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PAYNE–POLYA–WEINBERGER TYPE INEQUALITIES FOR EIGENVALUES OF NONELLIPTIC OPERATORS

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In this paper we consider the eigenvalue problems for some nonelliptic operators which include the real Kohn-Laplacian in the Heisenberg and generalized Baouendi-Grushin operator. Some interest inequalities for eigenvalues are given by establishing the identities and inequalities for noncommutative vector fields.

1. Introduction.

Let $\triangle$ denote the Laplacian in the Euclidean space. The classic upper estimates, independent of the domain, for the gaps of eigenvalues of $-\triangle, (-\triangle)^2$ and $(-\triangle)^k (k \geq 3)$ were studied extensively by many mathematicians, cf. Payne, Polya and Weinberger [**16**], Hile and Yeh [**10**], Chen and Qian [**2**], Guo [**8**] etc.. The asymptotic behaviors of eigenvalues for degenerate elliptic operators were considered by Beals,Greiner and Stanton [**1**], Menikoff and Sjöstrand [**15**], Fefferman and Phone [**3**, **4**], Garofalo and Shen [**7**], respectively.

In this paper, we are concerned with the following eigenvalue problem

$$-\triangle_{H_n} u = \lambda u, \text{ in } \Omega, \ u = 0, \text{ on } \partial\Omega \tag{1.1}$$

$$(-\triangle_{H_n})^2 u = \lambda u, \text{ in } \Omega, \ u = \frac{\partial u}{\partial \nu} = 0, \text{ on } \partial\Omega \tag{1.2}$$

$$(-\triangle_{H_n})^k u = \lambda u, \text{ in } \Omega, \ u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{K-1} u}{\partial \nu^{k-1}} = 0, \text{ on } \partial\Omega \tag{1.3}$$

where Ω is a bounded domain in the Heisenberg group H_n, with smooth boundary $\partial\Omega, n \geq 1, \nu$ is the unit outward normal to $\partial\Omega, k \geq 3$ is a positive integer. Let $\triangle_{H_n}$ denote the real Kohn-Laplacian in the Heisenberg group $\sum_{j=1}^n (X_j^2 + Y_j^2)$, where $X_j = \frac{\partial}{\partial x_j} + \frac{y_j}{2}\frac{\partial}{\partial t}, Y_j = \frac{\partial}{\partial y_j} - \frac{x_j}{2}\frac{\partial}{\partial t}, j = 1, \ldots, n, T = \frac{\partial}{\partial t}$. The existence of eigenvalue for (1.1) has been proved by Luo and Niu [**12**, **13**, **14**] using the Kohn inequality (see [**11**]) for the vector fields $\{X_j, Y_j\}$ together with the spectral properties of compact operators. In what follows we let

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots \to +\infty \tag{1.4}$$

denote the successive eigenvalues for (1.1) with corresponding eigenfunctions $u_1, u_2, \ldots, u_m, \ldots$ in $S_0^{1,2}(\Omega)$. Here, $S^{1,2}(\Omega)$ denotes the Hilbert space of the functions $u \in L^2(\Omega)$ such that $X_j u, Y_j u \in L^2(\Omega)$, and $S_0^{1,2}(\Omega)$ denotes closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{S^{1,2}}^2 = \int_\Omega (|\bigtriangledown_{H_n} u|^2 + |u|^2) dx\, dy\, dt$$

where $\bigtriangledown_{H_n} u = (x_1 u, \ldots, X_n u, Y-1u, \ldots, Y_n u)$ normalized so that

$$\langle u_i,\, u_j\rangle = \int_\Omega u_i\, u_j\, dx\, dy\, dt = \delta_{ij},\ i, j = 1, 2 \ldots.$$

For simplicity we will leave out Ω and $dxdydt$ in all integrals in the sequel and denote $L = -\triangle_{H_n}$.

It is clear that these statements are also valid for the problems (1.2) and (1.3) and the eigenfunctions belong to $S_0^{2,2}(\Omega)$ and $S_0^{k,2}(\Omega)$ respectively.

We will derive some upper estimates which are independent of the domain, for the eigenvalues of (1.1), (1.2), and (1.3), respectively. The noncommutativity of vector fields $\{X_j, Y_j\}$ makes the discussion of these problems more complicated than one in the case of Euclidean-Laplacian. Furthermore, we will also consider the eigenvalues of generalized Baouendi-Grushin operators [**6**].

The paper is constructed as follows: Section 2 presents some identities and inequalities based on the vector fields $\{x_j, Y_j, T\}, j = 1, \ldots, n$, which show the reason that the problems (1.1), (1.2) and (1.3) are treated separately. The main estimates for the eigenvalues of (1.1), (1.2) and (1.3) are given in Section 3, Section 4 and Section 5 respectively. We conclude Section 6 by estimating eigenvalues of the generalized Baouendi-Grushin operator.

2. Some preliminary lemmas.

We establish some properties for the commutative vector fields $\{x_j, Y_j, T\}$, $(j = 1, \ldots, n)$ which are of independent interest.

Lemma 2.1. *Given any positive integer $p, 1 \le p \le k$, we have*

(2.1)
$$L^p \begin{pmatrix} x_j u_i \\ y_j u_i \end{pmatrix} = \begin{pmatrix} x_j \\ y_j \end{pmatrix} L^p u_i - 2 \begin{pmatrix} X_j \\ Y_j \end{pmatrix} L^{p-1} u_i - 2 \sum_{q=1} p - 1 L^{p-q} \begin{pmatrix} X_j \\ Y_j \end{pmatrix} L^{q-1} u_i$$

$i = 1, \ldots, m, \ldots, j = 1, \ldots, n.$

Proof. Since

$$\begin{aligned} X_1(x_1 u_i) &= u_i + x_1 X_1 u_i, \\ X_j(x_1 u_i) &= x_1 X_j u_i, \qquad j = 2, \ldots, n, \\ Y_j(x_1 u_i) &= x_1 Y_j u_i, \qquad j = 2, \ldots, n, \end{aligned}$$

and

$$X_1^2(x_1u_i) = 2X_1u_i + x_1X_1^2u_i,$$
$$X_j^2(x_1u_i) = x_1X_j^2u_i, \qquad j = 2, \dots, n,$$
$$Y_j^2(x_1u_i) = x_1Y_j^2u_i, \qquad j = 2, \dots, n,$$

we obtain

$$L(x_1u_i) = x_1Lu_i - 2X_1u_i \tag{2.2}$$

and so

$$L^p(x_1u_i) = L^{p-1}(x_1Lu_i - 2X_1u_i) = L^{p-1}(x_1Lu_i) - 2L^{p-1}X_1u_i. \tag{2.3}$$

