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HOMOTOPY MINIMAL PERIODS FOR MAPS OF THREE DIMENSIONAL NILMANIFOLDS

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A natural number m is called the homotopy minimal period of a map $f : X \to X$ if it is a minimal period for every map g homotopic to f. The set $\mathrm{HPer}\,(f)$ of all minimal homotopy periods is an invariant of the dynamics of f which is the same for a small perturbation of f. In this paper we give a complete description of the sets of homotopy minimal periods of self-maps of nonabelian three dimensional nilmanifold which is a counterpart of the corresponding characterization for three dimensional torus proved by Jiang and Llibre. As a corollary we show that if $2 \in \mathrm{HPer}\,(f)$ then $\mathrm{HPer}\,(f) = \mathbb{N}$ for such a map.

0. Introduction.

One of the natural problems in dynamical systems is the study of the homotopy minimal periods of self-map $f : X \to X$ i.e., these periods which are also minimal periods for every map g homotopic to f. An aim is to give a complete characterization, of the set $\mathrm{HPer}\,(f)$ of all homotopy minimal periods, in terms of the homological information on f. Since the homotopy minimal period preserves under a small perturbation of a manifold map, one can say that the set of all homotopy minimal periods describes the rigid part of dynamics of f. A description of the set of all homotopy minimal periods of a map is difficult in general, however here are some results for the mappings of compact homogenous spaces of Lie groups by a discrete subgroup.

After the case of maps of the circle in [**4**] (Block, Guckenheimer, Misiurewicz and Young) in the second instance maps of two-dimensional torus ($X = T^2$) have been investigated in a series of papers [**1**] and [**2**] by Alsedá, Baldwin, Llibre, Swanson and Szlenk. In our notion they gave a complete description of the set of all homotopy minimal periods of a map of the circle or two torus respectively. The answer is given in terms of the linearization of map f, i.e., an integral matrix of the linear map induced by f. In the work of Jiang and Llibre [**12**] the qualitative description of this set was successfully studied for maps of r-dimensional torus, for an arbitrary $r \geq 1$. All of them use the Nielsen theory, which for the torus maps has very nice algebraic description ([**5**]) and prepossessing geometric properties ([**12**], [**17**] and [**18**]).

Using the general result of [**12**] Jiang and Llibre gave also a complete description of the set of all homotopy minimal periods (called them the minimal set of periods) of a map of the three torus. It can be done with relatively easy handling using algebraic integers of degree equal or less than three.

Recently the authors extended the main theorem of [**12**] onto the case of a map f of an arbitrary compact nilmanifold X with the similar qualitative statement ([**10**] Thm. A). The crucial step of the mentioned fact was a proof that $NP_n(f) = 0$ implies that $f \sim g$, where g has no periodic points of the minimal period n. Basing also on this theorem we give here a complete description of the set of minimal homotopy periods of a compact nonabelian three dimensional nilmanifold (Theorem 3.1). A preliminary version of this theorem has been presented already in [**10**] (Thm. C) but that statement does not contain all restrictions on the sets of homotopy minimal periods that appear in the discussed case. Here we make use of the classification of compact three dimensional nilmanifolds and the fact that every such nilmanifold X forms a fibration with S^1 as the fiber and T^2 as the base (cf. [**6**]). Moreover every self-map of X is homotopic to a fiber map of this fibration due the Fadell-Husseini theorem (cf. [**6**]). This means that the integral 3×3 matrix A corresponding to f is a direct sum of one-dimensional and two-dimensional summand which yields that its characteristic polynomial is the multiple of a two polynomials of degree one and two, corresponding to the fiber mapf_1 and the base map $\bar{f}$ respectively. It lets us to derive the set of homotopy minimal periods of f from the corresponding sets of the factors f_1, $\bar{f}$ (Theorem 3.1) by use of a formula (Theorem 3.5, Corollary 3.6). Due to this factorization we can use the previous classification done in [**2**] and [**4**], and do not need to cope with algebra. The main necessary topological ingredient, with except the mentioned Thm. A of [**10**], is a description of the form of automorphism of any nilpotent nonabelian group of rank 3 (Proposition 2.12). In particular this yields that the degree of base map $\bar{f}$ is equal to the degree of fiber map f_1 (Corollary 2.13).

As an application we specify our theorem to the case of a homeomorphism of such a nilmanifold (Theorem 4.1).

Here is the scheme of the paper. In Section 1 we recall the formula for the homotopy minimal periods of self-maps of S^1 and $\mathbb{T}^2$ ([**4**], [**1**] and [**2**]). In Section 2 the necessary information about nilmanifolds and theorem on $\mathrm{HPer}\, f$ for the self-maps of nilmanifolds of [**10**] are recalled. Also general form of an automorphism of any nilpotent nonabelian group of rank 3 is given. This gives a necessary and sufficient condition on 3×3 matrix to be the linearization of a self-map of such a manifold. Then in Section 3 we show how to reduce the 3-nilmanifold case to S^1 and $\mathbb{T}^2$. This let us to prove the main result (Theorem 3.1). As an application we present a theorem of Šarkovskii type (Corollary 3.9) that says that for a self-map of nonabelian three nilmanifold the existence of homotopy period 2 implies the existence of all homotopy minimal periods. Finally we show that for a homeomorphism f of such manifold if $\mathrm{HPer}\,(f) \neq \emptyset$ then $\mathrm{HPer}\,(f) = \mathbb{N}$ with except two special cases when $\mathrm{HPer}\,(f) = \mathbb{N} \setminus 2\mathbb{N}$.

1. Homotopy minimal periods of self-maps of S^1 and T^2.

In this section we recall the explicit formulae of the homotopy minimal periods of self-maps of S^1 and T^2 presented in [**4**], [**1**] and [**2**]. First we recall the basic definitions used in [**12**] and [**10**]. Remaining a standard terminology, let $f : X \to X$ be a self-map of a compact connected polyhedron X, and n be a natural number. Let $\mathrm{Fix}(f)$ be the fixed point set of f, $P^m(f) := \mathrm{Fix}(f^m)$ and let

$$P_m(f) \; := \; P^m(f) \setminus \bigcup_{n|m,\, n<m} P^n(f),$$

denote the set of periodic points with least period m.

