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ON REGULAR HOLONOMIC SYSTEMS WITH SOLUTIONS RAMIFIED ALONG $y^k = x^n$

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We classify the holonomic systems of (micro) differential equations of multiplicity one along the conormal of the hypersurface $y^k = x^n$. We show that their solutions are related to ${}_kF_{k-1}$ hypergeometric functions on the Riemann sphere.

1. Introduction.

Holonomic $\mathcal{D}$-modules characterize multivalued holomorphic functions in the same way that polynomials characterize algebraic numbers. When we replace systems of differential equations by systems of microdifferential equations we concentrate on the singularities of their multivalued holomorphic solutions modulo holomorphic functions. This point of view was introduced by Riemann with his study of the hypergeometric differential equation and was extended to the several variables case by [**19**].

The mathematical community isolated a class of multivalued holomorphic functions on the Riemann sphere that arise in many different problems, the *special functions.* It is an experimental fact that most of the sheaves of solutions of the differential equations that characterize these functions have the remarkable property of being determined by its local monodromies, which can be computed from the indicial equations of these differential equations at their singular points. The local sheaves with the above property are called *rigid.* We can find in [**9**] a program of systematic study of special functions based on the concept of rigid local system (see also [**22**]).

We now have several combinatorial descriptions of germs of regular holonomic $\mathcal{D}$-modules in two variables (see [**13**], [**14**], [**12**]). These remarkable works are not very useful to the specialist in PDE's. He would like to know the equations that define these systems. On the other hand there are too many systems and we cannot expect to obtain the equations for all of them. Pedro C. Silva suggests in [**21**] a notion of rigidity for germs of holonomic systems of microdifferential equations based on the results of [**15**], isolating a special class of holonomic systems. The holonomic systems presented in this paper are rigid in the sense of [**21**]. We initiate in this paper a systematic study of special functions on several variables from a microlocal point of view.

Let X be a complex manifold of dimension m. Let $\pi : T^*X \to X$ be the cotangent bundle of X. Let Λ be a germ of a conic Lagrangian subvariety of $T^*X \setminus X$. By Theorem 8.3 of [**18**], if Λ is singular and irreducible and Λ is contained in an involutive submanifold of $T^*X \setminus X$ of codimension $m-1$, Λ can be identified with the conormal of the hypersurface $y^k = x^n$. These are the singular Lagrangian varieties with milder singularities. We can find in [**18**], Theorem 8.6, the classification of systems of microdifferential equations with simple characteristics along Λ. The purpose of this paper is to classify the systems of microdifferential equations of multiplicity one along Λ. As a consequence we obtain a classification theorem for $\mathcal{D}$-modules.

Let k, n be integers such that $2 \leq k \leq n-1$ and $(k,n) = 1$. Set $\vartheta = x\partial_x + \frac{n}{k} y\partial_y$. Given complex numbers λ_i, $i \in \mathbb{Z}$, such that

$$(1) \qquad \lambda_{i+k} = \lambda_i, \quad \alpha_i := \lambda_i - \lambda_{i+1} + \frac{n-k}{k} \quad \text{is a nonnegative integer,}$$

we will denote by $\mathcal{M}_{(\lambda_i)}$ the $\mathcal{E}_{\mathbb{C}^m}$-module given by the generators u_i, $i \in \mathbb{Z}$, and relations

$$(2) \qquad u_{i+k} = u_i, \quad (\vartheta - \lambda_i)u_i = 0, \quad \partial_x u_i = -\frac{n}{k} x^{\alpha_i} \partial_y u_{i+1}, \quad \partial_{t_j} u_i = 0,$$

where j runs over the set $\{1, \ldots, m-2\}$. Notice that

$$(3) \qquad \alpha_{i+k} = \alpha_i \quad \text{and} \quad \sum_{i=0}^{k-1} \alpha_i = n-k.$$

We will denote by $\mathcal{L}_{(\lambda_i)}$ the $\mathcal{D}_{\mathbb{C}^m}$-module given by the same sets of generators and relations. If $2k+1 \leq n$ and there are $i, j \in \mathbb{Z}$, such that $i \not\equiv j \pmod{k}$ and $\alpha_i \alpha_j \neq 0$, the system $\mathcal{M}_{(\lambda_i)}$ is not with simple characteristics. These are, as far as the authors know, the first examples of systems of multiplicity one that are not with simple characteristics (see Theorem 2.7 and Proposition 4.5). Moreover, the solutions of the systems $\mathcal{L}_{(\lambda_i)}$ are pullbacks of ${}_kF_{k-1}$ hypergeometric functions on the Riemann sphere twisted by $y^{\lambda \frac{k}{n}}$.

The main results of this paper are stated in Section 4. We study the $\mathcal{O}_{\Lambda}(0)$-module $\mathcal{N}/\mathcal{N}(-1)$, where $\mathcal{N}$ is the canonical lattice of a holonomic system $\mathcal{M}$ with multiplicity one along Λ. The module $\mathcal{N}/\mathcal{N}(-1)$ gives us a first approximation to the structure of $\mathcal{M}$. We introduce a new invariant, a module over the semigroup of the Legendrian curve $\Lambda/\mathbb{C}^\times$, essential to distinguish between systems generated by sections with principal symbols of the same degree of homogeneity. We realized later that the same type of invariant had been used by G.-M. Greuel and G. Pfister (see [**4**]) in a different context. Theorem 3.5 is the tool that allows the extension of the results on the $\mathcal{O}_{\Lambda}(0)$-module $\mathcal{N}/\mathcal{N}(-1)$ to the $\mathcal{E}_X$-module $\mathcal{M}$. In Section 5 we apply the results of Section 4 to $\mathcal{D}$-modules. We obtain a classification theorem for regular holonomic systems with solutions ramified along the hypersurface $y^k = x^n$ verifying some natural conditions and we study their solutions.

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2. Systems of microdifferential equations.

Let X be a complex manifold. Let $\mathcal{D}_X$ denote the ring of differential operators on X. Let $\mathcal{E}_X$ denote the sheaf of microdifferential operators on X (see [**19**], [**20**] or [**2**]). Given $k \in \mathbb{Z}$, $\mathcal{E}_{X(k)}$ denotes the sheaf of microdifferential operators of order smaller than or equal to k. Let $\mathcal{M}$ be an $\mathcal{E}_X$-module and $\mathcal{N}$ an $\mathcal{E}_{X(0)}$-submodule of $\mathcal{M}$. Set $\mathcal{N}_{(k)} = \mathcal{E}_{X(k)}\mathcal{N}$.

A coherent $\mathcal{E}_X$-*module* $\mathcal{M}$ is called a *system of microdifferential equations*. The support of a system of microdifferential equations is an involutive variety (see [**19**]). A system of microdifferential equations is called *holonomic* if its support is Lagrangian.

Let $\mathcal{L}$ be a coherent $\mathcal{O}_X$-module and let Y be an irreducible component of the support of $\mathcal{L}$. For each $x \in Y$ set $S_{Y,x} = \{f \in \mathcal{O}_{X,x} : f^{-1}(0) \not\supset Y\}$. The length of the $S_{Y,x}^{-1}\mathcal{O}_{X,x}$-module $S_{Y,x}^{-1}\mathcal{L}_x$ is finite and does not depend on x. We call *multiplicity of* $\mathcal{L}$ *along* Y to the length of $S_{Y,x}^{-1}\mathcal{L}_x$.

Let Ω be a conic open subset of $T^*X \setminus X$. Let $\mathcal{M}$ be a coherent $\mathcal{E}_X|_\Omega$-module and let Λ be an irreducible component of the support of $\mathcal{M}$. Set $I_\Lambda = \{f \in \mathcal{O}_\Omega : f|_\Lambda = 0\}$. For each $p \in \Lambda$, there is a coherent $\mathcal{E}_X|_{U(0)}$-submodule $\mathcal{N}$ of $\mathcal{M}|_U$ defined on some conic open neighborhood U of p such that $\mathcal{E}_X|_U\mathcal{N} = \mathcal{M}|_U$. Set $\overline{\mathcal{M}} = \mathcal{N}/\mathcal{N}_{(-1)}$. The multiplicity along Λ of the coherent $\mathcal{O}_U$-module

$$\mathcal{O}_U \otimes_{\mathcal{O}_{U(0)}} \overline{\mathcal{M}} \tag{4}$$

does not depend on $\mathcal{N}$. We call *multiplicity of* $\mathcal{M}$ *along* Λ to the multiplicity along Λ of (4) (see [**5**]).

Let $\mathcal{L}$ be a coherent $\mathcal{D}_X$-module. We call *characteristic variety* of $\mathcal{L}$ to the support of the coherent $\mathcal{E}_X$-module

$$\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{L}. \tag{5}$$

We call *multiplicity of* $\mathcal{L}$ *along* Λ to the multiplicity of (5) along Λ.

Set $\mathcal{I}_\Lambda = \{P \in \mathcal{E}_{X(1)}|_\Omega : \sigma_1(P) \in I_\Lambda\}$. The morphism $\sigma_0 : \mathcal{E}_{X(0)} \to \mathcal{O}_{T^*X(0)}$ induces an isomorphism

$$\left(\mathcal{E}_{X(0)}|_\Omega\right) \,/\, \mathcal{I}_{\Lambda(-1)} \xrightarrow{\sim} \mathcal{O}_{\Lambda(0)}. \tag{6}$$

Proposition 2.1. *If* $\mathcal{M}$ *is a holonomic* $\mathcal{E}_X|_\Omega$*-module and* $\mathcal{M}'$ *is a coherent* $\mathcal{E}_X|_\Omega$*-submodule of* $\mathcal{M}$ *such that* $\mathrm{mult}_\Lambda\mathcal{M} = \mathrm{mult}_\Lambda\mathcal{M}'$ *for each irreducible component of the support of* $\mathcal{M}$*,* $\mathcal{M} = \mathcal{M}'$*.*

Proof. This is an immediate consequence of the fact that all irreducible components of the support of a coherent $\mathcal{E}_X|_\Omega$-module have dimension greater than or equal to $\dim X$. □

Definition 2.2. Let Ω be a conic open subset of $T^*X \setminus X$. Let Λ be a conic Lagrangian subvariety of Ω. Let $\mathcal{M}$ be coherent $\mathcal{E}_X|_\Omega$-module with support Λ. We say that $\mathcal{M}$ is *regular holonomic at a point* $a \in \Lambda$ if there is a conic open neighborhood U of a and a coherent $\mathcal{E}_{X(0)}|_U$-submodule $\mathcal{N}$ of $\mathcal{M}|_U$ such that $\mathcal{E}_X|_U\mathcal{N} = \mathcal{M}|_U$ and $\mathcal{I}_\Lambda|_U\mathcal{N} = \mathcal{N}$. We say that $\mathcal{M}$ is *regular holonomic* if the set of points $a \in \Lambda$ such that $\mathcal{M}$ is not regular at a is nowhere dense in Λ.

The following result was obtained taking $c = 0$ in Theorem 5.1.6 of [7].

Theorem 2.3. *Let $\mathcal{M}$ be a regular holonomic system of microdifferential equations with support $\Lambda \subset T^*X \setminus X$. Then $\mathcal{M}$ has a canonical coherent $\mathcal{E}_{X(0)}$-submodule $\mathcal{N}$ such that*

1) $\mathcal{E}_X\mathcal{N} = \mathcal{M}$ *and* $\mathcal{I}_\Lambda\mathcal{N} = \mathcal{N}$.
2) *The support of a section of the sheaf $\mathcal{M}/\mathcal{N}$ is an analytic set of codimension smaller than or equal to the dimension of X.*

We call $\mathcal{N}$ the *canonical lattice* of $\mathcal{M}$.

