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Examples are given to show that some compact contractible 4-manifolds can be knotted in the 4-sphere. It is then proved that any finitely presented perfect group with a balanced presentation is a knot group for an embedding of some contractible 4-manifold in S^4.

1. Introduction.

A construction will here be described that can produce a compact contractible 4-manifold M embedded piecewise linearly (or smoothly) in S^4 with the fundamental group of its complement being nontrivial. Then, another embedding of M in S^4 will be produced which has simply connected complement. Several examples of this will be given. Of course, the construction emphasizes that contractible spaces do not behave entirely as do single points. It is important to note that these embeddings are piecewise linear or smooth; they are certainly not wild. The famous construction of the Alexander wild horned sphere gives a wild embedding of a 3-ball in S^3 that has its complement not simply connected. However the boundary of a contractible compact 3-manifold is just a 2-sphere, so by the piecewise linear 3-dimensional Schönflies theorem, if such a manifold can be embedded *piecewise linearly* in S^3, each of the manifold and its complement must be a 3-ball.

Recall the general definition of knotting, when all maps and spaces are in the piecewise linear category: A polyhedron X knots in a polyhedron Y if there are two embeddings, e_0 and e_1 of X in Y, that are homotopic but not ambient isotopic. The embeddings are ambient isotopic if there exist homeomorphisms $F_t : Y \to Y$, for each $t \in [0,1]$, such that $(y,t) \mapsto (F_t(y),t)$ defines a piecewise linear homeomorphism from $Y \times [0,1]$ to itself, F_0 is the identity and $F_1 e_0 = e_1$. Thus to be ambient isotopic the complements of the images of the two embeddings must certainly be homeomorphic. The knotting phenomenon explores the possibility of moving between embeddings along a path of embeddings as opposed to moving along a path of maps. The examples given here are of contractible 4-manifolds that can knot in S^4 for, just as in classical knot theory, the fundamental group of complements is used to show embeddings are not ambient isotopic. Examples of knots usually rely on the entwining of some nontrivial cycle, but here there is none. In fact, in higher dimensions, if X and Y are piecewise linear manifolds with $\dim Y - \dim X \geq 3$, there are theorems of Hudson [4] that assert that there is no knotting of X in Y provided these spaces are sufficiently highly connected.

It should be noted that when M is a contractible 4-manifold, piecewise linearly contained in S^4, the Alexander duality theorem implies that $S^4 - M$ has the same homology as a point. Thus $\pi_1(S^4 - M)$ is a perfect group in contrast to the situation of classical knot theory. It will be proved in Theorem 3 that for any finitely presented perfect group with a balanced presentation (that is, a presentation with the same number of generators as relators) there are embeddings e_0 and e_1 of some contractible 4-manifold M into S^4 so that $\pi_1(S^4 - e_1 M)$ is the given group and $\pi_1(S^4 - e_0 M)$ is trivial.

The author is grateful to Simon Norton for making a helpful remark and to Charles Livingston for a correction. Livingston has extended the method of this paper to obtain, for *certain* contractible 4-manifolds, infinitely many knots in S^4.

2. Examples of the embedding construction.

The theorem that now follows is really an example describing the main simple idea of the construction of this paper. The second theorem amplifies it to more general circumstances.

Theorem 1. *There are two piecewise linear (or smooth) embeddings, e_0 and e_1, of a certain compact contractible 4-manifold M into S^4 such that $\pi_1(S^4 - e_1 M)$ is nontrivial and $S^4 - e_0 M$ is contractible.*

Proof. Firstly, construct a compact 4-manifold X by adding three 1-handles and three 2-handles onto a 4-ball in the following way. The handles are to be chosen so that $\pi_1(X)$ has the presentation

$$\langle a, b, c : \; b^{-1}c^{-2}bc^3, \; c^{-1}a^{-2}ca^3, \; a^{-1}b^{-2}ab^3 \rangle,$$

where based loops encircling the three 1-handles represent a, b and c, and the attaching circles of the three 2-handles give the three relators. This situation is shown in Figure 1 in the notation common in considerations of the 'Kirby calculus' (see [2] for example).

