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We show that, given a weak compactness condition which is always satisfied when the underlying space does not contain an isomorphic copy of c_0, all the operators in the weakly closed algebra generated by the real and imaginary parts of a family of commuting scalar-type spectral operators on a Banach space will again be scalar-type spectral operators, provided that (and this is a necessary condition with even only two operators) the Boolean algebra of projections generated by their resolutions of the identity is uniformly bounded.

1. Introduction.

The problem we address, raised by Dunford [**8**] in 1954, is to find conditions under which the sum and product of a pair of commuting scalar-type spectral operators on a Banach space is also a scalar-type spectral operator.

Two difficulties arise when working on an arbitrary Banach space, as opposed to a Hilbert space: the unit ball of the algebra of bounded linear operators need not be weakly compact; and the Boolean algebra generated by two uniformly bounded Boolean algebras of projections need not be bounded [**15**].

In view of this we must restrict ourselves to the case where the Boolean algebra generated by the resolutions of the identities is uniformly bounded.

Previous treatments of this problem [to show that the sum of two commuting scalar-type spectral operators is a scalar-type spectral operator] have focussed on identifying the resolution of the identity of the sum [**11**, **16**, **20**]. These methods have worked essentially only when X contains no copy of c_0. However, this is precisely the case when one can exploit Grothendieck's theorem on the automatic weak compactness of linear mappings from a C*-algebra into X, and prove somewhat more: that all operators in the weakly closed involutory algebra generated by them are scalar-type spectral operators. An advantage of this approach is that one does not have to identify the resolutions of the identity of the sums, or products, or limits, directly.

2. C*-algebras on Banach spaces.

The properties of scalar-type spectral operators and the involutory algebras they generate seem best explained in the context of numerical range, of hermitian operators, and of C*-algebras. For the sake of completeness, and the convenience of the reader, we present a résumé of the key results.

Consider a complex Banach space X; write $L(X)$ for the Banach algebra of bounded linear operators on X, endowed with the operator norm.

We write A_1 for the unit ball of a subset A of a normed space.

We write $\langle x,\, x'\rangle$ for the value of the functional x' in X' at x in X. Let $\boldsymbol{\omega}$ be the linear span of the functionals $\omega_{x,x'} : L(X) \to \mathbb{C} : T \mapsto \langle Tx,\, x'\rangle$. Let Π be the set

$$\bigl\{(x, x') \in X \times X' : \bigl\langle x,\, x'\bigr\rangle = \|x\| = \bigl\|x'\bigr\| = 1\bigr\}$$

and let $\boldsymbol{\omega}_\Pi$ be the set of functionals

$$\{\omega_{x,x'} : (x, x') \in \Pi\}.$$

The *strong operator topology* and *weak operator topology* on $L(X)$ are of paramount importance: important here too are the BWO *topology* and BSO *topology,* the strongest topologies coinciding with the weak and strong topologies on bounded subsets of $L(X)$ — see [**9**, VI, 9].

The *ultraweak operator topology* on $L(X)$ is the topology generated by the seminorms $T \mapsto |\sum_n \langle Tx_n,\, x'_n\rangle|$ where $\{x_n\}$ and $\{x'_n\}$ range over pairs of sequences in X and X' subject to $\sum_n \|x_n\|\,\|x'_n\| < \infty$. The *ultrastrong operator topology* on $L(H)$ is the topology generated by the seminorms $T \mapsto \Bigl\{\sum_n \|Tx_n\|^2\Bigr\}^{\frac{1}{2}}$ where $\{x_n\}$ ranges over sequences for which $\sum_n \|x_n\|^2 < \infty$.

The BWO topology coincides with the ultraweak topology, the BSO topology with the ultrastrong topology, on $L(H)$, when H is a Hilbert space.

The (*spatial*) *numerical range* $V(T)$ of an operator T is defined to be

$$V(T) \triangleq \bigl\{\, \bigl\langle Tx,\, x'\bigr\rangle : (x, x') \in \Pi\bigr\}.$$

An operator R on X is *hermitian* if its numerical range is real i.e., if $V(R) \subset \mathbb{R}$; equivalently, if

$$\{\, \|\exp(irR)\| : r \in \mathbb{R}\}$$

is bounded. The set of hermitian operators is closed in the norm, strong and weak operator topologies.

The following result is crucial:

Theorem 2.1 (Vidav-Palmer Theorem)**.** *Suppose that $\mathcal{A}$ is a unital subalgebra of $L(X)$ [the unit being the identity operator on X]. Let $\mathcal{H}$ be the set of hermitian elements of $\mathcal{A}$. Then $\mathcal{A} = \mathcal{H} + i\mathcal{H}$ if and only if $\mathcal{A}$ is a pre-*C*-algebra under the operator norm and the natural involution*

$$* : \mathcal{A} \to \mathcal{A} : R + iJ \mapsto R - iJ \qquad (R,\, J \in \mathcal{H}).$$

It then follows that $\mathcal{B} \triangleq \overline{\mathcal{A}}$ is a C*-algebra on X, containing the identity I_X on X. (See [3, §38] for a discussion of these topics.)

When $\mathcal{B}$ is a C*-algebra on X the family $\boldsymbol{\omega}_\Pi$ is a *separating* family of *states* on $\mathcal{B}$.

We shall use the following terminology: a *von Neumann algebra* is a weakly closed C*-algebra of operators on a Hilbert space, while a W*-*algebra* is a C*-algebra which has a realisation as a von Neumann algebra [equivalently, is a dual space of a Banach space].