Recall that $\mathrm{Per}(f)$ denotes the set of all minimal periods of f i.e.,

$$\mathrm{Per}(f) := \{m \in \mathbb{N};\, P_m(f) \neq \emptyset\}.$$

When a map $g : X \to X$ is homotopic to f, we shall write $g \simeq f$. Define the *set of homotopy minimal periods* to be the set

$$\mathrm{HPer}\,(f) := \bigcap_{g \simeq f} \mathrm{Per}(g). \tag{1.1}$$

Boju Jiang and Llibre use the name “the minimal set of periods” but we hope that what we use here more emphasizes that $n \in \mathrm{HPer}\,(f)$ iff n is a minimal period for every g homotopic to f.

We begin with $X = S^1$ which was studied by Block and co-authors in [**4**]. The meaning of letters (E), (F), (G) as well as the definition of matrix A and the set $T_A \subset \mathbb{N}$ in the theorem given below are given in the next section (Theorem 2.3).

Theorem 1.2 ([**4**]). *Let $f : S^1 \to S^1$ be a map of the circle and $d \in \mathbb{Z} = \mathcal{M}_{1\times 1}(\mathbb{Z})$ be the matrix corresponding to f i.e., the degree of f.*

There are three types for the minimal homotopy periods of f:

- (E) $\mathrm{HPer}\,(f) = \emptyset$ *if and only if* $d = 1$.
- (F) $\mathrm{HPer}\,(f)$ *is nonempty and finite if and only if* $d = -1$ *or* $d = 0$. *We have* $\mathrm{HPer}\,(f) = \{1\}$ *then. Moreover the sets* T_A *are equal to* $\mathbb{N} \setminus 2\mathbb{N}$ *and* $\mathbb{N}$ *correspondingly.*
- (G) $\mathrm{HPer}\,(f)$ *is equal to* $\mathbb{N}$ *for the remaining* d, *i.e.,* $|d| > 1$, *with the exception of one special case* $d = -2$ *where* $T_A = \mathbb{N}$ *but* $\mathrm{HPer}\,(f) = \mathbb{N} \setminus \{2\}$.

The case $X = T^2$ had been completely described by Alseda and co-authors in [**1**] and [**2**]. A reformulation of it is the following:

Theorem 1.3 ([2]). *Let $f : T^2 \to T^2$ be a map of the torus, $A \in \mathcal{M}_{2\times 2}(\mathbb{Z})$ the linearization of f, and $\chi_A(t) = t^2 - at + b$ be its characteristic polynomial.*

There are three types for the minimal homotopy periods of f:

(E) $\mathrm{HPer}(f) = \emptyset$ *if and only if* $-a + b + 1 = 0$.

(F) $\mathrm{HPer}(f)$ *is nonempty and finite for* 6 *cases corresponding to one of the six pairs* (a, b) *listed below*

$$(0, 0),\ (-1, 0),\ (-2, 1),\ (0, 1),\ (-1, 1),\ (1, 1).$$

We have $\mathrm{HPer}(f) \subset \{1, 2, 3\}$ *then. Moreover the sets* T_A *and* $\mathrm{HPer}(f)$ *are the following:*

	Cases of Type (F)	
$(\ a,\ \ b)$	T_A	$\mathrm{HPer}(f)$
$(\ 0,\ \ 0)$	$\mathbb{N}$	$\{1\}$
$(\ 0,\ \ 1)$	$\mathbb{N} \setminus 4\mathbb{N}$	$\{1, 2\}$
$(-1,\ \ 0)$	$\mathbb{N} \setminus 2\mathbb{N}$	$\{1\}$
$(-1,\ \ 1)$	$\mathbb{N} \setminus 3\mathbb{N}$	$\{1\}$
$(-2,\ \ 1)$	$\mathbb{N} \setminus 2\mathbb{N}$	$\{1\}$
$(\ 1,\ \ 1)$	$\mathbb{N} \setminus 6\mathbb{N}$	$\{1, 2, 3\}$

(G) $\mathrm{HPer}(f)$ *is infinite for the remaining* a, *and* b. *Furthermore,* $\mathrm{HPer}(f)$ *is equal to* $\mathbb{N}$ *for all pairs* $(a, b) \in \mathbb{Z}^2$ *with the exception of the following special cases listed below. We say that a pair* $(a, b) \in \mathbb{Z}^2$ *satisfies condition*
1^0 *if* $a \neq 0$ *and* $a + b + 1 = 0$,
2^0 *if* $a + b = 0$,
3^0 *if* $a + b + 2 = 0$ *respectively,*
and (a, b) *is not one of the pairs of case* (E) *and* (F).

We have the following table of special cases:

	Special Cases of Type (G)	
$(\ a,\ \ b)$	T_A	$\mathrm{HPer}(f)$
$(-2,\ \ 2)$	$\mathbb{N}$	$\mathbb{N} \setminus \{2, 3\}$
$(-1,\ \ 2)$	$\mathbb{N}$	$\mathbb{N} \setminus \{3\}$
$(\ 0,\ \ 2)$	$\mathbb{N}$	$\mathbb{N} \setminus \{4\}$
$(\ a,\ \ b)$, (a, b) satisfies 1^0	$\mathbb{N} \setminus 2\mathbb{N}$	$\mathbb{N} \setminus 2\mathbb{N}$
$(\ a,\ \ b)$, (a, b) satisfies 2^0	$\mathbb{N}$	$\mathbb{N} \setminus \{2\}$
$(\ a,\ \ b)$, (a, b) satisfies 3^0	$\mathbb{N}$	$\mathbb{N} \setminus \{2\}$

2. Nilmanifolds.

A compact manifold M is a nilmanifold iff it is of the form G/Γ where G is a simply connected nilpotent Lie group of dimension r and Γ is a lattice of rank r of G i.e., a discrete, torsion free, subgroup of G of rank r ([**14**] and [**16**]). Then the fundamental group of M is Γ and Γ uniquely determines M up to homeomorphism.

Fadell and Husseini in [**6**] show that every self map on M can be inductively fibered on an orientable fibration into a map on torus and a map on a lower dimensional nilmanifold ([**6**, Thm. 3.3]). This enables the proof of the following theorem (cf. [**9**] and [**13**], see also [**10**] for an exposition of it):

Theorem 2.1. *Let $f : X \to X$ be a map of a compact nilmanifold X of dimension r. Then there exists an $r \times r$ matrix A with integral coefficients such that*

$$L(f^m) = \det(\mathrm{I} - A^m)$$

for every $m \in \mathbb{N}$.