Given a left ideal $\mathcal{I}$ of $\mathcal{E}_X$, let $\overline{\mathcal{I}}$ be the ideal of $\mathcal{O}_{T^*X}$ generated by the principal symbols of the microdifferential operators $P \in \mathcal{I}$. We call $\overline{\mathcal{I}}$ the *ideal of symbols* of the ideal $\mathcal{I}$.

Definition 2.4. Let $\mathcal{M}$ be a holonomic system of microdifferential equations with support Λ. Let U be a conic open neighborhood of $a \in \Lambda$. Let u be a generator of $\mathcal{M}|_U$. Let $\mathcal{I}$ denote the annihilator of u. The section u is called *a local generator of $\mathcal{M}$ with simple characteristics* if $\overline{\mathcal{I}}$ equals the defining ideal of $\Lambda \cap U$. The module $\mathcal{M}$ has *simple characteristics along* Λ if it admits a generator with simple characteristics in a neighborhood of each point of Λ.

It follows from Theorem 2.5 that a holonomic system $\mathcal{M}$ has multiplicity one along a Lagrangian variety Λ if and only if $\mathcal{M}$ has simple characteristics along the regular part of Λ.

Theorem 2.5. *Let Ω be an open subset of $T^*X \setminus X$. Let $\mathcal{M}$ be a holonomic $\mathcal{E}_X|_\Omega$-module with support Λ. The following holds.*

1) *If $\mathcal{M}$ has simple characteristics along Λ then $\mathcal{M}$ is regular holonomic with multiplicity one along the irreducible components of Λ.*
2) *If $\mathcal{M}$ has multiplicity one along the irreducible components of Λ then $\mathcal{M}$ has simple characteristics along $\Lambda \setminus \operatorname{Sing}(\Lambda)$. Moreover, $\mathcal{M}$ is regular holonomic.*

Proof. The first statement follows from Theorem I. 6.3.2 of [**20**].

Given $q \in \Lambda$ there is a conic open neighborhood U of q and a section u of $\mathcal{M}|_U$ that does not vanish on $\Lambda \cap U$. Since $\mathrm{mult}_{\Lambda\cap U}(\mathcal{E}_X u) \geq 1$ and $\mathcal{E}_X u \subset \mathcal{M}|_U$, $\mathcal{E}_X u = \mathcal{M}|_U$ by Proposition 2.1. Set $\overline{\mathcal{M}} = \mathcal{E}_X(0)u/\mathcal{E}_X(-1)u$. Since

$$\mathrm{mult}_\Lambda \mathcal{M} = \sum_{k\geq 0} \mathrm{mult}_\Lambda \left(I_\Lambda^k \otimes \overline{\mathcal{M}} / I_\Lambda^{k+1} \otimes \overline{\mathcal{M}} \right)$$

(see [**20**], Appendix D) and $I_\Lambda^k \otimes \overline{\mathcal{M}} = I_\Lambda^{k+1} \otimes \overline{\mathcal{M}} \Rightarrow I_\Lambda^{k+1} \otimes \overline{\mathcal{M}} = I_\Lambda^{k+2} \otimes \overline{\mathcal{M}}$, $\mathrm{mult}_\Lambda \left(\mathcal{O}_{T^*X} \otimes \overline{\mathcal{M}}/I_\Lambda \otimes \overline{\mathcal{M}}\right) = 1$ and there is a dense Zariski open subset U_0 of U such that $I_\Lambda \otimes \overline{\mathcal{M}} \mid_{U_0} = I_\Lambda^2 \otimes \overline{\mathcal{M}} \mid_{U_0}$. Hence $I_\Lambda \otimes \overline{\mathcal{M}} \mid_{U_0} = \cap_{k\geq 1} I_\Lambda^k \otimes \overline{\mathcal{M}} \mid_{U_0} = 0$. Let $q_0 \in U_0$ be a nonsingular point of Λ. We can assume that $q_0 = (0, dx_1)$ and $\Lambda = \{x_1 = \xi_2 = \cdots = \xi_n = 0\}$. There are $R_1, \ldots, R_n \in \mathcal{E}_{X,q_0}(-1)$ such that

$$(x_1 - R_1)u_{q_0} = (\partial_{x_2}\partial_{x_1}^{-1} - R_2)u_{q_0} = \cdots = (\partial_{x_n}\partial_{x_1}^{-1} - R_n)u_{q_0} = 0.$$

Hence the ideal of symbols of the annihilator of u_{q_0} equals I_{Λ,q_0}. We have shown in this way that $\mathcal{M}$ has simple characteristics at a generic point of Λ, hence is regular holonomic. By the classification theorem for regular holonomic $\mathcal{E}_X$-modules with smooth support (see for instance Remark 6.7 of [**8**]), $\mathcal{M}$ has simple characteristics along $\Lambda \setminus \mathrm{Sing}(\Lambda)$. $\square$

The following result is a consequence of Theorem 5.1.6 of [**7**] and Theorem I.6.3.3 of [**20**].

Corollary 2.6. *Let $\mathcal{M}$ be a holonomic systems of microdifferential equations with simple characteristics along Λ. Let $\mathcal{N}$ be the canonical lattice of M. Given $a \in \Lambda$ there is a conic open neighborhood U of a and a section u of $\mathcal{M}$ on U such that u is a generator of $\mathcal{M}|_U$ with simple characteristics along Λ and $\mathcal{N}|_U = \mathcal{E}_X(0)u$.*

Theorem 2.7. *Let Λ be the germ at a point p of an irreducible Lagrangian variety of $T^*X \setminus X$. Let M be the fiber at p of a regular holonomic $\mathcal{E}_X$-module. Let N be the canonical lattice of M.*

1) *The module M has multiplicity one along Λ if and only if $N/N(-1)$ is a finitely generated torsion free $\mathcal{O}_{\Lambda,p}(0)$-module of rank one.*
2) *The module M has simple characteristics along Λ if and only if $N/N(-1)$ is a free $\mathcal{O}_{\Lambda,p}(0)$-module of dimension one.*

Proof. Since N is a coherent $\mathcal{E}_{X,p}$-module, $N/N(-1)$ is finitely generated. By Theorem 2.5, the support of $I_{\Lambda,p}(0)(N/N(-1))$ is contained in the singular locus of Λ. By Theorem 2.3, $I_{\Lambda,p}(0)(N/N(-1)) = 0$. Hence $N/N(-1)$ is an $\mathcal{O}_{\Lambda,p}(0)$-module. By Theorem 2.3 $N/N(-1)$ is a torsion free $\mathcal{O}_{\Lambda,p}(0)$-module.

Set $S_{\Lambda,p}(0) = S_{\Lambda,p} \cap \mathcal{O}_{T^*X,p}(0)$. The $\mathcal{E}_{X,p}$-module M has multiplicity one along Λ if and only if

$$S^{-1}_{\Lambda,p}\left(\mathcal{O}_{T^*X,p} \otimes_{\mathcal{O}_{T^*X,p}(0)} N/N(-1)\right) \tag{7}$$

is a simple $S^{-1}_{\Lambda,p}\mathcal{O}_{T^*X,p}$-module. Since $S^{-1}_{\Lambda,p}\mathcal{O}_{T^*X,p}$ is faithfully flat over $(S_{\Lambda,p}(0))^{-1}\,\mathcal{O}_{T^*X,p}(0)$, (7) is simple if and only if $(S_{\Lambda,p}(0))^{-1}\,(N/N(-1))$ is a simple $(S_{\Lambda,p}(0))^{-1}\,\mathcal{O}_{T^*X,p}(0)$-module. Since $(S_{\Lambda,p}(0))^{-1}\,\mathcal{O}_{\Lambda,p}(0)$ is the quotient field of $\mathcal{O}_{\Lambda,p}(0)$, M has multiplicity one along Λ if and only if $N/N(-1)$ has rank one.

Assume that $\mathcal{M}$ has simple characteristics along Λ. By Corollary 2.6 there is a local generator u of $\mathcal{M}$, with simple characteristics along Λ, such that $N = \mathcal{E}_{X,p}(0)u$. Hence $N/N(-1)$ is a torsion free $\mathcal{O}_{\Lambda,p}(0)$-module generated by $u + N(-1)$.

Assume that $N/N(-1)$ is a free $\mathcal{O}_{\Lambda,p}(0)$-module generated by a section v. Let u be a section of N such that $v = u + N(-1)$. Since $N = \mathcal{E}_X(0)u + \mathcal{E}_X(-k)N$ for all $k \geq 0$, it follows from a theorem of [**20**] that $N = \mathcal{E}_X(0)u$. Let $f \in I_{\Lambda,p}(0)$. Since $fv = 0$, there is $Q \in \mathcal{E}_{X,p}(0)$ such that $\sigma(Q) = f$ and $Qu \in \mathcal{E}_{X,p}(-1)u$. Hence there is $P \in \mathcal{E}_{X,p}(0)$ such that $\sigma(P) = f$ and $Pu = 0$. Hence u is a generator with simple characteristics along Λ. $\square$

3. A microdifferential Cauchy problem.

Throughout this section we will assume $m = 2$. We shall denote by $\mathbb{N}$ the set of nonnegative integers.

Let X be an open set of $\mathbb{C}^2$ containing the origin. Let (x, y) be a system of coordinates of $\mathbb{C}^2$. Let (x, y, ξ, η) be the associated system of coordinates of $T^*\mathbb{C}^2$ such that $\omega = \xi dx + \eta dy$ is the canonical one-form.

Let d be a positive integer. Given $f \in \mathrm{M}_d(\mathcal{O}_{T^*X}(-j))$, $g \in \mathrm{M}_d(\mathcal{O}_{T^*X}(-i))$, $i, j, \alpha, \beta \in \mathbb{N}$, we set (see [**17**]),

$$(f, g)^{(\alpha,\beta)} = \frac{1}{\alpha!\beta!}\frac{\partial^{\alpha+\beta} f}{\partial\xi^\alpha\partial\eta^\beta}\,\frac{\partial^{\alpha+\beta} g}{\partial x^\alpha \partial y^\beta}.$$

Let $P, Q \in \mathrm{M}_d\left(\mathcal{E}_X(0)(\Omega)\right)$ for some open neighborhood Ω of $(0, dy)$ such that $P = \sum_{l\geq 0} p_l\eta^{-l}$ and $Q = \sum_{l\geq 0} q_l\eta^{-l}$ with $p_l, q_l \in \mathrm{M}_d(\mathbb{C}\{x, y, p\})$. The product of the matrices P, Q is given by the Leibniz rule

$$PQ = \sum_{l\geq 0}\ \sum_{\mu+\nu+\alpha+\beta=l} (p_\mu\eta^{-\mu}, q_\nu\eta^{-\nu})^{(\alpha,\beta)}. \tag{8}$$

We will consider in Ω the coordinate system (x_0, y_0, p_0, ζ), where $x_0 = -x$, $y_0 = -y$, $p_0 = -\xi/\eta$ and $\zeta = \eta$. Remark that $\partial_x = -\partial_{x_0}$, $\partial_y = -\partial_{y_0}$, $\partial_\xi = -\zeta^{-1}\partial_{p_0}$ and $\partial_\eta = \partial_\zeta - \zeta^{-1}p_0\partial_{p_0}$. Hence, $x\partial_x = x_0\partial_{x_0}$, $y\partial_y = y_0\partial_{y_0}$,

$\xi\partial_\xi = p_0\partial_{p_0}$ and $\eta\partial_\eta = \zeta\partial_\zeta - p_0\partial_{p_0}$. Given φ, $\psi \in \mathbb{C}\{x_0, y_0, p_0\}$,

$$\partial_\xi^\alpha(\varphi\zeta^{-\mu}) = (-1)^\alpha\left(\partial_{p_0}^\alpha\varphi\right)\zeta^{-\mu-\alpha},$$

$$\partial_\eta^\beta(\varphi\zeta^{-\mu}) = (-1)^\beta\left(\prod_{0\le k<\beta}(\mu+k+p_0\partial_{p_0})\varphi\right)\zeta^{-\mu-\beta},$$

$$\partial_x^\alpha(\psi\zeta^{-\nu}) = (-1)^\alpha\left(\partial_{x_0}^\alpha\psi\right)\zeta^{-\nu},$$

$$\partial_y^\beta(\psi\zeta^{-\nu}) = (-1)^\beta\left(\partial_{y_0}^\beta\psi\right)\zeta^{-\nu}.$$

