Pacific Journal of Mathematics

PHRAGMÈN–LINDELÖF THEOREM FOR MINIMAL SURFACE EQUATIONS IN HIGHER DIMENSIONS

Chun-Chung Hsieh, Jenn-Fang Hwang, and Fei-Tsen Liang

Volume 207 No. 1 November 2002

PACIFIC JOURNAL OF MATHEMATICS
Vol. 207, No. 1, 2002

PHRAGMÈN–LINDELÖF THEOREM FOR MINIMAL SURFACE EQUATIONS IN HIGHER DIMENSIONS

Chun-Chung Hsieh, Jenn-Fang Hwang, and Fei-Tsen Liang

Here we prove that if u satisfies the minimal surface equation in an unbounded domain which is properly contained in a half space of $\mathbb{R}^n$, with $n \geq 2$, then the growth rate of u is of the same order as that of the shape of Ω and the boundary value of u.

1. Introduction.

Consider the minimal surface equation

$$\operatorname{div} Tu = 0,$$

where

$$Tu = \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \qquad \text{and} \qquad \nabla u = (u_{x_1}, \ldots, u_{x_n}).$$

In 1965, Nitsche [**7**] announced the following result: "Let $\Omega_\alpha \subset \mathbb{R}^2$ be a sector with angle $0 < \alpha < \pi$. If u satisfies the minimal surface equation with vanishing boundary value in Ω_α, then $u \equiv 0$". Hwang extends this result in [**4**], [**5**], [**6**] and proves that, in an unbounded domain Ω properly contained in the half plane in $\mathbb{R}^2$, if u satisfies the minimal surface equation, then, the growth property of u is determined completely by the shape of Ω and the boundary value of u. In this respect, the Phragmèn-Lindelöf theorem for the minimal surface equation is better than that for the Laplace equation. (Indeed, if u satisfies the Laplace equation in an unbounded domain Ω, the growth property of u cannot be determined completely by the shape of Ω and the boundary data of u alone (cf. [**10**]).)

The purpose of this paper is to generalize the two-dimensional Phragmèn-Lindelöf theorems in [**4**], [**5**] and [**6**], to higher dimensions. In §2, we review the statements of the Phragmèn-Lindelöf theorem of [**4**], [**5**] and [**6**]. The higher-dimensional version is similar in content, but proof is different. In §3, based on an argument of [**2**], we established the suitable comparison principle. In §4, we compute the mean curvature of our comparison function, and use it to finish the proof of our main theorems in §5.

2. Preliminary.

The main purpose of this paper is to generalize the two-dimensional Phragmèn-Lindelöf theorem in [**4**], [**5**], [**6**] to higher dimensions. We may, first of all, recall some results in these papers and consider functions

$$f : [0, \infty) \to [0, \infty), \; f \in C^2([0, \infty)), \; f' \equiv \frac{df(y)}{dy} > 0,$$

from which we define

$$p(f) = 1 - \frac{ff''}{(f')^2}.$$

In particular, for $f(y) = y^m$, m being a positive constant, we have

$$p(f) = \frac{1}{m},$$

which is precisely the reciprocal of the order of f, while for $f(y) = e^y$, we have

$$p(f) = 0;$$

moreover, in case f grows faster than the exponential function, we can assume $p(f) \geq -\epsilon$ for some small positive constant ϵ, essentially (cf. [**5**, Remark 2.7]). Accordingly, we may proceed to solve the ordinary differential equation in $[-1, 1]$

$$(*) \qquad (1 - p(f))(h - th')(1 + {h'}^2) + h''(h^2 + t^2) = 0$$

with initial values

$$(**) \qquad h(-1) = 0 \quad \text{and} \quad h'(-1) = \tan\left((1 - p(f))\frac{\pi}{2}\right),$$

and then denote its solution, if exists, by h_m if $f(y) = y^m$ (and hence $p(f) = \frac{1}{m}$), and by h_∞ if $f(y) = e^y$ (and hence $p(f) = 0$). In general, $(*)$ and $(**)$ cannot be solved explicitly; but, for some specific m, its solution can be written out explicitly. For example, we have

$$h_2 = \frac{1 - t^2}{2},$$

and also

$$h_\infty = \sqrt{1 - t^2}.$$

It is useful to know some interesting properties of h_m, $0 < m \leq \infty$, in the following:

Lemma 1 ([**6**]). *For $1 < m$, $m' \leq \infty$ and $t \in (-1, 1)$, then we have*

$$\text{(i)} \qquad h_m(t) > h_{m'}(t), \qquad \textit{whenever} \quad m > m',$$

and

$$\text{(ii)} \qquad h_m(t) < h_m(t'), \qquad \textit{whenever} \quad |t| > |t'|.$$

The Phragmèn-Lindelöf theorems in [**5**], [**6**] can now be formulated as follows.

Theorem 2. *Let $\Omega \subseteq \{(x,y) \in \mathbb{R}^2 | \quad -ay^m < x < ay^m, y > 0\} \subseteq \mathbb{R}^2$ be an unbounded domain, where a and m are positive constants, $m \geq 1$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and suppose that*

$$\begin{cases} \operatorname{div} Tu \geq 0 & \text{in} \quad \Omega \\ u \leq ay^m h_m(\frac{x}{ay^m}) & \text{on} \quad \partial\Omega. \end{cases}$$

Then we have $u \leq ay^m h_m(\frac{x}{ay^m}) \leq ay^m h_\infty(\frac{x}{ay^m}) = \sqrt{a^2y^{2m} - x^2}$ in Ω.

Theorem 2*. *Let $\Omega \subseteq \{(x,y) \in \mathbb{R}^2 | \quad -ae^{by} < x < ae^{by}, y > 0\}$ be an unbounded domain where a, b are positive constants. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and suppose that*

$$\begin{cases} \operatorname{div} Tu \geq 0 & \text{in} \quad \Omega \\ u \leq \sqrt{a^2e^{2by} - x^2} & \text{on} \quad \partial\Omega. \end{cases}$$

Then we have $u \leq \sqrt{a^2e^{2by} - x^2}$ in Ω.

