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We derive the existence of a specific two-variable p-adic L-function by means of a method provided by Washington. This two-variable function is a generalization of the one-variable p-adic L-function of Kubota and Leopoldt, yielding the one-variable function when the second variable vanishes.

1. Introduction.

In [5] Kubota and Leopoldt prove the existence of meromorphic functions, $L_p(s;\chi)$, defined over the p-adic number field, that serve as p-adic equivalents of the Dirichlet L-series. These p-adic L-functions interpolate the values

$$L_p(1-n;\chi) = -\frac{1}{n}\left(1-\chi_n(p)p^{n-1}\right)B_{n,\chi_n},$$

whenever n is a positive integer. Here, $B_{n,\chi}$ denotes the n^{th} generalized Bernoulli number associated with the primitive Dirichlet character χ, and $\chi_n = \chi\omega^{-n}$, with ω the Teichmüller character. Since the time of that publication, a number of individuals have derived the existence of these functions by various means. In particular, Washington [8] derives the functions by elementary means and expresses them in an explicit form:

Theorem 1. *Let F be a positive integral multiple of q and f_χ, and let*

$$L_p(s;\chi) = \frac{1}{s-1}\frac{1}{F}\sum_{\substack{a=1\\(a,p)=1}}^{F}\chi(a)\langle a\rangle^{1-s}\sum_{m=0}^{\infty}\binom{1-s}{m}\left(\frac{F}{a}\right)^m B_m.$$

Then $L_p(s;\chi)$ is analytic for $s\in\mathfrak{D}$ when $\chi\neq 1$, and meromorphic for $s\in\mathfrak{D}$, with a simple pole at $s=1$ having residue $1-1/p$, when $\chi=1$. Furthermore, for each $n\in\mathbf{Z}$, $n\geq 1$,

$$L_p(1-n;\chi) = -\frac{1}{n}\left(1-\chi_n(p)p^{n-1}\right)B_{n,\chi_n}.$$

Thus, $L_p(s;\chi)$ vanishes identically if $\chi(-1)=-1$.

Recently, a particular two-variable extension, $L_p(s,t;\chi)$, of the p-adic L-functions was produced—one in which interpolating values of the two-variable functions yield expressions in terms of the generalized Bernoulli polynomials [3]. For positive integers n, these functions satisfy

$$L_p(1-n,t;\chi) = -\frac{1}{n}\left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right),$$

with the restriction that $t \in \mathbf{C}_p$, $|t|_p \le 1$. It has been shown that these interpolating values share certain congruence properties with the corresponding interpolating values of the one-variable functions [2]. By applying the method that Washington used to derive Theorem 1, we obtain $L_p(s,t;\chi)$ by elementary means and express the functions in an explicit form.

Theorem 2. *Let F be a positive integral multiple of q and f_χ, and let*

$$L_p(s,t;\chi) = \frac{1}{s-1}\frac{\chi(-1)}{F}\sum_{\substack{a=1\\(a,p)=1}}^{F}\chi(a)\langle a-qt\rangle^{1-s}\sum_{m=0}^{\infty}\binom{1-s}{m}\left(\frac{F}{a-qt}\right)^m B_m.$$

Then $L_p(s,t;\chi)$ is analytic for $t \in \mathbf{C}_p$, $|t|_p \le 1$, provided $s \in \mathfrak{D}$, except $s \ne 1$ when $\chi = 1$. Also, if $t \in \mathbf{C}_p$, $|t|_p \le 1$, this function is analytic for $s \in \mathfrak{D}$ when $\chi \ne 1$, and meromorphic for $s \in \mathfrak{D}$, with a simple pole at $s = 1$ having residue $1 - 1/p$, when $\chi = 1$. Furthermore, for each $n \in \mathbf{Z}$, $n \ge 1$,

$$L_p(1-n,t;\chi) = -\frac{1}{n}\left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right).$$

Thus, $L_p(s,0;\chi) = L_p(s;\chi)$ for each $s \in \mathfrak{D}$, with $s \ne 1$ if $\chi = 1$.