The diagram of Figure 1 shows curves in the 3-sphere, the boundary of the 4-ball. Open regular neighbourhoods, of three standard disjoint discs in the 4-ball, are to be removed from the 4-ball to create a ball with the three 1-handles added. The boundaries of these discs are the circles, decorated with dots, labeled a, b and c. That this is in order can be checked as follows. A ball with 1-handles added can be changed back to a ball by adding 2-handles to cancel the 1-handles; removing those 2-handles consists of removing neighbourhoods of the discs that are the co-cores of the 2-handles. Thus a 4-ball, with 1-handles added, is the same as a 4-ball from

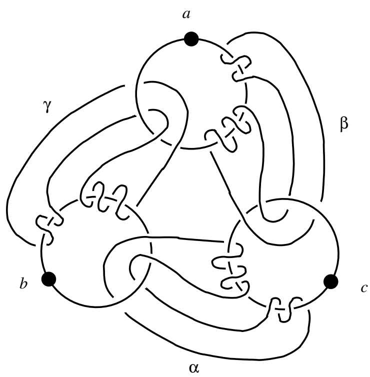

Figure 1

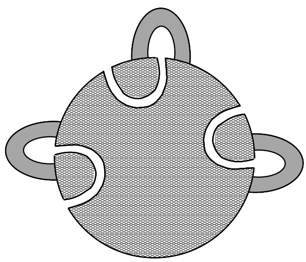

Figure 2.

which standard 2-handles have been removed. A 1-handle can be regarded as $D^1 \times D^3$ with $\partial D^1 \times \star$ being the attaching sphere and $\star \times \partial D^3$ being the belt sphere (where each $\star$ is a base point). In Figure 1 a belt sphere consists of the union of a disc spanning a dotted circle, less a regular neighbourhood of that circle, and a disc in the boundary of the 2-handle that has been removed. Meridians encircling the three dotted circles represent generators, to be called a, b and c, of the fundamental group of the ball with 1-handles, and a based closed curve represents a word, in a, b and c, corresponding to its signed intersections with the three belt spheres. In this way the curves shown, labeled α, β and γ, represent $b^{-1}c^{-2}bc^3$, $c^{-1}a^{-2}ca^3$ and $a^{-1}b^{-2}ab^3$. Thus adding 2-handles, with these curves as attaching spheres (choose the zero framings), gives the 4-manifold X with the required presentation for $\pi_1(X)$. It has been shown by Rapaport [6] that this is the presentation of a *nontrivial group*. There are, of course, very many ways that attaching curves can be chosen for the 2-handles in order to achieve this presentation (and the choice will be explored further in Theorem 3), but the one shown is about the simplest and is the one that will now be considered. The situation is shown schematically in Figure 2.

The manifold X is a 4-ball, B_1 say, with 2-handles removed and 2-handles added. Regard this 4-ball as being contained in S^4 and consider the complementary 4-ball B_2. The 2-handles removed from B_1 can be thought of as added to B_2. The other 2-handles, that were added to B_1, were added with zero framing along unknotted, unlinked curves (labeled α, β and γ), so they can be regarded as standard 2-handles removed from B_2. Thus the closure of $S^4 - X$ is the 4-ball B_2 with three 2-handles removed (creating added 1-handles) and three 2-handles added and *this* is to be the required 4-manifold M. The situation for M is again represented by Figure 1, except that now the dots should be removed from the curves labeled a, b and c and placed on those labeled α, β and γ. However, the α-curve bounds a disc that meets only the c-curve; there are similar discs for the β- and γ-curves. The words in α, β and γ, coming from the intersections of the a, b and c curves with these discs, make it clear that $\pi_1(M)$ is presented by

$$\langle \alpha, \beta, \gamma : \alpha^3\alpha^{-2},\ \beta^3\beta^{-2},\ \gamma^3\gamma^{-2}\rangle$$

which, very obviously, presents the trivial group. Thus M is simply connected. A count of the handles shows that the Euler characteristic of M is 1, hence $H_2(M) = 0$. Furthermore $H_r(M) = 0$ for $r > 2$, as there are no r-handles for $r > 2$, and so, by the Hurewicz isomorphism theorem, M has all homotopy groups trivial and hence is contractible. Note that $\pi_1(\partial M) \neq \{1\}$, as otherwise $\pi_1(X) \cong \pi_1(S^4)$ by the Van Kampen theorem. Hence M is not a 4-ball. The inclusion of M, as so defined in S^4, is the embedding e_1.

The above *presentation* given for $\pi_1(M)$ coming from the handle structure of M is almost trivial. It certainly reduces to the trivial presentation by Andrews-Curtis moves (see below). Any such M has the property that $M \times [0,1] \cong B^5$ where B^5 is a 5-ball. To show that in this instance, it is necessary only to realise that $M \times [0,1]$ has the same handle structure as does M. The extra dimension means that, when a 2-handle is attached (to the boundary of a 5-manifold) only the homotopy class of the attaching map is significant (a homotopy of attaching circles can be changed to an isotopy by using the fourth dimension to prevent the circles from crossing each other). Now let e_0 be the inclusion of $M \times \{0\}$ in the 4-sphere $\partial(M \times [0,1])$. The complement of $M \times \{0\}$ in this sphere is $(\partial M \times [0,1]) \cup M \times \{1\}$ and this is just another copy of the contractible manifold M.