Moreover, $(\varphi\zeta^{-\mu}, \psi\zeta^{-\nu})^{(\alpha,\beta)}$ equals

$$\frac{1}{\alpha!\beta!}\left(\partial_{p_0}^\alpha\prod_{0\le k<\beta}(\mu+k+p_0\partial_{p_0})\varphi\right)\left(\partial_{x_0}^\alpha\partial_{y_0}^\beta\psi\right)\zeta^{-\mu-\nu-\alpha-\beta}. \tag{9}$$

A formal expression

$$U = \sum_{l\ge 0} u_l\zeta^{-l}, \quad u_l \in \mathbb{C}\{x_0, y_0, p_0\},$$

defines a microdifferential operator near $q = (0, dy)$ if and only if there exists an open neighborhood Γ of $(0,0,0)$ where the u_l's converge and

$$\limsup_{l\to\infty}\sqrt[l]{\sup_{z\in\Gamma}|u_l(z)|/l!} < \infty. \tag{10}$$

Given formal expressions $\overline{U} = \sum_{l\ge 0}\overline{u}_l\zeta^{-l}$ and $U = \sum_{l\ge 0}u_l\zeta^{-l}$ where $\overline{u}_l, u_l \in \mathbb{C}\{x_0, y_0, p_0\}$ we denote $\overline{U} \gg U$ if there is $N \in \mathbb{Z}$, $K \in \mathbb{R}$, $N, K > 0$, such that $K^l\overline{u}_l \gg u_l$ (i.e., $K^l\overline{u}_l$ estimates u_l) for all $l \ge N$. In view of (10) if $\overline{U} \gg U$ and $\overline{U}$ defines a microdifferential operator near $(0, dy)$, U also defines a microdifferential operator near $(0, dy)$. We extend the notation to matrices in the obvious way. In order to prove that a formal expression as above defines a convergent microdifferential operator, we introduce a microlocal version of the majorant method. The following lemma follows immediately from (9):

Lemma 3.1. *Let $\varphi_1, \overline{\varphi}_1, \varphi_2, \overline{\varphi}_2 \in M_d(\mathbb{C}\{x_0, y_0, p_0\})$ and let $\alpha, \beta, \mu, \nu \in \mathbb{N}$. Set $\overline{\varphi}\zeta^{-\mu-\nu-\alpha-\beta} = (\overline{\varphi}_1\zeta^{-\mu}, \overline{\varphi}_2\zeta^{-\nu})^{(\alpha,\beta)}$, $\varphi\zeta^{-\mu-\nu-\alpha-\beta} = (\varphi_1\zeta^{-\mu}, \varphi_2\zeta^{-\nu})^{(\alpha,\beta)}$ with $\overline{\varphi}, \varphi \in M_d(\mathbb{C}\{x_0, y_0, p_0\})$. If $\overline{\varphi}_1 \gg \varphi_1$ and $\overline{\varphi}_2 \gg \varphi_2$ then $\overline{\varphi} \gg \varphi$.*

The two following results are particular cases of the main result in [**16**] and its proof.

Lemma 3.2. *Let $c_1, \dots, c_t$ be complex numbers such that 0 does not belong to its convex hull. Let $b, f \in \mathbb{C}\{z_1, \dots, z_t\}$ be analytic functions at the origin.*

Assume that $b(0) \notin c_1\mathbb{N} + \cdots + c_t\mathbb{N}$ *or* $b(0) = 0$. *There is* $u \in \mathbb{C}\{z_1, \ldots, z_t\}$ *solution of*

$$\Big(c_1 z_1 \frac{\partial}{\partial z_1} + \cdots + c_t z_t \frac{\partial}{\partial z_t} - b\Big) u = f, \tag{11}$$

if and only if $f \in (z_1, \ldots, z_t, b)$. *In the former case the solution* u *is unique. In the later case there is a one to one correspondence between the solutions of* (11) *and the Cauchy data* $u(0)$.

Lemma 3.3. *Let* $c_1, \ldots, c_t$ *be complex numbers such that 0 does not belong to its convex hull and* λ' *a complex number such that* $\{\lambda'\} \cap (c_1\mathbb{N} + \cdots + c_t\mathbb{N}) \subset \{0\}$. *Then there exists an* $\varepsilon > 0$ *verifying,*

$$|l_1 c_1 + \cdots + l_t c_t - \lambda'| \geq \varepsilon(l_1 + \cdots + l_t), \quad \forall l_1, \ldots, l_t \in \mathbb{N}.$$

Given $a, b \in \mathbb{R}$, $0 < a < b$, and $c \in (\mathbb{C} \setminus \mathbb{R}) \cup]0,1[$, set $P = c x_0 \partial_{x_0} + y_0 \partial_{y_0} + (1-c) p_0 \partial_{p_0}$ and $\overline{P} = a x_0 \partial_{x_0} + b y_0 \partial_{y_0} + (b-a) p_0 \partial_{p_0}$.

Lemma 3.4. *Given* $\lambda' \in \mathbb{C}$ *such that* $\{\lambda'\} \cap (c\mathbb{N} + (1-c)\mathbb{N}) \subset \{0\}$ *there exists* $\delta = \delta(a, b, c, \lambda') > 0$, *such that for* $l \in \mathbb{N}$, $l \geq 1$, $\varphi, \overline{\varphi} \in \mathbb{C}\{x_0, y_0, p_0\}$,

$$(\overline{P} + bl)\overline{\varphi} \gg (P + l - \lambda')\varphi \quad \Rightarrow \quad \overline{\varphi} \gg \delta\varphi. \tag{12}$$

Proof. Write

$$\varphi = \sum_{\alpha,\beta,\gamma \geq 0} \varphi_{\alpha,\beta,\gamma} {x_0}^\alpha {y_0}^\beta {p_0}^\gamma, \qquad \overline{\varphi} = \sum_{\alpha,\beta,\gamma \geq 0} \overline{\varphi}_{\alpha,\beta,\gamma} {x_0}^\alpha {y_0}^\beta {p_0}^\gamma.$$

The left-hand side of (12) is equivalent to say that

$$((\alpha + \beta + l)a + (\beta + \gamma + l)(b - a))\overline{\varphi}_{\alpha,\beta,\gamma}$$

estimates

$$|(\alpha + \beta + l)c + (\beta + \gamma + l)(1 - c) - \lambda'||\varphi_{\alpha,\beta,\gamma}|,$$

for all $\alpha, \beta, \gamma \geq 0$ and for all $l \geq 1$. By the hypothesis on c and λ' along with Lemma 3.3, there is an $\varepsilon > 0$ such that for all $\alpha, \beta, \gamma \geq 0$ and for all $l \geq 1$,

$$|(\alpha + \beta + l)c + (\beta + \gamma + l)(1 - c) - \lambda'| \geq \varepsilon\Big((\alpha + \beta + l) + (\beta + \gamma + l)\Big).$$

Choosing $\delta \in \mathbb{R}$, $0 < \delta < \varepsilon / \max\{a, b - a\}$,

$$|(\alpha + \beta + l)c + (\beta + \gamma + l)(1 - c) - \lambda'|$$

estimates

$$\delta((\alpha + \beta + l)a + (\beta + \gamma + l)(b - a)) > 0,$$

which implies the right-hand side of (12). ☐

In the sequel we shall denote the identity matrix of order t by I_t.

Theorem 3.5. *Consider the microdifferential Cauchy problem*

$$\left\{\begin{array}{l}[\vartheta', U]-[A_0, U]-\mathcal{A}_{-1}U=0,\\ \sigma_0(U)((0,dy))=I_d,\end{array}\right. \tag{13}$$

where $\vartheta' = cx\partial_x + y\partial_y$, $c \in (\mathbb{C}\setminus\mathbb{R})\cup]0,1[$, $A_0 \in M_d(\mathbb{C})$ *and* $\mathcal{A}_{-1} \in M_d(\mathcal{E}_{X}(-1))$. *Assume that* A_0 *is semisimple with eigenvalues* $\lambda'_0,\dots,\lambda'_{d-1}$ *verifying*

$$\{\lambda'_i-\lambda'_j : 0\le i,j\le d-1\}\cap\Big(c\mathbb{N}+(1-c)\mathbb{N}\Big)\subset\{0\}. \tag{14}$$

Then there exists, in a neighborhood Ω *of* $(0,dy)$, *one and only one invertible matrix* $U\in M_d(\mathcal{E}_{X}(0))(\Omega)$ *solution of* (13).

Proof. We can assume that $A_0=\operatorname{diag}(\lambda'_0,\dots,\lambda'_{d-1})$ and $\mathcal{A}_{-1}=\sum_{l\ge1}a_l\zeta^{-l}$ with $a_l\in\mathbb{C}\{x_0,y_0,p_0\}$ for all l.

Let $f=\sigma(\vartheta')=cx\xi+y\zeta$ be the principal symbol of ϑ' and let $H_f = c\,x\partial_x+y\partial_y-c\,p\partial_p-\eta\partial_\eta = c\,x_0\partial_{x_0}+y_0\partial_{y_0}+(1-c)\,p_0\partial_{p_0}-\zeta\partial_\zeta = P-\zeta\partial_\zeta$ be the corresponding Hamiltonian vector field. Let us find a formal series

$$U=\sum_{l\ge0}u_l\zeta^{-l},\quad u_l\in \mathrm{M}_d(\mathbb{C}\{x_0,y_0,p_0\}),$$

such that $H_f(u_l\zeta^{-l})-[A_0,u_l]\zeta^{-l}=\omega_l$, $\forall l\ge0$, where

$$\omega_l = \sum_{\substack{\nu+\mu+\alpha+\beta=l\\ \mu\ge1}}(a_\mu\zeta^{-\mu},u_\nu\zeta^{-\nu})^{(\alpha,\beta)},\qquad \alpha,\beta,\nu,\mu\in\mathbb{N}. \tag{15}$$

The matrices u_j are solution of the system of differential equations

$$(P+l)u_l-[A_0,u_l]=\omega_l,\quad l\ge0. \tag{16}$$

Since $\omega_0=0$, $u_0=I_d$ is the solution of (16) for $l=0$ with Cauchy data $u_0(0)=I_d$ (see Lemma 3.2). Assume that $l\ge1$. Set $\lambda'_{i,j}=\lambda'_i-\lambda'_j$, $i,j=0,\dots,d-1$. Writing $u_l=(u_{l;i,j})_{i,j}$ and $\omega_l=(\omega_{l;i,j})_{i,j}$, the system (16) is equivalent to the system of d^2 first order linear differential equations with degenerate principal symbols

$$(P+l-\lambda'_{i,j})u_{l;i,j}=\omega_{l;i,j},\qquad i,j\in\{0,\dots,d-1\}. \tag{17}$$

It follows from (15) that $\omega_{l;i,j}$ only depends on the entries $u_{\nu;i,j}$ for $\nu<l$.

By the hypothesis zero does not belong to the convex hull of $\{c,1-c,1\}$. Since $l\ge1$, $\lambda'_{i,j}-l\ne0$. By Lemma 3.2 we can find, recursively, unique analytic functions $u_{l;i,j}$, $i,j=0,\dots,d-1$, verifying (17). In order to finish the proof of Theorem 3.5 it is enough to show that $U=\sum_{l\ge0}u_{l;i,j}\zeta^{-l}$, $i,j\in\{0,\dots,d-1\}$, is the symbol of a convergent microdifferential operator.