By an analysis of the formula for $L_p(s;\chi)$ given in Theorem 1, one can obtain Diamond's formula for the value of $L_p'(0;\chi)$ (see [6, p. 393]):

Theorem 3. *Let χ be a primitive Dirichlet character, and let F be a positive integral multiple of q and f_χ. Then*

$$L_p'(0;\chi) = \sum_{\substack{a=1\\(a,p)=1}}^{F}\chi_1(a)G_p\left(\frac{a}{F}\right) - L_p(0;\chi)\log_p(F).$$

Here, the function G_p is the Diamond function, and $\log_p$ is the p-adic logarithm function of Iwasawa.

Young [9] derives a similar formula for $(\partial/\partial s)L_p(0,t;\chi)$ by means of a p-adic integral representation of $L_p(s,t;\chi)$. However, his work is restricted to those characters χ such that the conductor of χ_1 is not a power of p. The explicit formula given in Theorem 2 enables one to derive a formula for $(\partial/\partial s)L_p(0,t;\chi)$, similar to that obtained by Young, but for all primitive Dirichlet characters χ.

Theorem 4. *Let χ be a primitive Dirichlet character, and let F be a positive integral multiple of q and f_χ. Then for any $t \in \mathbf{C}_p$, $|t|_p \leq 1$,*

$$\frac{\partial}{\partial s} L_p(0, t; \chi) = \chi(-1) \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi_1(a) G_p\left(\frac{a - qt}{F}\right) - L_p(0; \chi) \log_p(F).$$

Note that, if $\chi(-1) = -1$, then the function $L_p(s; \chi)$ vanishes identically. However, $L_p(s, t; \chi)$ is not identically 0 for any character χ. Thus, $L_p(s, t; \chi)$ provides us with a p-adic L-function that does not vanish identically for those χ such that $\chi(-1) = -1$. This may prove to be of use in the study of structures associated with such characters.

2. Preliminaries.

Let χ be a Dirichlet character, defined modulo its conductor f_χ. Then $\chi(a)^{\phi(f_\chi)} = 1$ for any $a \in \mathbf{Z}$ with $(a, f_\chi) = 1$, and $\chi(a) = 0$ otherwise. For two such characters χ and ψ, having conductors f_χ and f_ψ, respectively, let $\chi\psi$ denote the primitive character associated to the product of the characters. The conductor $f_{\chi\psi}$ then divides $\mathrm{lcm}(f_\chi, f_\psi)$.

The generalized Bernoulli polynomials associated with χ, $B_{n,\chi}(t)$, are defined by the generating function

$$\sum_{a=1}^{f_\chi} \frac{\chi(a) x e^{(a+t)x}}{e^{f_\chi x} - 1} = \sum_{n=0}^{\infty} B_{n,\chi}(t) \frac{x^n}{n!}. \tag{1}$$

The corresponding generalized Bernoulli numbers can then be defined by $B_{n,\chi} = B_{n,\chi}(0)$. With this definition, the generalized Bernoulli polynomials are expressed more precisely in terms of the expansion

$$B_{n,\chi}(t) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m,\chi} t^m,$$

which is derived from (1).

The classical Bernoulli polynomials, $B_n(t)$, are defined by

$$\frac{x e^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!},$$

and the classical Bernoulli numbers by $B_n = B_n(0)$. This yields the values $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $\ldots$, with $B_n = 0$ for odd $n \geq 3$. The Bernoulli numbers are rational numbers, and the von Staudt-Clausen theorem states that for even $n \geq 2$,

$$B_n + \sum_{\substack{p \text{ prime} \\ (p-1)|n}} \frac{1}{p} \in \mathbf{Z}.$$

Thus, the denominator of each B_n must be square-free. We also have the relation

$$B_n(t) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m} t^m. \tag{2}$$

The classical Bernoulli polynomials are related to the generalized Bernoulli polynomials in that $B_{n,1}(t) = (-1)^n B_n(-t)$, where $\chi = 1$ is the unique character having conductor 1 and satisfying $\chi(a) = 1$ for each $a \in \mathbf{Z}$.