Changing u_i in (2.2) to Lu_i yields

$$L(x_1Lu_i) = x_1L^2u_i - 2X_1Lu_i$$

and so

$$L^{p-1}(x_1Lu_i) = L^{p-2}(x_1L^2u_i) - 2L^{p-2}X_1Lu_i.$$

Substituting into (2.3) and repeating these steps, we prove the first formula in (2.1) for $L^P(x_1u_i)$. □

Remark 2.1. When $p = 1, 2$, we have

$$L\begin{pmatrix} x_ju \\ y_ju \end{pmatrix} = \begin{pmatrix} x_j \\ y_j \end{pmatrix} Lu_i - 2\begin{pmatrix} X_j \\ Y_j \end{pmatrix} u_i \tag{2.4}$$

$$L^2\begin{pmatrix} x_ju_i \\ y_ju_i \end{pmatrix} = \begin{pmatrix} x_j \\ y_j \end{pmatrix} L^2u_i - 2\begin{pmatrix} X_j \\ Y_j \end{pmatrix} Lu_i - 2L\begin{pmatrix} X_j \\ Y_j \end{pmatrix} u_i \tag{2.5}$$

respectively. It yields the main difference in the following estimations.

Let

$$0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_m \le \dots$$

denote the eigenvalues of (1.1) ((1.2) and (1.3), respectively) with corresponding orthogonal normalized eigenfunctions $u_1, u_2, \dots, u_m, \dots$ in $S_0^{1,2}(\Omega)$ ($S_0^{2,2}(\Omega)$ and $S_0^{k,2}(\Omega)$, respectively).

Take the trial functions

$$\varphi_{ix_j} = x_ju_i - \sum_{i=1}^m a_{ilx_j}u_l, \quad \varphi_{iy_j} = y_ju_i - \sum_{i=1}^m a_{ily_j}u_l \tag{2.6}$$
$$i = 1, \dots, m, j = 1, \dots, n,$$

where $a_{ilx_j} = \int x_ju_iu_l$, $a_{ily_j} = \int y_iu_iu_l$. It is easy to know that each function of (2.6) is orthogonal to $u_1, \dots, u_m$ and vanishing on $\partial\Omega$.

Lemma 2.2. *For $m \ge 1$, it holds*

$$\sum_{i=1}^m \int \varphi_{ix_j}X_ju_i = \sum_{i=1}^m \int \varphi_{iy_j}Y_ju_i = -\frac{m}{2},\ j = 1, \dots, n. \tag{2.7}$$

Proof. We only prove the first equality in (2.7). Since $a_{ilx_j} = a_{lix_j}$, $\int_\Omega u_l X_j u_i = -\int_\Omega u_i X_j u_l$ and thus $\sum_{i,l} a_{ilx_j} \int u_l X_j u_i = 0$, one has

$$\begin{aligned}\sum_i \int \varphi_{ix_j} X_j u_i &= \sum_i \int (x_j u_i - \sum_i a_{ilx_j} u_l) X_j u_i \\ &= \sum_i \int x_j u_i X_j u_i - \sum_i \int X_j (x_j u_i) u_i \\ &= -\sum_i \int u_i^2 - \sum_i \int x_j u_i X_j u_i \\ &= -m - \sum_i \int x_j u_i X_j u_i.\end{aligned}$$

The desired equality is derived immediately. □

For simplicity, we let $u_i \in C_0^\infty(\Omega)$ in what follows. Obviously, by the density property, the following results are valid for $u_i \in S_0^{1,2}(\Omega)$.

Lemma 2.3. *For any $p \geq 1$, we have*

$$\left(\int_\Omega |\nabla_L^p u_i|^2\right)^{\frac{1}{p}} \leq \left(\int_\Omega |\nabla_L^{p+1} u_i|^2\right)^{\frac{1}{p+1}}, \ i = 1, \dots, m, \dots, \tag{2.8}$$

where $\nabla_L u = (X_1 u, \dots, X_n u, Y_1 u, \dots, Y_n u)$.

Proof. Evidently, for $p \geq 1$

$$\int |\nabla_L u_i|^2 = -\int u_i L u_i \leq \left(\int u_i^2\right)^{\frac{1}{2}} \left(\int (L u_i)^2\right)^{\frac{1}{2}} = \left(\int |\nabla_L^2 u_i|^2\right)^{\frac{1}{2}}$$

by the induction assumption it follows that

$$\begin{aligned}\int |\nabla_L^p u_i|^2 &= -\int \nabla_L^{p-1} u_i \, \nabla_L^{p+1} u_i \\ &\leq \left(\int |\nabla_L^{p-1} u_i|^2\right)^{\frac{1}{2}} \left(\int |\nabla_L^{p+1} u_i|^2\right)^{\frac{1}{2}} \\ &\leq \left(\int |\nabla_L^{p} u_i|^2\right)^{\frac{p-1}{2p}} \left(\int |\nabla_L^{p+1} u_i|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore the assertion of the Lemma is proved. □

As a consequence, we have:

Corollary 2.1. *For k in (1.3), we have*

$$\int |\nabla_L u_i|^2 \leq \left(\int |\nabla_L^k u_i|^2\right)^{\frac{1}{k}}. \tag{2.9}$$

Lemma 2.4. *For $p \geq 1$,*

$$L^p \begin{pmatrix} X_j \\ Y_j \end{pmatrix} = \sum_{s=0}^{p} C_p^s \begin{pmatrix} (-1)^{[\frac{s+1}{2}]} A_j^s \\ (-1)^{[\frac{s}{2}]} A_j^{s+1} \end{pmatrix} L^{p-s} (2T)^s, \tag{2.10}$$

where $j = 1, \ldots, n$; $[\bullet]$ denote the largest integer part of $\bullet$, $C_p^s = \frac{p!}{s!(p-s)1}$, $A_j^s = X_j$, if $s = 0, 2, 4, \ldots$; $A_j^s = Y_j$, if $s = 1, 3, \ldots$.

