Pacific Journal of Mathematics

SOME PLANAR ALGEBRAS RELATED TO GRAPHS

Brian Curtin

Volume 209 No. 2 April 2003

PACIFIC JOURNAL OF MATHEMATICS
Vol. 209, No. 2, 2003

SOME PLANAR ALGEBRAS RELATED TO GRAPHS

Brian Curtin

Let X denote a finite nonempty set, and let W denote a matrix whose rows and columns are indexed by X and whose entries belong to some field $\mathbb{K}$. We study three planar algebras related to W. Briefly, a planar algebra is a graded vector space $\mathcal{V} = \cup_{n\in\mathbb{Z}^+\cup\{+,-\}}\mathcal{V}_n$ which is closed under "planar" operators.

The first planar algebra which we study, $\mathcal{F}^W = \cup\mathcal{F}^W_n$, is defined by the group theoretic properties of W. For $n \in \mathbb{Z}^+$, $\mathcal{F}^W_n$ is the vector space of functions from X^n to $\mathbb{K}$ which are constant on the $\mathfrak{Aut}(W)$-orbits of X^n, and $\mathcal{F}^W_+$, $\mathcal{F}^W_-$ are identified with $\mathbb{K}$. The second planar algebra, $\mathcal{P}^W = \cup\mathcal{P}^W_n$, is the planar algebra generated W. We define it combinatorially: $\mathcal{P}^W_n$ is spanned by functions from X^n to $\mathbb{K}$ defined via statistical mechanical sums on certain planar open graphs. The third planar algebra, $\mathcal{O}^W = \cup\mathcal{O}^W_n$, differs from $\mathcal{P}^W$ only in that the open graphs defining the functions need not be planar.

It turns out that $\mathcal{P}^W \subseteq \mathcal{O}^W \subseteq \mathcal{F}^W$. We show that $\mathcal{P}^W = \mathcal{O}^W$ if and only if $\mathcal{P}^W_4$ contains a single special function known as the "transposition". We show that $\mathcal{O}^W = \mathcal{F}^W$ whenever $|X|!$ is not divisible by the characteristic of $\mathbb{K}$.

1. Introduction.

Planar algebras were introduced by V.F.R. Jones [15] to study the structure of subfactors. A planar algebra is a graded vector space $\mathcal{V} = \cup_{n\in\mathbb{Z}^+\cup\{+,-\}}\mathcal{V}_n$ over some field $\mathbb{K}$ which is closed under certain operators. True to its operator algebra origins, an emphasis is placed upon the interactions of the operators. These operators are defined diagrammatically by objects known as planar tangles. We recall relevant definitions in Section 2. The study of a planar algebra via the dependencies of these operators has a knot theoretic flavor, very much like Conway's tangles and skein relations [4]. This is no coincidence, as planar algebras were influenced by the deep relationship between subfactors and knots [12] and [14].

When $\dim\mathcal{V}_n$ is finite for all n, it is natural to ask for the exact value. We shall consider this problem for some combinatorial planar algebras. In our examples, $\mathcal{V}_n$ is a vector space of functions from X^n to $\mathbb{K}$ for some fixed, finite, nonempty set X. The action of the planar tangles is defined via the statistical mechanical construction known as a partition function.

In Section 3 we introduce three planar algebras related to a matrix W whose rows and columns are indexed by X and entries are in $\mathbb{K}$. In the first planar algebra defined from W, $\mathcal{F}^W = \cup \mathcal{F}_n^W$, the vector space $\mathcal{F}_n^W$ consists of those functions from X^n to $\mathbb{K}$ which are constant on the orbits of X^n under the action of $\mathfrak{Aut}(W)$. The second is a singly generated planar algebra, $\mathcal{P}^W = \cup \mathcal{P}_n^W$, whose vector space $\mathcal{P}_n^W$ is spanned by functions from X^n to $\mathbb{K}$ defined via statistical mechanical state sums on the planar graphs derived from planar tangles. The third planar algebra, $\mathcal{O}^W = \cup \mathcal{O}_n^W$, differs from $\mathcal{P}^W$ only in that the graphs defining the functions need not be planar.

It turns out that $\mathcal{P}_n^W \subseteq \mathcal{O}_n^W \subseteq \mathcal{F}_n^W$. It is easy to compute $\dim \mathcal{F}_n^W$ using the Cauchy-Frobenius-Burnside formula for group characters. However, we are more interested in $\dim \mathcal{P}_n^W$, and it is generally very difficult to compute. Thus we consider when $\mathcal{P}_n^W = \mathcal{F}_n^W$ for all n. The planar algebra $\mathcal{O}^W$ plays an important role in this problem. In Section 4, we show that $\mathcal{P}^W = \mathcal{O}^W$ if and only if $\mathcal{P}_4^W$ contains a particular element called the "transposition". The proof of this result is essentially skein theoretic in nature. Then in Section 5, show that $\mathcal{O}^W = \mathcal{F}^W$ whenever $|X|!$ is not divisible by the characteristic of $\mathbb{K}$. Since this condition holds in characteristic zero, the most important case is thus treated. To prove this result, we encode $\mathcal{O}_n^W$ into polynomials and then appeal to results concerning the polynomial invariants of a finite group.