We denote by $(e)_d$ the matrix of type $d\times d$ with all the entries equal to e. Notice that $(e)_d(e')_d=d(ee')_d$. There is a convergent microdifferential operator $Q=\sum_{l\ge1}q_l\zeta^{-l}$, defined in a small neighborhood of $(0,dy)$, such that $(q_l)_d \gg a_l$ for all $l \geq 1$ (see for instance the proof of Theorem 3.2 of [17]). Set $\overline{\vartheta} = ax\partial_x + by\partial_y$ where $a, b \in \mathbb{R}$, $0 < a < b$. Set $\overline{f} := \sigma(\overline{\vartheta}) = ax\xi + by\eta$. Then $H_{\overline{f}} = \overline{P} - b\zeta\partial_\zeta$, where $\overline{P} = a\, x_0\partial_{x_0} + b\, x_0\partial_{y_0} + (b-a)p_0\partial_{p_0}$. Let $V = \sum_{l\geq 0} v_l\zeta^{-l} \in \mathcal{E}_{X(0)}$ be the unique invertible microdifferential operator such that

$$V^{-1}(\overline{\vartheta} - Q)V = \overline{\vartheta}, \qquad \sigma_0(V)(0, dy) = 1 \tag{18}$$

(see Lemma 3.2 in [17]). Actually $v_0 = 1$ is the unique solution of the Cauchy problem with degenerate principal symbols, $\overline{P}v_0 = 0$, $v_0(0) = 1$, and v_l, $l \geq 1$, is determined recursively by $(\overline{P} + bl)v_l = \overline{\omega}_l$ where

$$\overline{\omega}_l\zeta^{-l} = \sum_{\substack{\nu+\mu+\alpha+\beta=l\\ \mu\geq 1}} (q_\mu\zeta^{-\mu}, v_\nu\zeta^{-\nu})^{(\alpha,\beta)}.$$

Fix $\delta \in \mathbb{R}$, $0 < \delta \ll 1$ in conditions of Lemma 3.4 for all $\lambda'_{i,j}$'s. Set $\varepsilon = \delta^{-1}d$. Let us prove by induction that $\varepsilon^l(v_l)_d \gg u_l$ for all l, that is, $(V)_d \gg U$. By the hypothesis, $(v_0)_d = (1)_d \gg u_0 = I_d$. Assume that $\varepsilon^\nu(v_\nu)_d \gg u_\nu$ for $\nu \leq l-1$. Since $(q_\mu)_d \gg a_\mu$, Lemma 3.1 yields

$$\sum_{\substack{\nu+\mu+\alpha+\beta=l\\ \mu\geq 1}} \Big((q_\mu\zeta^{-\mu})_d, \varepsilon^\nu(v_\nu\zeta^{-\nu})_d\Big)^{(\alpha,\beta)} \gg \sum_{\substack{\nu+\mu+\alpha+\beta=l\\ \mu\geq 1}} \Big(a_\mu\zeta^{-\mu}, u_\nu\zeta^{-\nu}\Big)^{(\alpha,\beta)}.$$

Since $\varepsilon \geq 1$, the left-hand side of the previous relation is estimated by

$$\sum_{\substack{\nu+\mu+\alpha+\beta=l\\ \mu\geq 1}} \Big((q_\mu\zeta^{-\mu})_d, \varepsilon^{l-1}(v_\nu\zeta^{-\nu})_d\Big)^{(\alpha,\beta)},$$

that equals

$$d\varepsilon^{l-1}\left(\sum_{\substack{\nu+\mu+\alpha+\beta=l\\ \mu\geq 1}} \Big(q_\mu\zeta^{-\mu}, v_\nu\zeta^{-\nu}\Big)^{(\alpha,\beta)}\right)_d.$$

Therefore by definition of ω_l and $\overline{\omega}_l$ we have $d\varepsilon^{l-1}\overline{\omega}_l \gg \omega_{l;i,j}$ for all i, j hence

$$d\varepsilon^{l-1}(\overline{P} + bl)v_l \gg (P + l - \lambda'_{i,j})u_{l;i,j},$$

for all $i, j \in \{0, \ldots, d-1\}$. By Lemma 3.4, $\varepsilon^l v_l \gg u_{l;i,j}$ for all i, j. $\square$

4. Main results.

In the following we shall denote the nilpotent Jordan block of order t by N_t.

Lemma 4.1. *Let $\alpha \in \mathbb{C}$ and assume that $Y \in M_{\mu\times\nu}(\mathbb{C}\{x\}[x^{-1}])$ verifies the differential equation*

$$\left(x\frac{d}{dx} - \alpha\right)Y = YN_\nu - N_\mu Y. \tag{19}$$

Then $Y = Cx^\alpha$, where $C \in M_{\mu\times\nu}(\mathbb{C})$ verifies $CN_\nu - N_\mu C = 0$. In particular, if $\alpha \notin \mathbb{Z}$, $Y = 0$.

Proof. We will identify a matrix $Y = (a_{i,j}) \in \mathrm{M}_{\mu\times\nu}$ with a family $(a_{i,j})$, $i, j \in \mathbb{Z}$, such that $a_{i,j} = 0$ if $i \notin \{0, \dots, \mu-1\}$ or $j \notin \{0, \dots, \nu-1\}$. We will show by recursion on t that there is a constant c_t such that $a_{i,i+t} = c_t x^\alpha$ if $0 \le i \le \mu - 1$, $0 \le i + t \le \nu - 1$. The statement is true for t small enough. Let us assume that it holds for a certain t. Then

$$\left(x\frac{d}{dx} - \alpha\right) a_{i,i+t+1} = a_{i,i+t} - a_{i+1,i+t+1} = 0. \tag{20}$$

Hence there are complex numbers $c_{i,t+1}$ such that $a_{i,i+t+1} = c_{i,t+1}x^\alpha$. Moreover,

$$\left(x\frac{d}{dx} - \alpha\right) a_{i,i+t+2} = \Big(c_{i,t+1} - c_{i+1,t+1}\Big)x^\alpha.$$

Hence $c_{i,t+1}$ does not depend on i. □

Theorem 4.2. *Let L be a free $\mathbb{C}\{x\}$-module of dimension k. Let ∇ be a $\mathbb{C}$-linear endomorphism of L such that*

$$\nabla(fu) = x\frac{df}{dx}u + f\nabla u,\ \ f \in \mathbb{C}\{x\},\ \ u \in L.$$

Let p be a $\mathbb{C}\{x\}$-linear endomorphism of L such that $[\nabla, p] = ((n-k)/k)p$ and $p^k = (n/k)^k x^{n-k}$. There are complex numbers λ_i, $i \in \mathbb{Z}$, and a system of generators of L, u_i, $i \in \mathbb{Z}$, such that (1) holds, $(u_0, \dots, u_{k-1})$ is a basis of L and

$$u_{i+k} = u_i, \qquad \nabla u_i = \lambda_i u_i, \qquad pu_i = \frac{n}{k}\,x^{\alpha_i}u_{i+1}, \qquad i \in \mathbb{Z}.$$

Proof. Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be respectively the matrices of the actions of ∇ and p with respect to a basis $(u_0, \dots, u_{k-1})$ of L. Set $u_j = u_i$ if $j - i \equiv 0 \pmod{k}$, for each $j \in \mathbb{Z}$. Since $\nabla(pu_i) = [\nabla, p]u_i + p\nabla u_i$,

$$\sum_{j=0}^{k-1} x\frac{db_{j,i}}{dx}u_j + \sum_{j=0}^{k-1} b_{j,i}\sum_{l=0}^{k-1} a_{l,j}u_l = \frac{n-k}{k}\sum_{j=0}^{k-1} b_{j,i}u_j + \sum_{j=0}^{k-1} a_{j,i}\sum_{l=0}^{k-1} b_{l,j}u_l.$$

Therefore,

$$\left(x\frac{d}{dx} - \frac{n-k}{k}\right)B = [B, A]. \tag{21}$$

Assume the additional hypothesis *A is constant.* We can assume that A is the direct sum of l Jordan blocks A_r of size m_r and eigenvalue λ_r, $0 \le r \le l-1$, where $1 \le l \le k$. Consider the block decomposition $B = (B_{r,s})$, $0 \le r, s \le l-1$, $B_{r,s} \in \mathrm{M}_{m_r\times m_s}(\mathbb{C}\{x\})$. Let $A_r = \lambda_r I_{m_r} + N_{m_r}$ be the decomposition of the Jordan block A_r into semisimple and nilpotent parts.

We get a block decomposition $[B,A] = (B_{r,s}A_s - A_rB_{r,s})$, $B_{r,s}A_s - A_rB_{r,s} = (\lambda_s - \lambda_r)B_{r,s} + B_{r,s}N_{m_s} - N_{m_r}B_{r,s}$. Hence

$$\left(x\frac{d}{dx} + \lambda_r - \lambda_s - \frac{n-k}{k}\right)B_{r,s} = B_{r,s}N_{m_s} - N_{m_r}B_{r,s}.$$

By Lemma 4.1 there are matrices $C_{r,s} \in \mathrm{M}_{m_r\times m_s}(\mathbb{C})$, $0 \le r,s \le l-1$, such that

$$B_{r,s} = C_{r,s}x^{\lambda_s - \lambda_r + \frac{n-k}{k}}. \tag{22}$$

Hence $B_{r,s} = 0$ or $\lambda_s - \lambda_r + \frac{n-k}{k} \in \mathbb{Z}$. Since we have assumed $2 \le k \le n-1$ and $(n,k) = 1$,

$$B_{r,r} = 0 \qquad \text{and} \qquad B_{r_0,r_1}, B_{r_1,r_2}, \ldots, B_{r_{t-1},r_t} \neq 0 \Rightarrow B_{r_0,r_t} = 0, \tag{23}$$

for $t = 2,\ldots,l-1$. In particular $l \ge 2$. Since $B^k = (n/k)^k x^{n-k} I_k$, there are integers $i_0,\ldots,i_{k-1}$, s.t. $0 \le i_j \le l-1$ and $B_{i_0,i_1}B_{i_1,i_2}\ldots B_{i_{k-1},i_0} \neq 0$. If $0 \le r < s \le k-1$, $i_r \neq i_s$. Otherwise there would exist a constant matrix $C \neq 0$ s.t. $B_{i_r,i_{r+1}}\ldots B_{i_{s-1},i_s} = x^{(s-r)(n-k)/k}C$ and $(s-r)(n-k)/k \in \mathbb{Z}$. Hence $l = k$ and the map $j \mapsto i_j$ defines a circular permutation of $\{0,\ldots,k-1\}$. We can assume $B_{j,i} \neq 0$ if $j \equiv i+1 \pmod{k}$, $i = 0,\ldots,k-1$. By (23) $B_{j,i} = 0$ if $j \not\equiv i+1 \pmod{k}$, $i = 0,\ldots,k-1$. Therefore A is a diagonal matrix with eigenvalues λ_i, $0 \le i \le k-1$. Moreover, $pu_i = C_{i+1,i}x^{\alpha_i}u_{i+1}$, where

$$\alpha_i = \lambda_i - \lambda_{i+1} + \frac{n-k}{k},\ i \in \mathbb{Z}, \tag{24}$$

and $\lambda_j = \lambda_i$ if $j \equiv i \pmod{k}$. By Lemma 4.1, $\alpha_i \in \mathbb{Z}$. Since $(u_0,\ldots,u_{k-1})$ is a basis of the $\mathbb{C}\{x\}$-module L, $\alpha_i \ge 0$. Up to a $\mathbb{C}$-linear change of basis, $C_{i+1,i} = n/k$ for $0 \le i \le k-1$.