Proof. A direct calculation gives

$$LX_j = X_j L - 2TY_j,\ LY_j = Y_j L - 2TX_j,\ j - 1, \ldots, n.$$

By the induction assumption that

$$L^{p-1} X_j = \sum_{s=0} p - 1 C_{p-1} s (-1)^{[\frac{s+1}{2}]} A_j^s L^{p-s-1} (2T)^s,$$

it follows

$$\begin{aligned}
L^p X_j &= L \left[\sum_{s=0}^{p-1} C_{p-1} s (-1)^{[\frac{s+1}{2}]} A_j^s L^{p-s-1} (2T)^s \right] \\
&= X_j L^P - 2TY_j L^{p-1} \\
&\quad + \sum_{s=2 (s \,\text{even})}^{p-1} C_{p-1} s (-1)^{[\frac{s+1}{2}]} (X_j L - 2TY_j) L^{p-s-1} (2T)^s \\
&\quad + \sum_{s=1 (s \,\text{odd})}^{p-1} C_{p-1} s (-1)^{[\frac{s+1}{2}]} (Y_j L - 2TX_j) L^{p-s-1} (2T)^s \\
&= X_j L^P - 2TY_j L^{p-1} \\
&\quad + \sum_{s=2 (s \,\text{even})}^{p-1} C_{p-1} s (-1)^{[\frac{s+1}{2}]} X_j L^{p-s} (2T)^s \\
&\quad - \sum_{s=2 (s \,\text{even})}^{p-1} C_{p-1} s (-1)^{[\frac{s+1}{2}]} Y_j L^{p-s-1} (2T)^{s+1} \\
&\quad + \sum_{s=1 (s \,\text{odd})}^{p-1} C_{p-1} s (-1)^{[\frac{s+1}{2}]} X_j L^{p-s} (2T)^s \\
&\quad - \sum_{s=1 (s \,\text{odd})}^{p-1} C_{p-1} s (-1)^{[\frac{s+1}{2}]} Y_j L^{p-s-1} (2T)^{s+1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{s=0(s\,\mathrm{even})}^{p-1} \Big[C_{p-1}s(-1)^{[\frac{s+1}{2}]} X_j L^{p-s}(2T)^s \\
&\qquad + C_{p-1}^{s-1}(-1)^{[\frac{s}{2}]} X_j L^{p-s}(2T)^s \Big] \\
&\quad + \sum_{s=1(s\,\mathrm{odd})}^{p-1} \Big[-C_{p-1}s-1(-1)^{[\frac{s}{2}]} Y_j L^{p-s}(2T)^s \\
&\qquad + C_{p-1}^{s}(-1)^{[\frac{s+1}{2}]} Y_j L^{p-s}(2T)^s \Big] \\
&= \sum_{s=0}^{p} C_p^s (-1)^{[\frac{s}{2}]} A_j^s L^{p-s}(2T)^s,
\end{aligned}
$$

where we have used that $C_{p-1}s + C_{p-1}s - 1 = C_p^s$ in the last equality; $\left[\frac{s+1}{2}\right] = \left[\frac{s}{2}\right]$ if s is even and $\left[\frac{s+1}{2}\right] = \left[\frac{s}{2}\right] + 1$ if s is odd. This proves the first equality in (2.10). Another equality is proved similarly. □

Lemma 2.5. *For $p \geq 1$, it has*

$$
\int TL^{p+1}u_i \cdot TL^p u_i \leq \frac{1}{2n-1} \int L^{p+2}u_i \cdot L^{p+1}u_i, \tag{2.11}
$$

$$
\int TL^p u_i \cdot TL^p u_i \leq \frac{1}{2(2n-1)} \left[\int L^{p+2}u_i \cdot L^{p+1}u_i + \int L^{p+1}u_i \cdot L^p u_i \right]. \tag{2.12}
$$

Proof. Since $tu_i = Y_j X_j u_i - X_j Y_j u_i$, $j = 1, \ldots, n$, we get

$$
\begin{aligned}
&2n \int TL^{p+1}u_i \cdot TL^p u_i \\
&= 2 \int \sum_{j=1}^{n} (Y_j X_j - X_j Y_j) L^{p+1}u_i \cdot TL^p u_i \\
&= 2 \sum_{j=1}^{n} \int (TX_j L^p u_i \cdot Y_j L^{p+1}u_i - TY_j L^p u_i \cdot X_j L^{p+1}u_i) \\
&\leq \sum_{j=1}^{n} \int \left[(TX_j L^p u_i)^2 + (Y_j L^{p+1}u_i)^2 + (TY_j L^p u_i)^2 + (X_j L^{p+1}u_i)^2 \right] \\
&= \sum_{j=1}^{n} \int \Big[-TX_j^2 L^p u_i \cdot TL^p u_i - Y_j^2 L^{p+1}u_i \cdot L^{p+1}u_i \\
&\qquad - TY_j^2 L^p u_i \cdot TL^p u_i - X_j 2L^{p+1}u_i \cdot L^{p+1}u_i \Big] \\
&= \int TL^{p+1}u_i \cdot TL^p u_i + \int L^{p+2}u_i \cdot L^{p+1}u_i
\end{aligned}
$$

and (2.11) is proved. As for (2.12), one has

$$\begin{aligned} 2\int TL^p u_i \cdot TL^p u_i &= 2\int T \bigtriangledown_L L^p u_i \cdot T \bigtriangledown_L L^{p-1} u_i \\ &\leq \int (T \bigtriangledown_L L^p u_i)^2 + \int (T \bigtriangledown_L L^{p-1} u_i)^2 \\ &= \int TL^{p+1} u_i \cdot TL^p u_i + \int TL^p u_i \cdot TL^{p-1} u_i \\ &\leq \frac{1}{2n-1}\left[\int L^{p+2} u_i \cdot L^{p+1} u_i + \int L^{p+1} u_i \cdot L^p u_i\right] \end{aligned}$$

and the conclusion is obtained. □

Corollary 2.2. *For positive integers $a, p \geq 1$, the following inequalities hold:*

$$\int T^a L^{p+1} u_i \cdot t^a L^p u_i \leq \frac{1}{(2n-1)^a} \int L^{p+a+1} u_i \cdot L^{p+a} u_i \tag{2.13}$$

(2.14)
$$\int (T^a L^{p+1} u_i)^2 \leq \frac{1}{2(2n-1)^a}\left[\int L^{p+a+1} u_i \cdot L^{p+a} u_i + L^{p+a} u_i \cdot L^{p+a-1} u_i\right].$$

Proof. It is easy to obtain from (2.11)

$$\begin{aligned} \int T^a L^{p+1} u_i \cdot t^a L^p u_i &\leq \frac{1}{2n-1} \int T^{a-1} L^{p+2} u_i \cdot t^{a-1} L^{p+1} u_i \leq \dots \\ &\leq \frac{1}{(2n-1)^a} \int L^{p+a+1} u_i \cdot L^{p+a} u_i \end{aligned}$$

and from (2.12) and (2.13)

$$\begin{aligned} \int (T^a L^p u_i)^2 &\leq \frac{1}{2(2n-1)}\left[\int T^{a-1} L^{p+2} u_i \cdot t^{a-1} L^{p+1} u_i \right. \\ &\qquad \left. + \int T^{a-1} L^{p+1} u_i \cdot t^{a-1} L^p u_i\right] \\ &\leq \frac{1}{2(2n-1)^a}\left[\int L^{p+a+1} u_i \cdot L^{p+a} u_i + \int L^{p+a} u_i \cdot L^{p+a-1} u_i\right]. \end{aligned}$$