These results are related to Theorem 4.3 of [**16**] concerning a certain planar algebra $\mathcal{P}^\sigma$ which contains $\mathcal{P}^W$. This result asserts that any planar subalgebra of $\mathcal{P}^\sigma$ which contains a transposition is the set of elements of $\mathcal{P}^\sigma$ which are fixed under the action of some group $\mathfrak{G}$ such that $\mathfrak{Aut}(W) \subseteq \mathfrak{G} \subseteq \mathfrak{S}_X$. This result relies on the theory of subfactors, and so it is only applicable when the ground field is the real or complex numbers and when the matrix W is symmetric. By introducing the intermediate planar algebra $\mathcal{O}^W$ we have extended this result (as applied to $\mathcal{P}^W$) to almost any field and to any matrix. In this case we also know precisely which group is involved. Moreover, the proof given here is combinatorial in nature, where the original was very non-combinatorial.

2. Planar algebras: Definitions.

Planar algebras were introduced to study the structure of subfactors. True to their operator algebra origins, planar algebras are defined in terms of operators on vector spaces. These operators are defined diagrammatically by objects known as planar tangles. A planar tangle can be presented in several ways. We shall use a slight variation of the operadic definition of [**15**] (see also [**16**]). From this point of view, a planar tangle consists of a collection of disjoint disks which are joined by disjoint smooth curves, together with a coloring of the regions formed by the strings and disks. Various constraints on this collection arise from their subfactor origins; however, no knowledge of subfactors is necessary to proceed.

We begin with a definition of a planar tangle. Let D_0 denote the unit disk. Pick disjoint disks $D_1, D_2, \dots, D_n$ in the interior of D_0. Form a finite collection of disjoint "strings" (simple smooth curves) in the interior of $D_0 \setminus \cup_{i=1}^{n} D_i$, all of whose endpoints meet the boundary of some disk transversally. There may be some closed loops which touch no disk. Further assume that an even number of strings touch each disk, say $2k_i$ touching D_i. Color the regions interior to $D_0 \setminus \cup_{i=1}^{n} D_i$ formed by the strings black and white so that regions on either side of a string have opposite colors. Call the points on the boundary of each disk where a string touches "marked". The marked points divide the boundary of each disk into intervals. On each disk, one of the intervals which touches a white region is chosen to be "privileged". The entire boundary of a disk with no marked points is either privileged or not according to whether it touches a white region or a black region. Specifying the privileged intervals makes the coloring data redundant. It will sometimes be convenient to number the marked points on each disk consecutively in a clockwise direction where the marked point at the clockwise end of the privileged interval is numbered one.

The smooth isotopy class of this collection of disks, strings, coloring, and privileged intervals is called a *planar* k_0*-tangle*. There is a natural composition for planar tangles. If S is a planar k-tangle with an internal disk D_i with $2k_i$ marked points and T is a planar k_i-tangle, then we may replace D_i with a rescaled and isotoped version of T without its unit disk, by matching corresponding marked points (first to first, etc.) and smoothing the connections of the strings. The coloring conventions are preserved by composition. The collection of planar tangles with this composition is called the *planar operad*.

A *general planar algebra* is a graded vector space $\mathcal{V}_k$ for $k > 0$ and two vector spaces $\mathcal{V}_+$ and $\mathcal{V}_-$ such that every element T of the planar operad determines a multilinear map from a tensor product of these vector spaces, one for each internal disk of T, to the vector space corresponding to the boundary of T. We require a natural *homomorphism property*. Given planar tangles T_1, T_2, T_3 which admit compositions of T_2 into T_1 and T_3 into this composition, the net result of these compositions in the planar operad does not depend upon the order in which they are carried out: The same must be true for the corresponding multilinear maps on the planar algebra. We also impose a condition on $\mathcal{V}_+$ and $\mathcal{V}_-$. We view these two vector spaces as corresponding to the two colorings of any planar 0-tangle–$\mathcal{V}_+$ to those colored black next to the unit disk and $\mathcal{V}_-$ to those colored white next to the unit disk. Observe that surrounding the interior component with a closed string reverses its coloring. We require that surrounding the interior components of a planar 0-tangle with two closed strings yields a multilinear map to $\mathcal{V}_+$ or $\mathcal{V}_-$ which differs from the original multilinear map only by a fixed scalar multiple.

Let $\mathcal{V}$ denote a planar algebra. Then $\mathcal{V}$ is said to be *finite dimensional* whenever the vector spaces $\mathcal{V}_+$, $\mathcal{V}_-$, and $\mathcal{V}_k$ $(k > 0)$ are all finite dimensional. All of the examples that we shall consider in this paper are finite dimensional. In fact, $\mathcal{V}_+$ and $\mathcal{V}_-$ will both be one-dimensional, making the examples *planar algebras* in the sense of [**15**]. Given that $\mathcal{V}$ is finite dimensional, it is natural to compute $\dim \mathcal{V}_k$ for all k. This problem motivates the results of this paper. We are interested in a singly generated planar algebra $\mathcal{P}^W$ which is contained in a planar algebra $\mathcal{F}^W$ whose dimensions we can compute. We shall consider when these two planar algebras are equal. We now describe the planar algebras which we shall study.

3. Planar algebras: Examples.

3.1. The planar algebra of functions on a finite set. We present a very simple planar algebra. We are more interested in some of its planar subalgebras, but we take this opportunity to describe the multilinear map corresponding to each planar tangle with no other distractions. This correspondence will be the same in all planar algebras which follow.