Let us prove the proposition without the additional hypothesis A *is constant*. If the eigenvalues of $A(0)$ do not differ by a nonzero integer we can assume $A = A(0)$ (see [**24**]) and the proposition is proved. In order to finish the proof it is enough to show that two eigenvalues of $A(0)$ cannot differ by a nonzero integer. Assume otherwise. Let S be an invertible $\mathbb{C}\{x\}[x^{-1}]$-linear transformation of $\mathbb{C}\{x\}[x^{-1}]\otimes_{\mathbb{C}\{x\}}L$. We say that S is a *shearing transformation* if there is a $\mathbb{C}\{x\}[x^{-1}]$-linear decomposition $\mathbb{C}\{x\}[x^{-1}]\otimes_{\mathbb{C}\{x\}}L = \oplus_{i\in I}L_i$ and a family of integers m_i, $i \in I$, such that $S|_{L_i} = x^{m_i}$. Up to a shearing transformation of $\mathbb{C}\{x\}[x^{-1}] \otimes_{\mathbb{C}\{x\}} L$, we can assume that one of the eigenvalues of $A(0)$ has multiplicity bigger than one and that two eigenvalues of $A(0)$ do not differ by a nonzero integer. Hence we can assume $A = A(0)$. Therefore there are $i,j \in \mathbb{Z}$, $1 \le j \le k-1$, such that $\lambda_i = \lambda_{i+j}$. By (24),

$$j\frac{n-k}{k} = \alpha_i - \alpha_{i+j} \in \mathbb{Z}.$$

□

Let R denote the $\mathbb{C}$-algebra

$$\mathbb{C}\{x, p, t_1, \dots, t_{m-2}\}/\left(p^k - \left(\frac{n}{k}\right)^k x^{n-k}\right). \tag{25}$$

The derivation $x\partial_x + ((n-k)/k)p\partial_p$ of $\mathbb{C}\{x, p, t_1, \dots, t_{m-2}\}$ leaves invariant the ideal $(p^k - (n/k)^k x^{n-k})$, inducing a derivation Δ of R.

Theorem 4.3. *Let L be a finitely generated torsion-free R-module of rank one. Let ∇ be a $\mathbb{C}$-linear endomorphism of L such that*

$$\nabla(fu) = \Delta(f)u + f\nabla u, \;\; f \in R, \;\; u \in L. \tag{26}$$

If ∂_{t_j}, $1 \le j \le m-2$, act on L as $\mathbb{C}$-linear endomorphisms and

$$\partial_{t_j}(fv) = \frac{\partial f}{\partial t_j}v + f\partial_{t_j}v, \qquad 1 \le j \le m-2, \tag{27}$$

for $f \in R$ and $v \in L$, L is a free $\mathbb{C}\{x, t_1, \dots, t_{m-2}\}$-module of rank k. Moreover, there is a system of generators of L, v_i, $i \in \mathbb{Z}$, and complex numbers λ_i, $i \in \mathbb{Z}$, such that $(v_0, \dots, v_{k-1})$ is a basis of L as a $\mathbb{C}\{x, t_1, \dots, t_{m-2}\}$-module, relation (1) *holds and*

(28)
$$v_{i+k} = v_i, \quad \nabla v_i = \lambda_i v_i, \quad pv_i = \frac{n}{k}x^{\alpha_i}v_{i+1}, \quad \partial_{t_j}v_i = 0, \quad 1 \le j \le m-2.$$

Proof. Assume in the first place that $m = 2$. Since L is a finitely generated R-module and R is a finitely generated $\mathbb{C}\{x\}$-module, L is a finitely generated $\mathbb{C}\{x\}$-module. Since $\mathbb{C}\{x\}$ is a principal ideal domain and L is a torsion-free $\mathbb{C}\{x\}$-module, L is a finitely free $\mathbb{C}\{x\}$-module. Let l be the dimension of the $\mathbb{C}\{x\}$-module L.

Let K be the quotient field of R. Since the rings $\mathbb{C}\{x\}[x^{-1}] \otimes_{\mathbb{C}\{x\}} R$ and $\mathbb{C}\{x\}[x^{-1}][p]/(p^k - (n/k)^k x^{n-k})$ are isomorphic and $p^k - (n/k)^k x^{n-k}$ is an irreducible polynomial over the field $\mathbb{C}\{x\}[x^{-1}]$, $\mathbb{C}\{x\}[x^{-1}] \otimes_{\mathbb{C}\{x\}} R$ is a field. Hence

$$\mathbb{C}\{x\}[x^{-1}] \otimes_{\mathbb{C}\{x\}} R \xrightarrow{\sim} K. \tag{29}$$

Since L has rank one, K is a $\mathbb{C}\{x\}[x^{-1}]$-vector space of dimension k. By (29), $\mathbb{C}\{x\}[x^{-1}] \otimes_{\mathbb{C}\{x\}} L$ is a $\mathbb{C}\{x\}[x^{-1}]$-vector space of dimension k. Therefore $l = k$.

The action of p on L induces a $\mathbb{C}\{x\}$-linear endomorphism of L such that $p^k = (n/k)^k x^{n-k}$. Since $[\nabla, p] = ((n-k)/k)p$, the proof in the case $m = 2$ follows from Theorem 4.2. Let us prove the theorem when $m \ge 3$.

Let (t) denote the ideal of R generated by $t_1, \dots, t_{m-2}$. Set $\widetilde{L} = L/(t)L$. The $\mathbb{C}\{x, p\}$-module $\widetilde{L}$ verifies the assumptions of the theorem with $m = 2$. Let v_i, $i \in \mathbb{Z}$, be a system of generators of $\mathbb{C}\{x\}$-module $\widetilde{L}$ in the conditions of Theorem 4.3. Take $w_i \in L$ such that $w_{i+k} = w_i$ and $v_i = w_i + (t)L$ for all i. Let M be the $\mathbb{C}\{x, t_1, \dots, t_{m-2}\}$-module generated by the w_i's. Since

$L = M + (t)^l L$ for all l, $M = \cap_l \big(M + (t)^l L\big) = L$ (see [20], Proposition II.1.1.3).

We will denote (25) by R_{m-2}. We will show by induction that $w_0, ..., w_{k-1}$ are linearly independent over R_q, $q = 0, \ldots, m-2$. The statement is true for $q = 0$. Assume that it holds for a certain q, $2 \leq q \leq m-3$. Assume that there are $a_i \in R_{q+1}$ such that $\sum_{i=0}^{k-1} a_i w_i = 0$ and some of the a_i's do not vanish. There are a nonnegative integer l and $i_0 \in \{0, \ldots, k-1\}$ such that $a_i \in (t_{q+1})^l$, $0 \leq i \leq k-1$, and $a_{i_0} \notin (t_{q+1})^{l+1}$. Since L is torsion free,

$$\sum_{i=0}^{k-1} (a_i / t_{q+1}^l) w_i = 0.$$

Hence $\sum_{i=0}^{k-1} ((a_i/t_{q+1}^l) + (t_{q+1}))(w_i + (t_{q+1})L) = 0$, contradicting the induction hypothesis. Hence L is a free $\mathbb{C}\{x, t_1, \ldots, t_{m-2}\}$-module of rank k.

Let $c_{l,j,\nu} \in \mathbb{C}\{x, t_1, \ldots, t_{m-2}\}$ such that $\partial_{t_j} w_l = \sum_{\nu=0}^{k-1} c_{l,j,\nu} w_l$. Given $v = \sum_{l=0}^{k-1} a_l w_l$, ∂_{t_1} annihilates v if and only if

$$\frac{\partial a_l}{\partial t_1} + \sum_{\nu=0}^{k-1} a_\nu c_{\nu,1,l} = 0, \qquad l = 0, \ldots, k-1. \tag{30}$$

Let $a_{1,0,l}, \ldots, a_{1,k-1,l} \in \mathbb{C}\{x, t_1, \ldots, t_{m-2}\}$ be the solution of (30) with Cauchy data $a_{1,\nu,l}(0) = \delta_{\nu,l}$. Here $\delta_{\nu,l}$ denotes the Kronecker symbol. Replacing w_l by $\sum_{\nu=0}^{k-1} a_{1,\nu,l} w_\nu$, $l = 0, \ldots, k-1$, we can assume from the beginning that $\partial_{t_1} w_l = 0$ for all l. Now assume that $\partial_{t_j} w_l = 0$ for $l = 0, \ldots, k-1$, $j = 1, \ldots, q$ ($q \leq m-3$). Let $c_{l,q+1,\nu} \in \mathbb{C}\{x, t_1, \ldots, t_{m-2}\}$ such that $\partial_{t_{q+1}} w_l = \sum_{\nu=0}^{k-1} c_{l,q+1,\nu} w_\nu$. Since $\partial_{t_j} w_l$ vanishes for all $j = 1, \ldots, q$,

$$\frac{\partial c_{l,q+1,\nu}}{\partial t_j} = 0, \qquad l, \nu = 0, \ldots, k-1, \quad j = 1, \ldots, q.$$

By the argument above we can replace, w_l, $l = 0, ..., k-1$, by $\sum_{\nu=0}^{k-1} a_{q+1,\nu,l} w_\nu$ with $a_{q+1,\nu,l} \in \mathbb{C}\{x, t_{q+1}, \ldots, t_{m-2}\}$, $a_{q+1,\nu,l}(0) = \delta_{\nu,l}$. Hence we can assume that $\partial_{t_j} w_l = 0$ for $l = 0, \ldots, k-1$, $j = 1, \ldots, m-2$.

There are $\varepsilon_i, \delta_i \in (t)L$ such that

$$\nabla w_i = \lambda_i w_i + \varepsilon_i, \quad p w_i = \frac{n}{k} x^{\alpha_i} w_{i+1} + \delta_i, \qquad 0 \leq i \leq k-1.$$

Since $\partial_{t_j} w_i$ vanishes,

$$\partial_{t_j} \varepsilon_i = \partial_{t_j} (\lambda_i w_i + \varepsilon_i) = \partial_{t_j} (\nabla w_i) = \nabla \partial_{t_j} w_i = 0,$$

for $i \in \mathbb{Z}$, $j = 1, \ldots, m-2$. A similar argument shows that $\partial_{t_j} \delta_i = 0$, $i \in \mathbb{Z}$, $j = 1, \ldots, m-2$. □

The map that takes x into t^k, p into $\frac{n}{k} t^{n-k}$ and t_j into t_j for all j, identifies the integral closure of R with the power series ring $\mathbb{C}\{t, t_1, \ldots, t_{m-2}\}$. Set $\mathcal{K} = \mathbb{C}\{t, t_1, \ldots, t_{m-2}\}[t^{-1}]$. Let $v : \mathcal{K} \to \mathbb{Z} \cup \{+\infty\}$ denote the canonical valuation by the order of the zero in the variable t. The semigroup of R is by definition the additive sub-semigroup $\Gamma = (v(R) \setminus \{+\infty\})$ of $\mathbb{Z}$. A subset Σ of $\mathbb{Z}$ is called a Γ-*module* if $\Gamma + \Sigma = \Sigma$. Two subsets Σ_1, Σ_2 of $\mathbb{Z}$ such that $\Gamma + \Sigma_i = \Sigma_i$ are isomorphic as Γ-modules if and only if there is an integer σ verifying $\Sigma_2 = \sigma + \Sigma_1$.

Given a torsion free R-module L of rank one let $\phi : \mathcal{K} \otimes_R L \to \mathcal{K}$ be an isomorphism of $\mathcal{K}$-modules. Let Σ_ϕ denote the Γ-module defined by the intersection of $\mathbb{Z}$ with the image of the map $L \xrightarrow{-\otimes 1} \mathcal{K} \otimes_R L \xrightarrow{\phi} \mathcal{K} \xrightarrow{v} (\mathbb{Z} \cup \{+\infty\})$. The set Σ_ϕ depends on ϕ but its isomorphism class as a Γ-module does not. We can choose ϕ in a way such that the minimum of Σ_ϕ equals 0. We will denote this set by $\Sigma(L)$.