Let p be a fixed prime. We will use $\mathbf{Z}_p$ to represent the p-adic integers, and $\mathbf{Q}_p$ the p-adic rationals. Let $\mathbf{C}_p$ denote the completion of the algebraic closure of $\mathbf{Q}_p$ under the p-adic absolute value $|\cdot|_p$, normalized so that $|p|_p = p^{-1}$. Fix an embedding of the algebraic closure of $\mathbf{Q}$ into $\mathbf{C}_p$. Since each value of a Dirichlet character χ is either 0 or a root of unity, we may consider the values of χ as lying in $\mathbf{C}_p$.

Denote $q = 4$ if $p = 2$ and $q = p$ otherwise. Let ω denote the Teichmüller character, having conductor $f_\omega = q$. For an arbitrary character χ we then define the character $\chi_n = \chi\omega^{-n}$, where $n \in \mathbf{Z}$, in the sense of the product of characters.

Let $\langle a \rangle = \omega^{-1}(a)a$ whenever $(a,p) = 1$. Then $\langle a \rangle \equiv 1 \pmod{q\mathbf{Z}_p}$ for these values of a. For our purposes, we extend this by defining $\langle a + qt \rangle = \omega^{-1}(a)(a + qt)$ for all $a \in \mathbf{Z}$, with $(a,p) = 1$, and $t \in \mathbf{C}_p$ such that $|t|_p \leq 1$. Then $\langle a + qt \rangle = \langle a \rangle + q\omega^{-1}(a)t$, so that $\langle a + qt \rangle \equiv 1 \pmod{q\mathbf{Z}_p[t]}$.

The p-adic logarithm function [4], $\log_p$, is the unique function mapping $\mathbf{C}_p^\times \to \mathbf{C}_p$ that satisfies $\log_p(1 + x) = \sum_{n=1}^\infty (-1)^{n-1} x^n/n$ for $|x|_p < 1$, $\log_p(xy) = \log_p(x) + \log_p(y)$ for all $x, y \in \mathbf{C}_p^\times$, and $\log_p(p) = 0$. Note that these conditions imply that this function vanishes at any rational power of p. The Diamond function [1] is defined by

$$G_p(x) = \left(x - \frac{1}{2}\right)\log_p(x) - x + \sum_{j=2}^{\infty} \frac{B_j}{j(j-1)} x^{1-j}.$$

The domain of this function is $|x|_p > 1$, with the p-adic convergence of this sum being for each x in this domain.

Recall that whenever $m \in \mathbf{Z}$, $m \geq 0$, the power of p that divides $m!$ is given by the sum

$$\sum_{j=1}^{\infty} \left[\frac{m}{p^j}\right] \leq \frac{m}{p-1},$$

where $[x]$ is the unique integer n satisfying $n \leq x < n + 1$. The bound on this sum then implies that $|m!|_p \geq |p|_p^{m/(p-1)}$.

For each $n \in \mathbf{Z}$, $n \geq 0$, the quantity $\binom{x}{n}$ is defined in like manner as the binomial coefficients, denoting $\binom{x}{0} = 1$ and

$$\binom{x}{n} = \frac{1}{n!}x(x-1)\dots(x-(n-1))$$

for $n > 0$. Note that each such quantity is a polynomial in x.

Consider the following result from [8] (see also Chapter 5 of [7]):

Lemma 5. *Let* $A_j(X) = \sum_{n=0}^{\infty} a_{n,j}x^n$, $a_{n,j} \in \mathbf{C}_p$, $j = 0, 1, \dots,$ *be a sequence of power series, each of which converges in a fixed subset* D *of* $\mathbf{C}_p$, *such that:*

1. $a_{n,j} \to a_{n,0}$ *as* $j \to \infty$ *for each* n*; and*
2. *for each* $s \in D$ *and* $\epsilon > 0$, *there exists* $n_0 = n_0(s, \epsilon)$ *such that* $|\sum_{n \geq n_0} a_{n,j}s^n|_p < \epsilon$ *for all* j.

Then $\lim_{j\to\infty} A_j(s) = A_0(s)$ *for all* $s \in D$.