The integral matrix A is the basic object in study minimal and homotopy minimal periods of a self-map $f : X \to X$. Note that if $X = T^r$ is the torus then A is the unique homomorphism of $\Gamma = Z^r$ which corresponds to f and A is called the linearization of f (cf. [**9**] and [**13**]). As matter of fact the spectrum of matrix A, or equivalently the characteristic polynomial $\chi_A(t) \in \mathbb{Z}[t]$ determines the set $\mathrm{HPer}\,(f)$. For given $A \in \mathcal{M}_{r\times r}(\mathbb{Z})$ we set

$$T_A := \{\, n \in \mathbb{N} \,|\, \det(\mathrm{I} - A^n) \neq 0\}. \tag{2.2}$$

In the case if $A = A_f$ is the matrix associated to a self-map $f : X \to X$ of a compact nilmanifold X we call T_A *the set of algebraic periods of* f. The main result of [**9**] says the following:

Theorem 2.3 ([**10**], Thm. A)**.** *Let $f : X \to X$ be a map of a compact nilmanifold X of dimension r, A the matrix associated with f and $T_A \subset \mathbb{N}$ the set of algebraic periods of f.*

Then $\mathrm{HPer}\,(f) \subset T_A$ *and it is in one of the following three* (*mutually exclusive*) *types, where the letters E, F, and G are chosen to represent "empty", "finite" and "generic" respectively:*

- (E) $\mathrm{HPer}\,(f)$ *is empty if and only if* $N(f) = L(f) = 0$, *i.e., if and only if* 1 *is an eigenvalue of* A;
- (F) $\mathrm{HPer}\,(f)$ *is nonempty but finite if and only if all the eigenvalues of* A *are either zero or roots of unity different from* 1;
- (G) $\mathrm{HPer}\,(f)$ *is infinite and* $T_A \setminus \mathrm{HPer}\,(f)$ *is finite.*

Moreover, for every dimension r of X, there are finite sets $P(r)$, $Q(r)$ of integers such that $\mathrm{HPer}\,(f) \subset P(r)$ *in Type F and* $T_A \setminus \mathrm{HPer}\,(f) \subset Q(r)$ *in Type* (G)*.*

Theorem 2.3 generalizes the corresponding result of Boju Jiang and Llibre ([**12**, Thm. B]) from the torus map onto the case of any compact nilmanifold. The last was used by Jiang and Llibre to give a complete description of all homotopy minimal periods of an arbitrary map of three torus in the terms of characteristic polynomial $\chi_A(t)$ of its linearization ([**12**, Thm. C]). The corresponding result for three dimensional nonabelian nilmanifolds was given in a not complete form in [**10**] (Thm. C). Now we would like to present a complete version of this description. To do this we need a little bit more information about three nilmanifolds.

We would like to remind the reader that the simplest nontrivial examples of compact nilmanifolds are *Iwasawa manifolds* $\mathbb{N}_n(\mathbb{R})/\mathbb{N}_n(\mathbb{Z})$ and $\mathbb{N}_n(\mathbb{C})/\mathbb{N}_n(\mathbb{Z}[\mathbf{i}])$, where $\mathbb{Z}[\mathbf{i}]$ is the ring of Gaussian integers and for any ring $\mathcal{R}$ with unity $\mathbb{N}_n(\mathcal{R})$ denotes the group of all unipotent upper triangular matrices whose entries are elements of the ring $\mathcal{R}$. The Iwasawa 3-manifold $\mathbb{N}_3(\mathbb{R})/\mathbb{N}_3(\mathbb{Z})$, called also "Baby Nil" is the simplest example of compact three dimensional nonabelian nilmanifold, since $\mathbb{N}_3(\mathbb{Z}) \neq \mathbb{Z}^3$. Generalizations of the Iwasawa manifolds are compact nilmanifolds $\mathbb{N}_3(\mathbb{R})/\Gamma_{p,q,r}$, where the subgroup $\Gamma_{p,q,r}$, with fixed p, q, $r \in \mathbb{N}$ consists of all matrices of the form

$$(2.4)\qquad \begin{bmatrix} 1 & \frac{k}{p} & \frac{m}{p\cdot q\cdot r} \\ 0 & 1 & \frac{l}{q} \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{where} \quad k,\, l,\, m \in \mathbb{Z}.$$

Since the group $\mathbb{N}_3(\mathbb{R})$ are named the Heisenberg group, the nilmanifolds $\mathbb{N}_3(\mathbb{R})/\Gamma_{p,q,r}$, are also called the Heisenberg nilmanifolds. The groups $\pi_1(X)$ for all compact nilmanifolds X are precisely all finitely generated torsion-free nilpotent groups (see [**3**], [**7**], [**14**] and [**16**]).

This leads to the following well-known classification theorem ([**7**, 4.1, Cor. 2]):

Theorem 2.5. *Let X be a compact nilmanifold of dimension* 3. *Then X is diffeomorphic to T^3 or to $\mathbb{N}_3(\mathbb{R})/\Gamma_{1,1,r}$ with some $r \in \mathbb{N}$.*

Proof. The point is that any finitely generated nilpotent group of rank 3 is isomorphic to Z^3 or to the group $\Gamma_{1,1,r}$ with some $r \in \mathbb{N}$ (cf. [**7**, 4.1, Cor. 2]).

We explain it briefly. In fact the correspondence

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & px & pqz \\ 0 & 1 & qy \\ 0 & 0 & 1 \end{bmatrix}$$

is an isomorphism of $\mathbb{N}_3(\mathbb{R})$ sending $\Gamma_{p,q,r}$ onto $\Gamma_{1,1,r}$. It is sufficient to show [!] that any discrete subgroup of rank 3 of $\mathbb{R}^3$, or $\mathbb{N}_3(\mathbb{R})$ is equal, up to isomorphism, to $\mathbb{Z}^3$, or $\Gamma_{p,q,r}$ respectively.

Since a nilmanifold is the quotient of a simply connected nilpotent Lie group by its uniform (hence discrete) subgroup, it remains to know that any three dimensional simply connected non-commutative nilpotent Lie group is isomorphic to the Heisenberg group $\mathbb{N}_3(\mathbb{R})$. The last follows from the fact that there is one non-commutative nilpotent Lie algebra of dimension three, up to isomorphism.