Theorem 3. *Let $f \in C^2([0,\infty)), f > 0, f' > 0$ in $(0,\infty)$ and $p(f) \geq p_0$, where p_0 is a negative constant, and let $f_1 \in C^0([0,\infty))$ and $f_1 > 0$ in $(0,\infty)$. For a given unbounded open domain*

$$\Omega \subset \{(x,y) \in \mathbb{R}^2 \mid -f_1(y) < x < f_1(y),\ y > 0\},$$

and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with

$$\begin{cases} \operatorname{div} Tu \geq 0 & \text{in} \quad \Omega \\ u \leq a\sqrt{f^2 - x^2} & \text{on} \quad \partial\Omega, \end{cases}$$

where $f^2 \geq \frac{(a^2-1)(2-p_0)}{(a^2-(1-p_0))} f_1^2$ and a is a positive constant satisfying

$$a^2 - 1 + p_0 > 0.$$

Then, we have

$$u \leq a\sqrt{f^2 - x^2} \qquad \text{in} \quad \Omega.$$

Remark. In Theorem 3, since $p_0 < 0$ and $a > 0$, we have

$$\frac{(a^2-1)(2-p_0)}{(a^2-(1-p_0))} = \left(\frac{a^2-1}{a^2-(1-p_0)}\right)(2-p_0) > 2.$$

Thus, in case $u \leq 0$ on $\partial\Omega$, our estimates are not good enough since we use worse boundary conditions, whereas the best estimates remain unknown.

These theorems will be generalized to higher dimensions in §5.

3. A comparison principle.

To establish the higher-dimensional Phragmèn-Lindelöf theorem, we shall need the following comparison principle.

Lemma 4. *Let Ω be an unbounded domain in $\mathbb{R}^n$, and let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Suppose that*

$$\begin{cases} \text{div } Tu - \text{div } Tv \geq C & \textit{in} \quad \Omega, \\ u \leq v & \textit{on} \quad \partial\Omega, \end{cases}$$

where C is a positive constant. Then we have $u \leq v$ in Ω.

Proof. The idea of proof is analogous to that of [**2**].

Suppose that this lemma fails to hold. There then exists a positive constant ϵ such that

$$\Omega' = \{x \in \Omega \mid \quad u(x) > v(x) + \epsilon\}$$

is not empty; by Sard's theorem, we may further assume that $\partial\Omega' \cap \Omega$ is smooth. For every $R > 0$, set

$$\begin{aligned} B_R &= \{x \in \mathbb{R}^n \mid \quad |x| < R\}, \\ \Omega_R &= B_R \cap \Omega', \\ \Gamma_R &= \partial B_R \cap \partial\Omega_R, \end{aligned}$$

and

$$|\Gamma_R| = \text{the Hausdorff } (n-1) - \text{dimensional measure of } \Gamma_R.$$

Also, let

$$\begin{aligned} g(R) &= \oint_{\partial\Omega_R} \tan^{-1}(u - v - \epsilon)(Tu - Tv) \cdot \nu \\ &= \int_{\Gamma_R} \tan^{-1}(u - v - \epsilon)(Tu - Tv) \cdot \nu \end{aligned} \tag{1}$$

where ν is the unit outward normal of $\partial\Omega_R$.

Then we have

$$\begin{aligned} g(R) = &\iint_{\Omega_R} \frac{(\nabla u - \nabla v) \cdot (Tu - Tv)}{1 + (u - v - \epsilon)^2} \\ &+ \iint_{\Omega_R} \tan^{-1}(u - v - \epsilon)(\text{div } Tu - \text{div } Tv). \end{aligned} \tag{2}$$

Since the integrand of the right-hand side of (1) is nonnegative, Fubini's theorem tells us that $g'(R)$ exists for almost all $R > 0$, and whenever it exists, we have, by (2),

$$
\begin{aligned}
(3)\qquad g'(R) &= \int_{\Gamma_R} \frac{(\nabla u - \nabla v)\cdot(Tu - Tv)}{1+(u-v-\epsilon)^2} \\
&\quad + \int_{\Gamma_R} \tan^{-1}(u-v-\epsilon)(\operatorname{div} Tu - \operatorname{div} Tv) \\
&\geq C \int_{\Gamma_R} \tan^{-1}(u-v-\epsilon), \qquad \text{(by assumption)} \\
&\geq \frac{C}{2} \int_{\Gamma_R} \tan^{-1}(u-v-\epsilon)\ |Tu - Tv|, \\
&\qquad\qquad (\text{since}\quad |Tu| < 1 \quad \text{and} \quad |Tv| < 1) \\
&\geq \frac{C}{2}\, g.
\end{aligned}
$$

Since g is an increasing function of R and $g \geq 0$, it is easy to see that Lemma 4 holds in the case that $g \equiv 0$. If, on the other hand, $g \not\equiv 0$, there would exist a positive constant R_0 such that $g(R) > 0$ for all $R \geq R_0$, and hence, for every $R > R_0$, in virtue of (3)

$$\int_{R_0}^{R} \frac{g'(r)}{g(r)}\, dr \geq \frac{C}{2}\ (R - R_0),$$

i.e.,

$$\log g(r)\Big|_{R_0}^{R} \geq \frac{C}{2}\,(R - R_0),$$

and therefore,

$$(4)\qquad g(R) \geq g(R_0)\ e^{\frac{c}{2}(R - R_0)}.$$

However, we have, by (1)

$$g(R) \leq \int_{\Gamma_R} \frac{\pi}{2} \cdot 2 \leq \pi|\Gamma_R|;$$

since $\Gamma_R \subset \partial B_R$, this yields a positive constant C_1 completely determined by n such that

$$g(R) \leq C_1 R^{n-1},$$

which contradicts (4) and yields the truth of Lemma 4. □

Remark. The above proof works well and so the lemma is valid if $v = +\infty$ on some parts of $\partial\Omega$.

4. An estimation of the growth of solutions.

Henceforth, we will denote Ω as an unbounded domain in $\mathbb{R}^n, n \geq 2$, such that, for some $f \in C^2\ ([0,\infty)),\ f > 0,\ f' > 0$ and $f'' > 0$ in $(0,\infty)$, we have

$$\Omega \subset \{(x,y) \in \mathbb{R}^2 \mid -f(y) < x < f(y),\ y > o\} \times \mathbb{R}^{n-2} \subset \mathbb{R}^n.$$

We shall extend the results in §2 to such a domain Ω.