For a second example consider $\langle a, b :\ ab^2ab^{-1},\ a^4ba^{-1}b\rangle$, a presentation of the perfect group G of 120 elements that is the fundamental group of the Poincaré homology 3-sphere. The method of the proof of Theorem 1 constructs a 4-manifold $X \subset S^4$ with $\pi_1(X) \cong G$ and with the fundamental group of the corresponding M presented by $\langle \alpha, \beta : \alpha^2\beta^3,\ \alpha^{-1}\beta^{-2}\rangle$. Again this easily reduces to the trivial presentation by Andrews-Curtis moves so that $M \times I$ is a 5-ball.

The above construction works easily for the presentations

$$\langle a_1, a_2, \ldots, a_m : r_1, r_2, \ldots, r_m \rangle$$

for every $m \geq 4$ when $r_i = a_i^{-1} a_{i+1}^{-1} a_i a_{i+1}^2$ for $i = 1, 2, \ldots m$ modulo m, and also when $r_i = a_i^{-1} a_{i+1}^{-2} a_i a_{i+1}^3$. These are known to be presentations of infinite groups (see [3] and [5]).

Note that $\langle a, b : a^{-1} b^{-2} a b^3, b^{-1} a^{-2} b a^3 \rangle$ is a presentation of the trivial group. If M is constructed from this presentation for X it is not clear whether the embedding of M is in any sense knotted.

3. A reminder of Andrews-Curtis moves.

The above proof makes a brief mention of the Andrews-Curtis moves. These moves are elementary changes that can be made to a group presentation that do not alter the group that is presented. The moves are also called 'extended Nielsen transformations' in [1], they are called 'Q-transformations' in [6] and they are sometimes also called 'Markov operations'. The permitted changes to a presentation $\langle a_1, a_2, \ldots, a_m : r_1, r_2, \ldots, r_n \rangle$ are the following moves and the inverses of these moves:

(i) Change r_i to $r_i a_j a_j^{-1}$ or $r_i a_j^{-1} a_j$.
(ii) Change r_i to a cyclic permutation of r_i.
(iii) Change r_i to r_i^{-1}.
(iv) Change r_i to $r_i r_j$ where $j \neq i$.
(v) Add a new generator a_{m+1} and a new relator $a_{m+1} w$ where w is a word in $a_1, a_2, \ldots, a_m$.

These are precisely the moves that can easily be imitated on a 5-manifold comprised of 0-handles, 1-handles and 2-handles only. If the handles of such a 5-manifold correspond to a presentation of the trivial group that can be reduced to the trivial presentation (that is, the empty presentation) by the above moves and their inverses, then it is shown in [1] that the manifold is the 5-ball. It is this result that is used in the above proof. The Andrews-Curtis conjecture [1] is that *any* presentation of the trivial group be reducible to the trivial presentation by the above moves and their inverses. This is popularly thought to be false, $\langle a, b : a^{-1} b^{-2} a b^3, b^{-1} a^{-2} b a^3 \rangle$ being one of many proposed counter-examples. The truth of the Andrews-Curtis conjecture would imply the truth of another conjecture that asserts that any 5-dimensional regular neighbourhood of a contractible 2-complex be a 5-ball (such a neighbourhood is known to be unique).

4. Arbitrary finitely presented perfect groups.

A few simple remarks lead up to an elementary, but possibly surprising, little lemma about finitely presented perfect groups and presentations of the trivial group. Suppose that an *abelian* group E is freely generated as an *abelian* group (with additive notation) by the generators $e_1, e_2, \dots, e_m$. The quotient of E by the subgroup generated by the n elements $\{\sum_{j=1}^m a_{ij}e_j : i = 1, 2, \dots, n\}$ is said to be presented by the $n \times m$ integer matrix $A = \{a_{ij}\}$. When $m = n$ the quotient is the trivial group if and only if A is unimodular, that is, $\det A = \pm 1$. A presentation P of any group (in multiplicative notation) leads at once to a presentation of the abelianisation of that group, by just deciding that all symbols commute. It is then sensible in each relator to assemble together all occurrences of a generator and its inverse, cancelling where possible, to obtain from the resulting exponents in each relator a presentation matrix A of the abelianisation of the group. The following lemma considers such things in the reverse order, showing that, if A presents the trivial abelian group, then P can be chosen to present the trivial group.