Then a three nilmanifold different than torus is of the form $\mathbb{N}_3(\mathbb{R})/\Gamma$ where Γ is a uniform subgroup in $\mathbb{N}_3(\mathbb{R})$ hence $\Gamma = \Gamma_{p,q,r}$ for some $p, q, r \in \mathbb{N}$. We notice that $\mathbb{N}_3(\mathbb{R})/\Gamma_{p,q,r} = \mathbb{N}_3(\mathbb{R})/\Gamma_{1,1,r}$. □

Next we point out that the Fadell-Husseini toral fibration of a three-dimensional compact nilmanifold has a special form. Since the commutator

$$G_1 = \left\langle [\mathbb{N}_3(\mathbb{R}), \mathbb{N}_3(\mathbb{R})] = \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; : \; z \in \mathbb{R} \right\rangle \tag{2.6}$$

is one dimensional, the quotient space $G_1/\Gamma \cap G_1 \approx S^1$. By the dimensional reasons the base space must be 2-torus and the fibration becomes $S^1 \subset \mathbb{N}_{1,1,r} \to T^2$.

The above gives the following statement:

Proposition 2.7. *Let $f : X \to X$ be a map of compact nilmanifold X of dimension 3 not diffeomorphic to T^3, and $f_1 : S^1 \to S^1$, $\bar{f} : T^2 \to T^2$ a pair of maps associated with f considered as a fiber map. Then the matrix A corresponding to f by Theorem 2.1 has the form*

$$\begin{bmatrix} d & 0 \\ 0 & A \end{bmatrix} = A_1 \oplus \overline{A},$$

where $A_1 = [d]$, with $d := \deg(f_1)$ the degree of the fiber map f_1, and $\overline{A} \in \mathcal{M}_{2\times 2}(\mathbb{Z})$ is the matrix corresponding to the map $\bar{f}$ of base T^2.

Consequently the characteristic polynomial of f is equal to $\chi_A(t) = \chi_{A_1}(t) \cdot \chi_{\overline{A}}(t) = (t-d)(t^2 - at + b)$, where $d \in \mathbb{Z}$, $t - d = \chi_{A_1}(t)$, $a, b \in \mathbb{Z}$ and $t^2 - at + b = \chi_{\overline{A}}(t)$ is the characteristic polynomial of $\overline{A}$. Moreover $a = \operatorname{tr} \overline{A}$ and $b = \det \overline{A} = \deg(\bar{f})$.

Proof. All with the exception of the last equality are obvious algebraically. The equality $\det \overline{A} = \deg(\bar{f})$ is well-known for the torus map induced by an integral matrix. □

The above proposition gives a natural restriction on an integral 3×3 matrix of the linearization any map of such a manifold. Now we formulate next algebraic restriction that comes from the geometry of the discussed spaces. First we recall a more general fact:

Proposition 2.8. *Let $\Gamma = \pi_1(X)$ be the fundamental group of a compact nilmanifold $X = G/\Gamma$. Then every map $f : X \to X$ is homotopic to a* *map given by a homomorphism* $\Phi : G \to G$ *and the induced homomorphism* $\pi_1(\Phi) : \Gamma \to \Gamma$ *is equal to* $\Phi_{|\Gamma}$.

Inversely, for every homomorphism $\phi : \Gamma \to \Gamma$ *there exist a map* $f : X \to X$ *such that* $\pi_1(f) = \phi$.

Proof. The statement follows from the fact that X is the $K(\Gamma, 1)$-space (cf. [**6**]), the fact that every endomorphism ϕ of Γ has a unique extension to an endomorphism Φ of G (cf. [**16**]), and that for a map $f : X \to X$ given by a homomorphism Φ of G the induced map of the fundamental group $\pi_1(f)$ is equal to $\Phi|\Gamma$. □

With respect to Theorem 2.5 and Proposition 2.8 and the below it is enough to determine the set of matrices of linearization of all endomorphisms of $\Gamma = \Gamma_{1,1,r}$. We begin with a description of $\Gamma_{1,1,r}$. Then we give a description of all endomorphisms of $\Gamma_{1,1,r}$. We follow the approach of [**8**] where the case of $\Gamma_{1,1,1}$ was discussed.

Assigning to any matrix

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{where} \quad x,\, y,\, z \in \mathbb{R} \quad \text{the vector} \quad (x,\, y,\, z)$$

we get the homeomorphism between $\mathbb{N}_3(\mathbb{R})$ and $\mathbb{R}^3$. In these coordinates the multiplication has form

$$(x,\, y,\, z) * (x',\, y',\, z') = (x + x',\, y + y',\, z + z' + xy').$$

Using the coordinates we see that $\Gamma_{1,1,r} \subset \mathbb{N}_3(\mathbb{R})$ is generated by the matrices

$$a := (1,\, 0,\, 0),\;\; b := (0,\, 1,\, 0),\;\; c := (0,\, 0,\, 1/r),$$

since $(m,\, p,\, q/r) = a^m b^p c^{q-mp}$. Moreover the only relations are

$$aba^{-1}b^{-1} = c^r,\;\; aca^{-1}c^{-1} = e,\;\; bcb^{-1}c^{-1} = e. \tag{2.9}$$

Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ be a map and let

$$\phi(a) = (\alpha_1,\, \beta_1,\, \gamma_1),\;\; \phi(b) = (\alpha_2,\, \beta_2,\, \gamma_2),\;\; \phi(c) = (\alpha_3,\, \beta_3,\, \gamma_3).$$

We look for a necessary and sufficient condition on ϕ to extend to homomorphism of $\Gamma_{1,1,r}$. Suppose that ϕ extends to a such homomorphism. Then for some integer k

$$\phi(c) = c^k, \tag{2.10}$$

because the cyclic group generated by c is equal to the center of $\Gamma_{1,1,r}$, consequently $\alpha_3 = 0$ $\beta_3 = 0$ $\gamma_3 = k$. Using the first equality of (2.9) and (2.10), deriving $\phi(a)\phi(b)\phi(a)^{-1}\phi(b)^{-1}$, and comparing the coordinates we get

$$k = \alpha_1\beta_2 - \alpha_2\beta_1. \tag{2.11}$$

Note that $\phi(c) = c^k$ implies that the second and third relations of (2.9) are preserved, because $\phi(c)$ is in the center of $\Gamma_{1,1,r}$. Notice that γ_1 , γ_2 may be arbitrary. Since (2.9) are the only relations we get the following fact:

Proposition 2.12. *A map $\phi : \Gamma_{1,1,r} \to \Gamma_{1,1,r}$ defined in the coordinate system by its values on the generators a, b, c as*

$$\phi(a) = (\alpha_1, \beta_1, \gamma_1), \ \phi(b) = (\alpha_2, \beta_2, \gamma_2), \ \phi(c) = (\alpha_3, \beta_3, \gamma_3)$$

extends to an automorphism of $\Gamma_{1,1,r}$ iff $\alpha_3 = \beta_3 = 0$, and $\gamma_3 = \alpha_1\beta_2 - \alpha_2\beta_1$.