This completes the proof. □

3. Estimates of eigenvalues for (1.1).

Theorem 3.1. *For $m \geq 1$,*

$$\lambda_{m+1} - \lambda_m \leq \frac{2}{mn}\left(\sum_{i=1}^{m} \lambda_i\right). \tag{3.1}$$

Proof. By the choice of trial function φ_{ix_1} (see (2.6)) and the Rayleigh-Ritz principle for $\triangle_{H_n}$, we have

$$\lambda_{m+1} \leq \frac{\int |\bigtriangledown_{H_n} \varphi_{ix_1}|^2}{\int |\varphi_{ix_1}|^2}, \quad i = 1, \ldots, m,$$

and then

$$\lambda \sum_{i=1}^{m} \int |\varphi_{ix_1}|^2 \leq \sum_{i=1}^{m} \int |\bigtriangledown_{H_n} \varphi_{ix_1}|^2 = \sum_{i=1}^{m} \int L\varphi_{ix_1} \cdot \varphi_{ix_1}.$$

By (2.4), the right-hand side becomes $-2\int \varphi_{ix_1} X_1 u_i + \lambda_i \int \varphi_{ix_1}^2$ and we obtain

$$(\lambda_{m+1} - \lambda_m) \sum_{i=1}^{m} \int \varphi_{ix_1}^2 \leq -2 \sum_{i=1}^{m} \int \varphi_{ix_1} X_1 u_i.$$

Repeating the argument to $\varphi_{ix_j} (j = 1, \ldots, n)$ yields

$$\begin{aligned} (3.2) \qquad (\lambda_{m+1} - \lambda_m) \sum_{i=1}^{m} \sum_{j=1}^{n} \int \left(\varphi_{ix_j}^2 + \varphi_{iy_j}^2 \right) \\ \leq -2 \sum_{i=1}^{m} \sum_{j=1}^{n} \int \left(\varphi_{ix_j} X_j u_i + \varphi_{iy_j} Y_j u_i \right). \end{aligned}$$

By Lemma 2.2 and Hölder's inequality

$$\begin{aligned} mn &= - \sum_{i=1}^{m} \sum_{j=1}^{n} \int \left(\varphi_{ix_j} X_j u_i + \varphi_{iy_j} Y_j u_i \right) \\ &\leq \left[\sum_{i,j} \int \left(\varphi_{ix_j}^2 + \varphi_{iy_j}^2 \right) \right]^{\frac{1}{2}} \left[\sum_{i,j} \int (X_j u_i)^2 + (Y_j u_i)^2 \right]^{\frac{1}{2}} \\ &- \left(\sum_{i} \lambda_i \right)^{\frac{1}{2}} \left[\sum_{i,j} \int \left(\varphi_{ix_j}^2 + \varphi_{iy_j}^2 \right) \right]^{\frac{1}{2}} \end{aligned}$$

hence we have

$$\sum_{i,j} \int \left(\varphi_{ix_j}^2 + \varphi_{iy_j}^2 \right) \geq (mn)^2 \left(\sum_{i} \lambda_i \right)^{-1}.$$

Returning to (3.2), the result is proved. □

Remark 3.1. (3.1) is a generalization of Payne-Polya-Weinberger theorem for Dirichlet eigenvalues of Euclidean Laplacian to our context here.

Introducing a parameter $\alpha > \lambda_m$, we have

$$(3.3)\quad (\lambda_{m+1}-\alpha)\sum_{i,j}\int\left(\varphi_{ix_j}^2+\varphi_{iy_j}^2\right) \le -2\sum_{i,j}\int\left(\varphi_{ix_j}X_ju_i+\varphi_{iy_j}Y_ju_i\right) - \sum_{i,j}(\alpha-\lambda_i)\int\left(\varphi_{ix_j}^2+\varphi_{iy_j}^2\right)$$

and for some $\delta > 0$,

$$\begin{aligned} mn &= -\sum_{i,j}\int\left(\varphi_{ix_j}X_ju_i+\varphi_{iy_j}Y_ju_i\right)\\ &\le \frac{\delta}{2}\sum_{i,j}(\alpha-\lambda_i)\int\left(\varphi_{ix_j}^2+\varphi_{iy_j}^2\right)\\ &\quad+\frac{1}{2\delta}\sum_{i,j}(\alpha-\lambda_i)^{-1}\int\left[(X_ju_i)^2+(Y_ju_i)^2\right]. \end{aligned}$$

It is easy to see

$$(\lambda_{m+1}-\alpha)\sum_{i,j}\int\left(\varphi_{ix_j}^2+\varphi_{iy_j}^2\right)\le 2mn-m^2n^2\left(\sum_i\frac{\lambda_i}{\alpha-\lambda_i}\right)^{-1}.$$

So an extension of Hile-Protter theorem is easily obtained:

Theorem 3.2. *For $m \ge 1$, one has*

$$\sum_{i=1} m\frac{\lambda_i}{\lambda_{m+1}-\lambda_i}\ge\frac{mn}{2}.$$

4. Estimates of eigenvalues for (1.2).

We denote the eigenvalues of (1.2) by

$$0<\lambda_1\le\lambda_2\le\cdots\le\lambda_m\le\cdots\to\infty$$

with corresponding eigenfunctions $u_1, u_2, \dots, u_m, \dots$ in $S_0^{2,2}(\Omega)$.

Theorem 4.1. *Let $m \ge 1$. then the following estimate holds*

$$(4.1)\qquad \lambda_{m+1}-\lambda_m\le\frac{4(n+1)}{m^2n^2}\left(\sum_{i=1}^m\sqrt{\lambda_i}\right)^{\frac{1}{2}}.$$

Proof. Following the argument of (3.3) and noting (2.5), we have

$$(4.2)\qquad (\lambda_{m+1}-\alpha)S\le J-T,$$

where α is a parameter, $\alpha > \lambda_m$,

$$S = \sum_{i,j} \int \left(\varphi_{ix_j}^2 + \varphi_{iy_j}^2 \right),$$

$$J = -2 \sum_{i,j} \int \left[(LX_j + X_j L) u_i \cdot \varphi_{ix_j} + (LY_j + Y_j L) u_i \cdot \varphi_{iy_j} \right],$$

$$T = \sum_{i,j} (\alpha - \lambda_j) \int \left(\varphi_{ix_j}^2 + \varphi_{iy_j}^2 \right).$$

□

We need two propositions:

Proposition 4.1.