Given a family of nonnegative integers (α_i), $i \in \mathbb{Z}$, such that (3) holds, denote by $L_{(\alpha_i)}$ the R-module generated by v_i, $i \in \mathbb{Z}$, with relations $v_{i+k} = v_i$ and $pv_i = \frac{n}{k} x^{\alpha_i} v_{i+1}$.

Proposition 4.4. *Two R-modules $L_{(\alpha_i)}$ and $L_{(\beta_i)}$ are isomorphic if and only if there is $\nu \in \mathbb{Z}$, $0 \le \nu \le k-1$, such that $\alpha_{i+\nu} = \beta_i$ for all $i \in \mathbb{Z}$.*

Proof. The *if part* is clear. Let us prove the *only if part*. We can compute $\Sigma(L_{(\alpha_i)})$ in the following way. Set $\gamma_i = v(\phi(v_i))$. Since $v_{i+k} = v_i$ and $\phi(v_{i+1}) = \phi(v_i)\phi(p)\phi\big(\frac{n}{k}x^{\alpha_i}\big)^{-1}$,

$$\gamma_{i+k} = \gamma_i, \qquad \gamma_{i+1} = \gamma_i + n - k - k\alpha_i, \qquad i \in \mathbb{Z}. \tag{31}$$

Hence $\gamma_i \equiv \gamma_j \pmod{k}$ if and only if $i \equiv j \pmod{k}$. After performing a translation of $\mathbb{Z}$ and the replacement of ϕ by another isomorphism from $\mathcal{K} \otimes_R L$ onto $\mathcal{K}$, we can assume that $\gamma_0 = \min\{\gamma_i : i \in \mathbb{Z}\} = 0$. Under this assumption

$$\gamma_i = (n-k)i - k(\alpha_0 + \cdots + \alpha_{i-1}), \qquad i \in \mathbb{Z}. \tag{32}$$

If $w \in L_{(\alpha_i)} \setminus \{0\}$ there are $m_i \in \mathbb{Z}$, $f_i \in \mathcal{K}$, $0 \le i \le k-1$, such that $m_i \ge 0$ and $w = \sum_{i=0}^{k-1} x^{m_i} f_i v_i$. Since $v(x^{m_i} f_i)$ is a multiple of k or $v(x^{m_i} f_i) = +\infty$, $v(\phi(x^{m_i} f_i v_i)) \ne v(\phi(x^{m_j} f_j v_j))$ if $i \ne j$. Therefore

$$\gamma_l = \min\{j \in \Sigma(L(\alpha_i)) \ : \ j \equiv (n-k)l \pmod{k}\}. \tag{33}$$

It follows from (32) and (33) that we can recover the family (α_i) from $\Sigma(L_{(\alpha_i)})$. $\square$

Proposition 4.5. *A torsion free R-module L of rank 1 is free if and only if $\Sigma(L) = \Gamma$. In particular, $L_{(\alpha_i)}$ is free if and only if there is one and only one $i \in \{0, \dots, k-1\}$ such that $\alpha_i \ne 0$.*

Proof. If the R module L is generated by u, Σ_ϕ is generated by an integer l. Let u be an element of L with valuation l. Let us show that L equals Ru. We will identify $\mathcal{K}$ with $\mathbb{C}\{x, p, t_1, \dots, t_{m-2}\}[x^{-1}]$. We will identify L with its image on $\mathcal{K}$. If $w \in L$, there are $a_i \in \mathbb{C}\{x, t_1, \ldots, t_{m-2}\} \setminus (x)$ and $r_i \in \mathbb{Z}$, $0 \leq i \leq k-1$, such that $w = \sum_{i=0}^{k-1} a_i p^i x^{r_i} u$. Hence

$$v(w) = v(u) + \inf\{(n-k)i + kr_i : a_i \neq 0\}.$$

Since $v(w) \in \Sigma_\phi$, $r_i \geq 0$. Hence $w \in L$. $\square$

Theorem 4.6. *Let X be a complex manifold of dimension m. Let Λ be the germ at $q \in T^*X$ of an irreducible conic Lagrangian variety contained in an involutive submanifold of $T^*X \setminus X$ of codimension $m-1$. Given a system of microdifferential equations $\mathcal{M}$ of multiplicity one along Λ, there are complex numbers λ_i, $i \in \mathbb{Z}$ such that (1) holds and after a convenient quantized contact transformation the germ at q of $\mathcal{M}$ is isomorphic to $\mathcal{M}_{(\lambda_i)}$. Two systems $\mathcal{M}_{(\lambda_i)}$ and $\mathcal{M}_{(\mu_i)}$ are isomorphic if and only if there is $\nu \in \{0, \ldots, k-1\}$, such that*

$$\beta_i = \alpha_{i+\nu}, \quad i \in \mathbb{Z}, \qquad \textit{and} \qquad \mu_0 \equiv \lambda_\nu \left(\operatorname{mod} \frac{n}{k}\right), \tag{34}$$

where $\beta_i = \mu_i - \mu_{i+1} + (n-k)/k$.

Proof. Let X be a copy of $\mathbb{C}^m$ with coordinates $(x, y, t_1, \ldots, t_{m-2})$. On a neighborhood of $(0, dy)$ the canonical 1-form θ of T^*X equals $\xi dx + \eta dy + \sum_{i=1}^{m-2} \tau_i dt_i = \eta(dy - (pdx + q_1 dt_1 + \cdots + q_{m-2} dt_{m-2}))$. The conormal of the hypersurface of $\mathbb{C}^m$ with equation $y^k - x^n = 0$ equals the Lagrangian variety defined by the equations

$$y - \frac{k}{n} xp = p^k - \left(\frac{n}{k}\right)^k x^{n-k} = q_1 = \cdots = q_{m-2} = 0. \tag{35}$$

Following [**18**], Theorem 8.3, we can assume that the support of $\mathcal{M}$ equals (35). Let $\mathcal{N}$ be the canonical lattice of $\mathcal{M}$. The fiber at $(0, dy)$ of $\mathcal{O}_\Lambda(0)$ equals R. Let M, N and L denote, respectively, the fibers at $(0, dy)$ of the sheaves $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{N}/\mathcal{N}(-1)$. By Proposition 2.7 L is a finitely generated torsion free R-module of rank one.

Since $[\vartheta, \mathcal{E}_X(0)] \subset \mathcal{E}_X(0)$, $[\vartheta, \mathcal{I}_\Lambda(-1)] \subset \mathcal{I}_\Lambda(-1)$ and (6) holds, the operator ϑ acts on R as a derivation by

$$\vartheta(f) = \sigma_0([\vartheta, P]) = \{\sigma(\vartheta), \sigma_0(P)\} = H_{\sigma(\vartheta)}(f),$$

where $P \in \mathcal{E}_X(0)$ such that $\sigma_0(P) = f$. Here $\{\cdot, \cdot\}$ denotes the Poisson brackets. Since

$$H_{\sigma(\vartheta)} = x\partial_x + \frac{n}{k} y\partial_y + \frac{n-k}{k} p\partial_p - \eta\partial_\eta,$$

$H_{\sigma(\vartheta)}$ acts on $\mathbb{C}\{x, p, t_1, \ldots, t_{m-2}\}$ as the $x\partial_x + \frac{n-k}{k} p\partial_p$-derivation. Hence ϑ acts on R as the derivation Δ. Moreover,

$$[\vartheta, p] = H_{\sigma(\vartheta)}(p) = \frac{n-k}{k} p.$$

By the regularity conditions $\vartheta\mathcal{N}(k) \subset \mathcal{N}(k)$ and $\partial_{t_j}\mathcal{N}(k) \subset \mathcal{N}(k)$ for $k \in \mathbb{Z}$ and $1 \leq j \leq m-2$. If $u \in \mathcal{N}$, $v = u + \mathcal{N}(-1)$, $P \in \mathcal{E}_X(0)$ and $f = \sigma_0(P)$, $\vartheta P u = [\vartheta, P]u + P\vartheta u$. Setting $\nabla = \vartheta$ we deduce that (26) holds. A similar argument shows that (27) holds.

We have shown that L verifies the hypothesis of Theorem 4.3. Let v_i, $i \in \mathbb{Z}$, be a system of generators of L verifying (28). Choose $\widetilde{v}_i \in N$, $i \in \mathbb{Z}$, such that $\widetilde{v}_{i+k} = \widetilde{v}_i$ and $v_i = \widetilde{v}_i + N(-1)$. The $\widetilde{v}_i$'s generate the $\mathcal{E}_{X,(0,dy)}(0)$-module N. For $i \in \mathbb{Z}$ and $l = 1, \ldots, m-2$, set

$$\omega_{i,l} = \partial_{t_l}\widetilde{v}_i, \qquad \omega_i = (\partial_x\partial_y^{-1})\widetilde{v}_i - \left(-\frac{n}{k}\right)x^{\alpha_i}\widetilde{v}_{i+1}.$$

By the microdifferential Cauchy Theorem I. 6.1.1 of [**20**] we can assume that $\omega_{i,l} = 0$ for $i \in \mathbb{Z}$, $l = 1, \ldots, m-2$. $l \geq 0$ $N(-l) = \mathcal{E}_{X,(0,dy)}(-l)N$. We will show that $\omega_i \in N(-l)$ for all $l \geq 1$, $i \in \mathbb{Z}$. We know that $\omega_i \in N(-1)$ for all i. Assume that $\omega_i \in N(-l)$. Since L is a free $\mathbb{C}\{x, t_1, \ldots, t_{m-2}\}$-module of rank k, there are $a_{i,j} \in \mathbb{C}\{x, t_1, \ldots, t_{m-2}\}$, $\varpi_i \in N(-l-1)$ such that

$$\omega_i = \sum_{j=0}^{k-1} a_{i,j}\partial_y^{-l}\widetilde{v}_j + \varpi_i.$$

Since $\partial_{t_s}\omega_i = \sum_{j=0}^{k-1}(\partial a_{i,j}/\partial t_s)\partial_y^{-l}\widetilde{v}_j + \partial_{t_s}\varpi_i$ vanishes for all s, $a_{i,j}$ lies in $\mathbb{C}\{x\}$ for all i, j. In one hand there is $\varpi_i' \in N(-l-1)$ such that

$$\begin{aligned}(36)\qquad \vartheta\omega_i &= \left(\partial_x\partial_y^{-1}\right)\left(\lambda_i + \frac{n-k}{k}\right)\widetilde{v}_i - \left(-\frac{n}{k}\right)x^{\alpha_i}\left(\lambda_{i+1} + \alpha_i\right)\widetilde{v}_{i+1} \\ &= \left(\lambda_i + \frac{n-k}{k}\right)\left((\partial_x\partial_y^{-1})\widetilde{v}_i - \left(-\frac{n}{k}\right)x^{\alpha_i}\widetilde{v}_{i+1}\right) \\ &= \left(\lambda_i + \frac{n-k}{k}\right)\left(\sum_{j=0}^{k-1} a_{i,j}\partial_y^{-l}\widetilde{v}_j\right) + \varpi_i'.\end{aligned}$$

On the other hand there is $\varpi_i'' \in N(-l-1)$ such that

$$(37)\qquad \vartheta\omega_i = \sum_{j=0}^{k-1}\left(x\frac{\partial}{\partial x} + \frac{n}{k}l + \lambda_i\right)a_{i,j}\partial_y^{-l}\widetilde{v}_j + \varpi_i''.$$

It follows from (36) and (37) that $a_{i,j} \in \mathbb{C}\{x\}$ is annihilated by the differential operator $x\frac{\partial}{\partial x} + 1 + \frac{n(l-1)}{k}$. Hence $a_{i,j}$ vanishes for all i, j. This implies that $\omega_i \in N(-l-1)$.