Let X denote a finite, nonempty set, and let $\mathbb{K}$ denote a field. For each positive integer k, let $\mathcal{X}_k$ be the vector space of all functions from X^k to $\mathbb{K}$ $(k > 0)$, with $\mathcal{X}_+$, $\mathcal{X}_-$ identified with $\mathbb{K}$. Then $\mathcal{X} = \cup \mathcal{X}_k$ is a planar algebra. Let T denote a planar k_0-tangle with internal disks D_1, D_2, $\ldots$, D_n with respectively $2k_1, 2k_2, \ldots , 2k_n$ many marked points. Then T defines a multilinear map $\bigotimes_{i=1}^n \mathcal{X}_{k_i} \to \mathcal{X}_{k_0}$ as follows. Index the black regions of T by 1, 2, $\ldots$, m. For all i $(0 \le i \le n)$ and for all j $(1 \le j \le k_i)$, let S_{ij} be the index of the j^{th} black incident with D_i when traversing the boundary of D_i clockwise so that the privileged interval is traversed last. Given $f_i \in \mathcal{X}_{k_i}$ $(1 \le i \le n)$, define a function $Z_T^{(f_1,f_2,\ldots,f_n)} : X^{k_0} \to \mathbb{C}$ which, when evaluated at $(x_1, x_2, \ldots , x_{k_0})$, returns

$$\sum\nolimits_\sigma \prod\nolimits_{i=1}^n f_i(\sigma(S_{i1}), \sigma(S_{i2}), \ldots, \sigma(S_{ik_i})), \tag{1}$$

where σ runs over all maps from $\{1, 2, \ldots , m\}$ to X with $\sigma(S_{0j}) = x_j$.

Extend Z_T multilinearly to a map $\bigotimes_{i=1}^n \mathcal{X}_i \to \mathcal{X}_{k_0}$. The homomorphism property of planar algebras follows since the function only depends upon the incidences of the black regions and composition merges regions with the same color. Enclosing a planar 0-tangle with two closed strings preserves the color of the interior (reverses it twice) but adds an isolated black band. This modified tangle gives a multilinear map which is $|X|$ times the multilinear map corresponding to the original tangle. Thus $\mathcal{X}$ is a planar algebra.

3.2. Planar algebras constructed via finite group action. Let X denote a finite, nonempty set. Let $\mathfrak{S}_X$ denote the symmetric group on X. For each positive integer k, extend the action of $\mathfrak{S}_X$ to X^k in the natural fashion: For all $\mathfrak{g} \in \mathfrak{S}_X$ and for all $(x_1, x_2, \ldots, x_k) \in X^k$, let $\mathfrak{g}(x_1, x_2, \ldots, x_k) = (\mathfrak{g}(x_1), \mathfrak{g}(x_2), \ldots, \mathfrak{g}(x_k))$. Let $\mathfrak{G} \subseteq \mathfrak{S}_X$ denote a subgroup so that $\mathfrak{G}$ is a permutation group on X. By a *$\mathfrak{G}$-orbit of X^k*, we mean a nonempty subset $Y \subseteq X^k$ such that $\vec{x}, \vec{y} \in Y$ if and only if there exists $\mathfrak{g} \in \mathfrak{G}$ such that $\vec{x} = \mathfrak{g}(\vec{y})$.

Let $\mathbb{K}$ denote a field. For each positive integer k, let $\mathcal{F}_k(\mathfrak{G}, X)$ denote the vector space of functions from $X^k \to \mathbb{K}$ which depend only upon the $\mathfrak{G}$-orbit of their inputs. We identify the vector spaces $\mathcal{F}_+(\mathfrak{G}, X)$ and $\mathcal{F}_-(\mathfrak{G}, X)$ with the field $\mathbb{K}$ (constant functions). Together, these vector spaces form a planar algebra $\mathcal{F}(\mathfrak{G}, X)$ with the same planar structure as $\mathcal{X}$. That is to say, (1) defines a map $\bigotimes_{i=1}^n \mathcal{F}_{k_i}(\mathfrak{G}, X) \to \mathcal{F}_{k_0}(\mathfrak{G}, X)$. To see that this is so, pick $f_i \in \mathcal{F}_{k_i}(\mathfrak{G}, X)$ $(1 \le i \le n)$, and define $f_0 : X^{k_0} \to \mathbb{K}$ by (1). To see that f_0 is constant on each $\mathfrak{G}$-orbit of X^{k_0}, consider replacing each map σ in (1) by $\mathfrak{g} \circ \sigma$. The effect of this change on the boundary leads (1) to return $f_0(\mathfrak{g}x_1, \mathfrak{g}x_2, \ldots, \mathfrak{g}x_{k_0}) = f_0(x_1, x_2, \ldots, x_{k_0})$ since $f_i \in \mathcal{F}_{k_i}(\mathfrak{G}, X)$ $(1 \le i \le n)$. Thus $f_0 \in \mathcal{F}_{k_0}(\mathfrak{G}, X)$. Hence $\mathcal{F}(\mathfrak{G}, X) = \cup \mathcal{F}_k(\mathfrak{G}, X)$ is a planar algebra. $\mathcal{F}(\mathfrak{G}, X)$ is called the *fixed-point planar algebra of $\mathfrak{G}$ acting on X*. This planar algebra is discussed in [**15**].

The vector spaces of $\mathcal{F}_k(\mathfrak{G}, X)$ are finite dimensional, and their dimension can be computed via the Cauchy-Frobenius-Burnside formula for the characters of the group, which we briefly recall now. See [**11**], for example. The *permutation representation* of $\mathfrak{G}$ acting on X is the map $\mathfrak{g} \mapsto R(\mathfrak{g}) \in \mathcal{M}_X(\mathbb{C})$ with (x, y)-entry equal to 1 if $y = \mathfrak{g} \cdot x$ and 0 otherwise $(x, y \in X)$. The *permutation character* of $\mathfrak{G}$ acting on X is the map $\pi : \mathfrak{G} \to \mathbb{C}$ given by $\pi(\mathfrak{g}) = \operatorname{Tr} R(\mathfrak{g})$ $(\mathfrak{g} \in \mathfrak{G})$. Fix a positive integer k. Then the number of orbits of X^k under the action of $\mathfrak{G}$ is

$$\dim \mathcal{F}_k(\mathfrak{G}, X) = \frac{1}{|\mathfrak{G}|} \sum_{\mathfrak{g} \in \mathfrak{G}} (\pi(\mathfrak{g}))^k. \tag{2}$$