$$J \le 4(n+1) \sum_{i=1}^{m} \sqrt{\lambda_i}. \tag{4.3}$$

Proof. Noting (2.6), one has

$$\begin{aligned}
&\sum_i \int (LX_j + X_j L) u_i \cdot \varphi_{ix_j} \\
&= \sum_i \int (LX_j + X_j L) u_i \left(x_j u_i - \sum_l a_{ilx_j} u_l \right) \\
&= \sum_i \int \left(X_j u_i L(x_j u_i) - Lu_i \cdot X_j (x_j u_i) \right) \\
&\quad - \sum_{i,l} \int (a_{ilx_j} X_j u_i L u_l - a_{ilx_j} L u_i \cdot X_j u_l) \\
&\triangleq M_1 + M_2.
\end{aligned}$$

Since $a_{ilx_j} = a_{lix_j}$, $i, l = 1, \ldots, m$, we see $M_2 = 0$. On the other hand, (2.4) implies

$$\begin{aligned}
&\sum_i \int (LX_j + X_j L) u_i \cdot \varphi_{ix_j} \\
&= M_1 = \sum_i \int \left[X_j u_i (-2X_j u_i + x_j L u_i) - L u_i (u_i + x_j X_j u_i) \right] \\
&= \sum_i \int (2X_j^2 u_i \cdot u_i - u_i L u_i).
\end{aligned}$$

Similarly, we have

$$\sum_i \int (LY_j + Y_j L) u_i \cdot \varphi_{iy_j} = \sum_i \int (2Y_j^2 u_i - u_i L u_i).$$

Therefore by Hölder's inequality we obtain

$$\begin{aligned} J &= -2\sum_i \left[\int -2Lu_i \cdot u_i - 2n\int u_i L u_i\right] \\ &= 4(n+1)\sum_i \int u_i LU_i \\ &\leq 4(n+1)\sum_i \left(\int u_i^2\right)^{\frac{1}{2}} \left(\int (Lu_i)^2\right)^{\frac{1}{2}} \\ &= 4(n+1)\sum_i \sqrt{\lambda_i}. \end{aligned}$$

□

Proposition 4.2.

$$T \geq m^2 n^2 \left(\sum_i \frac{\sqrt{\lambda_i}}{\alpha - \lambda_i}\right)^{-1}. \tag{4.4}$$

Proof. By Lemma 2.2 and Hölder's inequality we have

$$\begin{aligned} mn &= -\sum_{i,j}\int (\varphi_{ix_j} X_j u_i + \varphi_{iy_i} Y_j u_i) \\ &\leq \sum_i \left[\sum_j \int \left(\varphi_{ix_j}^2 + \varphi_{iy_j}^2\right)\right]^{\frac{1}{2}} \left[\sum_i \int \left((X_j u_i)^2 + (Y_j u_i)^2\right)\right]^{\frac{1}{2}} \\ &\leq \frac{\delta}{2}\sum_{i,j}(\alpha - \lambda_i)\int \left(\varphi_{ix_j}^2 + \varphi_{iy_j}^2\right) + \frac{1}{2\delta}\sum_i \frac{1}{\alpha - \lambda_i}\int u_i L u_i \\ &\leq \frac{\delta}{2} T + \frac{1}{2\delta}\sum_i \frac{\sqrt{\lambda_i}}{\alpha - \lambda_i}, \end{aligned} \tag{4.5}$$

where δ is some positive parameter. After minimizing the right-hand side of (4.5), the result is proved. □

Proof of Theorem 4.1. Substituting (4.3) and (4.4) into (4.2) yields

$$(\lambda_{m+1} - \alpha)S \leq 4(n+1)\sum_i \sqrt{\lambda_i} - m^2 n^2 \left(\sum_i \frac{\sqrt{\lambda_i}}{\alpha - \lambda_i}\right)^{-1}.$$

The inequality (4.1) is obtained with the similar discussion in [**10**]. □

5. Estimates of eigenvalues for (1.3).

Theorem 5.1. *If $k \geq 3$ is odd, then*

(5.1)
$$\lambda_{m+1}-\lambda_m \leq \frac{\sum_{i=1}^m \lambda_i^{\frac{1}{k}}}{m^2n^2}\left[(2n+4)k\sum_{i=1}^m \lambda_i^{\frac{k-1}{k}}+C_1(n,k)\sum_{i=1}^m\left(\lambda_i+\lambda_i^{\frac{k-2}{k}}\right)\right]$$

and if $k \geq 4$ is even, then

$$\lambda_{m+1}-\lambda_m \leq \frac{\sum_{i=1}^m \lambda_i^{\frac{1}{k}}}{m^2n^2}\left[(2n+4)k\sum_{i=1}^m \lambda_i^{\frac{k-1}{k}}+C_2(n,k)\sum_{i=1}^m \lambda_i^{\frac{k-1}{k}}\right], \tag{5.2}$$

where $C_1(n,k)$ and $C_2(n,k)$ are the constants depending on n and k.

Proof. Using the trial function φ_{ix_1} (see (2.6)) and the Rayleigh-Ritz inequality, we have

$$\begin{aligned}
\lambda_{m+1} &\leq \int \varphi_{ix_1} L^K \varphi_{ix_1} \\
&= \int \varphi_{ix_1} L^k \left(x_1 u_i - \sum_{l=1}^m a_{ilx_1} u_l\right) \\
&= \int \varphi_{ix_1}\left(\lambda_i x_1 u_i - 2\sum_{q=1}^k L^{k-q}X_1L^{q-1}u_i\right) \\
&= \lambda_i \int \varphi_{ix_1}^2 - 2\int\left(\sum_{q=1}^k L^{k-q}X_1L^q-1u_i\right)\varphi_{ix_1}, i=1,\dots,m.
\end{aligned}$$

Introducing a parameter β, we have

$$(\lambda_{m+1}-\beta)\sum_i\int\varphi_{ix_1}^2 \leq \sum_i(\alpha_i-\beta)\int\varphi_{ix_1}^2-2\sum_{i,q}\int L^{k-q}X_1L^{q-1}u_i\cdot\varphi_{ix_1}.$$