Consequently a 3×3 integral matrix A is the linearization matrix of a map of X given by an endomorphism of $\Gamma_{1,1,r}$ iff it is of the form

$$A = A_1 \oplus \overline{A} = \begin{bmatrix} k & 0 & 0 \\ 0 & \alpha_1 & \beta_1 \\ 0 & \alpha_2 & \beta_2 \end{bmatrix}$$

where $\det \overline{A} = k$.

Finally we formulate a topological consequence of Proposition 2.12:

Corollary 2.13. *Let $X \to X$ be a map of three dimensional nilmanifold not diffeomorphic to the torus.*

Then there exists $k \in \mathbb{Z}$ such that $\deg f = k^2$. In particular if $\deg f \neq 0$ then f preserves the orientation.

Proof. Note that for a fiber-map $f = (f_1, \bar{f})$ we have $\deg f = \deg f_1 \deg \bar{f}$. On the other hand we have just shown that for a map induced by a homomorphism, thus for every map, we have $\deg f_1 = d = \det \overline{A} = \deg \bar{f}$, by Proposition 2.12. □

3. The main theorem.

Theorem 3.1. *Let $f : X \to X$ be a map of three-dimensional compact nilmanifold X not diffeomorphic to T^3. Let $A = A_1 \oplus \overline{A} \in \mathcal{M}_{3\times 3}(\mathbb{Z})$ be the matrix induced by the fibre map $f = (f_1, \bar{f})$ (Theorem 2.1) and $\chi_A(t) = \chi_{A_1}(t) \cdot \chi_{\overline{A}}(t) = (t - d)(t^2 - at + b)$ be its characteristic polynomial. Then $d = b$ and there are three types for the minimal homotopy periods of f:*

- (E) $\mathrm{HPer}\,(f) = \emptyset$ *if and only if or $d = 1$ or $-a + d + 1 = 0$.*
- (F) $\mathrm{HPer}\,(f)$ *is nonempty and finite only for 2 cases corresponding to*

$$d = 0$$

combined with one of the two pairs (a, b)

$$(0, 0), \quad \textit{and} \quad (-1, 0).$$

We have $\mathrm{HPer}\,(f) = \{1\}$ then. Moreover the sets T_A and $\mathrm{HPer}\,(f)$ are the following:

Map	Cases of Type (F)	
$(d,\ a,\ b)$	T_A	$\mathrm{HPer}\,(f)$
$(0,\ 0,\ 0)$	$\mathbb{N}$	$\{1\}$
$(0,\ -1,\ 0)$	$\mathbb{N} \setminus 2\mathbb{N}$	$\{1\}$

(G) HPer (f) *is infinite for the remaining* $(d,\,a,\,b\,=\,d)$. *Furthermore,* HPer (f) *is equal to* $\mathbb{N}$ *for all triples* $(d,\,a,\,b = d) \in \mathbb{Z}^3$ *with the exception of the following special cases listed below:*

	Special Cases of Type (G)	
$(\ d,\ a,\ b)$	T_A	$\mathrm{HPer}\,(f)$
$a+d+1=0$, with $a \neq 0$, and $d \notin \{-2,\,-1,\,0,\,1\}$	$\mathbb{N} \setminus 2\mathbb{N}$	$\mathbb{N} \setminus 2\mathbb{N}$
$(\ 0,\ -2,\ 0)$	$\mathbb{N}$	$\mathbb{N} \setminus \{2\}$
$(-1,\ 1,\ -1)$	$\mathbb{N} \setminus 2\mathbb{N}$	$\mathbb{N} \setminus 2\mathbb{N}$
$(-1,\ -1,\ -1)$	$\mathbb{N} \setminus 2\mathbb{N}$	$\mathbb{N} \setminus 2\mathbb{N}$
$(-2,\ 1,\ -2)$	$\mathbb{N} \setminus 2\mathbb{N}$	$\mathbb{N} \setminus 2\mathbb{N}$
$(-2,\ 0,\ -2)$	$\mathbb{N}$	$\mathbb{N} \setminus \{2\}$
$(-2,\ 2,\ -2)$	$\mathbb{N}$	$\mathbb{N} \setminus \{2\}$

Moreover for every pair subset $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathbb{N}$, *appearing as* HPer (f) *and* T_A *listed above there exists a map* $f : X \to X$ *such that* HPer $(f) = \mathcal{S}_1$ *and* $T_A = \mathcal{S}_2$.

To prove Theorem 3.1 we show an algorithm which allows us to express the homotopy minimal periods of a given self-map of a (nontrivial) 3-nilmanifold by the corresponding data of self-maps on S^1 and T^2. This will be obtained as a consequence of formulas deriving the set T_A of algebraic periods of $A = \oplus_1^l A_i$ from the sets T_{A_i} and analogously HPer (f) from HPer (f_i) for a map $f : X \to X$ where the sequence of torus maps $\{f_i\}_1^l$ come from consecutive applications of the Fadell-Husseini fibrations.

Let us start with some more general remarks. Recall that a matrix $A \in \mathcal{M}_{r\times r}(\mathbb{Z})$ of a self-map of compact nilmanifold X according to Theorem 2.1 is given by the following procedure: Suppose that a self-map $f : X \to X$, $\dim X = r$ is a fiber map given by the Fadell-Husseini theorem i.e., a map such that the diagram

$$\begin{array}{ccc}
T^{s_1} & \xrightarrow{f_1} & T^{s_1} \\
\iota\big\downarrow & & \iota\big\downarrow \\
X & \xrightarrow{f} & X \\
p\big\downarrow & & p\big\downarrow \\
\overline{X} & \xrightarrow{\bar{f}} & \overline{X}
\end{array}$$

commutes. Then $f_1 : T^{s_1} \to T^{s_1}$ is induced, up to homotopy, by a matrix $A_1 \in \mathcal{M}_{s_1\times s_1}(\mathbb{Z})$. By induction on the dimension, we can assume that with $\bar{f}$ is assigned a matrix $\overline{A} = \oplus_2^l A_j$ $\overline{A} \in \mathcal{M}_{(r-s_1)\times(r-s_1)}(\mathbb{Z})$, $A_j \in \mathcal{M}_{s_j\times s_j}(\mathbb{Z})$.

Put

$$A := A_1 \oplus \overline{A} = \oplus_{i=1}^l A_j, \tag{3.1}$$

where l is the length of the given tower of consecutive Fadell-Husseini fibrations.