First, for every positive constant y_0, since $f > 0,\ f' > 0$ and $f'' > 0$ in $(0,\infty)$, it is easy to see that there exists a positive constant δ_1, depending on y_0, such that $\{(f(y),y) \in \mathbb{R}^2 \mid \quad y > 0\} \cap \{(x,y) \in \mathbb{R}^2 \mid \quad y_0 - y + \frac{\delta_1}{2}x^2 = 0\}$ has exactly one point. And also, $\{(f(y),y) \in \mathbb{R}^2 \mid \quad y > 0\} \cap \{(x,y) \in \mathbb{R}^2 \mid \quad y_0 - y + \frac{\delta}{2}x^2 = 0\}$ has exactly two points, say $(f(y_1),y_1)$ and $(f(y_2),y_2)$ with $0 < y_1 < y_2$, for all δ with $0 < \delta < \delta_1$. In general, we have $y_1 = y_1(y_0,\delta)$, $y_2 = y_2(y_0,\delta)$ and also $\lim_{\delta\to 0} y_1(y_0,\delta) = y_0$. From now on, we always assume that the positive constant δ is less than the above δ_1.

To apply Lemma 4 to estimate the speed of growth of solutions in Ω, we may consider comparison functions of the following form

$$F_{y_0,\delta} = \frac{A(f^2(y) - x^2)^{\frac{1}{2}}}{y_0 - y + \frac{\delta}{2}x^2},$$

which is defined on

$$\Omega_{y_0,\delta} = \Omega \cap \left(\left\{ (x,y) \in \mathbb{R}^2 \mid y_0 - y + \frac{\delta}{2}x^2 > 0,\ 0 < y < y_1 \right\} \times \mathbb{R}^{n-2} \right),$$

where δ, y_0, and A are positive constants. We first proceed to calculate the mean curvature of $F_{y_0,\delta}$. For convenience of computation, we may set

$$F = A \cdot P^{\frac{1}{2}} Q^{-1},$$

where $P = f^2(y) - x^2$ and $Q = y_0 - y + \frac{\delta}{2}x^2$. We observe that

$$\text{(5)} \qquad \operatorname{div} TF = \frac{(1+F_x^2)F_{yy} - 2F_xF_yF_{xy} + (1+F_y^2)F_{xx}}{(1+F_x^2+F_y^2)^{\frac{3}{2}}}$$

$$= \frac{(\frac{1}{F^2} + \frac{F_x^2}{F^2})\frac{F_{yy}}{F} - 2\frac{F_x}{F}\frac{F_y}{F}\frac{F_{xy}}{F} + (\frac{1}{F^2} + \frac{F_y^2}{F^2})\frac{F_{xx}}{F}}{(\frac{1}{F^2} + (\frac{F_x}{F})^2 + (\frac{F_y}{F})^2)^{\frac{3}{2}}}.$$

Denoting

$$\mathrm{I} \ = \frac{F_x^2}{F^2}\frac{F_{yy}}{F} + \frac{F_y^2}{F^2}\frac{F_{xx}}{F} - 2\frac{F_x}{F}\frac{F_y}{F}\frac{F_{xy}}{F}$$

and

$$\mathrm{II} \ = \frac{F_{xx}}{F^3} + \frac{F_{yy}}{F^3},$$

we note that the numerator in (5) is the sum of these two expressions and we shall treat them seperately. For the first expression, we have

$$\begin{aligned}
\mathrm{I} &= \frac{F_x^2}{F^2}\left(\partial_y\left(\frac{F_y}{F}\right)+\left(\frac{F_y}{F}\right)^2\right)+\frac{F_y^2}{F^2}\left(\partial_x\left(\frac{F_x}{F}\right)+\left(\frac{F_x}{F}\right)^2\right)\\
&\quad -2\frac{F_x}{F}\frac{F_y}{F}\left[\partial_x\left(\frac{F_y}{F}\right)+\frac{F_xF_y}{F^2}\right]\\
&= \frac{F_x^2}{F^2}\left(\partial_y\left(\frac{F_y}{F}\right)\right)+\frac{F_y^2}{F^2}\left(\partial_x\left(\frac{F_x}{F}\right)\right)-2\frac{F_xF_y}{F^2}\left(\partial_x\left(\frac{F_y}{F}\right)\right)\\
&= \mathrm{I}^*+\mathrm{I}^{**}
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I}^* = \frac{F_x^2}{F^2}&\left(-\frac{1}{2}\frac{P_y^2}{P^2}+\frac{Q_y^2}{Q^2}\right)+\frac{F_y^2}{F^2}\left(-\frac{1}{2}\frac{P_x^2}{P^2}+\frac{Q_x^2}{Q^2}\right)\\
&-2\frac{F_xF_y}{F^2}\left(-\frac{1}{2}\frac{P_xP_y}{P^2}+\frac{Q_xQ_y}{Q^2}\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{I}^{**} = \frac{F_x^2}{F^2}&\left(\frac{1}{2}\frac{P_{yy}}{P}-\frac{Q_{yy}}{Q}\right)+\frac{F_y^2}{F^2}\left(\frac{1}{2}\frac{P_{xx}}{P}-\frac{Q_{xx}}{Q}\right)\\
&-2\frac{F_xF_y}{F^2}\left(\frac{1}{2}\frac{P_{xy}}{P}-\frac{Q_{xy}}{Q}\right).
\end{aligned}$$

By a direct computation,

$$\mathrm{I}^* = \frac{-1}{4}\frac{1}{P^2Q^2}(P_yQ_x-P_xQ_y)^2,$$

while

$$\begin{aligned}
\mathrm{I}^{**} = &\left(\frac{1}{2}\frac{P_x}{P}-\frac{Q_x}{Q}\right)^2\left(\frac{1}{2}\frac{P_{yy}}{P}-\frac{Q_{yy}}{Q}\right)\\
&+\left(\frac{1}{2}\frac{P_y}{P}-\frac{Q_y}{Q}\right)^2\left(\frac{1}{2}\frac{P_{xx}}{P}-\frac{Q_{xx}}{Q}\right)\\
&-2\left(\frac{1}{2}\frac{P_x}{P}-\frac{Q_x}{Q}\right)\left(\frac{1}{2}\frac{P_y}{P}-\frac{Q_y}{Q}\right)\left(\frac{1}{2}\frac{P_{xy}}{P}-\frac{Q_{xy}}{Q}\right).
\end{aligned}$$

