbb{R})$. Then we let V be the subspace of $M_n(\mathbb{R})$ obtained from V_1 by appending a row and column of 0's to each matrix in V_1 .

(ii) The claim is trivial if n is even, so suppose n is odd. As indicated in the introduction, there exists an $(n-1)$ -subspace V_1 of $S_{n-1}(\mathbb{R})$ of dimension $\rho\left(\frac{n-1}{2}\right) + 1$. Now V is obtained from V_1 as in Part (i). \square

Corollary 4.2. *Let $n \equiv 1 \pmod{4}$. Then the inequalities (4.3) and (4.8) are sharp, and the inequalities (4.2) and (4.7) are sharp provided that $n \neq 5, 9$.*

Proof. For $n \equiv 1 \pmod{4}$ the maximum of the right-hand sides of (4.2) and (4.7) are $\rho(n-1) - 1$ and $\rho(n-1)$, respectively. Let V be the $\rho(n-1)$ dimensional $(n-1)$ -subspace defined in the Proof of Lemma 4.2, Part (i). This shows the sharpness of (4.7) provided that $n \neq 5, 9$. The sharpness of (4.2) is obtained by considering an odd smooth map from $S^{\rho(n-1)-1}$ to $M_n^{(n-1)}(\mathbb{R})$ as in the proof of Corollary 4.1 Part (a).

The sharpness of (4.3) and (4.8) is proved similarly, using the $\rho\left(\frac{n-1}{2}\right) + 1$ dimensional $(n-1)$ -subspace V of $S_n(\mathbb{R})$ in Part (ii) of Lemma 4.2. \square

Remark. In Theorem 4.3 it is possible to replace the assumption that φ is an odd smooth map by the assumption that φ is an odd continuous map.

The proof of the remark is achieved as follows: First approximate $\varphi(\alpha)$ arbitrarily by a smooth odd $\tilde{\varphi}(\alpha)$ (which is in $S_n(\mathbb{R})$ if $\varphi(S^p) \subset S_n^{(n-1)}(\mathbb{R})$). We can assume that for each $\alpha \in S^p$, $\tilde{\varphi}(\alpha)$ has a simple eigenvalue $\lambda(\alpha)$ which is the smallest in absolute value among all eigenvalues of $\tilde{\varphi}(\alpha)$ (and is also real if $\varphi(S^p) \subset S_n^{(n-1)}(\mathbb{R})$). Clearly $\lambda(-\alpha) = -\lambda(\alpha) \forall \alpha \in S^p$. Let $\psi(\alpha)$ be the projection of $\tilde{\varphi}(\alpha)$ corresponding to $\lambda(\alpha)$. Then $\psi(\alpha)$ is a smooth odd function of rank at most 1, and it follows that $\tilde{\varphi}(\alpha) - \psi(\alpha)$ satisfies the conditions of Theorem 4.3, so (4.2) and (4.3) hold respectively.

5. Existence of certain $2n$ dimensional subspaces of $S_{2n-1}(\mathbb{R})$.

Let $n \geq 4$ be an even integer. It follows from [7] and this paper that $d(n, n-2, \mathbb{R}) \leq n+1$, with equality for $n=4$. Now suppose that $n \geq 5$ is an odd integer. We do not know if a similar result holds, although it seems plausible. In this section we show that if there exists an $n+1$ dimensional subspace L of $S_n(\mathbb{R})$ such that each nonzero matrix in L has rank $n-1$ or n , then there is a severe restriction on the possible inertias attained in L . We also derive here additional results on $n+1$ dimensional subspaces of $n \times n$ matrices where each nonzero matrix has rank $n-1$ or n .

Let \mathbb{F} be any field and let V be an $n+1$ dimensional subspace of $M_n(\mathbb{F})$ spanned by A_1, A_2, \dots, A_{n+1} . Sometimes we find it convenient to identify $\mathbb{P}\mathbb{F}^n$ with \mathbb{F}^{n+1} and $\mathbb{P}V$ with $\mathbb{P}\mathbb{F}^n$. That is, we identify $\tilde{\alpha} \in \mathbb{P}\mathbb{F}^n$ and $\tilde{A} \in \tilde{V} := \mathbb{P}V$ with lines spanned by $0 \neq \alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n+1})^t$ and $0 \neq A$ in \mathbb{F}^{n+1} and V respectively. Let $A(\alpha) = \sum_{i=1}^{n+1} \alpha_i A_i$. We identify $\tilde{\alpha}$ with $\widetilde{A(\alpha)}$.