We begin with the following consequence of Theorem 2.1:

Proposition 3.2. *For a given map f of a compact nilmanifold X the matrix A and consequently its characteristic polynomial*

$$\chi_A(t) = \prod_1^l \chi_{A_j}(t) \in \mathbb{Z}[t]$$

depends only on the homotopy class of f.

Definition 3.3. For a given map $f : X \to X$, let $(f_1, f_2, \dots, f_l)$ be a tower of torus maps given by the described above procedure. The number

$$s := \max_{1\le j\le l} s_j = \max_{1\le j\le l} \deg \chi_{A_j}(t)$$

we call *the size* of this tower.

Now we have to formulate a criterion to determine whether a natural number is a homotopy minimal period of a given map of nilmanifold (cf. [**8**] and [**12**] for the torus case, or [**9**] and [**10**] for the nilmanifold case).

Theorem 3.4. *Let $f : X \to X$ be a map of a compact nilmanifold X. Then $m \notin \mathrm{HPer}\,(f)$ if and only if either $N(f) = 0$ or $N(f^m) = N(f^{m/p})$ for some prime factor p of m.*

Consequently $m \in \mathrm{HPer}\,(f)$ if and only if:

a) $N(f^m) = |L(f^m)| = |\det(\mathrm{I} - A^m)| = |\chi_A^m(1)| \neq 0,$ *and*
b) *for every prime $p|m$ we have $N(f^m) > N(f^{m/p})$.*

Proof. Recall that $m \notin \mathrm{HPer}\,(f) \Leftrightarrow NP_m(f) = 0$ (by [**17**] and [**18**] for tori, by [**10**] for nilmanifolds). On the other hand $NP_m(f) = 0 \Leftrightarrow N(f) = 0$ or $N(f^m) = N(f^{m/p})$ for some prime factor p of m ([**12**] for tori, [**10**] for nilmanifolds). □

We are in position to formulate the formula which allows us to derive the sets T_A and $\mathrm{HPer}\,(f)$ of a map f with a given Fadell-Husseini tower $(f_1, f_2, \ldots, f_l)$.

Theorem 3.5. *Let $f : X \to X$ be a map of a compact nilmanifold X of dimension r. Let next $(f_1, \ldots, , f_l)$ be the tower of consecutive torus maps given by the Fadell-Husseini fibrations and $(A_1, \ldots, A_l)$ the sequence of their linearizations and $A = \oplus_1^l A_j$ the matrix corresponding to f.*

Then

$$T_A = \cap_1^l T_{A_j} \qquad \textit{and}$$

$$T_A \cap (\cup_1^l \mathrm{HPer}\,(f_j)) \ \subset \ \mathrm{HPer}\,(f).$$

Proof. By the definition, $m \in T_A$ iff $\det(\mathrm{I} - A^m) = \chi_{A^m}(1) \neq 0$. But $\chi_A(1) = \prod_1^l \chi_{A_j}(1)$ which proves the first equality.

To prove the second formula, first note that $|\chi_{A^n}(1)|$ divides $|\chi_{A^m}(1)|$ if $n|m$ (provided $\chi_{A^n}(1) \neq 0$) for every integral matrix A. Consequently, by Theorem 3.4 it follows that $m \in \mathrm{HPer}\,(f)$ if $m \in T_A$ and there exists $1 \leq j_0 \leq l$ such that $|\chi_{A_{j_0}^m}(1)| > |\chi_{A_{j_0}^{m/p}}(1)|$ for every prime $p|m$, since $|\chi_{A_j^m}(1)| \geq |\chi_{A_j^{m/p}}(1)|$ for the remaining j. This shows the statement. □

As a consequence of the above theorem we have the following fact:

Corollary 3.6. *Let $f : X \to X$ be as in Theorem 3.5 and $m = p^a$ a prime power. Then $m \in \mathrm{HPer}\,(f)$ if and only if $m \in T_A \cap (\cup_1^l \mathrm{HPer}\,(f_j))$.*

Proof. By the argument of Theorem 3.5, since p is the only prime dividing m there exists $1 \leq j \leq l$ such that $|\chi_{A_j^m}(1)| > |\chi_{A_j^{m/p}}(1)|$. But this means that $m = p^a \in \mathrm{HPer}\,(f_j)$ in respect of Theorem 3.4. □

The next theorem reduces the computation of $\mathrm{HPer}\,(f)$ to $\mathrm{HPer}\,(\bar{f})$ and T_A which are given by Theorem 3.1.

Theorem 3.7. *Let X be a three dimensional compact nilmanifold different from a torus. Let $f : X \to X$ induces the pair of $(f_1, \bar{f})$ in the resulting*

Fadell-Husseini $S^1 \subset X \to T^2$ for X. Let $d = \deg f_1$. Then

$$\mathrm{HPer}\,(f) = \begin{cases} T_{\overline{A}} & \textit{for} \quad d \notin \{0,-1,+1,-2\} \\ \emptyset & \textit{for} \quad d = 1 \\ \mathrm{HPer}\,(\bar{f}) & \textit{for} \quad d = 0 \\ \mathrm{HPer}\,(\bar{f}) \setminus 2N & \textit{for} \quad d = -1 \\ T_{\overline{A}} \setminus \{2\} & \textit{for} \quad d = -2 \quad \textit{and} \quad 2 \notin \mathrm{HPer}\,(\bar{f}) \\ T_{\overline{A}} & \textit{for} \quad d = -2 \quad \textit{and} \quad 2 \in \mathrm{HPer}\,(\bar{f}). \end{cases}$$

Proof. We consider the following cases:

1. Let $d \notin \{0,-1,+1,-2\}$. Then $\mathrm{HPer}\,(f_1) = T_{A_1} = \mathbb{N}$. We will show that $\mathrm{HPer}\,(f) = T_{\overline{A}}$. $\subset$ is evident since $\mathrm{HPer}\,(f) \subseteq T_A = T_{\overline{A}} \cap T_{A_1} = T_{\overline{A}}$. On the other hand, Theorem 3.5 implies $\mathrm{HPer}\,(f) \supset T_A \cap \{\mathrm{HPer}\,(\bar{f}) \cup \mathrm{HPer}\,(f_1)\} = T_A = T_{\overline{A}}$, which gives $\supset$.