Thus, in particular, we have

$$\mathrm{I}^* \leq 0. \tag{6}$$

As for I^{**} and II, we recall that

$$P = f^2(y) - x^2 \qquad \text{and} \qquad Q = y_0 - y + \frac{\delta}{2}\, x^2,$$

and hence

$$P_x = -2x, \quad P_y = 2f'f, \quad Q_x = \delta x, \quad Q_y = -1;$$

moreover

$$P_{xx} = -2, \quad P_{xy} = 0, \quad P_{yy} = 2(f''f + f'f')$$

and

$$Q_{xx} = \delta, \quad Q_{xy} = 0, \quad Q_{yy} = 0.$$

Thus, we have

$$\begin{aligned}
\mathrm{I}^{**} = {} & \frac{1}{P^3}[x^2(f''f + f'^2) - f^2 f'^2] \\
& + \frac{(-2)}{P^2 Q}\left[-\delta x^2(f''f + f'^2) + f'f + \frac{\delta}{2} f^2 f'^2\right] \\
& + \frac{1}{PQ^2}[\delta^2 x^2(f''f + f'^2) - 1 - 2\delta f f'] - \delta Q^{-3},
\end{aligned}$$

and also

$$\begin{aligned}
\mathrm{II} & = \frac{Q^2}{A^2 P}\left(\partial_x\left(\frac{F_x}{F}\right) + \frac{F_x^2}{F^2} + \partial_y\left(\frac{F_y}{F}\right) + \frac{F_y^2}{F^2}\right) \\
& = \frac{Q^2}{A^2 P}\left[\partial_x\left(\frac{1}{2}\frac{P_x}{P} - \frac{Q_x}{Q}\right) + \partial_y\left(\frac{1}{2}\frac{P_y}{P} - \frac{Q_y}{Q}\right)\right. \\
& \qquad \left. + \left(\frac{1}{2}\frac{P_x}{P} - \frac{Q_x}{Q}\right)^2 + \left(\frac{1}{2}\frac{P_y}{P} - \frac{Q_y}{Q}\right)^2\right] \\
& = \frac{Q^2}{A^2}\left\{\frac{1}{P^3}[f^2(f''f - 1) - x^2(ff'' + f'^2)]\right. \\
& \qquad \left. + \frac{1}{P^2 Q}(2ff' - \delta f^2 + 3\delta x^2) + \frac{2}{PQ^2}(\delta^2 x^2 + 1)\right\}.
\end{aligned}$$

Thus the numerator of div TF is

$$\begin{aligned}
\mathrm{I} + \mathrm{II} = {} & \mathrm{I}^* + \mathrm{I}^{**} + \mathrm{II} \\
= {} & \frac{1}{P^3}\left[x^2(ff'' + f'^2) - f^2 f'^2 + \frac{Q^2}{A^2}((f^2(ff'' - 1) - x^2(ff'' + f'^2)))\right] \\
& + \frac{1}{P^2 Q}\left[2\delta x^2(ff'' + (f')^2) - 2ff' - \delta f^2 f'^2\right. \\
& \qquad \left. + \frac{Q^2}{A^2}(-\delta f^2 + 2ff' + 3\delta x^2)\right] \\
& + \frac{1}{PQ^2}\left[\delta^2 x^2(f''f + (f')^2) - 1 - 2\delta f f' + 2\frac{Q^2}{A^2}(\delta^2 x^2 + 1)\right] \\
& - \delta Q^{-3} - \frac{1}{4}P^{-2}Q^{-2}(P_x Q_y - Q_x P_y)^2.
\end{aligned} \tag{7}$$

We want to choose δ and A to make the third bracket of the right-hand side of (7) negative. For this, substituting the expression for Q in the bracket and rewriting it as

(8)
$$\mathrm{III} = -1 + \left(\frac{2}{A^2}\right)\left(y_0 - y + \frac{\delta x^2}{2}\right)^2 (1 + \delta^2 x^2) + \delta^2 x^2 (f'' f + (f')^2) - 2\delta f f'.$$

For any given λ, $0 < \lambda < \frac{1}{4}$, if we take δ such that

$$0 < \delta < \inf_{y \in (0, y_1)} \min \left\{ \frac{\lambda y_0}{f^2}, \frac{\lambda}{f}, \frac{\lambda}{f^2}, \frac{\lambda}{f(f'' f + (f')^2)^{\frac{1}{2}}} \right\} \tag{9}$$

and $A = 4\sqrt{2} y_0$, then we have

$$\mathrm{III} \le -1 + \frac{1}{4(2y_0)^2}\left(y_0 + \frac{\lambda y_0}{2}\right)^2 (1 + \lambda^2) + \lambda^2 \le -1 + \frac{1 + \lambda^2}{4} + \lambda^2 < 0.$$

As of the second bracket of the right-hand side of (7), to make it negative, it clearly suffices to make the following expression negative, namely

$$\mathrm{IV} = \left(\frac{Q^2}{A^2}\right)\left(f f' + \frac{3}{2}\delta x^2\right) - f f' + \delta x^2 (f f'' + (f')^2). \tag{10}$$

For this, we observe that, as $x^2 < f^2$ in Ω and $f > 0$, $f' > 0$ and $f'' > 0$ in $(0, \infty)$,

$$\frac{3}{2}\delta x^2 + f f' \le \frac{3}{2}\delta f^2 + f f' = f f' \left(1 + \frac{3}{2}\delta \frac{f^2}{f f'}\right),$$

while

$$\begin{aligned} -f f' + \delta x^2 (f f'' + (f')^2) &\le -f f' + \delta f^2 (f f'' + (f')^2) \\ &= -f f' \left(1 - \delta \frac{f^2 (f f'' + (f')^2}{f f'}\right) \end{aligned}$$

and furthermore, if we require that

$$\delta < \inf_{y \in (0, y_1)} \min \left\{ \frac{\lambda f f'}{f^2 (f f'' + (f')^2)}, \frac{\lambda f f'}{2 f^2} \right\} \tag{9*}$$

it follows from (9) that

$$\frac{Q^2}{A^2} \le \frac{1}{4}. \tag{11}$$

And also, we have

$$\mathrm{IV} \le f f' \left(\frac{Q^2}{A^2}(1 + \lambda) + \lambda - 1\right) \le f f' \left(\frac{1}{4}(1 + \lambda) + \lambda - 1\right) \le \frac{-1}{4} f f'.$$

Thus, the condition that $f > 0$, $f' > 0$ in $(0, \infty)$ ensures us of the negativity of (10). It remains to consider the first bracket of the right-hand side of (7).