Let

$$\begin{aligned}
S &= \sum_{i=1}^m\sum_{j=1}^n\int\left(\varphi_{ix_j}^2+\varphi_{iy_j}^2\right), \\
I_1 &= \sum_{i=1}^m\sum_{j=1}^n(\beta-\lambda_j)\int\left(\varphi_{ix_j}^2+\varphi_{iy_j}^2\right), \\
I_2 &= -2\sum_{i=1}^m\sum_{j=1}^n\sum_{q-1}^k\int\left(\varphi_{ix_j}L^{k-q}X_jL^{q-1}u_i+\varphi_{iy_j}L^{k-q}X_jL^{q-1}u_i\right),
\end{aligned}$$

we have

$$(\lambda_{m+1}-\beta)S+I_1\le I_2. \tag{5.3}$$

Applying Lemma 2.2 yields

$$m\le\left[\sum_{i=1}^m\int\left(\varphi_{ix_j}^2+\varphi_{iy_j}^2\right)\right]^{\frac12}\left[\sum_{i=1}^m\int\left[(X_ju_i)^2+(Y_ju_i)^2\right]\right]^{\frac12}$$

and summing over j gives

$$2mn\le\delta I_1+\frac{1}{\delta}\sum_{i=1}^m\frac{1}{\beta-\lambda_i}\int|\bigtriangledown_L u_i|^2,$$

where δ is a positive number. Then we have by Corollary 2.1

$$2mn\le\delta I_1+\frac{1}{\delta}\left(\sum_{i=1}^m\frac{\lambda_i^{\frac1k}}{\beta-\lambda_i}\right).$$

After choosing the minimum of the right-hand side, we derive a lower bound of I_1:

$$I_1\ge m^2n^2\left(\sum_{i=1}^m\frac{\lambda_i^{\frac1k}}{\beta-\lambda_i}\right)^{-1}. \tag{5.4}$$

Now we estimate I_2. Notice the following relation by (2.1)

$$-\sum_{i,l,q}a_{ilx_j}\int u_lL^{q-1}X_jL^{q-1}u_i=\frac12\sum_{i,L}a_{ilx_j}\int\left(u_lL^k(x_ju_i)-x_ju_lL^ku_i\right)=0$$

and

$$-\sum_{i,l,q}a_{ily_j}\int u_lL^{q-1}Y_jL^{q-1}u_i=0,\quad i,l=1,\dots,m,q=1,\dots,k.$$

Therefore we have

(5.5)

$$\begin{aligned}I_2&=-2\sum_{i,j,q}\int(x_ju_iL^{k-q}X_jL^{q-1}u_i+y_ju_iL^{k-q}Y_jL^{q-1}u_i)\\&=-2\sum_{i,j,q}\int\left(L^{k-q}(x_ju_i)X_jL^{q-1}u_i+L^{k-q}(y_ju_i)Y_JL^{q-1}U_i\right)\\&=-2\sum_{i,j,q}\Big\{\int\Big(\Big(x_jL^{k-q}u_i-2\sum_{r=1}^{k-q-1}L^{k-q-r}X_jL^{r-1}u_i\Big)X_jL^{q-1}u_i\\&\qquad-2X_jL^{k-q-1}u_i\cdot X_jL^{q-1}u_i\Big)\end{aligned}$$

$$+\int\Big(y_j L^{k-q}u_i - 2\sum_{r=1}^{k-q-1} L^{k-q-r}Y_jL^{q-1}u_i - 2Y_jL^{k-q-1}u_i\cdot Y_jL^{q-1}u_i\Big)\Big\}.$$

Since

$$\sum_{q=1}^{k}\int \begin{pmatrix} x_j \\ y_j\end{pmatrix} L^{k-q}u_i\cdot \begin{pmatrix} X_j \\ Y_j\end{pmatrix} L^{q-1}u_i$$
$$= -\sum_{q-1}^{k}\left[\int L^{k-q}u_i\cdot L_{q-1}u_i + \begin{pmatrix} x_jX_j \\ y_jY_j\end{pmatrix} L^{k-q}u_i\cdot L^{q-1}u_i\right]$$

we have

$$\sum_{q=1}^{k}\int \begin{pmatrix} x_j \\ y_j\end{pmatrix} L^{k-q}u_i\cdot \begin{pmatrix} X_j \\ Y_j\end{pmatrix} L^{q-1}u_i$$
$$= -\frac{1}{2}\sum_{q-1}^{k}\int L^{k-q}u_i\cdot L_{q-1}u_i$$
$$= -\frac{k}{2}\int L^{k-1}u_i\cdot u_i.$$

On the other hand

$$\sum_j\int\Big(X_jL^{k-q-1}U_i\cdot X_jL^{q-1}U_i + Y_jL^{k-q-1}u_i\cdot Y_jL^{q-1}u_i\Big)$$
$$= \int L^{k-1}u_i\cdot u_i$$

so it follows

(5.6)

$$I_2 = (2n+4)k\sum_{i=1}^{m}\int L^{k-1}u_i\cdot u_i$$
$$+4\sum_{i,j,q}\int\Big(\sum_{r=1}^{k-q-1} L^{k-q-r}X_jL^{r-1}u_i\cdot X_jL^{q-1}u_i$$
$$+\sum_{r=1}^{k-q-1} L^{k-q-r}Y_jL^{r-1}u_i\cdot Y_jL^{q-1}u_i\Big)$$
$$= (2n+4)k\sum_{i=1}^{m}\int L^{k-1}u_i\cdot u_i$$
$$+4\sum_{i,j,q}\sum_{r=1}^{k-q-r}\Big[\sum_{s=0}^{k-q-r} C^s_{k-r-q}(-1)^{[\frac{s+1}{2}]}A^s_j$$
$$\cdot L^{k-q-r-s}(2T)^sL^{r-1}u_i\cdot X_jL^{q-1}u_i$$

$$+\sum_{s=0}^{k-q-r} C_{k-q-r}^{s}(-1)^{[\frac{s}{2}]}A_j^{s+1}L^{k-q-r-s}(2T)^sL^{r-1}u_i\cdot Y_jL^{q-1}u_i\Bigg]$$

$$\triangleq (2n+4)k\sum_{i=1}^{m}\int L^{k-1}u_i\cdot u_i+4\sum_{i,j,q}\sum_{r=1}^{k-q-1}(I_3+I_4)$$

where we have used Lemma 2.4. We obtain that

(5.7)

$$\begin{aligned}
&I_{3(s\,\mathrm{odd})}+I_{4(s\,\mathrm{odd})}\\
&=\sum_{s\le k-q-r,s\,\mathrm{odd}}\Bigg[\int C_{k-q-r}^{s}\left(-(-1)^{[\frac{s}{2}]}\right)Y_jL^{k-q-r-s}(2T)^sL^{r-1}u_i\cdot X_jL^{q-1}u_i\\
&\qquad+\int C_{k-q-r}^{s}(-1)^{[\frac{s}{2}]}X_jL^{k-q-r-s}(2T)^sL^{r-1}u_i\cdot Y_jL^{q-1}u_i\Bigg]\\
&=-\sum_{s\le k-q-r,s\,\mathrm{odd}}\int C_{k-q-r}^{s}(-1)^{[\frac{s}{2}]}(Y_JX_j-X_jY_j)\\
&\qquad\qquad\cdot L^{k-q-r-s}(2T)^sL^{r-1}u_i\cdot L^{q-1}u_i\\
&=-\sum_{s\le k-q-r,s\,\mathrm{odd}}2^sC_{k-q-r}^{s}(-1)^{[\frac{s}{2}]}\int T^{s+1}L^{k-q-s-1}u_i\cdot L^{q-1}u_i\\
&=-\sum_{s\le k-q-r,s\,\mathrm{odd}}2^sC_{k-q-r}^{s}(-1)^{[\frac{s}{2}]}\int T^{s+1}u_i\cdot L^{k-s-2}u_i
\end{aligned}$$