3.3. A planar algebra $\mathcal{P}^W$. Fix a field $\mathbb{K}$. Let X denote a finite, nonempty set. Let $\mathcal{M}_X(\mathbb{K})$ denote the set of matrices with rows and columns indexed by X and entries in $\mathbb{K}$. Pick $W \in \mathcal{M}_X(\mathbb{K})$. Given $k > 0$, we describe a rule using W which maps any planar k-tangle all of whose internal disks have exactly 4 marked points to a function $X^k \to \mathbb{C}$. The vector space $\mathcal{P}_k^W$ spanned by these functions will be part of the grading of a planar algebra $\mathcal{P}^W$. A similar rule gives the vector spaces $\mathcal{P}_+^W$ and $\mathcal{P}_-^W$, which turn out to be isomorphic to $\mathbb{K}$.

Let T denote planar k-tangle in the unit disk D_0 with n internal disks $D_1, D_2, \ldots, D_n$ each having exactly 4 marked points. As in Subsection 3.1, label the black regions of T with indices 1, 2, $\ldots$, m and for all i

$(0 \leq i \leq n)$ and for $j = 1, 2$, let S_{ij} denote the index of the j^{th} black region incident with D_i when traversing the boundary of D_i clockwise so that the privileged interval is traversed last. Define a function $Z_T^W : X^k \to \mathbb{K}$ which, when evaluated at $(x_1, x_2, \ldots, x_k)$, returns

$$Z_T^W(x_1, x_2, \ldots, x_k) = \sum_{\sigma} \prod_{i=1}^{n} W(\sigma(S_{i1}), \sigma(S_{i2})), \tag{3}$$

where σ runs over all maps from $\{1, 2, \ldots, m\}$ to X with $\sigma(S_{0j}) = x_j$. Let $\mathcal{P}_k^W$ denote the $\mathbb{K}$-linear span of all functions which arise in this fashion from a planar k-tangle. For $k = 0$, we use the same rule to define functions, but place them in $\mathcal{P}_+^W$ or $\mathcal{P}_-^W$ when the color of the 0-tangle near the unit circle is black or white, respectively.

Now $\mathcal{P}^W = \cup \mathcal{P}_n^W$ is a planar subalgebra of $\mathcal{X}$ with closure under (1) assured since the composition of planar tangles yields a planar tangle. The planar algebra $\mathcal{P}^W$ is called the *planar algebra generated by* W. This planar algebra is also discussed in [**15**]. Singly generated planar algebras are considered in [**3**] as well. Not all of the planar algebras considered in [**3**] are generated by a matrix, however.

By an *automorphism* of W, we mean a permutation $\mathfrak{s}$ of X such that $W(u, v) = W(\mathfrak{s}(u), \mathfrak{s}(v))$ for all $u, v \in X$. Let $\mathfrak{Aut}(W)$ denote the full group of automorphisms of W. Observe that $\mathcal{P}^W \subseteq \mathcal{F}(\mathfrak{Aut}(W), X)$ since the definition of the functions in $\mathcal{P}^W$ depend only upon the structure of W. Thus (2) gives an upper bound for $\dim \mathcal{P}_k^W$. Our main result concerns the case of equality. The proof compares $\mathcal{P}^W$ and $\mathcal{F}(\mathfrak{Aut}(W), X)$ to an intermediate planar algebra which we now describe.

3.4. A planar algebra $\mathcal{O}^W$. We use the language of graph theory to generalize the construction of $\mathcal{P}^W$ of the previous subsection. We begin with some graph theoretic terminology.

By a *multi-digraph*, we mean a pair $\Delta = (V, E)$, where V is a nonempty set and E is a multiset of ordered pairs of (not necessarily distinct) elements of V. Let $\Delta = (V, E)$ be a multi-digraph. The elements of V are called the *vertices* of Δ, and an ordered pair $(u, v) \in E$ is called a (directed) *edge* from u to v. We say that there are *multiple edges* from u to v whenever the multiset E contains two or more copies of (u, v). Throughout this paper we shall assume that all multi-digraphs have finite vertex and edge sets. Fix a nonnegative integer n. By an *open graph of boundary size* n, we mean a triple $\Gamma = (V, E, \vec{b})$, where (V, E) is a multi-digraph and $\vec{b}$ is an n-tuple of elements of V, called the *boundary vector* of the open graph. Let $\mathcal{O}_n$ denote the set of all open graphs of boundary size n.

Let $\Gamma = (V, E, \vec{b})$ denote an open graph of boundary size n. Γ is said to be *planar* if the multi-digraph (V, E) has a plane embedding (no crossing edges) into the interior of an n-gon with clockwise ordered vertices b'_1, b'_2,

…, b'_n such that each b_i can be joined to b'_i in a planar way. A planar open graph may be viewed as a patch which has been cut out of a plane embedded planar graph: The boundary vertices may have neighbors in the larger graph, while all neighbors of non-boundary vertices must appear in the open graph. Let $\mathcal{P}_n$ denote the set of all planar open graphs of boundary size n.