To make it negative, it suffices to make negative the following expression

$$\mathrm{V} = x^2(ff'' + (f')^2) - f^2 f'^2 + \frac{Q^2}{A^2} f^2(ff'' - 1),$$

or, in view of (11),

$$\mathrm{V} \le x^2(ff'' + (f')^2) - f^2 f'^2 + \frac{1}{4} f^2(ff''). \tag{12}$$

Recall that for given function f as above, we define

$$p(f) = 1 - \frac{ff''}{(f')^2}.$$

For §5, and from now on, we assume that $-1 \le p(f) \le 1$, following a remark concerning $p(f)$ for our interesting functions, [**5**, Remark 2.7]. And so, in particular for $f(y) = (y+z)^m$, $p(f) = \frac{1}{m}$ and for $f(y) = ae^{by}$, $p(f) = 0$ where z, $m > 1$, a, and b are positive constants, and also it is easy to see that for $f(y) = e^{y^\alpha}$, with $\alpha > 1$, then $p(f) \longrightarrow 0^-$ as $y \longrightarrow +\infty$.

Rewriting (12) in terms of $p(f)$, and noticing that $(\frac{3}{4} + \frac{p}{4}) \times (\frac{1}{2-p}) \ge \frac{1}{6}$, we have

$$\mathrm{V} \le (2-p)(f')^2 \left(x^2 - \frac{\frac{3}{4} + \frac{p}{4}}{2-p} f^2 \right) \le (2-p)(f')^2 \left(x^2 - \frac{1}{6} f^2 \right), \tag{13}$$

and so if we assume furthermore that

$$\Omega \subseteq \left\{ (x,y) \in \mathbb{R}^2 \mid -\frac{1}{\sqrt{6}} f(y) < x < \frac{1}{\sqrt{6}} f(y), y > 0 \right\} \times \mathbb{R}^{n-2} \subseteq \mathbb{R}^n, \tag{14}$$

then $\mathrm{V} \le 0$ and get the following conclusion about the estimation of our comparison function: If $F = \sqrt{f^2(y) - x^2} \dfrac{4\sqrt{2} y_0}{(y_0 - y + \frac{\delta}{2} x^2)}$ with δ as in our assumptions, (9), (9*), then $\operatorname{div} TF \le 0$ in $\Omega_{y_0,\delta}$, where Ω is assumed as in (14). Now we state what we achieved as follows:

Proposition 5. *Let $f_1 : [0,\infty) \longrightarrow [0,\infty)$, and $f_1 \in C^2([0,\infty))$ with $f_1 > 0$, ${f_1}' > 0$, ${f_1}'' > 0$ on $[0,\infty)$, and $-1 \le p(f_1) \le 1$. Suppose that $\Omega \subseteq \{(x,y) \in \mathbb{R}^2 \mid -f_1(y) < x < f_1(y), y > 0\} \times \mathbb{R}^{n-2} \subseteq \mathbb{R}^n$ and that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and for some constant β with $0 < \beta < 1$ satisfying*

$$\begin{cases} \operatorname{div} Tu \ge 0 & \text{in } \Omega \\ u \le 4\sqrt{2}\beta\sqrt{6f_1^2(y) - x^2} & \text{on } \partial\Omega. \end{cases}$$

Then $u \le 4\sqrt{2}\sqrt{6{f_1}^2(y) - x^2}$ in Ω.

Proof. Set $f(y) = \sqrt{6} f_1(y)$ and define $F(x,y) = 4\sqrt{2} y_0 \dfrac{(f^2(y) - x^2)^{\frac{1}{2}}}{(y_0 - y + \frac{\delta}{2} x^2)}$ as above, where $y_0 > 0$ and $\delta > 0$, small as in (9) and (9*) and we also require that $\delta \leq \dfrac{(2-2\beta)y_0}{\beta(f(y_1))^2}$. Then following the computation as above, in particular that of (7), and also noticing that the first three brackets of the right-hand side of (7) are negative in $\Omega_{y_0,\delta}$ as shown above, it is easy to see that

$$\text{div } TF = \frac{(\frac{1}{F^2}+\frac{F_y^2}{F^2})\frac{F_{xx}}{F} - 2\frac{F_x}{F}\frac{F_y}{F}\frac{F_{xy}}{F} + (\frac{1}{F^2}+\frac{F_x^2}{F^2})\frac{F_{yy}}{F}}{\frac{1}{F^3}(1+|\bigtriangledown F|^2)^{\frac{3}{2}}}$$

and

$$\left(\frac{1}{F^2}+\frac{F_x^2}{F^2}\right)\frac{F_{yy}}{F} - 2\frac{F_x}{F}\frac{F_y}{F}\frac{F_{xy}}{F} + \left(\frac{1}{F^2}+\frac{F_y^2}{F^2}\right)\frac{F_{xx}}{F} < -\delta\left(y_0 - y + \frac{\delta}{2}x^2\right)^{-3},$$

when (x,y) is close to $\{(x,y) \in \mathbb{R}^2 \mid y_0 - y + \frac{\delta}{2}x^2 = 0\}$.