Lemma 2. *Suppose that A is a unimodular $n \times n$ matrix of integers. Then there exists a presentation P of the trivial group that has A as its abelianised presentation matrix. Furthermore, P is equivalent to the trivial presentation by Andrews-Curtis moves.*

Proof. Starting from the identity $n \times n$ matrix, the unimodular matrix A can be created by a sequence of row operations in which either a row is multiplied by -1 or a row is added to another row. These moves can be mimicked by changes to a presentation $\langle a_1, a_2, \dots, a_n : r_1, r_2, \dots, r_n \rangle$ of the trivial group, where initially $r_i = a_i$ for each i. If the ith row of the matrix is multiplied by -1, change r_i to r_i^{-1}; if row i is added to row j then change r_j to $r_j r_i$. At each stage the presentation is of the trivial group and at each stage the matrix is the *corresponding* presentation matrix of the abelianised group. Of course the moves used on the presentation are all Andrews-Curtis moves.

Theorem 3. *Let G be any finitely presented perfect group having a balanced presentation. Then there is a compact contractible 4-manifold M contained in S^4 such that $\pi_1(S^4 - M) \cong G$ and $M \times I$ is a 5-ball (so that, if G is non-trivial, there is a distinct second embedding of M in S^4 having contractible complement).*

Proof. Let $\langle a_1, a_2, \dots, a_n : r_1, r_2, \dots, r_n \rangle$ be a presentation P of G. The construction proceeds as in the proof of Theorem 1. Remove from the 4-ball B^4 neighbourhoods of n standard disjoint spanning discs to create a ball with n 1-handles added. The boundaries of the discs form a set of n unlinked simple closed ('dotted') curves in $\partial B^4 = S^3$, which are labeled $a_1, a_2, \dots, a_n$. In the following way construct, as the boundaries of disjoint discs $D_1, D_2, \dots, D_n$ contained in S^3, simple closed curves $\alpha_1, \alpha_2, \dots, \alpha_n$, corresponding to $r_1, r_2, \dots, r_n$, which are to be the attaching circles for n 2-handles. Begin with small, disjoint, oriented discs $\Delta_1, \Delta_2, \dots, \Delta_n$ in the complement of $a_1 \cup a_2 \cup \cdots \cup a_n$. For each letter $a_i^{\pm 1}$ in the word r_1 take a small meridian disc of the curve a_i, oriented according to the exponent on the letter, and to construct D_1, join the boundaries of these meridian discs by thin bands to the boundary of Δ_1, in the order around $\partial\Delta_1$ specified by r_1. The discs $D_2, \ldots, D_n$ are constructed similarly from $r_2, \ldots, r_n$ and there is no difficulty in ensuring that the D_i are embedded and mutually disjoint.

As in Theorem 1, form a 4-manifold $X \subset S^4$ by adding n 2-handles with zero framing along $\alpha_1, \alpha_2, \ldots, \alpha_n$ to the ball with n 1-handles. Then $\pi_1(X) \cong G$. The key point to note now is that the meridian discs described above (for all the D_i together) can be taken *in any order* around a_i. Different choices of order probably give different manifolds X and M, where again M is the closure of $S^3 - X$. Also note that the given presentation of G can be amended by the insertion of any number of copies of $a_i a_i^{-1}$, for any i, into any r_j without changing G nor the presentation matrix A of the (trivial) abelianisation of G coming from that presentation. Now $\pi_1(M)$ has a presentation Π of the form $\langle \alpha_1, \alpha_2, \ldots, \alpha_n : \rho_1, \rho_2, \ldots, \rho_n \rangle$ where the relators record in order the occurrence of the meridian discs around $a_1, a_2, \ldots, a_n$ (each signed intersection of a_i with a meridian disc contained in D_j producing an $\alpha_j^{\pm 1}$ entry in ρ_i). The abelian presentation matrix coming from Π is the transpose of A; it is certainly unimodular. Thus using Lemma 2, the ordering along the a_i of those meridional discs making up each D_j can be chosen, after inserting any necessary pairs of discs corresponding to $a_i a_i^{-1}$, so that, with respect to the new choice, Π becomes a presentation of the trivial group. Again from Lemma 2, Π is equivalent by Andrews-Curtis moves to the trivial presentation and so $M \times I$ is a 5-ball.

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