This lemma is used by Washington to show that each of the functions $\langle a \rangle^s$ and $\sum_{m=0}^{\infty} \binom{s}{m}(F/a)^m B_m$, where F is a multiple of q and f_χ, is analytic in $\mathfrak{D} = \{s \in \mathbf{C}_p : |s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$. This, along with an identity concerning generalized Bernoulli polynomials, enables the proof of the main theorem of [8].

By the same means, we derive a similar result for a two-variable p-adic L-function $L_p(s, t; \chi)$. This function is defined for $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and it interpolates the values

$$L_p(1-n, t; \chi) = -\frac{1}{n}\left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right),$$

where $n \in \mathbf{Z}$, $n \geq 1$. It is related to the one-variable function $L_p(s; \chi)$ in that $L_p(s, 0; \chi) = L_p(s; \chi)$ for each s in the domain of $L_p(s; \chi)$.

3. The two-variable p-adic L-function.

This now brings us to our main result. We will construct our function $L_p(s, t; \chi)$ for $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and in the process derive an explicit formula for this function. Before we begin this derivation, we need the following result concerning generalized Bernoulli polynomials:

Lemma 6. *Let* g *be a positive integral multiple of* f_χ. *Then for each* $n \in \mathbf{Z}$, $n \geq 0$,

$$B_{n,\chi}(t) = (-1)^n g^{n-1} \sum_{a=0}^{g-1} \chi(-a) B_n\left(\frac{a-t}{g}\right).$$

A version of this result appears in Chapter 2 of [4], and can be derived by a manipulation of the appropriate generating functions.

Proof of Theorem 2. Let $a \in \mathbf{Z}$, $(a,p)=1$. For $t \in \mathbf{C}_p$, $|t|_p \le 1$, the same argument as that given in the proof of the main theorem of [8] can be applied to show that each of the functions $\sum_{m=0}^{\infty} \binom{s}{m} (F/(a-qt))^m B_m$ and $\langle a-qt\rangle^s = \sum_{m=0}^{\infty} \binom{s}{m} (\langle a-qt\rangle - 1)^m$ is analytic for $s \in \mathfrak{D}$. This method can also be used to show that the function $\sum_{m=0}^{\infty} \binom{s}{m} (F/(a-qt))^m B_m$ is analytic for $t \in \mathbf{C}_p$, $|t|_p < |q|_p^{-1}$, whenever $s \in \mathfrak{D}$. It readily follows that $\langle a-qt\rangle^s = \langle a\rangle^s \sum_{m=0}^{\infty} \binom{s}{m} (-a^{-1}qt)^m$ is analytic for $t \in \mathbf{C}_p$, $|t|_p \le 1$, when $s \in \mathfrak{D}$. Thus, since $(s-1)L_p(s,t;\chi)$ is a finite sum of products of these two functions, it must also be analytic for $s \in \mathfrak{D}$ given $t \in \mathbf{C}_p$, $|t|_p \le 1$, and for $t \in \mathbf{C}_p$, $|t|_p \le 1$, whenever $s \in \mathfrak{D}$. Note that

$$\lim_{s \to 1} (s-1) L_p(s,t;\chi) = \frac{\chi(-1)}{F} \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi(a) = \begin{cases} 1-p^{-1}, & \text{if } \chi = 1, \\ 0, & \text{if } \chi \ne 1. \end{cases}$$

Thus, our conclusions on when $L_p(s,t;\chi)$ is analytic or meromorphic follow.

Now let $n \in \mathbf{Z}$, $n \ge 1$, and fix $t \in \mathbf{C}_p$, $|t|_p \le 1$. Since F must be a multiple of f_{χ_n}, Lemma 6 implies that

$$B_{n,\chi_n}(qt) = (-1)^n F^{n-1} \sum_{a=0}^{F-1} \chi_n(-a) B_n\left(\frac{a-qt}{F}\right).$$