Given any $0 \neq y \in \mathbb{F}^n$, let

$$(5.1) \quad L_y = \{A \in V : Ay = 0\}.$$

Since $\dim V \geq n + 1$ it is clear that $\dim L_y \geq 1$ for every $y \neq 0$. Let

$$(5.2) \quad M(y) = [A_1y, A_2y, \dots, A_{n+1}y] \in M_{n,n+1}(\mathbb{F}).$$

For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n+1})^t$ it is clear that $(\sum_{i=1}^{n+1} \alpha_i A_i)y = 0$ if and only if $M(y)\alpha = 0$. It follows that

$$\dim L_y = 1 \text{ if and only if } \text{rank } M(y) = n.$$

Observe that if $\text{rank } M(y) = n$ then, by Lemma 4.1, $\ker M(y)$ is spanned by

$$(5.3) \quad \alpha(y) = \left(-\det M(y)^{(1)}, \det M(y)^{(2)}, \dots, (-1)^{n+1} \det M(y)^{(n+1)}\right)^t.$$

Let

$$(5.4) \quad T = \{\tilde{y} \in \mathbb{P}\mathbb{F}^{n-1} : \alpha(y) = 0\}.$$

Hence, T consists of those \tilde{y} for which $\text{rank } M(y) < n$.

Given $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n+1})$ we consider the homogeneous polynomial

$$\psi(\alpha) = \det \left(\sum_{i=1}^{n+1} \alpha_i A_i \right),$$

and let

$$(5.5) \quad \tilde{Z}(\psi) = \text{the zero set of } \psi(\alpha), \text{ considered as a subset of } \mathbb{P}\mathbb{F}^n.$$

Lemma 5.1. *Let V be an $n + 1$ dimensional subspace of $M_n(\mathbb{C})$, spanned by A_1, A_2, \dots, A_{n+1} , and such that:*

- (i) $\exists A \in V$ such that $\det A \neq 0$.
- (ii) $\exists B \in V$ such that $\text{rank } B = n - 1$ and $\ker B$ is spanned by a vector $0 \neq u \in \mathbb{C}^n$ with $\dim L_u = 1$.

Then the following holds:

Let $H \subset \mathbb{P}\mathbb{C}^n$ be the irreducible component of the hypersurface $\tilde{Z}(\psi)$ passing through \tilde{B} . Then there exists a birational map

$$\varphi : H \dashrightarrow \mathbb{P}\mathbb{C}^{n-1}, \quad \theta = \varphi^{-1} : \mathbb{P}\mathbb{C}^{n-1} \dashrightarrow H$$

defined as follows:

- (a) For each $\tilde{\alpha} \in H$ with $\text{rank } C = n - 1$, where $C = \sum_{i=1}^{n+1} \alpha_i A_i$, $\varphi(\tilde{\alpha}) = \tilde{y}$, where $y \in \mathbb{C}^n$ is the basis of $\ker C$.
- (b) For each $\tilde{y} \in \mathbb{P}\mathbb{C}^{n-1}$ such that $\dim L_y = 1$, $\theta(\tilde{y}) = \tilde{\alpha}$, where $\alpha = \alpha(y)$.

Proof. The maps φ and θ are rational. For y in the neighborhood of u , $\dim L_y = 1$. Hence θ is holomorphic in the neighborhood of \tilde{u} . Let $B = \sum_{i=1}^{n+1} \beta_i A_i$. The assumptions (ii) imply that φ is holomorphic in the neighborhood of $\tilde{\beta}$ and $\varphi \cdot \theta$ is the identity map (as a rational map). Hence

$$\theta : \mathbb{P}\mathbb{C}^{n-1} - - \rightarrow H.$$

□

Remark. As $\theta : \mathbb{P}\mathbb{C}^{n-1} - - \rightarrow H$ is a rational map, it is not holomorphic on a variety of codimension at least 2 (cf. [9, Chapter 4, Section 2]). Hence $\text{codim } T \geq 2$.

Lemma 5.2. *Let V be an $n + 1$ dimensional subspace of $M_n(\mathbb{R})$, such that each nonzero matrix in V has rank $n - 1$ or n . Then:*

- (i) $\exists A \in V$ such that $\det A \neq 0$.
- (ii) $\exists B \in V$ such that $\text{rank } B = n - 1$ and $\ker B$ is spanned by a vector $0 \neq y \in \mathbb{R}^n$ with $\dim L_y = 1$.