where $T=Y_jX_j-X_jY_j$. Similarly,

(5.8)

$$I_{3(s\,\mathrm{even})}+I_{4(s\,\mathrm{even})}=\sum_{s\le k-q-r,s\,\mathrm{even}}2^sC_{k-q-r}^{s}(-1)^{[\frac{s}{2}]}\int T^su_i\cdot L^{k-s-1}u_i.$$

First let k be odd. If s is odd, then by (2.14)

$$\begin{aligned}
(5.9)\qquad &(-1)^{1+[\frac{s}{2}]}\int T^{s+1}u_i\cdot L^{k-s-2}u_i\\
&=(-1)^{\frac{s+1}{2}+[\frac{s}{2}]+1}\int\left(T^{\frac{s+1}{2}}L^{\frac{k-s-2}{2}}u_i\right)^2\\
&\le\frac{1}{2(2n-1)^{\frac{s+1}{2}}}\left[\int L^{\frac{k-1}{2}+1}u_i\cdot L^{\frac{k-1}{2}}u_i+\int L^{\frac{k-1}{2}}u_i\cdot L^{\frac{k-1}{2}-1}u_i\right]\\
&=\frac{1}{2(2n-1)^{\frac{s+1}{2}}}\left[\int L^ku_i\cdot u_i+\int L^{k-2}u_i\cdot u_i\right],
\end{aligned}$$

and if s is even, then

$$(5.10)\qquad \begin{aligned}&(-1)^{[\frac{s}{2}]}\int T^s u_i\cdot L^{k-s-1}u_i\\ &=(-1)^{\frac{s}{2}+[\frac{s}{2}]}\int\left(T^{\frac{s}{2}}L^{\frac{k-s-1}{2}}u_i\right)^2\\ &\le\frac{1}{2(2n-1)^{\frac{s}{2}}}\left[\int L^{\frac{k-1}{2}+1}u_i\cdot L^{\frac{k-1}{2}}u_i+\int L^{\frac{k-1}{2}}u_i\cdot L^{\frac{k-1}{2}-1}u_i\right]\\ &=\frac{1}{2(2n-1)^{\frac{s}{2}}}\left[\int L^k u_i\cdot u_i+\int L^{k-2}u_i\cdot u_i\right](k\ge 3).\end{aligned}$$

Summing (5.9) and (5.10) yields

(5.11)

$$\begin{aligned}I_2\le\ &(2n+4)k\sum_{i=1}^m\int L^{k-1}u_i\cdot u_i\\ &+4\sum_{i,j,q}\sum_{r=1}^{k-q-1}\left[\sum_{s\le k-q-r(s\,\mathrm{odd})}2^sC^s_{k-q-r}\frac{1}{2(2n-1)^{\frac{s+1}{2}}}\right.\\ &\qquad\cdot\left(\int L^k u_i\cdot u_i+\int L^{k-2}u_i\cdot u_i\right)\\ &\left.+\sum_{s\le k-q-r(s\,\mathrm{even})}2^sC^s_{k-q-r}\frac{1}{2(2n-1)^{\frac{s}{2}}}\left(\int L^k u_i\cdot u_i+\int L^{k-2}u_i\cdot u_i\right)\right]\\ \le\ &(2n+4)k\sum_{i=1}^m\int L^{k-1}u_i\cdot u_i\\ =\ &2\sum_{i,j,q}\sum_{r=1}^{k-q-1}\left(\sum_{s\le k-q-r(s\,\mathrm{odd})}\frac{2^sC^s_{k-q-r}}{(2n-1)^{\frac{s+1}{2}}}+\sum_{s\le k-q-r(s\,\mathrm{even})}\frac{2^sC^s_{k-q-r}}{(2n-1)^{\frac{s}{2}}}\right)\\ &\qquad\cdot\left(\int L^k u_i\cdot u_i+\int L^{k-2}u_i\cdot u_i\right)\\ \le\ &(2n+4)k\sum_{i=1}^m\left(\int L^k u_i\cdot u_i\right)^{\frac{k-1}{k}}\\ &+C_1(n,k)\sum_{i=1}^m\left[\lambda_i+\left(\int L^k u_i\cdot u_i\right)^{\frac{k-2}{k}}\right]\\ \le\ &(2n+4)k\sum_{i=1}^m\lambda_i^{\frac{k-1}{k}}+C_1(n,k)\sum_{i=1}^m\left[\lambda_i+\lambda_i^{\frac{k-2}{k}}\right],\end{aligned}$$

where we have used Lemma 2.3. Note that

$$C_1(n,k) \leq 2n \sum_{q=1}^{k} \sum_{r=1}^{k-q-1} \left(1 + \frac{2}{(2n-1)^{\frac{1}{2}}}\right)^{k-q-r}.$$

Substituting (5.11) in (5.3) leads to

$$\begin{aligned}(\lambda_{m+1} - \beta)s \leq (2n+4)k \sum_{i=1}^{m} \lambda_i^{\frac{k-1}{k}} \\ + C_1(n,k) \sum_{i=1}^{m} \left(\lambda_i + \lambda_i^{\frac{k-1}{k}}\right) - m^2 n^2 \left(\sum_{i=1}^{m} \frac{\lambda_i^{\frac{1}{k}}}{\beta - \lambda_i}\right)^{-1}.\end{aligned}$$

Along with the method in [**10**], we obtained (5.1).