And so, noticing that $P = f^2(y) - x^2$, $Q = y_0 - y + \frac{\delta}{2}x^2$ and $A = 4\sqrt{2}y_0$, when (x,y) is close to $\{(x,y) \in \mathbb{R}^2 \mid y_0 - y + \frac{\delta}{2}x^2 = 0\}$, we have

$$\begin{aligned}
\text{div } TF &\leq -\delta Q^{-3}\left(\frac{1}{F^2} + \left|\frac{\bigtriangledown F}{F}\right|^2\right)^{\frac{-3}{2}} \\
&\leq -\delta Q^{-3}\left(\frac{Q^2}{A^2P} + \left|\frac{1}{2}\frac{\bigtriangledown P}{P} - \frac{\bigtriangledown Q}{Q}\right|^2\right)^{\frac{-3}{2}} \\
&\leq -\delta\left(\frac{Q^4}{A^2P} + \left|\frac{1}{2}\frac{Q}{P}\bigtriangledown P - \bigtriangledown Q\right|^2\right)^{\frac{-3}{2}} \\
&\leq -\delta\left(\frac{Q^4}{A^2P} + \frac{1}{4}\frac{Q^2}{P^2}|\bigtriangledown P|^2 + |\bigtriangledown Q|^2 - \frac{Q}{P}\bigtriangledown P \cdot \bigtriangledown Q\right)^{\frac{-3}{2}} \\
&\leq -\frac{\delta}{2}(1+\delta^2x^2)^{\frac{-3}{2}}, \\
&\qquad \text{since } \frac{-Q}{P}\bigtriangledown P \cdot \bigtriangledown Q \geq 0 \text{ and } |\bigtriangledown Q|^2 = 1 + \delta^2 x^2.
\end{aligned}$$

But the bounded connected component of the closure of $\{(x,y) \in \mathbb{R}^2 \mid -f_1(y) < x < f_1(y), y > 0\} \cap \{(x,y) \in \mathbb{R}^2 \mid y_0 - y + \frac{\delta}{2}x^2 > 0\}$, which is denoted as $\overline{\Omega^*}$, is compact. And we have $\Omega_{y_0,\delta} \subseteq \Omega^* \times \mathbb{R}^{n-2}$, and so, there exists a positive constant c, such that

$$\begin{cases}
\text{div } TF \leq -c & \text{in } \Omega_{y_0,\delta}, \\
F \geq u & \text{on } \partial\Omega_{y_0,\delta} \cap \{(x,y) \in \mathbb{R}^2 \mid y_0 - y + \frac{\delta}{2}x^2 > 0\} \times \mathbb{R}^{n-2}, \\
F = +\infty & \text{on } \partial\Omega_{y_0,\delta} \cap \{(x,y) \in \mathbb{R}^2 \mid y_0 - y + \frac{\delta}{2}x^2 = 0\} \times \mathbb{R}^{n-2}.
\end{cases}$$

Now, by Lemma 4, we have $u \leq F$ in $\Omega_{y_0,\delta}$, which is

$$u(x,y,z_1,\ldots,z_{n-2}) \leq 4\sqrt{2}y_0\frac{(6f_1^2(y) - x^2)^{\frac{1}{2}}}{(y_0 - y + \frac{\delta}{2}x^2)} \quad \text{in } \Omega_{y_0,\delta}.$$

Now, let $\delta \longrightarrow 0$ and then let $y_0 \longrightarrow +\infty$, we get the conclusion of the proof. $\square$

5. Phragmèn-Lindelöf theorem in higher dimensions.

First, let's generalize Theorem 2 as follows:

Theorem 6. *Let* $\Omega \subseteq \{(x,y) \in \mathbb{R}^2 \mid -ay^m < x < ay^m, y > 0\} \times \mathbb{R}^{n-2} \subseteq \mathbb{R}^n$ *be an unbounded domain, where* $m \geq 1$ *and* a *are positive constants. Let* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *and suppose that*

$$\begin{cases} \text{div } Tu \geq 0 & \text{in } \Omega \\ u \leq ay^m h_m(\frac{x}{ay^m}) & \text{on } \partial\Omega. \end{cases}$$

Then we have $u \leq ay^m h_m(\frac{x}{ay^m}) \leq ay^m h_\infty(\frac{x}{ay^m}) = \sqrt{a^2y^{2m} - x^2}$ *in* Ω.

Proof. For every given positive constant $\epsilon > 0$, we now set $f_\epsilon(x,y) = a(y+\epsilon)^{m+\epsilon}$, $F_\epsilon(x,y,z_1,z_2,\dots,z_{n-2}) = a(y+\epsilon)^{m+\epsilon} h_{m+\epsilon}(\frac{x}{a(y+\epsilon)^{m+\epsilon}})$, where $(x,y,z_1,z_2,\dots,z_{n-2}) \in \Omega$.

Since $-ay^m < x < ay^m$, $y > 0$, we have

$$\left|\frac{x}{a(y+\epsilon)^{m+\epsilon}}\right| \leq \frac{y^m}{(y+\epsilon)^{m+\epsilon}} \longrightarrow 0 \quad \text{as } y \longrightarrow +\infty.$$

By Lemma 1, $h_{m+\epsilon}(\frac{x}{a(y+\epsilon)^{m+\epsilon}}) \longrightarrow h_{m+\epsilon}(0)$ uniformly as $y \longrightarrow +\infty$, and so it is easy to see that there exists a large constant y_3 such that $F_\epsilon(x,y) \geq \sqrt{200a^2y^{2m} - x^2}$ for $y \geq y_3$.

Next by [**6**, Theorem 2], setting $f_\epsilon(y) = a(y+\epsilon)^{m+\epsilon}$, $t = \frac{x}{f_\epsilon(y)}$ and recalling that $p(f_\epsilon) = \frac{1}{m+\epsilon}$, we have

$$\begin{aligned} &\text{div } TF_\epsilon \\ &= (1 + |\bigtriangledown F_\epsilon|^2)^{-\frac{3}{2}} \frac{(f'_\epsilon)^2}{f_\epsilon} \\ &\quad \cdot \left((1 - p(f_\epsilon))(h_{m+\epsilon} - t h'_{m+\epsilon})((h'_{m+\epsilon})^2 + 1) + h''_{m+\epsilon}(h^2_{m+\epsilon} + t^2) + \frac{h''_{m+\epsilon}}{(f'_\epsilon)^2} \right). \end{aligned}$$

Since $h_{m+\epsilon}(t)$ is the solution of $(*)$ and $(**)$ with $p(f_\epsilon) = \frac{1}{m+\epsilon}$, we have

$$\text{div } TF_\epsilon = (1 + |\bigtriangledown F_\epsilon|^2)^{-\frac{3}{2}} \frac{(f'_\epsilon)^2}{f_\epsilon} \cdot \frac{h''_{m+\epsilon}}{(f'_\epsilon)^2}$$

and so obviously that div $TF_\epsilon < 0$ on $\overline{\Omega'}$ where $\Omega' = \Omega \cap \{(x,y,z_1,\dots,z_{n-2}) \in \mathbb{R}^n | 0 < y < y_3\}$.