Proof. By Theorem 2 of [3], V cannot be an $(n - 1)$ -subspace or an n -subspace. Hence V contains nonsingular matrices and singular matrices. Let

$$(5.6) \quad \tilde{V}_s = \{ \tilde{A} \in \tilde{V} : \det A = 0 \}.$$

Let

$$\eta : \tilde{V}_s \rightarrow \mathbb{P}\mathbb{R}^{n-1}$$

be defined as follows:

Suppose that $B \in V$ is any singular matrix, and suppose $0 \neq y \in \mathbb{R}^n$ satisfies $By = 0$. Then let $\eta(\tilde{B}) = \tilde{y}$.

It is known that the set of singular points of \tilde{V}_s is a proper subvariety of \tilde{V}_s . As $\eta(\tilde{V}_s) = \mathbb{P}\mathbb{R}^{n-1}$, it follows that there exists a regular point \tilde{B} of \tilde{V}_s such that $D\eta(\tilde{B})$ has the rank $n - 1$. Let $y \in S^{n-1}$ be such that $By = 0$. Suppose that $\dim L_y \geq 2$. Then it is clear that $\dim \ker D\eta(\tilde{B}) \geq 1$.

It follows that the dimension of the tangent space to \tilde{B} in \tilde{V}_s has dimension $\geq n$, implying that $\dim \tilde{V}_s \geq n$, a contradiction. □

Lemma 5.3. *Let n be odd, $n \geq 3$, and let V be an $n + 1$ dimensional subspace of $S_n(\mathbb{R})$ such that each nonzero matrix in V has rank $n - 1$ or n . Suppose that $V^\perp = \{ B \in S_n(\mathbb{R}) : \text{tr}(AB) = 0 \forall A \in V \}$ does not contain rank one matrices. Let \tilde{V}_s be as in (5.6). Then, \tilde{V}_s is a smooth connected variety in $\mathbb{P}\mathbb{R}^n$.*

Proof. Let $A_k = \left(a_{ij}^{(k)} \right)_{i,j=1}^n$, $k = 1, 2, \dots, n + 1$ be a basis of V . Let $\psi(\alpha) = \det \left(\sum_{i=1}^{n+1} \alpha_i A_i \right)$. For $\tilde{\alpha} \in \tilde{V}_s$ we have $\psi(\alpha) = 0$, so $A(\alpha) = \sum_{i=1}^{n+1} \alpha_i A_i$ has

rank $n - 1$. Hence the $(n - 1)$ th compound of $A(\alpha), C_{n-1}(A(\alpha))$, has rank 1. So there exists $0 \neq x(\alpha) = (x_1(\alpha), x_2(\alpha), \dots, x_n(\alpha))^t$ and $\epsilon = \pm 1$ such that $C_{n-1}(A(\alpha)) = \epsilon x(\alpha)x^t(\alpha)$. Let $C_{n-1}(\alpha) = (c_{ij}(\alpha))_{i,j=1}^n$. Observe that

$$\frac{\partial \psi(\alpha)}{\partial \alpha_k} = \sum_{i,j=1}^n a_{ij}^{(k)} c_{ij}(\alpha) = \epsilon x^t(\alpha) A_k x(\alpha), \quad k = 1, 2, \dots, n + 1.$$

Since V^\perp does not contain a rank one matrix, we deduce that $\nabla \psi(\alpha) \neq 0$. Thus each $\tilde{\alpha} \in \tilde{V}_s$ is a smooth point and the local dimension of \tilde{V}_s at $\tilde{\alpha}$ is $n - 1$.

Let $V_{\mathbb{C}}$ be the complexification of V . Let $T_{\mathbb{C}}, T_{\mathbb{R}}$ be defined by (5.5) for $\mathbb{F} = \mathbb{C}, \mathbb{R}$, respectively. By the remark following Lemma 5.1, $\text{codim}_{\mathbb{C}} T_{\mathbb{C}} \geq 2$. Since $T_{\mathbb{R}} = T_{\mathbb{C}} \cap \mathbb{P}\mathbb{R}^{n-1}$, we have $\text{codim}_{\mathbb{R}} T_{\mathbb{R}} \geq 2$.