As in the previous subsection, we fix a field $\mathbb{K}$, a finite, nonempty set X, and a matrix $W \in \mathcal{M}_X(\mathbb{K})$. The planar tangles which define $\mathcal{P}^W$ can be interpreted as open graphs. Let T denote a planar k-tangle all of whose internal disks have exactly 4 marked points, and adopt the notation of Subsection 3.1. Define a multi-digraph whose vertex set V consists of the black regions of T, whose edge set E consists of the pairs (S_{i1}, S_{i2}) as i runs over indices of the internal disks, and whose boundary vector is $\vec{b} = (S_{01}, S_{02}, \ldots, S_{0k})$. It is not difficult to see that $(V, E, \vec{b})$ is a planar open graph and that every planar open graph arises in this way.

The data $\Gamma = (V, E, \vec{b})$ suffices to define the multilinear map $Z^W_\Gamma : X^k \to \mathbb{K}$ corresponding to the planar k-tangle T as in (3). The evaluation of Z^W_Γ at $(x_1, x_2, \ldots\ x_k) \in X^k$ is

$$Z^W_\Gamma(x_1, x_2, \ldots, x_k) = \sum_\sigma \prod_{(u,v)\in E} W(\sigma(u), \sigma(v)), \tag{4}$$

where σ runs over all maps from V to X with $\sigma(b_i) = x_i$ $(0 \le i \le n)$.

The construction (4) is well-known in statistical mechanics [**1**], [**2**], [**23**] and [**24**]. Thus we adopt the following terminology: The elements of X are called *spins*, and W is called the *Boltzman weight matrix*. A map $\sigma : V \to X$ with $\sigma(b_i) = x_i$ $(1 \le i \le n)$ is called a *state* compatible with the *boundary condition* $\sigma(b_i) = x_i$. The formula (4) is called the *partition function* of Γ with respect to W.

If we restrict Γ to planar open graphs, then (4) and (1) agree when Γ is produced from T as above. Thus $\mathcal{P}^W_k$ is the vector space spanned by the functions defined by (4) as Γ runs over all planar open graphs of boundary size k. However, the planar structure is not necessary in (4). Let $\mathcal{O}^W_k$ denote the vector space of functions $X^k \to \mathbb{K}$ spanned by the functions defined by (4) as Γ runs over all open graphs of boundary size k. Then $\mathcal{O}^W = \cup \mathcal{O}^W_k$ is a planar subalgebra of $\mathcal{X}$. The closure of $\mathcal{O}^W$ under the multilinear maps defined by planar tangles follows since such operations just combine graphs to form a new graph. We call $\mathcal{O}^W$ the *open graph planar algebra of* W.

By construction $\mathcal{P}^W_k \subseteq \mathcal{O}^W_k \subseteq \mathcal{F}_k(\mathfrak{Aut}(W), X)$. Our main results describe when $\mathcal{P}^W_k = \mathcal{O}^W_k$ and when $\mathcal{O}^W_k = \mathcal{F}_k(\mathfrak{Aut}(W), X)$. Before proceeding to these results, we present a graph theoretic interpretation of the partition function. Let $\Gamma = (V, E)$ and $\Xi = (X, R)$ denote graphs. By a *graph homomorphism* from Γ into Ξ, we mean a map $\sigma : V \to X$ such that if

$(u, v) \in E$ then $(\sigma(u), \sigma(v)) \in R$. (Graph homomorphisms are surveyed in [7].)

Lemma 3.1. *Suppose W is the adjacency matrix of a graph $\Xi = (X, R)$, and let $\Gamma = (V, E, \vec{b})$ denote an open graph of boundary size n for some fixed nonnegative integer n. Then for all $\vec{p} \in X^n$, $Z_\Gamma^\Xi(\vec{p})$ equals the number of graph homomorphisms from (V, E) to Ξ which map $\vec{b}$ to $\vec{p}$ coordinate-wise.*

Proof. Each state σ over which the sum in (4) runs maps $\vec{b}$ to $\vec{p}$ element-wise. If σ is not a graph homomorphism, then $(\sigma(u), \sigma(v)) \notin R$ for some $(u, v) \in E$, so $W(\sigma(u), \sigma(v)) = 0$ and the state contributes nothing to the partition function. If σ is a graph homomorphism, then $W(\sigma(u), \sigma(v)) = 1$ for all $(u, v) \in E$, so the state adds one to the partition function. □

The problem of determining if there is a graph homomorphism into a fixed graph H (the so-called *H-coloring problem*) is NP-complete in general [9] and [10]. In particular, one cannot expect to find a particularly efficient means of computing the partition functions of open graphs with respect to a fixed matrix W.

4. When $\mathcal{P}^W = \mathcal{O}^W$.

We consider when $\mathcal{P}^W = \mathcal{O}^W$. Of course this is the case when the partition function with respect to W of every open graph is a linear combination of the partition functions with respect to W of some planar open graphs. We give a more practical characterization involving just one special open graph.

Let Φ denote the (non-planar) open graph of boundary size 4 consisting two isolated vertices v_1 and v_2 and boundary vector (v_1, v_2, v_1, v_2). We call Φ the *transposition*. We picture Φ as a 4-tangle in Figure 1(b)–two black "ribbons" which cross, one above the other, without interacting. When drawing our tangles, we shall avail ourselves of the fact that they are determined only up to isotopy and draw the disks as squares. We mark the privileged interval on each with a $\diamond$, so it is unnecessary to draw the coloring of the regions.

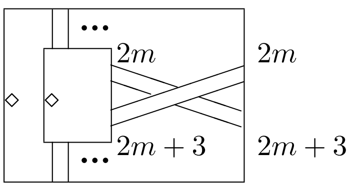
(a) open graph (b) tangle

Figure 1. Two views of Φ.

We now use the transposition Φ to build (non-planar) tangles which define operators on open graphs which transpose elements of the boundary vector.