We can assume that $\alpha_{-1} \neq 0$. Let $\iota : \mathbb{Z} \to \{i : \alpha_{i-1} \neq 0\}$ be the unique increasing bijection verifying $\iota(0) = 0$. For $s \in \mathbb{Z}$ set $m_s = \iota(s+1) - \iota(s)$, $\beta_s = \alpha_{\iota(s+1)-1}$ and $\lambda_s' = (k/n)\lambda_{\iota(s)}$. Set $d = \#\{i + k\mathbb{Z} \ : \ \alpha_{i-1} \neq 0\}$. Then $m_{s+d} = m_s$, $\beta_{s+d} = \beta_s$, $\sum_{s=0}^{d-1} m_s = k$, $\sum_{s=0}^{d-1}\beta_s = n-k$ and $\lambda_{s+1}' - \lambda_s' =$

$(1/n)((n-k)m_s - k\beta_s)$. Since ω_i vanish for all i, the $\mathcal{E}_{X,(0,dy)}(0)$-module N is generated by $\widetilde{v}_{\iota(s)}$, $s \in \mathbb{Z}$. Moreover, for $s \in \mathbb{Z}$, $l = 1, \ldots, m-2$,

$$(38) \qquad (\partial_x \partial_y^{-1})^{m_s} \widetilde{v}_{\iota(s)} = \left(-\frac{n}{k}\right)^{m_s} x^{\beta_s} \widetilde{v}_{\iota(s+1)}, \qquad \partial_{t_l} \widetilde{v}_{\iota(s)} = 0.$$

Let us prove that ϑ is given by a diagonal matrix with respect to a convenient system of generators of the $\mathcal{E}_{X,(0,dy)}(0)$-module N. Set $\vartheta' = (k/n)\vartheta$. Then $(\vartheta' - \lambda'_s)\, \widetilde{v}_{\iota(s)} \in N(-1)$ for all s. Let A_0 be the matrix $\mathrm{diag}(\lambda'_0, \ldots, \lambda'_{d-1})$. There is a matrix $\mathcal{A}_{-1} \in \mathrm{M}_d(\mathcal{E}_X(-1))$ such that $(\vartheta' - A_0 - \mathcal{A}_{-1})\widetilde{v}_{\iota(s)} = 0$. If $d = 1$ we can assume, by Lemma 8.8 of [**18**] (or Theorem 3.1 of [**17**]), that $\mathcal{A}_{-1}$ vanishes. Assume that $d > 1$. By construction $1 \leq m_s \leq k-1$, $1 \leq \beta_s \leq n-k-1$, $\sum_{s=0}^{d-1} m_s = k$, $\sum_{s=0}^{d-1} \beta_s = n-k$. Since

$$\lambda'_{s+l} - \lambda'_s = \frac{1}{n}\left((n-k)\sum_{j=s}^{s+l-1} m_j - k \sum_{j=s}^{s+l-1} \beta_j\right), \quad 0 \leq l \leq d-1,$$

$|\lambda'_{s+l} - \lambda'_s| \in (1/n)(k\mathbb{N} + (n-k)\mathbb{N})$ if and only if $l = 0$ (see [**3**], Lemma 10). By Theorem 3.5 we can assume that $\mathcal{A}_{-1}$ vanishes. For $i \in \mathbb{Z}$, $\iota(s) \leq i < \iota(s+1)$, set $u_i = \left(-\partial_x \partial_y^{-1}\right)^{i-\iota(s)} \widetilde{v}_{\iota(s)}$. The $\mathcal{E}_{X,(0,dy)}$-module M is generated by u_i, $i \in \mathbb{Z}$, and verifies the relations (1) and (2).

Let us prove the second statement of the theorem.

By Theorem 2.5 the restriction of $\mathcal{M}_{(\lambda_i)}$ to the regular locus of Λ has simple characteristics. We are going to compute the degree of homogeneity of the principal symbol of this restriction (see §3 of [**18**] for details).

We will identify the regular locus of Λ with $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^{m-2}$ by the parametrization $\gamma : \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^{m-2} \to \Lambda$ given by

$$(39) \quad \gamma(t, \tau, t_1, \ldots, t_{m-2}) = (t^k, t^n, t_1, \ldots, t_{m-2}; -(n/k)t^{n-k}\tau, \tau, 0, \ldots, 0).$$

We have

$$\begin{aligned} t\partial_t &= kx\partial_x + ny\partial_y + (n-k)\xi\partial_\xi \\ \tau\partial_\tau &= \xi\partial_\xi + \eta\partial_\eta. \end{aligned}$$

Set $u = u_0$, $\lambda = \lambda_0$ and set, for $x \neq 0$,

$$P = kx\partial_x + ny\partial_y - k\lambda,$$

$$Q = x^{n-k}\left(\prod_{i=1}^{k}\left(x^{-\alpha_{k-i}}\partial_x\right) - \left(-\frac{n}{k}\right)^k \partial_y^k\right).$$

It follows from (2) that, outside of $\{x = 0\}$, the differential operators P, Q and ∂_{t_j}, $j = 1, \ldots, m-2$, annihilate u. Set $\alpha = \sum_{j=0}^{k-2}\sum_{i=0}^{j} \alpha_i$. Then

$$\sigma_k(Q) = \xi^k - \left(-\frac{n}{k}\right)^k x^{n-k}\eta^k, \qquad \sigma_{k-1}(Q) = -\frac{\alpha}{x}\xi^{k-1}.$$

The principal symbol of u is a solution of the homogeneous system defined by the operators

$$\begin{aligned}\widetilde{L}_P &= t\partial_t - n\tau\partial_\tau - k\lambda + \frac{1}{2}(1-k) - n \\ \widetilde{L}_Q &= \frac{\xi^{k-1}}{x}\left(t\partial_t - \alpha + \frac{1}{2}(n-k-1)(k-1)\right).\end{aligned}$$

Therefore the degree of homogeneity of the principal symbol of u equals

$$\frac{1}{n}\left(\alpha - \lambda k - \frac{1}{2}(n-k)(k-1)\right) \pmod{\mathbb{Z}},$$

that is, equals,

$$-\frac{1}{n}\sum_{i=0}^{k-1}\lambda_i \pmod{\mathbb{Z}}.$$

Actually,

$$\sum_{i=0}^{k-1}\lambda_i = k\lambda_0 - \alpha + \frac{n-k}{k}\sum_{i=1}^{k-1} i = k\lambda_0 - \alpha + \frac{(n-k)(k-1)}{2}.$$

By Theorem 4.1 of [8] the congruence class (modulo $\mathbb{Z}$) of the degree of homogeneity of the section $u \neq 0$ determines the structure of the restriction of $\mathcal{M}_{(\lambda_i)}$ to the regular locus of Λ. Hence this congruence class does not depend on the choice of the generator u. Therefore, if $\mathcal{M}_{(\lambda_i)} \simeq \mathcal{M}_{(\mu_i)}$,

$$\alpha - k\lambda_0 \equiv \beta - k\mu_0 \pmod{n\,\mathbb{Z}}, \tag{40}$$

where $\beta = \sum_{j=0}^{k-2}\sum_{i=0}^{j}\beta_i$. The module $\mathcal{N} = \sum_i \mathcal{E}_{X(0)}u_i$ satisfies the conditions of Theorem 2.3. Therefore the R-module canonically associated by Theorem 2.3 to the system $\mathcal{M}_{(\lambda_i)}$ equals the R-module $L_{(\alpha_i)}$ introduced in Proposition 4.4. By Theorem 2.3 the R-modules $L_{(\alpha_i)}$ and $L_{(\beta_i)}$ are isomorphic. By Proposition 4.4 we can assume that there is a $\nu \in \mathbb{Z}$, $0 \leq \nu \leq k-1$, such that $\beta_i = \alpha_{i+\nu}$ for all $i \in \mathbb{Z}$. Notice that

$$\sum_{j=0}^{k-2}\sum_{i=0}^{j}(\alpha_{i+\nu} - \alpha_i) = \nu(n-k) - k\sum_{i=0}^{\nu-1}\alpha_i = k(\lambda_\nu - \lambda_0). \tag{41}$$

By the previous relation along with (40), $\mu_0 - \lambda_\nu \equiv 0 \left(\bmod \frac{n}{k}\right)$.

Conversely, assume that $\beta_i = \alpha_{i+\nu}$ and $\mu_0 = \lambda_\nu + l\frac{n}{k}$. The inner automorphism of $\mathcal{E}_X$, $R \mapsto \partial_y^{-l}R\partial_y^{l}$, changes $\vartheta - \mu_i$ into $\vartheta - \lambda_i$ for all i. Hence $\mathcal{M}_{(\lambda_i)} \simeq \mathcal{M}_{(\mu_i)}$. ∎

5. $\mathcal{D}$-modules.

Let Λ be a germ at a point $q \in T^*X \setminus T^*_X X$ of a conic Lagrangian variety in generic position. Let $\mathcal{L}$ be a germ at $\pi(q)$ of a coherent $\mathcal{D}_X$-module with characteristic variety contained in the union of Λ with the zero section. We say that $\mathcal{L}$ belongs to the category $\mathcal{H}ol(\Lambda, \mathcal{D})$ if each germ at $\pi(q)$ of a vector field ω such that $\sigma(\omega)(q) \neq 0$ induces an isomorphism of complex vector spaces $u : \mathcal{L}_{\pi(q)} \to \mathcal{L}_{\pi(q)}$. Here $\sigma(\omega)(q)$ denotes the principal symbol of the differential operator ω. Let $\mathcal{H}ol(\Lambda, \mathcal{E})$ denote the category of germs at q of coherent $\mathcal{E}_X$-modules with characteristic variety contained in Λ.

Theorem 5.1 (See [**2**], Theorem 8.6.19)**.** *The functor $\mu_q : \mathcal{H}ol(\Lambda, \mathcal{D}) \to \mathcal{H}ol(\Lambda, \mathcal{E})$, defined by $\mu_p(\mathcal{L}) = \mathcal{E}_{X,q} \otimes_{\mathcal{D}_{X,\pi(q)}} \mathcal{L}$, is an equivalence of categories. Its quasi-inverse is the base change functor associated to the inclusion morphism $\mathcal{D}_{X,\pi(q)} \hookrightarrow \mathcal{E}_{X,q}$.*

Given a ring R and $\varepsilon \in R$, we use Pochhammer's notation

$$(\varepsilon)_j = \varepsilon(\varepsilon + 1)\dots(\varepsilon + j - 1).$$

Theorem 5.2. *Let k, n be integers such that $2 \leq k \leq n-1$ and $(k, n) = 1$. Let $\mathcal{L}$ be the germ at the origin of a coherent $\mathcal{D}_{\mathbb{C}^m}$-module with characteristic variety equal to the union of the conormal of the hypersurface $y^k = x^n$ with the zero section. Then $\mathcal{L}$ has multiplicity one along the conormal of $y^k = x^n$ and*

$$\partial_y : \mathcal{L}_0 \to \mathcal{L}_0 \tag{42}$$

is an isomorphism of complex vector spaces if and only if there are nonnegative integers α_i, $i \in \mathbb{Z}$, and complex numbers λ_i, $i \in \mathbb{Z}$, such that $\mathcal{L}$ is isomorphic to $\mathcal{L}_{(\lambda_i)}$ and

$$\lambda_i \notin \frac{n}{k}\{-1, -2, \dots\}, \qquad 0 \leq i \leq k-1. \tag{43}$$