Let $Y_r = \mathbb{P}\mathbb{R}^{n-1} \setminus T_{\mathbb{R}}$. Then Y_r is connected. Using (ii) of Lemma 5.2 we can define H and θ as in Lemma 5.1. Let $H_{\mathbb{R}} = H \cap \mathbb{P}\mathbb{R}^n$. Then $\theta(Y_r)$ is connected in $H_{\mathbb{R}}$, and this implies that $\overline{\theta(Y_r)} = H_{\mathbb{R}}$ is connected.

To finish the proof it suffices to show that $\tilde{V}_s = H_{\mathbb{R}}$. Since $\tilde{Y}_r = \mathbb{P}\mathbb{R}^{n-1}$ it follows that for any $\tilde{y} \in \mathbb{P}\mathbb{R}^{n-1}$ there exists $\tilde{\beta} \in H_{\mathbb{R}}$ such that $A(\beta)y = 0$. Let $\tilde{\gamma}$ be an arbitrary point in \tilde{V}_s . Then $\exists 0 \neq u \in \mathbb{R}^n$ such that $A(\gamma)u = 0$. If $\dim L_u = 1$, then $\tilde{\gamma} \in H_{\mathbb{R}}$. So suppose $\dim L_u \geq 2$. Then $\exists \tilde{\beta} \in H_{\mathbb{R}}$ such that $A(\beta) \in L_u$. Hence $\tilde{\beta}, \tilde{\gamma} \in \tilde{L}_u$, and clearly $\tilde{\beta}$ and $\tilde{\gamma}$ are connected in $\tilde{L}_u \subset \tilde{V}_s$. As \tilde{V}_s is a smooth variety, it follows that $\tilde{\gamma} \in H_{\mathbb{R}}$. Hence $\tilde{V}_s = H_{\mathbb{R}}$. □

Recall that the standard inner product in $S_n(\mathbb{R})$ is given by $\langle A, B \rangle = \text{tr}(AB)$, so with respect to this inner product a matrix A is normalized if and only if $\text{tr}(A^2) = 1$.

Corollary 5.1. *Let n and V satisfy the assumptions of Lemma 5.3. Let*

$$V_\nu = \{A \in V : \text{tr}(A^2) = 1\}$$

and

$$V_{\nu s} = \{A \in V_\nu : \det A = 0\}.$$

Then $V_{\nu s}$ is a smooth connected hypersurface in V_ν .

Proof. Assume that the basis A_1, A_2, \dots, A_{n+1} is orthonormal with respect to the standard inner product in $S_n(\mathbb{R})$. Let $\tilde{V}_s, T_{\mathbb{R}}$ and Y_r be as in Lemma 5.3. Identify \tilde{V}_s with $V_{\nu s}/\{\pm I\}$. It follows from Lemma 5.3 that $V_{\nu s}$ is smooth. To show that $V_{\nu s}$ is connected, it suffices to show we can connect some $A \in V_{\nu s}$ to $-A$ in $V_{\nu s}$.

Take $y \in S^{n-1}$ such that $\tilde{y} \in Y_r$. Connect y to $-y$ by a path \mathcal{J} (in S^{n-1}) such that $\tilde{\mathcal{J}} \subset Y_r$. For $z \in S^{n-1}$ such that $\tilde{z} \notin T_{\mathbb{R}}$ let

$$\hat{\alpha}(z) = \frac{1}{\|\alpha(z)\|_2} \alpha(z),$$

where $\alpha(z)$ is given by (5.3). Since α is odd, $A(\hat{\alpha}(z)), z \in \mathcal{J}$, connects $A(\hat{\alpha}(y))$ to $A(\hat{\alpha}(-y)) = -A(\hat{\alpha}(y))$. \square

Theorem 5.1. *Let n be odd, $n \geq 3$, and let V be an $n + 1$ dimensional subspace of $S_n(\mathbb{R})$ such that each nonzero matrix in V has rank $n - 1$ or n . Then each nonzero matrix in V has at least $\frac{n-1}{2}$ positive and negative eigenvalues.*

Proof. Let A_1, A_2, \dots, A_{n+1} be a basis of V . We distinguish two cases.