For all $n \geq 4$ and m ($1 \leq m \leq n$), form an n-tangle Φ_m^n with one interior disk D_1 with $2n$-marked points by joining the i^{th} marked point of D_1 to the i^{th} marked point of on the unit circle of Φ_m^n for all i except $2m$, $2m+1$, $2m+2$, and $2m+3$ (taken mod $2n$). The $2m^{\text{th}}$, $2m+1^{\text{st}}$, $2m+2^{\text{nd}}$, and $2m+3^{\text{rd}}$ marked points of D_1 are joined to the $2m+2^{\text{nd}}$, $2m+3^{\text{rd}}$, $2m^{\text{th}}$, and $2m+1^{\text{st}}$ marked points on the unit disk of Φ_m^n, respectively (see Figure 2). Observe that each Φ_m^n is formed by composing a planar n-tangle with Φ–simply cut out a disk around the transposition.

An examination of the tangle presentation of Φ_m^n reveals that it transposes the order in which the m^{th} and $m+1^{\text{st}}$ black regions are encountered when the unit disk is traversed clockwise versus their order on the interior disk. Taking composition of planar tangles as the product, the Φ_m^n generate the symmetric group on the n black regions incident with the unit disk. In particular, the various compositions of the Φ_m^n give rise to all permutations of the black regions when considering the order in which they appear around the unit disk versus the interior disk. Note that the construction (3) can be used to define an operator on open graphs from Φ_m^n. By the above observations, we see that the resulting open graph operator, swaps the m^{th} and $m+1^{\text{st}}$ (mod n) boundary vertices of its input.



$$\Phi_m^n(V, E, (b_1, b_2, \ldots, b_m, b_{m+1}, \ldots, b_n)) =$$
$$(V, E, (b_1, b_2, \ldots, b_{m+1}, b_m, \ldots, b_n))$$

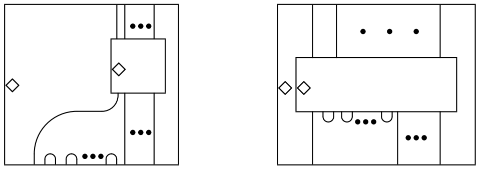
Figure 2. The transposition operator Φ_k^n.

Theorem 4.1. *Let W denote a matrix over any field. Then the following are equivalent:*

(i) $\mathcal{P}^W = \mathcal{O}^W$.
(ii) *There exists $n \geq 4$ such that $\mathcal{P}_n^W = \mathcal{O}_n^W$.*
(iii) $\mathcal{P}_4^W = \mathcal{O}_4^W$.
(iv) $Z_\Phi^W \in \mathcal{P}_4^W$.

Proof. (i) $\Rightarrow$ (ii): Clear.

(ii) $\Rightarrow$ (iii): Let Γ denote an open graph of boundary size 4. Form an open graph Γ' of boundary size n by extending the boundary vector of Γ by repeating the last vertex $n-4$ times. This is a planar operation corresponding to the planar tangle of Figure 3(a) (this is not the preferred inclusion of [**15**]). Composing Figure 3(a) into Figure 3(b) returns the original planar tangle along with some closed loops with white interiors, which can be removed with no effect in our planar algebra. By (ii), there exist open graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_k \in \mathcal{P}_n$ such that $Z^W_{\Gamma'} = \sum_{j=1}^k \alpha_j Z^W_{\Gamma_j}$ for some scalars α_j. The same is true of Γ since we may apply the restriction to these functions.



(a) An inclusion (b) Its inverse restriction

Figure 3. Two planar tangles.

(iii) $\Rightarrow$ (iv): Clear.

(iv) $\Rightarrow$ (i): Fix a nonnegative integer n and pick $\Gamma \in \mathcal{O}_n$. We shall show that there are open graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_k \in \mathcal{P}_n$ such that $Z^W_\Gamma = \sum_{j=1}^k \alpha_j Z^W_{\Gamma_j}$ for some scalars α_j. This will prove that $\mathcal{O}^W_n = \mathcal{P}^W_n$.

In order for an open graph to be planar, it must be possible to embed it in the plane so that the positions of its boundary vertices are incident with the exterior and ordered clock-wise as they appear in the boundary vector. This may not be the case for Γ. However, by permuting the boundary vector this can be corrected. By (iv) and the remarks at the beginning of the section, there exist transposition operators such that

$$\Gamma = \Phi^n_{m_1}(\Phi^n_{m_2}(\ldots \Phi^n_{m_j}(\hat{\Gamma}) \ldots)),$$

where $\hat{\Gamma}$ is an open graph with the same vertex and edge sets as Γ and boundary vector a re-ordering of that of Γ so that all repetitions occur in cyclically successive positions. It is now possible to embed $\hat{\Gamma}$ with the desired boundary property. There remains the possibility that $\hat{\Gamma}$ has crossing edges in any plane embedding with the boundary vertices incident with the exterior face.

In light of (iv), we now only need to prove that $Z^W_{\hat{\Gamma}} \in \mathcal{P}^W_n$. Indeed, suppose that this is the case. Then there exists a set of planar open graphs $\hat{\Gamma}_1, \hat{\Gamma}_2, \ldots, \hat{\Gamma}_{\hat{\jmath}}$ such that $Z^W_{\hat{\Gamma}} = \sum_{i=1}^{\hat{\jmath}} \beta_j Z^W_{\hat{\Gamma}_i}$. Now by (iv) there exists a set of planar open graphs $\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \ldots, \widetilde{\Gamma}_{\widetilde{\jmath}}$ such that

$$Z^W_{\Phi^n_{m_j}(\hat{\Gamma}_i)} = \sum_{\ell=1}^{\widetilde{\jmath}} Z^W_{\widetilde{\Gamma}_\ell} \in \mathcal{P}^W_n.$$

Proceeding by induction, we find that $Z^W_\Gamma \in \mathcal{P}^W_n$.