If $\chi_n(p) \ne 0$, then $(p, f_{\chi_n}) = 1$, so that F/p is a multiple of f_{χ_n}. Therefore,

$$\begin{aligned} &\chi_n(p) p^{n-1} B_{n,\chi_n}\left(p^{-1}qt\right) \\ &= (-1)^n \chi_n(p) F^{n-1} \sum_{a=0}^{F/p-1} \chi_n(-a) B_n\left(\frac{a-p^{-1}qt}{Fp^{-1}}\right) \\ &= (-1)^n F^{n-1} \sum_{\substack{a=0 \\ p \mid a}}^{F-1} \chi_n(-a) B_n\left(\frac{a-qt}{F}\right). \end{aligned}$$

The difference of these quantities yields

$$B_{n,\chi_n}(qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(p^{-1}qt\right) = \chi(-1) F^{n-1} \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi_n(a) B_n\left(\frac{a-qt}{F}\right).$$

By using (2), we can rewrite the Bernoulli polynomial $B_n(t)$ in this expression as

$$B_n\left(\frac{a-qt}{F}\right) = F^{-n}(a-qt)^n \sum_{m=0}^{n} \binom{n}{m} \left(\frac{F}{a-qt}\right)^m B_m.$$

Since $\chi_n(a) = \chi(a)\omega^{-n}(a)$ and $\omega^{-1}(a)(a - qt) = \langle a - qt\rangle$ for $(a, p) = 1$ and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, we obtain

$$\begin{aligned} &B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right) \\ &= \frac{\chi(-1)}{F} \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi(a)\langle a - qt\rangle^n \sum_{m=0}^{\infty} \binom{n}{m} \left(\frac{F}{a - qt}\right)^m B_m. \end{aligned}$$

Therefore,

$$-\frac{1}{n}\left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right) = L_p(1 - n, t; \chi),$$

completing the proof. □

Note that the proof of the main theorem of [8] infers the existence of the factor $\chi(-1)$ in the formula for $L_p(s; \chi)$. However, since $\chi(-1) \neq 1$ implies that $L_p(s; \chi)$ is identically 0, this quantity is not needed in the given expression. As $L_p(s, t; \chi)$ is not identically 0 for any character χ, the factor $\chi(-1)$ is needed in the expression corresponding to this function.

In [7], Washington modifies the derivation of $L_p(s; \chi)$ by first defining the function

$$H_p(s, a, F) = \frac{1}{s - 1}\frac{1}{F}\langle a\rangle^{1-s} \sum_{m=0}^{\infty} \binom{1 - s}{m} \left(\frac{F}{a}\right)^m B_m, \tag{3}$$

where $s \in \mathfrak{D}$, $s \neq 1$, $a \in \mathbf{Z}$ with $(a, p) = 1$, and F is a multiple of q. The function $L_p(s; \chi)$ can then be expressed as the sum

$$L_p(s; \chi) = \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi(a)H_p(s, a, F),$$

provided F is a multiple of both q and f_χ. The function $H_p(s, a, F)$ is meromorphic for $s \in \mathfrak{D}$ with a simple pole at $s = 1$, having residue $1/F$, and it interpolates the values

$$H_p(1 - n, a, F) = -\frac{1}{n}\omega^{-n}(a)F^{n-1}B_n\left(\frac{a}{F}\right),$$

where $n \in \mathbf{Z}$, $n \geq 1$.

It is obvious that we can express $L_p(s, t; \chi)$ in a similar manner. Using (3) to define $H_p(s, a - qt, F)$ for all $a \in \mathbf{Z}$, $(a, p) = 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, we obtain

$$L_p(s, t; \chi) = \chi(-1) \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi(a)H_p(s, a - qt, F).$$

From the proof of Theorem 2, it follows that $H_p(s, a-qt, F)$ is analytic for $t \in \mathbf{C}_p$, $|t|_p \leq 1$, when $s \in \mathfrak{D}$, $s \neq 1$, and meromorphic for $s \in \mathfrak{D}$, with a simple pole at $s = 1$, when $t \in \mathbf{C}_p$, $|t|_p \leq 1$.

4. The value of $(\partial/\partial s)L_p(0, t; \chi)$.

Let us now consider the values of the first partial derivatives of the function $L_p(s, t; \chi)$ at $s = 0$.