Unital *-isomorphisms of C*-algebras are isometric.

Theorem 2.2 (BWO Closure Theorem)**.** *Suppose that* $\mathcal{B}$ *is a* C*-*algebra on* X *and that its unit ball* $\mathcal{B}_1$ *is relatively weakly compact. Then the* BWO *closure of* $\mathcal{B}$*,*

$$\mathcal{B}^{\sim} \triangleq \bigcup_{n=1}^{\infty} n\overline{\mathcal{B}_1}^{w},$$

is a W*-*algebra; and* $(\mathcal{B}^{\sim})_1 = \overline{\mathcal{B}_1}^{w}$*. Moreover, any faithful representation of* $\mathcal{B}^{\sim}$ *as a concrete von Neumann algebra is* BWO *bicontinuous.*

The proof [24] rests on the fact that, by the identity of comparable compact Hausdorff topologies, the weak topology on $\overline{\mathcal{B}_1}^{w}$ is the weak topology induced by the states $\boldsymbol{\omega}_\Pi$.

It remains open, in general, to decide whether $\mathcal{B}^{\sim} = \overline{\mathcal{B}}^{w}$.

2.1. Commutative C*-algebras on X. The remaining results in this section apply to any commutative unital C*-subalgebra $\mathcal{B}$ of $L(X)$, and in particular to any algebra generated by a Boolean algebra of (hermitian) projections: see §3.

The operators in a commutative C*-subalgebra of $L(X)$ are called *normal* (sometimes *strongly normal*). *Abstractly,* they enjoy all the properties of normal operators on Hilbert spaces.

Let Λ be the maximal ideal space of $\mathcal{B}$ and Θ the *inverse Gelfand map*

$$\Theta : C(\Lambda) \to \mathcal{B}$$

which is a unital isometric *-isomorphism: Θ is also called the *functional calculus* for $\mathcal{B}$.

On restricting Θ to the C*-subalgebra generated by I, T (for any $T \in \mathcal{B}$) we obtain a functional calculus for a (strongly) normal T: a unital isometric *-isomorphism

$$\Theta_T : C(\mathrm{sp}(T)) \to \mathcal{B}$$

such that

$$\begin{aligned}\Theta_T(z \mapsto 1) &= I\\ \Theta_T(z \mapsto z) &= T\\ \Theta_T(z \mapsto \overline{z}) &= T^*\\ \|\Theta_T(f)\| &= \|f\|_{\mathrm{sp}(T)} \qquad \big(f \in C(\mathrm{sp}(T))\big).\end{aligned}$$

The following two lemmas demonstrate how to some extent normal operators on a Banach space mimic normal operators on a Hilbert space:

Lemma 2.3. *Let $\mathcal{B}$ be a commutative* C^**-algebra on X and let $\mathcal{H}$ be the set of hermitian elements of $\mathcal{B}$. Suppose that ${}^{H}_{K} \in \mathcal{H}$ and $0 \le H \le K$. Then*

$$\|Hx\| \le \|Kx\| \qquad \big(x \in X\big).$$

For any $\varepsilon > 0$ the operator $L = H/(K + \varepsilon I)$ is defined in $\mathcal{H}$, and, by the functional calculus, $0 \le L \le 1$; so $\|L\| \le 1$. It follows that $\|Hx\| = \|L(K + \varepsilon I)x\| \le \|(K + \varepsilon I)x\|$. Now let $\varepsilon \to 0$.

The next result, originally due to Palmer [18, Lemma 2.7], helps us extend the C^* structure from $\mathcal{B}$ to $\mathcal{C} \triangleq \overline{\mathcal{B}}^w$. The following short proof is taken from [4]:

Lemma 2.4. *For all $B \in \mathcal{B}$ and $x \in X$ we have*

$$\|Bx\| = \|B^*x\|.$$

Proof. For $\varepsilon > 0$ the functional calculus gives

$$\big\|B - B^2(B^*B + \varepsilon I)^{-1}B^*\big\| = \big\|\varepsilon B(B^*B + \varepsilon I)^{-1}\big\| \le \sqrt{\varepsilon}/2,$$

and

$$\big\|B^2(B^*B + \varepsilon I)^{-1}\big\| \le 1.$$

Thus, for any $x \in X$,

$$\|Bx\| = \lim_{\varepsilon \to 0} \big\|B^2(B^*B + \varepsilon I)^{-1}B^*x\big\| \le \|B^*x\|,$$

and then $\|B^*x\| \le \|B^{**}x\| = \|Bx\|$. □

The weak closure of a *commutative* C^*-algebra on X is also a C^*-algebra on X.

Theorem 2.5. *Let $\mathcal{B}$ be a commutative* C^**-algebra on X and let $\mathcal{H}$ be the set of hermitian elements of $\mathcal{B}$. Let $\overline{\mathcal{H}}^w$ be the weak operator topology closure of $\mathcal{H}$, and $\overline{\mathcal{B}}^w$ the weak operator topology closure of $\mathcal{B}$. Then*

$$\overline{\mathcal{B}}^w = \overline{\mathcal{H}}^w + i\overline{\mathcal{H}}^w$$

and so $\overline{\mathcal{B}}^w$ is a C^**-algebra. Moreover, $\big(\overline{\mathcal{B}}^w\big)_1 = \overline{\mathcal{B}_1}^w$. Hence $\mathcal{B}^\sim = \overline{\mathcal{B}}^w$.*

Proof. First note that the weak and strong closures coincide for $\mathcal{H}$ and $\mathcal{B}$ (they are both convex sets). Now Lemma 2.4 shows that $\overline{\mathcal{B}}^s = \overline{\mathcal{H}}^s + i\overline{\mathcal{H}}^s$, so $\overline{\mathcal{B}}^w$ is a C*-algebra.