To show that $Z^W_{\hat{\Gamma}} \in \mathcal{P}^W_n$, it is enough to show that $Z^W_\Delta \in \mathcal{P}^W_n$ for any open graph Δ whose boundary vector is such that all repetitions occur in cyclically successive positions. Embed Δ in the plane such that all of its vertices lie evenly spaced on a circle and its boundary vertices are ordered clock-wise as they appear in the boundary vector. If no edges cross in this embedding, we are done. Suppose that some edges cross. Among all vertices p and q which are incident with crossing edges (p, p') and (q, q') pick those which are cyclically nearest according to their positions on the circle. Observe that p and q partition the remaining vertices into two sets according to which sides of p and q they lie. Moreover, by the choice of p and q nearest, there are no edges between these two sets. By deforming the edges of this embedding, we can can make it so that all edges which cross (p, p') and (q, q') do so between p' and x or between q' and x without creating any new crossings, where x is the point in the plane where (p, p') and (q, q') cross. Factor this crossing as two non-crossing edges (through which all edges crossing (p, p') and (q, q') pass as if nothing has changed) and a transposition–see Figure 4. Now by (iv), the transposition belongs to $\mathcal{P}_4^W$. Thus there exist open graphs Δ_1, $\Delta_2, \ldots, \Delta_h$ such that

$$Z_\Delta^W = \sum_{\ell=1}^{h} \gamma_\ell Z_{\Delta_\ell}^W \in \mathcal{P}_n^W,$$

and which differ from Δ only in that under a similar embedding the crossing (p, p') and (q, q') has been replaced by a planar graph. Proceeding by induction (on the number of vertices between the endpoints of crossing edges as one takes the shortest path along the circle), we may remove all crossings in Δ. Thus $Z_\Delta^W \in \mathcal{P}_n^W$, as desired. $\square$

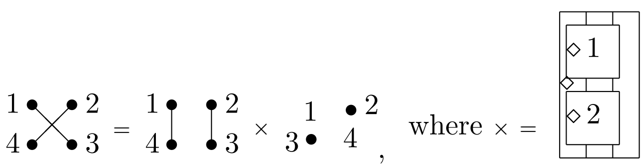


Figure 4. Factoring crossing edges.

The arguments of this section suggest the relations of the planar algebras $\mathcal{O}^W$ and $\mathcal{P}^W$ be interpreted as a graph rewriting system. Let $\Delta_1, \Delta_2, \ldots,$ Δ_k be open graphs with the same boundary size, and say $\sum_{i=1}^{k} \alpha_i \Delta_i = 0$ (modulo W) when $\sum \alpha_i Z_\Delta^W = 0$. The homomorphism property for planar algebras make this relation a "local rewriting rule". Suppose $\Gamma_1, \Gamma_2, \ldots,$ Γ_k are graphs which are identical everywhere except on patch where the subgraph of Γ_i is isomorphic to Δ_i. Then $\sum_{i=1}^{k} \alpha_i \Gamma_i = 0$ (modulo W). Moreover, by construction the linear extension of the partition function is an invariant of the associated graph rewriting system. This sort of graph relation is similar to the formal combinations of diagrams used by knot theorists, such as in Conway's tangles and skein relations [4] and the invariant is like a spin model [8] and [13]. Thus, planar algebras provide a foundation for a skein theoretic approach to certain graph rewriting problems (this not the standard notion of graph rewriting [20], [5] and [6], although open graphs are used in [17] to study graph rewriting).

5. When $\mathcal{O}^W = \mathcal{F}^W$.

Let $\mathbb{K}$ denote a field, let X denote a finite, nonempty set, and pick $W \in \mathcal{M}_{\mathbb{K}}(X)$. Write $\mathcal{F}^W$ in place of $\mathcal{F}(\mathfrak{Aut}(W), X)$. We show that $\mathcal{O}^W = \mathcal{F}^W$ whenever the characteristic of $\mathbb{K}$ does not divide $|X|!$. In particular, $\mathcal{O}^W = \mathcal{F}^W$ whenever $\mathbb{K}$ has characteristic zero.

Lemma 5.1. *Pick $W \in \mathcal{M}_{\mathbb{K}}(X)$, and fix a nonnegative integer n. Then the following are equivalent:*

(i) $\mathcal{O}_n^W = \mathcal{F}_n^W$.

(ii) *For all $\vec{p}$, $\vec{q} \in X^n$, $Z_\Delta^W(\vec{p}) = Z_\Delta^W(\vec{q})$ for all $\Delta \in \mathcal{O}_n$ implies that $\vec{p}$ and $\vec{q}$ belong to the same $\mathfrak{Aut}(W)$-orbit of X^n.*

Proof. For all $\vec{p}$, $\vec{q} \in X^n$, $\vec{p}$ and $\vec{q}$ belong to the same $\mathfrak{Aut}(W)$-orbit of X^n if and only if $f(\vec{p}) = f(\vec{q})$ for all $f \in \mathcal{F}_n^W$ by the definition of $\mathcal{F}_n^W$. The equivalence of (i) and (ii) follows since $\mathcal{O}_n^W \subseteq \mathcal{F}_n^W$ and $\mathcal{O}_n^W = \text{span}\{Z_\Gamma^W \mid \Gamma \in \mathcal{O}_n\}$. □

We shall prove that Condition (ii) of Theorem 5.1 holds whenever the characteristic of $\mathbb{K}$ does not divide $|X|!$. In fact, we only need to consider $\vec{p}$, $\vec{q} \in X^n$ which differ by a permutation of X.