In [3], it is shown that whenever $n \in \mathbf{Z}$, $n \geq 1$,

$$\frac{\partial^n}{\partial t^n} L_p(s, t; \chi) = n! q^n \binom{-s}{n} L_p(s+n, t; \chi_n)$$

for all $s \in \mathfrak{D}$, $s \neq 1$ if $\chi = 1$, and $t \in \mathbf{C}_p$ with $|t|_p \leq 1$. Furthermore, we have

$$\lim_{s \to 1-n} \binom{-s}{n} L_p(s+n, t; \chi) = -\frac{1}{n} \left(1 - \chi(p)p^{-1}\right) B_{0,\chi}.$$

Therefore,

$$\frac{\partial^n}{\partial t^n} L_p(1-n, t; \chi) = -(n-1)! q^n \left(1 - \chi_n(p)p^{-1}\right) B_{0,\chi_n}.$$

Since $B_{0,\chi} = 0$ whenever $\chi \neq 1$, this becomes

$$\frac{\partial^n}{\partial t^n} L_p(1-n, t; \chi) = \begin{cases} -(n-1)! q^n \left(1 - p^{-1}\right), & \text{if } \chi_n = 1, \\ 0, & \text{if } \chi_n \neq 1. \end{cases}$$

Thus, when $n = 1$, we have the value of $(\partial/\partial t)L_p(0, t; \chi)$.

The value of $(\partial/\partial s)L_p(0, t; \chi)$ is given in Theorem 4. The proof of this result follows in much the same manner as the proof of Theorem 3, given in [6, pp. 393-394].

Proof of Theorem 4. The value of $(\partial/\partial s)L_p(0, t; \chi)$ is the coefficient of s in the expansion of $L_p(s, t; \chi)$ about $s = 0$. We find this by determining the constant and linear terms in the corresponding expansions of each of three functions of s that make up the expression given in Theorem 2.

Expanding $1/(1-s)$ about $s = 0$ yields

$$\frac{1}{1-s} = 1 + s + \cdots,$$

while expanding $\langle a - qt \rangle^{1-s}$ about $s = 0$ yields

$$\langle a - qt \rangle^{1-s} = \langle a - qt \rangle \left(1 - s \log_p \langle a - qt \rangle + \cdots\right).$$

The expansion of $\binom{1-s}{m}$ about $s = 0$ is given by

$$\binom{1-s}{m} = \frac{(-1)^{m+1}}{m(m-1)} s + \cdots,$$

provided $m \geq 2$. Employing these expansions, along with some algebraic manipulations, we obtain

$$\frac{\partial}{\partial s} L_p(0,t;\chi) = \chi(-1) \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi_1(a) \left(\left(\frac{a-qt}{F} - \frac{1}{2} \right) \log_p \langle a - qt \rangle - \frac{a-qt}{F} + \sum_{m=2}^{\infty} \frac{1}{m(m-1)} \left(\frac{a-qt}{F} \right)^{1-m} B_m \right).$$

Since $\omega(a)$ is a root of unity for $(a,p) = 1$, we see that $\log_p \langle a - qt \rangle = \log_p(a - qt)$. Therefore,

$$\frac{\partial}{\partial s} L_p(0,t;\chi) = \chi(-1) \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi_1(a) \left(F^{-1} \log_p(F) \cdot a + G_p\left(\frac{a-qt}{F} \right) \right).$$

By evaluating the sum

$$F^{-1} \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi_1(a) a = (1 - \chi_1(a))\, B_{1,\chi_1} = -L_p(0;\chi),$$

we obtain the result. □

By means similar to those used in the proof of Theorem 4, one can derive the following formula for the value of $L_p(1,t;\chi)$, whenever $\chi \neq 1$:

$$L_p(1,t;\chi) = \frac{\chi(-1)}{F} \sum_{\substack{a=1 \\ (a,p)=1}}^{F} \chi(a) \left(-\log_p \langle a - qt \rangle + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \left(\frac{F}{a-qt} \right)^m B_m \right),$$

where F is a positive integral multiple of q and f_χ. This is a generalization of a similar formula for $L_p(1;\chi)$ (see [**6**, p. 85]).

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