Consider $H \in (\overline{\mathcal{H}}^w)_1$. Then $K = \big(I - [I - H^2]^{\frac{1}{2}}\big)/H \ \in \overline{\mathcal{H}}^w$, and $H = 2K/(I + K^2)$. Take a net K_α in $\mathcal{H}$ converging strongly to K: put $H_\alpha = 2K_\alpha/(I + K_\alpha^2)$. Then

$$H_\alpha - H = 2(I + K_\alpha^2)^{-1}\,(K_\alpha - K)\,(I + K^2)^{-1} + \frac{1}{2}H_\alpha(K - K_\alpha)H$$

so $H \in \overline{\mathcal{H}_1}^w$. By the Russo-Dye Theorem [**3**, §38] we have $(\overline{\mathcal{B}}^w)_1 \subseteq \overline{\mathcal{B}_1}^w$. □

Corollary 2.6. *If, further, the unit ball of $\mathcal{B}$ is relatively weakly compact, then $\overline{\mathcal{B}}^w$ is a* W**-algebra and any faithful representation of $\overline{\mathcal{B}}^w$ as a concrete von Neumann algebra on a Hilbert space is* BWO *bicontinuous* (*that is, weakly bicontinuous on bounded sets*).

Proof. Use Theorem 2.2. □

Remark 2.7. We show later (§4) that any such faithful representation is also BSO bicontinuous (that is, strongly bicontinuous on bounded sets). The proof (maybe the result) depends on being able to represent $\overline{\mathcal{B}}^w$ by a spectral measure: and the presence of c_0 as a subspace of X seems to be the natural obstruction to this: see §6 below.

3. Boolean algebras of projections & the algebras they generate.

Let X be a complex Banach space, and $\mathcal{E}$ a *bounded Boolean algebra of projections on* X:

$$\begin{array}{rcll} I \in \mathcal{E} & \subseteq & L(X) & \\ E \in \mathcal{E} & \Longrightarrow & E^2 = E & \\ E \in \mathcal{E} & \Longrightarrow & I - E \in \mathcal{E} & \\ E,\, F \in \mathcal{E} & \Longrightarrow & EF = FE \in \mathcal{E} & \\ \|E\| & \leq & K_{\mathcal{E}} & \qquad (E \in \mathcal{E}) \end{array}$$

for some constant $K_{\mathcal{E}}$. Write $\operatorname{aco}\mathcal{E}$ for the *absolutely convex hull* of $\mathcal{E}$ in $L(X)$.

It can be shown (see [**6**, 5.4]) that then

$$\mathcal{S} = \left\{ \sum_{\text{finite}} \lambda_j E_j : |\lambda_j| \leq 1, E_j \in \mathcal{E}, E_j E_k = 0 \ (j \neq k) \right\}$$

is a bounded multiplicative semigroup of operators on X. If we define

$$\|x\|_{\mathcal{E}} = \sup \{ \|Sx\| : S \in \mathcal{S} \} \qquad (x \in X)$$

we obtain a norm $\|\cdot\|_{\mathcal{E}}$ on X, equivalent to the original norm on X, with respect to which each element of $\mathcal{E}$ is hermitian. Thus, without loss of generality,

we shall assume that all elements of $\mathcal{E}$ are hermitian.

Remark 3.1. By Sinclair's Theorem $\|E\| = 1$ for any nonzero hermitian projection.

Theorem 3.2. *Let $\mathcal{E}$ be a Boolean algebra of hermitian projections on a complex Banach space X. Then $\mathcal{A}$, the linear span of $\mathcal{E}$, is the *-algebra generated by $\mathcal{E}$: $\mathcal{A}$ is a commutative unital algebra, and $\mathcal{A} = \mathcal{H} + i\mathcal{H}$, where $\mathcal{H}$ is the set of hermitian elements of $\mathcal{A}$. So $\mathcal{B}$, $\triangleq \overline{\mathcal{A}}$, is a commutative* C^**-algebra on X.*

Proof. Immediate from the Vidav-Palmer Theorem (Theorem 2.1). □

Lemma 3.3. *Let $S \in \mathcal{A}$ and suppose that $-I \leq S \leq I$. Then*

$$S \in 2\,\mathrm{aco}\,\mathcal{E}.$$

Proof. Suppose first that $0 \leq S \leq I$. Write S in *$\mathcal{E}$-step-form* as $S = \sum_{j=1}^{M} \lambda_j E_j$, where the E_j are pairwise disjoint. Then $0 \leq \lambda_j \leq 1$. Arrange the λ_j in descending order: then $\|S\| = \lambda_1$. Define $\lambda_{M+1} = 0$ and use Abel summation —

$$S = \sum_{j=1}^{M} \lambda_j E_j = \sum_{j=1}^{M} (\lambda_j - \lambda_{j+1}) \left(\sum_{h=1}^{j} E_h \right) \in \mathrm{aco}\,\mathcal{E}.$$