Case 1. Suppose that V^\perp does not contain a matrix of rank one. Let V_{ν_s} be as in Corollary 5.1. Then, each matrix in V_{ν_s} has rank $n - 1$, and by the connectedness of V_{ν_s} it follows that A and $-A$ have the same inertia for each $A \in V_{\nu_s}$. It follows that every singular matrix in $V \setminus \{0\}$ has $\frac{n-1}{2}$ positive eigenvalues, $\frac{n-1}{2}$ negative eigenvalues, and 1 zero eigenvalue. Continuity yields immediately that each $0 \neq A \in V$ has at least $\frac{n-1}{2}$ positive and negative eigenvalues.

Case 2. Suppose now that V^\perp does contain a rank one matrix. Note that we have $\dim \mathbb{P}(V^\perp) = \frac{n(n+1)}{2} - (n + 1) - 1$, while $\dim \mathbb{P}V_{1,n}(\mathbb{R}) = n - 1$. Hence, for any $\epsilon > 0$ there exist $A_{1,\epsilon}, A_{2,\epsilon}, \dots, A_{n+1,\epsilon} \in S_n(\mathbb{R})$ which satisfy the following three conditions:

- (i) $\|A_{i,\epsilon} - A_i\| < \epsilon, i = 1, 2, \dots, n + 1$.
- (ii) $V_\epsilon = \text{span}\{A_{1,\epsilon}, A_{2,\epsilon}, \dots, A_{n+1,\epsilon}\}$ is $n + 1$ dimensional.
- (iii) V_ϵ^\perp does not contain a rank one matrix.

For $\epsilon > 0$ small enough, we can apply Case 1 and conclude that any nonzero matrix in V_ϵ has at least $\frac{n-1}{2}$ positive and negative eigenvalues. Let $\epsilon \rightarrow 0$ and use continuity to finish the proof. \square

6. Concluding remarks and conjectures.

Theorem 6.1. *Let $1 \leq q \in \mathbb{Z}$. Then $\deg \mathbb{P}V_{n-q,n}(\mathbb{C})$ is odd if $n > q$ and*

$$(6.1) \quad n \equiv \pm q \pmod{2^{\lceil \log_2 2q \rceil}}.$$

Proof. Let $0 \neq q \in \mathbb{Z}$. Let $\mu(q) = \lceil \log_2 2|q| \rceil$ and $0 \leq \nu(q) \in \mathbb{Z}$ be the largest integer such that $2^{\nu(q)}$ divides q . Then $\nu(q) + 1 \leq \mu(q)$. Clearly if $\ell \equiv m \not\equiv 0 \pmod{2^\mu}$ for some $1 \leq \mu \in \mathbb{Z}$ then $\nu(\ell) = \nu(m)$. Let $\delta_{k,n} := \deg \mathbb{P}V_{k,n}(\mathbb{C})$. We claim that for n satisfying (6.1)

$$(6.2) \quad \nu(\delta_{n-q,n}) = \sum_{j=0}^{q-1} \nu \left(\binom{\hat{q} + j}{q - j} \right) - \nu \left(\binom{2j + 1}{j} \right), \quad \hat{q} = \pm q.$$

Here for any real x and $0 \leq k \in \mathbb{Z}$ we let

$$\binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!}.$$

In particular

$$(6.3) \quad \binom{-x}{k} = (-1)^k \binom{x+k-1}{k}, \quad k = 1, \dots$$

Observe that for $j \leq q-1$

$$\binom{n+j}{q-j} = \frac{(n+j)(n+j-1)\dots(n-q+2j+1)}{1 \cdot 2 \dots (q-j)}.$$

For n satisfying (6.1) it follows that $n + \ell \equiv \pm q + \ell \not\equiv 0 \pmod{2^{\mu(q)}}$ if $-q + 1 \leq \ell \leq q - 1$. Hence (6.2) holds.

Our theorem is equivalent to the statement that $\nu(\delta_{n-q,n}) = 0$ if (6.1) holds. We first consider the case $n \equiv -q \pmod{2^{\mu(q)}}$. Then

$$\begin{aligned} \nu(\delta_{n-q,n}) &= \sum_{j=0}^{q-1} \nu\left(\binom{-q+j}{q-j}\right) - \nu\left(\binom{2j+1}{j}\right) \\ &= \sum_{j=0}^{q-1} \nu\left((-1)^{q-j} \binom{2(q-j)-1}{q-j}\right) - \sum_{j=0}^{q-1} \nu\left(\binom{2j+1}{j}\right) \\ &= \sum_{j=0}^{q-1} \nu\left(\binom{2(q-j)-1}{q-j-1}\right) - \sum_{j=0}^{q-1} \nu\left(\binom{2j+1}{j}\right) = 0. \end{aligned}$$