2. Let $d = 1$. Then $\mathrm{HPer}\,(f) = \emptyset$ by Theorem 2.3.

3. Let $d = 0$. Then $\chi_k(t) = t\bar{\chi}_k(t)$ gives $N(f^k) = |\chi_k(1)| = |\bar{\chi}_k(1)| = N(\bar{f}^k)$ which implies

$$\mathrm{HPer}\,(f) = \mathrm{HPer}\,(\bar{f}).$$

4. Let $d = -1$. We will show that $\mathrm{HPer}\,(f) = \mathrm{HPer}\,(\bar{f}) \setminus 2\mathbb{N}$. We notice that $T_{A_1} = \mathbb{N} \setminus 2\mathbb{N}$ and $\mathrm{HPer}\,(f_1) = \{1\}$. Now $\chi_k(t) = (t - (-1)^k)\bar{\chi}_k(t)$ gives

$$N(f^k) = \begin{cases} 2N(\bar{f}^k) & \text{for} \quad \text{k} \quad \text{odd}, \\ 0 & \text{for} \quad \text{k} \quad \text{even}. \end{cases}$$

By Theorem 3.4 notice that no even number k belongs to $\mathrm{HPer}\,(f)$ since $N(f^k) = 0$ and that any odd number k will either belong to both of $\mathrm{HPer}\,(\bar{f})$ and $\mathrm{HPer}\,(f)$ or neither of these since $N(f^k) = 2N(\bar{f}^k)$. Consequently

$$\mathrm{HPer}\,(f) = \mathrm{HPer}\,(\bar{f}) \setminus 2\mathbb{N}$$

in this case.

5. Let $d = -2$. Now by definition $T_{A_1} = \mathbb{N}$ and from Theorem 1.2 $\mathrm{HPer}\,(f_1) = \mathbb{N} \setminus \{2\}$. Then $T_A = T_{\overline{A}}$, and by Theorem 3.5 $\mathrm{HPer}\,(f) \supset T_A \cap \{\mathrm{HPer}\,(f_1) \cup \mathrm{HPer}\,(\bar{f})\} = T_{\overline{A}} \cap \{(\mathbb{N} \setminus \{2\}) \cup \mathrm{HPer}\,(\bar{f})\}$.

Consequently we have

$$\mathrm{HPer}\,(f) = \begin{cases} T_{\overline{A}} \setminus \{2\} & \text{for} \quad 2 \notin \mathrm{HPer}\,(\bar{f}) \\ T_{\overline{A}} & \text{for} \quad 2 \in \mathrm{HPer}\,(\bar{f}). \end{cases}$$

$\square$

Proof of Theorem 3.1. We shall use Theorems 1.2, 1.3, 3.7, and Proposition 2.12. From Proposition 2.12 it follows that $d = b$. At first we notice that

$$\begin{aligned}\mathrm{HPer}\,(f) = \emptyset &\Longleftrightarrow \mathrm{HPer}\,(\bar{f}) = \emptyset \quad \text{or} \quad \mathrm{HPer}\,(f_1) = \emptyset\\ &\Longleftrightarrow \det(\overline{A}) = 0 \quad \text{or} \quad d = 1\\ &\Longleftrightarrow 1 - a + d = 0 \quad \text{or} \quad d = 1.\end{aligned}$$

We will assume now that $\mathrm{HPer}\,(f) \neq \emptyset$. Suppose first that $d \notin \{-2, -1, 0, 1\}$. From Theorem 1.3 it follows that $T_{A_1} = \mathbb{N}$ and consequently $\mathrm{HPer}\,(f) = T_A = T_{\overline{A}}$ by Theorem 3.7. Now we look for the case $T_{\overline{A}} \neq \mathbb{N}$ and $d \notin \{-2, -1, 0, 1\}$ in the tables of Theorem 1.3. The second condition does not hold if $\bar{f} \in$ (F). On the other hand if $\bar{f} \in$ (G) then the first condition holds iff (a, b) satisfies 1^0) i.e., $a \neq 0$, and $a + d + 1 = 0$. $\mathrm{HPer}\,(f) = T_A = T_{\overline{A}} = \mathbb{N} \setminus 2\mathbb{N}$ then. This gives the first row of the table of special cases (G) of the statement.

Let $d = 0$. Then $\mathrm{HPer}\,(f) = \mathrm{HPer}\,(\bar{f})$ by Theorem 3.7. Looking at the tables of Theorem 1.3 we get the two triples which gives the case (F) of the statement. Moreover, substituting $b = d = 0$ to the special cases 1^0, 2^0, 3^0 of (G) of Theorem 1.3, deriving a, and excluding pairs (a, b) that have been already listed we get the second row of the case (G) of the statement.

Let $d = -1$. Then $T_{A_1} = \mathbb{N} \setminus 2\mathbb{N}$ and $\mathrm{HPer}\,(f_1) = \{1\}$. Thus $T_A = T_{\overline{A}} \setminus 2\mathbb{N}$. On the other hand $\mathrm{HPer}\,(f) \supset T_A \cap \mathrm{HPer}\,(\bar{f}) = \mathrm{HPer}\,(\bar{f}) \setminus 2\mathbb{N}$. Now looking at the tables of Theorem 1.3 we notice that $b = d = -1$ may occur only in (G). But even then $\mathrm{HPer}\,(\bar{f}) \supset \mathbb{N} \setminus 2\mathbb{N}$. Thus $\mathbb{N} \setminus 2\mathbb{N} = T_A \supset \mathrm{HPer}\,(f) \supset \mathrm{HPer}\,(\bar{f}) \setminus 2\mathbb{N} \supset \mathbb{N} \setminus 2\mathbb{N}$ implies $\mathrm{HPer}\,(f) = \mathbb{N} \setminus 2\mathbb{N}$. On the other hand we notice that this occurs exactly for $(1, -1)$ and $(-1, -1)$. This gives the third and the fourth line of the exceptional cases in 3.1 (G).

Let $d = -2$. By Theorem 3.7

$$\mathrm{HPer}\,(f) = \begin{cases} T_{\overline{A}} \setminus \{2\} & \text{for } \ 2 \notin \mathrm{HPer}\,(\bar{f}) \\ T_{\overline{A}} & \text{for } \ 2 \in \mathrm{HPer}\,(\bar{f}). \end{cases}$$

Lemma 3.8 shows that $2 \notin \mathrm{HPer}\,(\bar{f})$ iff $a = 0, 1, 2$. In all remaining cases $(d = -2, \mathrm{HPer}\,(\bar{f}) \neq \emptyset)$ $\mathrm{HPer}\,(f) = T_{\overline{A}}$. In Theorem 1.3 we look for the cases $T_{\overline{A}} \neq \mathbb{N}$ (with $d = -2$). This is possible only for $(a, d) = (1, -2), (2, -2), (0, -2)$ the three exceptional cases discussed in Lemma 3.8. In all remaining cases $\mathrm{HPer}\,(f) = T_{\overline{A}} = \mathbb{N}$. Now the three last lines in the table in Theorem 3.1 (G) follow from Lemma 3.8.