If $-I \leq S \leq I$, split S into its positive and negative parts. □

Theorem 3.4. *Let $\mathcal{E}$ be a Boolean algebra of hermitian projections on a complex Banach space X, and let $\mathcal{B}$ be the* C^**-algebra it generates. Let $\mathcal{B}_1$ be the closed unit ball of $\mathcal{B}$. Then*

$$\mathcal{B}_1 \subseteq 4\,\overline{\mathrm{aco}}\,\mathcal{E}.$$

Proof. Consider an element $B \in \mathcal{B}$ such that $\|B\| < 1$. Given $\varepsilon > 0$ we can find $S = R + iJ$ in $\mathcal{A}$ such that $\|B - R - iJ\| \leq \min\{\varepsilon, 1 - \|B\|\}$. Now $\begin{smallmatrix}\|R\| \\ \|J\|\end{smallmatrix} \leq 1$, so that, by Lemma 3.3, $\begin{smallmatrix}R \\ J\end{smallmatrix} \in 2\,\mathrm{aco}\,\mathcal{E}$. □

Corollary 3.5. *The following are equivalent:*

1) $\mathcal{B}_1$ *is relatively weakly compact.*
2) $\mathrm{aco}\,\mathcal{E}$ *is relatively weakly compact.*
3) $\mathcal{E}$ *is relatively weakly compact.*

Proof. Use the Krein-Šmulian Theorem. □

We can now state the main theorem of this section.

Theorem 3.6. *Let $\mathcal{E}$ be a relatively weakly compact Boolean algebra of hermitian projections on a complex Banach space X, and let $\mathcal{B}$ be the* C*-*algebra generated by $\mathcal{E}$. Then $\overline{\mathcal{B}}^w$ is a* W*-*algebra and any faithful representation of $\overline{\mathcal{B}}^w$ as a concrete von Neumann algebra on a Hilbert space is* BWO *bicontinuous* (*that is, weakly bicontinuous on bounded sets*).

Proof. This follows from Corollary 3.5 and Theorem 2.2. □

4. σ-complete Boolean algebras of projections & spectral measures.

The fundamental results on Boolean algebras of projections on a Banach space were developed by Bade and are to be found in [**10**, XVII]. Much interesting material on this topic is also to be found in [**21**].

Following [**10**] we say that an abstract Boolean algebra $\mathcal{E}$ is (σ-)*complete* if each (countable) subset of $\mathcal{E}$ has a supremum and infimum in $\mathcal{E}$.

$\mathcal{E}$, a Boolean algebra of projections on X, is (σ-)*complete on* X if each (countable) subset $\mathcal{F}$ of $\mathcal{E}$ has a supremum and infimum in $\mathcal{E}$ such that

$$\left(\bigvee \mathcal{F}\right) X = \overline{\operatorname{lin}}\{F\,X \,:\, F \in \mathcal{F}\}, \qquad \left(\bigwedge \mathcal{F}\right) X = \bigcap_{F\in\mathcal{F}} F\,X.$$

It has been shown that $\mathcal{E}$ is (σ-)complete on X if and only if every bounded monotone (sequence) net in $\mathcal{E}$ converges strongly to a limit [**10**, XVII.3.4]. In this case $\mathcal{E}$ must be bounded [**10**, XVII.3.3].

On Hilbert space. On a Hilbert space $\mathcal{H}$ the following two *facts* are classical. We sketch their (elementary) proofs for the convenience of the reader.

Fact 4.1. Any monotone net of hermitian projections on $\mathcal{H}$ has a supremum, to which it converges strongly.

Proof. Let $\left(E_\alpha\right)_{\alpha\in A}$ be such a net. The generalized Cauchy-Schwarz inequality $\left\langle P^2\xi,\, \xi\right\rangle \le \langle P\xi,\, \xi\rangle \left\langle P^3\xi,\, \xi\right\rangle$, which holds for any positive operator P on $\mathcal{H}$ and any element $\xi\in\mathcal{H}$, shows that the net $\left(E_\alpha\right)_{\alpha\in A}$ is strongly Cauchy. Also, its limit must be the supremum. □

Fact 4.2. Suppose that $\left(E_\alpha\right)_{\alpha\in A}$ is a net of hermitian projections that converges *weakly* to a projection E. Then it converges *strongly.*

Proof. This is immediate from the calculation

$$\begin{aligned}\|(E-E_\alpha)\,\xi\|^2 &= \left\langle (E-E_\alpha)^2\,\xi,\, \xi\right\rangle \\ &= \left\langle E^2\xi,\, \xi\right\rangle - \langle E E_\alpha\,\xi,\, \xi\rangle - \langle E_\alpha E\,\xi,\, \xi\rangle + \left\langle E_\alpha^2\xi,\, \xi\right\rangle \\ &\to \left\langle (E-E^2)\,\xi,\, \xi\right\rangle = 0.\end{aligned}$$

□

It follows that *on a Hilbert space* every Boolean algebra $\mathcal{E}$ of hermitian projections can be extended to a *complete* one; that $\overline{\mathcal{E}}^s$ is the smallest such complete extension; and that $\overline{\mathcal{E}}^s = \overline{\mathcal{E}}^w \bigcap\{\text{projections on } \mathcal{H}\}$.

On a Banach space the situation is more delicate. It has been shown that if $\mathcal{E}$ is σ-complete on X then $\overline{\mathcal{E}}^s$ is complete on X [**10**, XVII.3.23], and that the family of projections in $\overline{\mathcal{E}}^w$ coincides with $\overline{\mathcal{E}}^s$. See Corollary 4.10 below for a proof [independent of Bade's original methods].