Proof. Let us show that Condition (43) is necessary. By Theorem 8.6.19 of [**2**] and Theorem 4.6, $\mathcal{L}$ is isomorphic to some $\mathcal{D}$-module $\mathcal{L}_{(\lambda_i)}$. Set $q = (0, dy) \in T^*\mathbb{C}^m$. Set $t = (t_1, \dots, t_{m-2})$. It follows from (2) that

$$y\partial_y u_l \;=\; \frac{k}{n}\lambda_l u_l + x^{\alpha_l+1}\partial_y u_{l+1}, \qquad \partial_{t_j} u_l = 0, \tag{44}$$

$l = 0, \dots, k-1$, $j = 1, \dots, m-2$. It follows from (44) and (2) that we have an isomorphism of complex vector spaces

$$\left(\mathcal{L}_{(\lambda_i)}\right)_0 \;\cong\; \bigoplus_{i=0}^{k-1} \left(\mathbb{C}\{x, t\}[\partial_y] \oplus y\mathbb{C}\{x, y, t\}\right) u_i. \tag{45}$$

Set $V = \oplus_{l=0}^{k-1} \left(\mathbb{C}\{x, t\}[\partial_y] \oplus y\mathbb{C}\{x, t\}[y]\right) u_l$. Set $\delta_{l,i} = i + \alpha_l + \dots + \alpha_{l+i-1}$, $0 \leq l \leq k-1$, $i \geq 0$. Assume that (42) is injective. We will show by induction in r that $\lambda_l \notin (n/k)\{-1,-2,\dots,-r\}$. Set

$$Q_{j,l} = x^{\delta_{l,j+1}} u_{l+j+1} + \sum_{i=1}^{j} k\frac{\lambda_{l+i}}{n} x^{\delta_{l,i}} R_{j-i,l+i}, \tag{46}$$

$$R_{j,l} = \left(k\frac{\lambda_l}{n} + j + 1\right)^{-1} \left(y^{j+1}u_l - Q_{j,l}\right), \tag{47}$$

for $0 \le l \le k-1$, $0 \le j \le r$. Since

$$\partial_y \left(y^{r+1}u_l - Q_{r,l}\right) = \left(k\frac{\lambda_l}{n} + r + 1\right) y^r u_l \tag{48}$$

and $Q_{r,l}$ is a $\mathbb{C}\{x,t\}$-linear combination of $y^j u_l$, $0 \le j \le r$, $0 \le l \le k-1$,

$$k\frac{\lambda_l}{n} + r + 1 \neq 0.$$

Hence we can define $R_{r+1,l}$ and $Q_{r+1,l}$ and iterate the procedure.

Assume that (43) holds. We can show by induction in $s \in \mathbb{N}$ that there are complex numbers $b_{l,r,s}$ such that

$$\left(\partial_y y\right)_s u_l = \left(k\frac{\lambda_l}{n} + 1\right)_s u_l + \sum_{r=1}^{s} b_{l,r,s} x^{\delta_{l,r}} \partial_y^r u_{l+r}. \tag{49}$$

Since $\left(\partial_y y\right)_s = \left(\partial_y\right)^s y^s$,

$$\left(k\frac{\lambda_l}{n} + 1\right)_s \partial_y^{-s} u_l = y^s u_l - \sum_{r=1}^{s} b_{l,r,s} x^{\delta_{l,r}} \partial_y^{r-s} u_{l+r}. \tag{50}$$

By (43) there are complex numbers $a_{l,r,s}$ such that

$$\partial_y^{-s} u_l = \sum_{r=0}^{s} a_{l,r,s} x^{\delta_{l,r}} y^{s-r} u_{l+r}. \tag{51}$$

Let W_{-s}, $s \in \mathbb{N}$, be the $\mathbb{C}\{x,y,t\}$-submodule of $\oplus_{l=0}^{k-1}\mathbb{C}\{x,y,t\}u_l$ generated by $\partial_y^{-s}u_l$, $l = 0,\dots,k-1$. Let V_{-s}, $s \in \mathbb{N}$, be the $\mathbb{C}\{x,t\}[y]$-submodule of V generated by $\partial_y^{-s}u_l$, $l = 0,\dots,k-1$. By (51),

$$\bigoplus_{l=0}^{k-1} \mathcal{E}_{\mathbb{C}^m,q}(-s)u_l \subset W_{-s} + \bigoplus_{l=0}^{k-1} \mathcal{E}_{\mathbb{C}^m,q}(-s-1)u_l,$$

for all $s \ge 0$. Hence

$$\bigoplus_{l=0}^{k-1} \mathcal{E}_{\mathbb{C}^m,q}(-s)u_l \subset W_0 + \bigoplus_{l=0}^{k-1} \mathcal{E}_{\mathbb{C}^m,q}(-s)u_l,$$

for all $s \geq 0$. By Proposition II.1.1.3 of [20],

$$\bigoplus_{l=0}^{k-1} \mathbb{C}\{x,y,t\}u_l = \bigoplus_{l=0}^{k-1} \mathcal{E}_{\mathbb{C}^m,q}(0)u_l.$$

Hence the inclusion

$$\left(\mathcal{L}_{(\lambda_i)}\right)_0 \hookrightarrow \left(\mathcal{M}_{(\lambda_i)}\right)_q \tag{52}$$

is surjective. Let Φ denote the $\mathbb{C}\{x,t\}$-linear endomorphism of V defined by $\Phi(\partial_y^{j+1}u_l) = \partial_y^j u_l$, $\Phi(y^j u_l) = R_{j,l}$, $j \geq 0$. Notice that (42) induces a $\mathbb{C}\{x,t\}$-linear endomorphism of V. Moreover,

$$\partial_y V_{-s} \subset V_{-s+1},\ \partial_y W_{-s} \subset W_{-s+1} \text{ and } \Phi(V_{-s}) \subset V_{-s-1}. \tag{53}$$

By (48), $\Phi(\partial_y y^{j+1}u_l) =$

$$= \Phi\left(\partial_y Q_{j,l} + \left(k\frac{\lambda_l}{n} + j + 1\right) y^j u_l\right)$$

$$= \Phi\left(\partial_y\left(x^{\delta_{l,j+1}}u_{l+j+1} + \sum_{i=1}^{j} k\frac{\lambda_{l+i}}{n}x^{\delta_{l,i}}R_{j-i,l+i})\right) + \left(k\frac{\lambda_l}{n} + j + 1\right) y^j u_l\right)$$

$$= x^{\delta_{l,j+1}}u_{l+j+1} + \sum_{i=1}^{j} k\frac{\lambda_{l+i}}{n}x^{\delta_{l,i}}R_{j-i,l+i} + y^{j+1}u_l - Q_{j,l}$$

$$= y^{j+1}u_l.$$

Therefore the kernel of (42) is contained in W_{-s} for all s. Hence (42) is injective. By (53) and (47) $\partial_y\Phi(y^j u_l) = y^j u_l$ for $0 \leq l \leq k-1$, $j \geq 0$. Hence $\partial_y\mathcal{L}_0 + W_{-s} = \mathcal{L}_0$ for all s. By Proposition II.1.1.3 of [20], (42) is surjective. The result follows from Theorem 5.1. $\square$

The higher hypergeometric series was introduced by Thomae (cf. [23]) as the series

$${}_kF_{k-1}(\varepsilon_0,\ldots,\varepsilon_{k-1},\theta_0,\ldots,\theta_{k-2}\,|\,z) = \sum_{j=0}^{\infty} \frac{(\varepsilon_0)_j\ldots(\varepsilon_{k-1})_j z^j}{(\theta_0)_j\ldots(\theta_{k-2})_j j!}.$$

Set $\delta_z = z\frac{d}{dz}$. Given $\widetilde{\alpha} = (\widetilde{\alpha}_i)$, $\widetilde{\beta} = (\widetilde{\beta}_i)$, $i = 0,\ldots,k-1$, set

$$D(\widetilde{\alpha},\widetilde{\beta}) = (\delta_z + \widetilde{\beta}_0 - 1)\ldots(\delta_z + \widetilde{\beta}_{k-1} - 1) - z(\delta_z + \widetilde{\alpha}_0)\ldots(\delta_z + \widetilde{\alpha}_{k-1}).$$

If the $\widetilde{\beta}_i$'s are distinct modulo $\mathbb{Z}$ then k independent solutions of $D(\widetilde{\alpha},\widetilde{\beta})\varphi = 0$ are given by

$$z^{1-\widetilde{\beta}_i}{}_kF_{k-1}(1+\widetilde{\alpha}_0-\widetilde{\beta}_i,\ldots,1+\widetilde{\alpha}_{k-1}-\widetilde{\beta}_i,1+\widetilde{\beta}_0-\widetilde{\beta}_i,\check{\ldots},1+\widetilde{\beta}_{k-1}-\widetilde{\beta}_i|z)$$

for $i = 0,\ldots,k-1$. Here $\check{\ldots}$ denotes the omission of $1+\widetilde{\beta}_i-\widetilde{\beta}_i$. Levelt computed the monodromy of the equations above (see [11], [1]).

For $i,j=0,\dots,k-1$, set $\widetilde{\alpha}_j=\left(\sum_{0\le l\le j-1}\alpha_l+j\right)/n$, $\widetilde{\beta}_j-1=\lambda/n-j/k$, $\varepsilon_{i,j}=1+\widetilde{\alpha}_j-\widetilde{\beta}_i$ and $\theta_{i,j}=1+\widetilde{\beta}_j-\widetilde{\beta}_i$.

Theorem 5.3. *We have k independent solutions of $\mathcal{L}_{(\lambda_i)}$ given by the analytic continuations of*

$$y^{\lambda_0\frac{k}{n}}\left(\frac{y^k}{x^n}\right)^{1-\widetilde{\beta}_i}{}_kF_{k-1}\left(\varepsilon_{i,0},\dots,\varepsilon_{i,k-1},\theta_{i,0},\dots,\check{\theta}_{i,i},\dots,\theta_{i,k-1}\,\middle|\,\frac{y^k}{x^n}\right),$$

for $i=0,\dots,k-1$.

Proof. Set $\lambda=\lambda_0$, $u=u_0$ and $v(x,y)=y^{-\lambda\frac{k}{n}}u(x,y)$. Since $\vartheta v=0$, v is constant along the integral curves of ϑ, that is, along the fibers of the map $\Phi:\mathbb{C}^2\setminus\{(0,0)\}\to\mathbb{P}^1$ defined by $\gamma(x,y)=(x^n:y^k)$. Since v is a multivalued holomorphic function ramified along $xy(y^k-x^n)=0$, there is a multivalued holomorphic function φ on $\mathbb{P}^1$, ramified along $0,1,\infty$, such that $v=\varphi\circ\Phi$. Hence,

$$u(x,y)=y^{\lambda\frac{k}{n}}\varphi\left(\frac{y^k}{x^n}\right).$$

Set $\delta_x=x\partial_x$ and $\delta_y=y\partial_y$. Notice that

$$\delta_x\varphi\left(\frac{y^k}{x^n}\right)=-n(\delta_z\varphi)\left(\frac{y^k}{x^n}\right),\qquad \delta_y\varphi\left(\frac{y^k}{x^n}\right)=k(\delta_z\varphi)\left(\frac{y^k}{x^n}\right).$$

Since $\delta_x u_i=-(n/k)x^{\alpha_i+1}\partial_y u_{i+1}$,

$$\left[\frac{y^k}{x^n}\prod_{j=0}^{k-2}\left(\delta_x-\sum_{i=0}^{j}\alpha_i-j-1\right)\delta_x-\left(-\frac{n}{k}\right)^k y^k\partial_y^k\right]u=0,$$

hence,

$$\left[\prod_{j=0}^{k-1}\left(\delta_z-\frac{j}{k}+\frac{\lambda}{n}\right)-z\delta_z\prod_{j=0}^{k-2}\left(\delta_z+\frac{1}{n}\sum_{i=0}^{j}(\alpha_i+j+1)\right)\right]\varphi=0.$$

Therefore $D(\widetilde{\alpha},\widetilde{\beta})\varphi=0$. □

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