We now consider the case $n \equiv q \pmod{2^{\mu(q)}}$. In this case it is enough to show the identity

$$(6.4) \quad \prod_{j=0}^{q-1} \frac{\binom{q+j}{q-j}}{\binom{2j+1}{j}} = 1, \quad q = 1, \dots$$

We prove the above identity by induction on q . For $q = 1$ (6.4) trivially holds. Assume that (6.4) holds for $q = m \geq 1$. Let $q = m + 1$. Use the identity

$$\binom{m+1+j}{m+1-j} = \frac{m+1+j}{m+1-j} \binom{m+j}{m-j}, \quad j = 0, 1, \dots, m,$$

to deduce (6.4) for $q = m + 1$ from $q = m$. □

Corollary 6.1. *Let $n > q$ and let (6.1) hold. Then*

$$d(n, n - q, \mathbb{R}) = d(n, n - q, \mathbb{C}) = \binom{q+1}{2} + 1.$$

Corollary 6.2. *Let $n \geq q \geq 2$ and assume that (6.1) holds. Then any $\binom{q+1}{2}$ dimensional subspace U of $S_n(\mathbb{R})$ contains a nonzero matrix with an eigenvalue of multiplicity q at least. Furthermore, there exists an $\binom{q+1}{2} - 1$ dimensional subspace $V \subset S_n(\mathbb{R})$ such that each eigenvalue of $0 \neq A \in V$ has a multiplicity less than q .*

Proof. For $n = q$ $U = S_n(\mathbb{R})$ and the eigenvalue 1 of I_q has multiplicity q . Assume that $V \subset S_q(\mathbb{R})$ does not contain I_q . Then each eigenvalue of $0 \neq A \in V$ has multiplicity less than q . Assume that $q < n$. Then the claim of the corollary follows from Corollary 6.1 and Proposition 3.3. \square

Clearly $\delta_{n-1,n} = n$ is odd if and only if n is odd. As $d(n, n - 1, \mathbb{R}) = \rho_s(n) + 1$ it follows that $d(n, n - 1, \mathbb{R}) > d(n, n - 1, \mathbb{C}) = 2$ if n is even.

Conjecture 6.1. *Let $n > q \geq 2$ and assume that (6.1) does not hold. Then $\delta_{n-q,n}$ is even. Furthermore $d(n, n - q, \mathbb{R}) > d(n, n - q, \mathbb{C})$.*

For $q = 2, 3, 4, 5$ it is straightforward to check that $\delta_{n-q,n}$ is odd if and only if (6.1) holds.

Finally, we return to the sequence $\{d(n, n - 2, \mathbb{R})\}_{n=3}^\infty$ and state two questions. The first unknown number in this sequence is $d(5, 3, \mathbb{R})$. As we have seen in Section 3, $d(5, 2, \mathbb{R}) = 7$, so $d(5, 3, \mathbb{R}) \leq 7$.

Question 1. Is $d(5, 3, \mathbb{R}) = 6$?

We think that the answer is yes. In [6] it is shown that $d(5, 3, \mathbb{R}) \leq 6$. Let L be the 5 dimensional subspace of $S_5(\mathbb{R})$ spanned by the matrices

$$\begin{aligned}
 B_1 &= \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 \end{bmatrix}, & B_2 &= \begin{bmatrix} 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 \end{bmatrix}, \\
 B_3 &= \begin{bmatrix} -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{bmatrix}, & B_4 &= \begin{bmatrix} 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 \end{bmatrix}, \\
 B_5 &= \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 \end{bmatrix}.
 \end{aligned}$$

A local minimization procedure using Matlab seems to indicate that every nonzero matrix in L has rank 4 or 5. This has yet to be confirmed by other means, but if it is correct, then $d(5, 3, \mathbb{R}) = 6$.

Note that for every field \mathbb{F} we have

$$(6.5) \quad d(n, 1, \mathbb{F}) \geq d(n, 2, \mathbb{F}) \geq \cdots \geq d(n, n-2, \mathbb{F}) \geq d(n, n-1, \mathbb{F}).$$

Question 2. Is there strict inequality everywhere in (6.5) if $\mathbb{F} = \mathbb{R}$?

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