We shall require the following result, proposed as an exercise in [**9**]:

Lemma 4.3. *If $\mathcal{S} \subset L(X)$ then $\mathcal{S}$ is relatively compact in the weak operator topology if and only if the sets $\mathcal{S}\,x$ are relatively weakly compact for all $x \in X$.*

Proof. See [**9**, VI.9.2]. □

4.1. Spectral measures. Let Σ be a σ-algebra of subsets of a set Ω and Γ a total subset of X'. A *spectral measure of class* (Σ, Γ) is a Boolean algebra homomorphism $\sigma \mapsto E(\sigma)$ from Σ into $L(X)$ such that $\langle E(\sigma)x,\, x'\rangle$ is countably additive for each $x \in X$ and $x' \in \Gamma$: by the Banach-Orlicz-Pettis theorem any spectral measure of class X' is strongly countably additive.

A σ-complete Boolean algebra of projections $\mathcal{E}$ on X can be identified with the range of a spectral measure of class X' on the Borel sets of the Stone space of $\mathcal{E}$ ([**5**, Chapter I]): then each vector measure $\mathcal{E}\,x$ is strongly countably additive.

Lemma 4.4. *If μ is a strongly countably additive vector measure with values in X then* $\operatorname{aco}\{\mu(\sigma) : \sigma \in \Sigma\}$ *is relatively weakly compact.*

Proof. Essentially this is a result of Bartle, Dunford and Schwartz [**1**, 2.3]: see also [**5**, I.2.7 & I.5.3]. □

Corollary 4.5. *If $\mathcal{E}$ is σ-complete then the set $\overline{\operatorname{aco}}^{\,w}(\mathcal{E}x)$ is weakly compact for each $x \in X$.*

Theorem 4.6. *Let $\mathcal{E}$ be a σ-complete Boolean algebra of hermitian projections. Then $\mathcal{C},\ \triangleq \overline{\mathcal{B}}^w$, the commutative* C**-algebra generated by $\mathcal{E}$ in the weak operator topology, is a* W**-algebra, and $\mathcal{C}_1 = \overline{\mathcal{B}_1}^w \subseteq 4\,\overline{\operatorname{aco}}^{\,w}\,\mathcal{E}$. Furthermore, any faithful representation of $\mathcal{C}$ as a von Neumann algebra on a Hilbert space is weakly bicontinuous on bounded sets.*

Proof. $\overline{\operatorname{aco}}^{\,w}(\mathcal{E}x)$ is weakly compact for each $x \in X$ (Corollary 4.5) so $\operatorname{aco}(\mathcal{E})$ is relatively weakly compact, by Lemma 4.3. Apply Theorem 3.6. □

Theorem 4.7. *Let $\mathcal{B}$ be a commutative* C**-algebra on X such that $\mathcal{B}_1$ is relatively weakly compact. Let $\mathcal{C} = \overline{\mathcal{B}}^w$. Then there is a representing spectral measure $E(\cdot)$ defined on the Borel sets of the Gelfand space Λ of $\mathcal{C}$ such that*

$$\Theta(f) = \int_\Lambda f(\lambda)E(d\lambda) \qquad (f \in C(\Lambda)).$$

Proof. Let $\pi : \mathcal{C} \to L(H)$ be a BWO continuous representation of $\mathcal{C}$ as a concrete W*-algebra. Let $\widetilde{E}(\cdot)$ be a representing spectral measure for $\pi(\mathcal{C})$:

$$\pi \circ \Theta(f) = \int_\Lambda f(\lambda)\widetilde{E}(d\lambda) \qquad (f \in C(\Lambda)).$$

Now define $E(\cdot) = \pi^{-1}\widetilde{E}(\cdot)$: this yields a spectral measure on X $\big[E(\cdot)$ is weakly countably additive, and so, by the Banach-Orlicz-Pettis theorem, strongly countably additive$\big]$: and then

$$\Theta(f) = \int_\Lambda f(\lambda)E(d\lambda) \quad (f \in C(\Lambda)).$$

□

It is immediate that for a bounded net $\big(T_\alpha\big)_{\alpha\in A}$ of operators *on a Hilbert space* we have

$$\big(T_\alpha\big)_{\alpha\in A} \to_{\text{strongly}} 0 \iff \big(T_\alpha^* T_\alpha\big)_{\alpha\in A} \to_{\text{weakly}} 0.$$

A similar result holds for normal operators on a Banach space provided that they belong to a *common* W*-algebra.

Theorem 4.8. *Let $\mathcal{C}$ be a commutative* W*-*-algebra on X. Suppose that $\big(S_\alpha\big)_{\alpha\in A}$ is a bounded net in $\mathcal{C}$. Then*

$$\big(S_\alpha\big)_{\alpha\in A} \to_{\text{strongly}} 0 \iff \big(S_\alpha^* S_\alpha\big)_{\alpha\in A} \to_{\text{weakly}} 0.$$

Proof. Clearly $S_\alpha \to_{\text{strongly}} 0$ implies that $S_\alpha^* S_\alpha \to_{\text{strongly}} 0$, whence $S_\alpha^* S_\alpha \to_{\text{weakly}} 0$.

Let $E(\cdot)$ be the representing spectral measure for $\mathcal{C}$ guaranteed by Theorem 4.7.