We are left with the task to prove that for every pair of sets listed as (T_A, and $\mathrm{HPer}\,(f)$) in the statement of Theorem 3.1 there exists a map $f : \mathbb{N}_3(\mathbb{R})/\Gamma_{p,q,r} \to \mathbb{N}_3(\mathbb{R})/\Gamma_{p,q,r}$ which gives this pair. Fix a, $b = d \in \mathbb{Z}$. For given a, b we define an integral matrix

$$A = \begin{bmatrix} b & 0 & 0 \\ 0 & a & b \\ 0 & -1 & 0 \end{bmatrix}.$$

By Proposition 2.12 A defines a homomorphisms of $\Gamma_{p,q,r}$ and hence a map of $\mathbb{N}_3(\mathbb{R})/\Gamma_{p,q,r}$ whose linearization is equal $A = A_1 \oplus \overline{A}$. We have $\det \overline{A} = b$, and $\operatorname{tr} \overline{A} = a$, which proves the theorem. □

An elementary consideration gives the lemma below, which verification is left to the reader.

Lemma 3.8. *If $d = -2$ and $\mathrm{HPer}\,(f) \neq \emptyset$ then $2 \notin \mathrm{HPer}\,(f) \iff a = 0, 1, 2$. Moreover:*

- $T_A = \mathrm{HPer}\,(f) = \mathbb{N} \setminus 2\mathbb{N}$ *for* $a = 1$,
- $T_A = \mathbb{N}$, $\mathrm{HPer}\,(f) = \mathbb{N} \setminus \{2\}$ *for* $a = 0$ *or* $a = 2$.

As a consequence of Theorem 3.1 we get the following:

Corollary 3.9. *If a self map of a 3-nilmanifold different than 3-torus is such that $3 \in \mathrm{HPer}\,(f)$ then $\mathbb{N} \setminus 2\mathbb{N} \subset \mathrm{HPer}\,(f) \subset \mathrm{Per}(f)$. If $2 \in \mathrm{HPer}\,(f)$ then $\mathbb{N} = \mathrm{HPer}\,(f) = \mathrm{Per}(f)$. In particular, the first assumption is satisfied if $L(f^3) \neq L(f)$ and the second if $L(f^2) \neq L(f)$.*

Proof. By Theorem 3.1, $\mathrm{HPer}\,(f)$ finite implies $\mathrm{HPer}\,(f) \subset \{1\}$. Thus $3 \in \mathrm{HPer}\,(f)$ implies case (G) hence $\mathrm{HPer}\,(f) \supset \mathbb{N} \setminus 2\mathbb{N}$. If $2 \in \mathrm{HPer}\,(f)$ then the special cases in Theorem 3.1 are excluded hence $\mathrm{HPer}\, f = \mathbb{N}$. □

Remark 3.10. It is easy to note that one may modify Theorem 3.1 to a nilmanifold of any dimension provided the size of its Fadell-Husseini tower is less or equal to two. If the size of tower is less or equal to three these approach should still work due to the complete description of the homotopy minimal periods of the three torus maps done by Jiang and Llibre in [**12**].

Remark 3.11. Roughly speaking Corollary 3.9 is a Šarkovskii type theorem. Instead of the existence of an orbit of a given length (here 2 or 3) we need a stronger assumption 2 or $3 \in \mathrm{HPer}\,(f)$. However the conclusion is also stronger, because it states the existence of homotopy minimal periods.

Remark 3.12. The next natural and possible to achieve case is a description of minimal homotopy periods of maps of some low dimensional compact solvmanifolds especially of dimension 4. The latter needs a slight modifications of theorems of [**10**] and some facts already proved in [**9**]. The possibility of non Nielsen number fibre uniformity on the associated Mostow fibrations for solvmanifolds makes the study more complicated.

4. Homeomorphisms of 3-nilmanifolds.

We will formulate a version of the last section for homeomorphisms of three dimensional nilmanifolds (see [**12**] for the corresponding theorem for a homeomorphism of the three dimensional torus).

Theorem 4.1. *Let $f : X \to X$ be a homeomorphism of three-dimensional compact nilmanifold X not diffeomorphic to T^3. Let $A = A_1 \oplus \bar{A} \in \mathcal{M}_{3\times 3}(\mathbb{Z})$ be the matrix induced by the fibre map $f = (f_1, \bar{f})$ (Proposition 2.7) and $\chi_A(t) = \chi_{A_1}(t) \cdot \chi_{\bar{A}}(t) = (t-d)(t^2 - at + b)$ its characteristic polynomial. Then $d = b = \pm 1$ and consequently $\mathrm{HPer}\,(f) = \emptyset$ iff $d = 1$ or ($d = -1$ and $a = 0$). For $d = -1$ and the remaining a we have $\mathrm{HPer}\,(f) = \mathbb{N}$ with the only two exceptions being when $a = 1$ or $a = -1$. For these special cases $T_A = \mathrm{HPer}\,(f) = \mathbb{N} \setminus 2\mathbb{N}$.*

Proof. The statement follows from Theorem 3.1 and the fact that $d = \pm 1$. □

As a direct consequence we get the following analog of the Šarkovskii type for a homeomorphisms of nonabelian three nilmanifolds:

Corollary 4.2. *Let $f : X \to X$ be a homeomorphism, or more general a homotopy equivalence, of a compact three dimensional nilmanifold X not diffeomorphic to the torus. If $\mathrm{HPer}\,(f) \neq \emptyset$ then $\mathbb{N}\setminus 2\mathbb{N} \subset \mathrm{HPer}\,(f)$. Moreover if $2 \in \mathrm{HPer}\,(f)$, e.g., if $L(f^2) \neq L(f)$, (or if any $2k \in \mathrm{HPer}\,(f)$) then $\mathrm{HPer}\,(f) = \mathbb{N}$.*

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Institute of Applied Mathematics
University of Agriculture
ul. Nowoursynowska 166
02-787 Warszawa
Poland
E-mail address: jezierski@alpha.sggw.waw.pl

Faculty of Mathematics and Computer Science
Adam Mickiewicz University of Poznań
ul. Matejki 48/49
60-769 Poznań
Poland
E-mail address: marzan@math.amu.edu.pl