Now let k be even. If s is odd, then by Corollary 2.2

$$\begin{aligned}(-1)^{[\frac{s}{2}]+1} \int T^{s+1} u_i L^{k-s-2} u_i = & \int T^{\frac{s+1}{2}} L^{\frac{k-s-3}{2}} u_i \cdot T^{\frac{s+1}{2}} L^{\frac{k-s-3}{2}+1} u_i \\ & \leq \frac{1}{(2n-1)^{\frac{s+1}{2}}} \int L^{\frac{k-2}{2}+1} u_i L^{\frac{k-2}{2}} u_i \\ & = \frac{1}{(2n-1)^{\frac{s+1}{2}}} \int L^{k-1} u_i \cdot u_i\end{aligned}$$

and if s is even, then

$$\begin{aligned}(-1)^{[\frac{s}{2}]} \int T^{s} u_i L^{k-s-1} u_i = & \int T^{\frac{s}{2}} L^{\frac{k-s-2}{2}} u_i \cdot T^{\frac{s}{2}} L^{\frac{k-s-2}{2}+1} u_i \\ & \leq \frac{1}{(2n-1)^{\frac{s}{2}}} \int L^{\frac{k-2}{2}} u_i L^{\frac{k-2}{2}+1} u_i \\ & = \frac{1}{(2n-1)^{\frac{s}{2}}} \int L^{k-1} u_i \cdot u_i.\end{aligned}$$

Summing two inequalities above yields

$$
\begin{aligned}
I_2^{'} \leq\ & (2n+4)K\sum_{i=1}^{m}\int L^{k-1}u_i\cdot u_i \\
& +4\sum_{i,j,q}\sum_{r=1}^{k-q-1}\left(\sum_{s\leq k-q-r(s\,\mathrm{odd})}\frac{2^sC_{k-q-r}^s}{(2n-1)^{\frac{s+1}{2}}}\right. \\
& \qquad\left.+\sum_{s\leq k-q-r(s\,\mathrm{even})}\frac{2^sC_{k-q-r}^s}{(2n-1)^{\frac{s}{2}}}\right)\int L^{k-1}u_i\cdot u_i \\
\triangleq\ & (2n+4)k\sum_{i=1}^{m}\int L^{k-1}u_i\cdot u_i+C_2(n,k)\sum_{i=1}^{m}\int L^{k-1}u_i\cdot u_i \\
\leq\ & (2n+4)k\sum_{i=1}^{m}\lambda_i^{\frac{k-1}{k}}+C_2(n,k)\sum_{i=1}^{m}\lambda_i^{\frac{k-1}{k}}.
\end{aligned}
$$

Note that

$$
C_2(n,k)\leq 4n\sum_{q=1}^{k}\sum_{r=1}^{k-q-1}\left(1+\frac{2}{(2n-1)^{\frac{1}{2}}}\right)^{k-q-r}.
$$

Then proceeding with the same way for proving (5.1) we deduce (5.2). □

6. Eigenvalues for generalized Baouendi-Grushin operator.

Consider the eigenvalue problem

$$
\begin{cases} -Lu=\lambda u, & \text{in } \ \Omega \\ u=0 & \text{on } \ \Omega \end{cases} \tag{6.1}
$$

where L denote the generalized Baouendi-Grushin operator

$$
L=\sum_{i=1}^{M}X_i^2+\sum_{j=1}^{N}Y_j^2 \tag{6.2}
$$

with the nonsmooth vector fields

$$
X_j=\frac{\partial}{\partial x_i},\ \ i=1,\dots,M,\quad Y_j=|x|^{\alpha}\frac{\partial}{\partial y_j},\ \ j=1,\dots,N, \tag{6.3}
$$

$\alpha\geq 1$, $\alpha\in R$, $M,N\geq 1$ and $M+N=n$. Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$.

The existence of eigenvalues for the problem (6.1) can be proved with the method in [**12**] and the embedding theorem in [**6**] and then (6.1) possesses a system of eigenfunctions $\{u_h\}$ that forms an orthonormal base, with the corresponding eigenvalues $\{\lambda_h\}\,:\,0<\lambda_1\leq\lambda_2\leq\dots$.

By choosing the trial functions

$$\varphi_{hx_i} = x_i u_h - \sum_{i=1}^{m} a_{hlx_j} u_l, \quad \varphi_{hy_i} = y_i u_h - \sum_{i=1}^{m} a_{hly_j} u_l,\ h = 1, \ldots, m, \tag{6.4}$$

we have

$$\begin{aligned}(\lambda_{m+1} - \lambda_m) &\sum_{h=1}^{m} \left(\sum_{i=1}^{M} \int \varphi_{hx_i}^2 + \sum_{j=1}^{N} \int \varphi_{hy_j}^2 \right) \\ &\leq -2 \sum_{l=1}^{m} \left(\sum_{i=1}^{M} \varphi_{hx_i} X_i u_h + \sum_{j=1}^{N} \int |x|^\alpha \varphi_{hy_j} Y_i u_h \right).\end{aligned} \tag{6.5}$$

Noting that

$$\begin{aligned}\sum_{i=1}^{M} \sum_{h=1}^{m} \int \varphi_{hx_i} X_i u_h &= -\frac{mM}{2}, \\ \sum_{j=1}^{N} \sum_{h=1}^{m} \int \varphi_{hy_j} Y_i u_h &= -\frac{N}{2} \sum_{h=1}^{m} \int |x|^{2\alpha} u_h^2\end{aligned}$$

and Schwarz's inequality, we get

$$\begin{aligned}\frac{mM}{2} &\leq \frac{mM}{2} + \frac{N}{2} \sum_{h=1}^{m} \int |x|^{2\alpha} u_h^2 \\ &\leq \sum_{h=1}^{m} \left[\sum_{i=1}^{M} \int |\varphi_{hx_i} X_i u_h| + \sum_{j=1}^{N} \int |x|^\alpha |\varphi_{hy_j} Y_j u_h| \right] \\ &\leq \max(1,\, d^\alpha) \left[\sum_{h=1}^{m} \left(\sum_{i=1}^{M} \int |\varphi_{hx_i}|^2 + \sum_{j=1}^{N} \int |\varphi_{hy_i}|^2 \right) \right]^{\frac{1}{2}} \\ &\qquad \cdot \left[\sum_{h=1}^{m} \int (-L u_h) u_h \right]^{\frac{1}{2}}\end{aligned}$$

where d is the diameter of Ω_y, the projection of Ω in the y-space. Then

$$\sum_{h=1}^{m} \left(\sum_{i=1}^{M} \int |\varphi_{hx_i}|^2 + \sum_{j=1}^{N} \int |\varphi_{hy_j}|^2 \right) \geq \frac{m^2 M^2}{4 \max(1, d^{2\alpha}) \sum_{h=1}^{m} \lambda_h}.$$

Returning to (6.5), we prove the following:

Theorem 6.1. *Let $m \geq 1$, then*

$$\lambda_{m+1} = \lambda_m \leq \frac{4n}{mM^2} \max(1, d^{4\alpha}) \sum_{h=1}^{m} \lambda_h.$$

Remark 6.1. This shows that the upper bounds for eigenvalues of (6.1) depends on the domain.

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