Suppose that $S_\alpha^* S_\alpha \to_{\text{weakly}} 0$. Let $f_\alpha = \Theta^{-1}S_\alpha$. Then

$$\lim_\alpha \big\langle S_\alpha^* S_\alpha x,\, x'\big\rangle = \lim_\alpha \int_\Lambda |f_\alpha|^2 \big\langle E(d\lambda)x,\, x'\big\rangle \qquad (x \in X,\ x' \in X').$$

Therefore $\lim_\alpha f_\alpha = 0$ in var $\langle E(\cdot)x,\, x'\rangle$ measure and $\lim_\alpha \int_\Lambda f_\alpha \langle E(d\lambda)x,\, x'\rangle = 0$. For fixed $x \in X$ the set $\{\langle E(\cdot)x,\, x'\rangle : \|x'\| \le 1\}$ is a relatively weakly compact set of measures [**9**, IV.10.2]: hence $\lim_\alpha \int_\Lambda f_\alpha \langle E(d\lambda)x,\, x'\rangle = 0$ uniformly for $\|x'\| \le 1$ [**14**, Théorème 2]. Therefore $\lim_\alpha \int_\Lambda f_\alpha E(d\lambda)x = 0$; that is, $S_\alpha \to_{\text{strongly}} 0$. □

Corollary 4.9. *Let $\mathcal{C}$ be a commutative* W*-*-algebra on X. Then any faithful concrete representation of $\mathcal{C}$ as a von Neumann algebra is weakly and strongly bicontinuous on bounded sets.*

Corollary 4.10. *Let $\mathcal{E}$ be a σ-complete Boolean algebra of hermitian projections, and let $\big(E_\alpha\big)_{\alpha\in A}$ be a monotone net of hermitian projections in* *the commutative* W^**-algebra* $\mathcal{C}$ *generated on* X *by* $\mathcal{E}$. *Then* $\left(E_\alpha\right)_{\alpha\in A}$ *converges strongly to a projection in* $\mathcal{C}$. *So* $\overline{\mathcal{E}}^s$ *is complete on* X. *What is more,* $\overline{\mathcal{E}}^s = \overline{\mathcal{E}}^w \bigcap\{\textit{projections in } \mathcal{C}\}$.

Proof. This follows immediately from the known results on Hilbert spaces and from the strong bicontinuity of faithful representations guaranteed by the theorem. □

The next corollary complements [**23**, Theorem 5] and [**12**, Theorems 1, 2].

Corollary 4.11. *Let* $\mathcal{E}$ *be a bounded Boolean algebra of projections on a Banach space* X *and suppose that* $\mathcal{E}$ *is relatively weakly compact. Then* $\mathcal{E}$ *has a* (σ-)*complete extension contained in* $\overline{\mathcal{E}}^s$.

Remark 4.12. This happens automatically when $X \not\supset c_0$ (see §6).

Corollary 4.13 ([**10**, XVII.3.7])**.** *Let* $\mathcal{E}$ *be a complete bounded Boolean algebra of projections on a Banach space* X. *Then* $\mathcal{E}$ *is strongly closed.*

Remark 4.14. The results of [**7**] overlap with ours.

5. Spectral operators.

An operator $T \in L(X)$ is *prespectral of class* Γ if there is a spectral measure $E(\cdot)$ of class (Σ_p, Γ) (here Σ_p is the family of Borel subsets of the complex plane) such that for all $\sigma \in \Sigma_p$:

$$T\,E(\sigma) = E(\sigma)\,T, \tag{1}$$

$$\mathrm{sp}(T\big|E(\sigma)X) \subseteq \overline{\sigma}. \tag{2}$$

The spectral measure $E(\cdot)$ is called a *resolution of the identity of class* Γ for T. If, further, $T = \int_{\mathrm{sp}(T)} \lambda E(d\lambda)$, then T is a *scalar-type operator of class* Γ.

Remark 5.1. Given a scalar-type spectral operator $T = \int_{\mathrm{sp}(T)} \lambda E(d\lambda)$ we can define its *real part* $\Re\, T = \int_{\mathrm{sp}(T)} \Re\,\lambda\; E(d\lambda)$, and its *imaginary part* $\Im\, T = \int_{\mathrm{sp}(T)} \Im\,\lambda\; E(d\lambda)$. By the (closed) *-algebra generated by T we mean the (closed) algebra generated by $\Re\, T$ and $\Im\, T$.

An operator $T \in L(X)$ is a *spectral operator* if it is prespectral of class X': that is, if there is a spectral measure $E(\cdot)$ of class X' satisfying Conditions (1) and (2) above, and if also

$$E(\cdot) \text{ is strongly countably additive on } \Sigma_p.$$

An important property of spectral operators is that if T is spectral and S commutes with T, then S commutes with the resolution of the identity of T [**6**, Theorem 6.6].

Scalar-type spectral operators have been characterised as follows:

Theorem 5.2 ([17] & [22, Theorem])**.** *The operator $T \in L(X)$ is a scalar-type spectral operator if and only if it satisfies the following two conditions:*

(1) *T has a functional calculus,* and
(2) *for every $x \in X$ the map $\Theta_x : C(\mathrm{sp}(T)) \to X : f \mapsto \Theta(f)x$ is weakly compact.*

Note that by Lemma 4.3 Property (2) is equivalent to:

(2′) *The functional calculus $\Theta : C(\mathrm{sp}(T)) \to L(X)$ is weakly compact* in the sense that $\Theta\left(\left\{f \in C(\mathrm{sp}(T)) : \|f\|_{\mathrm{sp}(T)} \le 1\right\}\right)$ is relatively compact in the weak operator topology of $L(X)$.