And so, there exists a positive constant $C_1 > 0$ such that

$$\text{div } TF_\epsilon \leq -C_1 \qquad \text{on } \overline{\Omega'}.$$

But, noticing that

$$u \le ay^m h_m\left(\frac{x}{ay^m}\right) \le \sqrt{a^2y^{2m}-x^2} \le 4\sqrt{2}\beta\sqrt{6a^2y^{2m}-x^2},$$

for some constant $\beta < 1$ on $\partial\Omega$, by Proposition 5, we also have

$$u \le 4\sqrt{2}\sqrt{6a^2y^{2m}-x^2} \le \sqrt{200\cdot a^2y^{2m}-x^2} \qquad \text{in } \ \Omega\setminus\Omega'.$$

By Lemma 4, we have

$$u \le F_\epsilon \qquad \text{in } \ \overline{\Omega'}.$$

In conclusion, we have

$$u \le F_\epsilon \qquad \text{in } \ \Omega,$$

and let $\epsilon \longrightarrow 0$, the proof is done. □

As a corollary of Theorem 6, we state a generalization of Nitsche's theorem [**7**] as follows.

Corollary. *Let $\Omega = \{(x,y)\in\mathbb{R}^2 \mid -ay < x < ay,\ y>0\}\times\mathbb{R}^{n-2}$ be a wedge domain, where a is a positive constant. Let $u\in C^2(\Omega)\cap C^0(\overline{\Omega})$ and suppose that*

$$\begin{cases}\operatorname{div} Tu = 0 & \text{in } \ \Omega,\\ u=0 & \text{on } \ \partial\Omega.\end{cases}$$

Then $u\equiv 0$ in Ω.

Proof. Apply Theorem 6 to functions u and $-u$, we have $u\le 0$ in Ω and $-u\le 0$ in Ω, and so $u\equiv 0$ as claimed. □

Next, let's generalize Theorem 2* as follows:

Theorem 6*. *Let $\Omega \subseteq \{(x,y)\in\mathbb{R}^2 \mid -ae^{by} < x < ae^{by}, y>0\}\times\mathbb{R}^{n-2}\subseteq\mathbb{R}^n$, where a, b are positive constants. Let $u\in C^2(\Omega)\cap C^0(\overline{\Omega})$ and suppose that*

$$\begin{cases}\operatorname{div} Tu \ge 0 & \text{in } \ \Omega\\ u\le\sqrt{a^2e^{2by}-x^2} & \text{on } \ \partial\Omega.\end{cases}$$

Then we have $u\le\sqrt{a^2e^{2by}-x^2}$ in Ω.

Proof. The proof is similar to that of Theorem 6.

For every $\epsilon>0$, we consider the following function

$$F_\epsilon(x,y,z_1,z_2,\ldots,z_{n-2}) = ae^{(b+\epsilon)y}h_\infty\left(\frac{x}{ae^{(b+\epsilon)y}}\right) = \sqrt{a^2e^{2(b+\epsilon)y}-x^2}$$

with $(x,y,z_1,\ldots,z_{n-2})\in\Omega$.

Since $-ae^{by}<x<ae^{by}$, $y>0$, we have

$$\left|\frac{x}{ae^{(b+\epsilon)y}}\right| \le \frac{ae^{by}}{ae^{(b+\epsilon)y}} \longrightarrow 0 \quad \text{as } \ y\longrightarrow+\infty$$

and notice that $F_\epsilon = ae^{(b+\epsilon)y}(1-\frac{x^2}{a^2e^{2(b+\epsilon)y}})^{\frac{1}{2}}$.

Hence, there exists a positive constant $y_3 > 0$ such that

$$F_\epsilon \geq \sqrt{200a^2e^{2by} - x^2} \qquad \text{for } \ y \geq y_3.$$

Next, by [**6**, Theorem 2], and setting $f_\epsilon(y) = ae^{(b+\epsilon)y}$, $t = \frac{x}{f_\epsilon(y)}$ and noticing that $p(f_\epsilon) = 0$, we have

$$\begin{aligned} \text{div } TF_\epsilon &= (1 + |\bigtriangledown F_\epsilon|^2)^{-\frac{3}{2}} \frac{(f'_\epsilon)^2}{f_\epsilon} \\ &\quad \cdot \left((h_\infty - th'_\infty)({h'_\infty}^2 + 1) + h''_\infty(h^2_\infty + t^2) + \frac{h''_\infty}{f'^2_\epsilon} \right) \\ &= (1 + |\bigtriangledown F_\epsilon|^2)^{-\frac{3}{2}} \frac{(f'_\epsilon)^2}{f_\epsilon} \frac{h''_\infty}{(f'_\epsilon)^2}. \end{aligned}$$

So, we have

$$\text{div } TF_\epsilon < 0 \qquad \text{on } \ \overline{\Omega'},$$

where

$$\Omega' = \Omega \cap \{(x, y, z_1, \ldots, z_{n-2}) \in \mathbb{R}^n \mid \quad 0 < y < y_3\},$$

and so there exists a positive constant $C_1 > 0$ such that

$$\text{div } TF_\epsilon \leq -C_1 \qquad \text{in } \ \overline{\Omega'}.$$

Finally, by Proposition 5, notice that

$$u \leq \sqrt{a^2e^{2by} - x^2} \leq 4\sqrt{2}\beta\sqrt{6a^2e^{2by} - x^2},$$

for some constant $\beta < 1$ on $\partial\Omega$, we also have

$$u \leq 4\sqrt{2}\sqrt{6a^2e^{2by} - x^2} \leq F_\epsilon \qquad \text{in } \ \Omega \setminus \Omega'.$$

So, by Lemma 4, we have

$$u \leq F_\epsilon \qquad \text{on } \ \overline{\Omega'},$$

and so obviously, we get

$$u \leq F_\epsilon \qquad \text{in } \ \Omega,$$

and let $\epsilon \longrightarrow 0$, the proof is finished. □

Finally, let's generalize Theorem 3 as follows:

Theorem 7. *Let $f_1 \in C^2([0, \infty))$ with $f_1 \geq 0$, $f'_1 > 0$, and $f''_1 \geq 0$ in $(0, \infty)$ such that $p(f_1) \geq p_0$, where p_0 is a constant with $-1 \leq p_0 \leq 0$. Suppose that $\Omega \subseteq \{(x, y) \in \mathbb{R}^2 \mid \quad - f_1(y) < x < f_1(y), y > 0\} \times \mathbb{R}^{n-2} \subseteq \mathbb{R}^n$ and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying*

$$\begin{cases} \text{div } Tu \geq 0 & \text{in } \ \Omega \\ u \leq a\sqrt{f^2(y) - x^2} & \text{on } \ \partial\Omega, \end{cases}$$

where $f = \left(\frac{(a^2-1)(2-p_0)}{(a^2-1+p_0)}\right)^{\frac{1}{2}} f_1$ and a is a positive constant with $a^2 - 1 + p_0 > 0$.

Then we have $u \leq a\sqrt{f^2(y) - x^2}$ in Ω.

Proof. For any given $\epsilon > 0$, we define $f_\epsilon(y) = e^{\epsilon y} f(y+\epsilon)$ and $F_\epsilon(x, y, z_1, \ldots, z_{n-2}) = a\sqrt{f_\epsilon^2(y) - x^2}$, then there exists $y_3 > 0$ such that

$$F_\epsilon \geq a(e^{2\epsilon y} f^2(y) - x^2)^{\frac{1}{2}} \geq (200 \cdot a^2 f^2(y) - x^2)^{\frac{1}{2}} \quad \text{for} \quad y > y_3.$$

Computing the mean curvature of F_ϵ and using the definition, $p(f_\epsilon) = 1 - \frac{f_\epsilon f_\epsilon''}{(f_\epsilon')^2}$, we have

$$\begin{aligned}\operatorname{div} TF_\epsilon &= (1 + |\bigtriangledown F_\epsilon|^2)^{-\frac{3}{2}} (f_\epsilon^2 - x^2)^{\frac{-3}{2}} \\ &\quad \cdot \Big(a(f_\epsilon')^2 [(a^2 - 1)(2 - p(f_\epsilon))x^2 - f_\epsilon^2 (a^2 - 1 + p(f_\epsilon))] - a f_\epsilon^2 \Big).\end{aligned}$$

Obviously, we have

$$f_\epsilon^2(y) \geq f^2(y) \geq \frac{(a^2-1)(2-p_0)}{(a^2-1+p_0)} f_1^2(y) \geq \frac{(a^2-1)(2-p(f_\epsilon))}{(a^2-1+p(f_\epsilon))} f_1^2(y),$$

and so, we have

$$\operatorname{div} TF_\epsilon \leq -a(1 + |\bigtriangledown F_\epsilon|^2)^{-\frac{3}{2}} f_\epsilon^2 < 0 \qquad \text{in} \quad \Omega,$$

and by compactness, there exists a positive constant $C_1 > 0$ such that

$$\operatorname{div} TF_\epsilon \leq -C_1 \qquad \text{in} \;\; \Omega_1 = \{(x, y, z_1, z_2, \ldots, z_{n-2}) \in \Omega | y < y_3\}.$$

But by Proposition 5, notice that

$$u \leq a\sqrt{f^2(y) - x^2} \leq 4\sqrt{2}\beta\sqrt{6a^2 f^2 - x^2},$$

for some constant $\beta < 1$ on $\partial\Omega$, we also have

$$u \leq \sqrt{200 a^2 f^2(y) - x^2} \leq F_\epsilon \qquad \text{in} \;\; \Omega \setminus \Omega_1.$$

By Lemma 4, we have

$$u \leq F_\epsilon \qquad \text{in} \;\; \Omega_1.$$

In conclusion, we have $u \leq F_\epsilon$ in Ω, and then let $\epsilon \longrightarrow \infty$.

We thus finish the proof. □

References

[1] R. Finn, *Equilibrium Capillary Surfaces*, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, 1986, MR 88f:49001, Zbl 0583.35002.

[2] R. Finn and J. Hwang, *On the comparison principle for capillary surfaces*, J. Fac. Sci. Univ. Tokyo Sect. IA, Math., **36** (1989), 131-134, MR 90h:35099, Zbl 0684.35007.

[3] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Second edition, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983, MR 86c:35035, Zbl 0562.35001.

[4] J. Hwang, *Phragmèn-Lindelöf theorem for the minimal surface equation*, Proc. Amer. Math. Soc., **104** (1988), 825-828, MR 89j:35016, Zbl 0787.35012.

[5] ———, *Growth property for the minimal surface equation in unbounded domains*, Proc. Amer. Math. Soc., **121** (1994), 1027-1037, MR 94j:35019, Zbl 0820.35010.

[6] ———, *Catenoid-like solutions for the minimal surface equation*, Pacific J. Math., **183** (1998), 91-102, MR 99d:58041, Zbl 0905.35029.

[7] J.C.C. Nitsche, *On new results in the theory of minimal surface*, Bull. Amer. Math. Soc., **71** (1965), 195-270, MR 30 #4200, Zbl 0135.21701.

[8] ———, *Vorlesungen über Minimalflächen*, Springer-Verlag, Berlin-Heidelbert-New York, 1975, MR 56 #6533, Zbl 0319.53003.

[9] R. Osserman, *A Survey of Minimal Surfaces*, Van Nostrand-Reinhold, New York, 1969, MR 41 #934, Zbl 0209.52901.

[10] M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, N.J., 1967, MR 36 #2935, Zbl 0153.13602.

Received February 27, 2001.

Mathematics Department
Academia Sinica
Nan-kang
Taipei, 115, Taiwan
E-mail address: macchsieh@ccvax.sinica.edu.tw

Mathematics Department
Academia Sinica
Nan-kang
Taipei, 115, Taiwan
E-mail address: majfh@ccvax.sinica.edu.tw

Mathematics Department
Academia Sinica
Nan-kang
Taipei, 115, Taiwan
E-mail address: liang@math.sinica.edu.tw