6. In the absence of c_0.

The following theorem goes back to Grothendieck, Bartle-Dunford-Schwartz, and others. See [5, VI, Notes] for an interesting discussion of its genesis and development.

Theorem 6.1. *If $\mathcal{B}$ is a* C*-*algebra, if $\Theta : \mathcal{B} \to X$ is a bounded operator, and X does not contain a subspace isomorphic to* c_0*, then Θ is a weakly compact mapping.*

Remarks on the proof. A stronger version of this theorem, where $\mathcal{B}$ may be any complete Jordan algebra of operators, not necessarily commutative, can be found in [25, Theorem 2]. That proof relies on James's characterisation of weakly compact sets and the Bessaga-Pełczyński result that X contains no copy of c_0 if and only if all series $\sum_n x_n$ in X with $\sum_n |\langle x_n, x'\rangle|$ convergent for all $x' \in X'$ are unconditionally norm convergent.

Corollary 6.2. *Let T be a normal operator on a Banach space X that does not contain a subspace isomorphic to* c_0*. Then T is a scalar-type spectral operator.*

Proof. T has a functional calculus (see §2) which, by the theorem, is weakly compact. Apply Theorem 6.1. □

We can now present a theorem which is stronger than any other known to us in this area.

Theorem 6.3. *Let $\mathcal{E}$ be a bounded Boolean algebra of hermitian projections on a Banach space X and suppose that X does not contain a subspace isomorphic to* c_0*. Then the weakly closed algebra $\overline{\mathcal{B}}^w$ generated by $\mathcal{E}$ is a* W*-*algebra and any faithful representation of $\overline{\mathcal{B}}^w$ as a concrete von Neumann algebra on a Hilbert space is* BWO *and* BSO *bicontinuous. Moreover, every operator in $\overline{\mathcal{B}}^w$ is a scalar-type spectral operator.*

Proof. Theorem 6.1 shows that $\mathcal{E}$ is relatively weakly compact. The result follows from Theorem 3.6, Corollary 4.9, and Corollary 6.2. □

Corollary 6.4. *Let $\mathcal{T}$ be a commuting family of scalar-type spectral operators on a Banach space X that does not contain a subspace isomorphic to c_0. Suppose that the Boolean algebra generated by the resolutions of the identity of T for each $T \in \mathcal{T}$ is uniformly bounded. Then every operator in the weakly closed *-algebra generated by $\mathcal{T}$ is a scalar-type spectral operator.*

It has recently been shown [**13**, Theorem 2.5] that on a Banach lattice the Boolean algebra generated by two commuting bounded Boolean algebras of projections is itself bounded. Hence:

Corollary 6.5. *Let X be a complex Banach lattice not containing a copy of c_0, and let $\mathcal{T}$ be a finite commuting family of scalar-type spectral operators on X. Then every operator in the weakly closed *-algebra generated by $\mathcal{T}$ is a scalar-type spectral operator.*

c_0 as the natural obstruction. If X contains c_0 then there is a strongly closed bounded Boolean algebra $\mathcal{F}$ of projections on X that is not complete [**12**, Theorem 2]. Then the weakly closed algebra generated by $\mathcal{F}$ cannot have relatively weakly compact unit ball, and there can be no BWO bicontinuous faithful representation of this algebra on a Hilbert space.

7. Boolean algebras with countable basis.

As remarked above, c_0 seems to be the natural essential obstruction to extending the results of the previous section. It is of course conceivable that closer analysis will lead to a proof that the sum and product of a pair of commuting scalar-type spectral operators must be scalar-type spectral operators so long as the Boolean algebra generated by their resolutions of the identity is bounded.

We shall say that a Boolean algebra $\mathcal{E}$ has a *countable basis* if it contains a countable orthogonal subfamily $\mathcal{F} = \big(F_m\big)_{m\in\mathbb{N}}$ such that every $E \in \mathcal{E}$ can be written as the strong *sum* of a subset of this family. Note that then $I = \sum_{m=1}^{\infty} F_m$, the sum being strongly convergent.

Lemma 7.1. *Let $\mathcal{C}$ be a commutative* C^**-algebra on X and $\big(F_m\big)_{m\in\mathbb{N}}$ a countable family of positive elements of $\mathcal{C}$ such that $\sum_{m=1}^{\infty} F_m$ converges in the strong topology. Let C_m be any sequence in $\mathcal{C}$ for which $0 \le C_m \le I$ $(\forall m)$. Then*

$$\sum_{m=1}^{\infty} C_m F_m$$

converges strongly.

Proof. Note that $0 \le C_m F_m \le F_m$ $(\forall m)$. Then, for $M < N$,

$$0 \le \sum_{m=M+1}^{N} C_m F_m \le \sum_{m=M+1}^{N} F_m;$$

so, by Lemma 2.3, the sequence $\big(C_m F_m\big)_{m=\in\mathbb{N}}$ is a strongly Cauchy sequence, and hence strongly convergent. □

The following theorem generalises [**13**, Theorem 3.6]:

Theorem 7.2. *Suppose that $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ are two commuting σ-complete Boolean algebras of projections on X and that the Boolean algebra $\mathcal{E}$ generated by $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ is bounded. Assume, further, that $\mathcal{E}^{(2)}$ has a countable basis $\mathcal{F} = \big(F_m\big)_{m\in\mathbb{N}}$. Then $\mathcal{E}$ has a σ-complete extension, and hence a complete extension.*

Proof. As remarked in §3 we may, and shall, assume that all the elements of $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ are hermitian. Let $\mathcal{C}$ be the weakly closed C*-algebra generated by $\mathcal{E}$.

For each sequence of projections $\big(E_m^{(1)}\big)_{m\in\mathbb{N}}$ taken from $\mathcal{E}^{(1)}$ we can, by Lemma 7.1, define $E = \sum\limits_{m=1}^{\infty} E_m^{(1)} F_m \in \mathcal{C}$. Each such E is a hermitian projection in $\mathcal{C}$ so has norm ≤ 1.

Consider

$$\mathcal{G} \triangleq \left\{ \sum_{m=1}^{\infty} E_m^{(1)} F_m : E_m^{(1)} \in \mathcal{E}^{(1)} \right\}.$$

It is clear that $F_m \in \mathcal{G}$ $(\forall m)$, so $\mathcal{E}^{(2)} \subseteq \mathcal{G}$. Note also that for any $E^{(1)} \in \mathcal{E}^{(1)}$ we have $E^{(1)} = \sum_m E^{(1)} F_m$, so $E^{(1)} \in \mathcal{G}$. Thus $\mathcal{E}^{(1)} \vee \mathcal{E}^{(2)} \subseteq \mathcal{G}$.

It is clear that $\mathcal{G}$ is closed under products. Further, for any

$$E = \sum_{m=1}^{\infty} E_m^{(1)} F_m \in \mathcal{G}$$

we have

$$I - E = \sum_{m=1}^{\infty} \big[I - E_m^{(1)}\big] F_m \in \mathcal{G},$$

and so $\mathcal{G}$ is a Boolean algebra of hermitian projections on X.

Note that for any such $E \in \mathcal{G}$ we have $E F_m = E_m^{(1)} F_m$ $(\forall m)$: thus any element of $\mathcal{G}$, which can be written, though not in a unique manner, as an (orthogonal) sum

$$E = \sum_{m=1}^{\infty} E_m^{(1)} F_m,$$

satisfies

$$E = \sum_{m=1}^{\infty} E_m^{(1)} F_m = \sum_{m=1}^{\infty} E F_m .$$

Now consider a sequence $\left(E_h\right)_{h \in \mathbb{N}}$ of pairwise orthogonal projections in $\mathcal{G}$:

$$E_h = \sum_{m=1}^{\infty} E_{h,m}^{(1)} F_m = \sum_{m=1}^{\infty} E_h F_m .$$

For each k and m define

$$G_{k,m} \triangleq \bigvee_{h=1}^{k} E_{h,m}^{(1)} \in \mathcal{E}^{(1)}$$

and then define

$$G_m \triangleq \bigvee_{k=1}^{\infty} G_{k,m} = \bigvee_{h=1}^{\infty} E_{h,m}^{(1)} \in \mathcal{E}^{(1)} .$$

Note that for each k and m

$$G_{k,m} F_m = \bigvee_{h=1}^{k} E_{h,m}^{(1)} F_m = \sum_{h=1}^{k} E_{h,m}^{(1)} F_m = \left(\sum_{h=1}^{k} E_h \right) F_m .$$

Suppose that $x \in X$ and $\varepsilon > 0$. Then there exists an M such that

$$\left\| x - \sum_{m=1}^{M} F_m\, x \right\| < \varepsilon$$

and so we can find N such that for $1 \leq m \leq M$ and $k \geq N$

$$\| (G_m - G_{k,m}) x \| < \varepsilon / M .$$

Suppose that $j < k$: then $0 \leq \sum_{h=j+1}^{k} E_h \leq I$, and so, by Lemma 2.3,

$$\begin{aligned}
\left\| \left(\sum_{h=j+1}^{k} E_h \right) x \right\| &\leq \left\| \left(\sum_{h=j+1}^{k} E_h \right) \left(x - \sum_{m=1}^{M} F_m\, x \right) \right\| \\
&\quad + \sum_{m=1}^{M} \left\| \left(\sum_{h=j+1}^{k} E_h \right) F_m x \right\| \\
&\leq \left\| x - \sum_{m=1}^{M} F_m\, x \right\| + \sum_{m=1}^{M} \| (G_{k,m} - G_{j,m})\, F_m\, x \| \\
&\leq \left\| x - \sum_{m=1}^{M} F_m\, x \right\| + \sum_{m=1}^{M} \| (G_{k,m} - G_{j,m})\, x \| \\
&\leq \varepsilon + \varepsilon = 2\varepsilon .
\end{aligned}$$

This shows that $\mathcal{G}$ is σ-complete. Then $\overline{\mathcal{E}}^s$ is complete (Corollary 4.10). □

From this we obtain the following results.

Theorem 7.3. *Let $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ be two commuting σ-complete Boolean algebras of hermitian projections on X. Suppose that the Boolean algebra $\mathcal{E}$ generated by $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ is bounded, and that $\mathcal{E}^{(2)}$ has a countable basis. Then the weakly closed *-algebra $\mathcal{C}$ generated by $\mathcal{E}$ is a* W*-*algebra.*

Corollary 7.4 (Extension of [**13**, 3.6])**.** *Let X be a Banach space and T_1, T_2 be commuting scalar-type spectral operators on X with resolutions of the identity $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$ such that $\mathcal{E}^{(1)} \vee \mathcal{E}^{(1)}$ is bounded. Suppose further that one of these operators has countable spectrum. Then all operators in the weakly closed *-algebra generated by T_1 and T_2 are scalar-type spectral operators.*

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