Lemma 5.2. *Pick $\vec{p}$, $\vec{q} \in X^n$. If $Z_\Delta^W(\vec{p}) = Z_\Delta^W(\vec{q})$ for all $\Delta \in \mathcal{O}_n$, then there exists $\mathfrak{s} \in \mathfrak{S}_X$ such that $\vec{p} = \mathfrak{s}\vec{q}$.*

Proof. Observe that there exists $\mathfrak{s} \in \mathfrak{S}_X$ with $\vec{p} = \mathfrak{s}\vec{q}$ precisely when $p_i = p_j$ if and only if $q_i = q_j$ $(1 \le i, j \le n)$. Suppose there exists some i, j $(1 \le i < j \le n)$ such that $p_i = p_j$ but $q_i \ne q_j$. Let Γ denote the open graph of boundary size n consisting of $n-1$ isolated vertices, each appearing once on the boundary except one that is both the i^{th} and j^{th} boundary vertex. Then $Z_\Gamma^W(\vec{p}) = 1$ and $Z_\Gamma^W(\vec{q}) = 0$. □

The idea behind the following argument is to fix some nonnegative integer n and some $\vec{p} \in X^n$ and then reconstruct W from the data $\{(\Gamma, Z_\Gamma^W(\vec{p})) \mid \Gamma \in \mathcal{O}_n\}$. This means that this information is sufficient to determine the $\mathfrak{Aut}(W)$-orbit of $\vec{p}$. We do this reconstruction by encoding this data as a set of polynomials and then showing that W is essentially the only simultaneous zero of these polynomials (at least when the characteristic of $\mathbb{K}$ does not divide $|X|!$).

Let $\overline{\mathbb{K}}$ denote the algebraic closure of $\mathbb{K}$. Let $\mathcal{L}$ denote the polynomial ring over $\overline{\mathbb{K}}$ in the variables ℓ_{uv} $(u, v \in X)$. We evaluate these polynomials over $\mathcal{M}_{\overline{\mathbb{K}}}(X)$ since the variables are indexed by $X \times X$. Let $L \in \mathcal{M}_{\mathcal{L}}(X)$ denote the matrix whose (u, v)-entry is the variable ℓ_{uv}. Observe that $\mathfrak{s} \in \mathfrak{S}_X$ acts on $\mathcal{L}$ by $\mathfrak{s}(\ell_{uv}) = \ell_{\mathfrak{s}(u)\mathfrak{s}(v)}$ $(u, v \in X)$. Similarly, $\mathfrak{s}$ acts on $\mathcal{M}_{\overline{\mathbb{K}}}(X)$ by $(\mathfrak{s}M)_{u,v} = M_{\mathfrak{s}u,\mathfrak{s}v}$ $(u, v \in X)$ for all $M \in \mathcal{M}_{\overline{\mathbb{K}}}(X)$.

For any nonnegative integer n and for all $\vec{p} \in X^n$, let $E(\vec{p}) = \{Z^L_\Delta(\vec{p}) - Z^W_\Delta(\vec{p}) \mid \Delta \in \mathcal{O}_n\}$. Let $Z(\vec{p})$ denote the affine variety over $\overline{\mathbb{K}}$ defined by $E(\vec{p})$ (the common zeros of all polynomials in $E(\vec{p})$). We view $Z(\vec{p})$ as a subset of $\mathcal{M}_{\overline{\mathbb{K}}}(X)$. Observe that $W \in Z(\vec{p})$.

There is a trivial symmetry of $E(\vec{p})$ and $Z(\vec{p})$ which arises because the polynomial $Z^L_\Delta(\vec{p})$ will not change if we permute the spins not in $\vec{p}$. Let $\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$ denotes the subgroup of $\mathfrak{S}_X$ which fixes the spins in $\vec{p}$ pointwise. Note that $\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$ is isomorphic to $\mathfrak{S}_{X\setminus\vec{p}}$. When $n = 0$, $\vec{p}$ is the empty vector and $\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p}) = \mathfrak{S}_X$.

The next result shows that Condition (ii) of Theorem 5.1 can be restated in terms of $Z(\vec{p})$ and $Z(\vec{q})$. In light of Lemma 5.2, we need only consider $\vec{q}$ of the form $\mathfrak{s}\vec{p}$ for some $\mathfrak{s} \in \mathfrak{S}_X$.

Lemma 5.3. *Pick $\mathfrak{s} \in \mathfrak{S}_X$ and $\vec{p} \in X^n$. The following are equivalent:*

(i) $Z^W_\Delta(\vec{p}) = Z^W_\Delta(\mathfrak{s}\vec{p})$ *for all* $\Delta \in \mathcal{O}_n$.
(ii) $\mathfrak{s}E(\vec{p}) = E(\mathfrak{s}\vec{p})$.
(iii) $\mathfrak{s}Z(\vec{p}) = Z(\mathfrak{s}\vec{p})$.

Moreover, (i)-(iii) *hold when $\mathfrak{s} \in \mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$ and when $\mathfrak{s} \in \mathfrak{Aut}(W)$.*

Proof. Observe that for all $\mathfrak{s} \in \mathfrak{S}_X$, $\mathfrak{s}Z^L_\Delta(\vec{p}) = Z^L_\Delta(\mathfrak{s}\vec{p})$ since the sum defining the partition function runs over all states satisfying the boundary condition. Thus $\mathfrak{s}(Z^L_\Delta(\vec{p}) - Z^W_\Delta(\vec{p})) = Z^L_\Delta(\mathfrak{s}\vec{p}) - Z^W_\Delta(\vec{p}) \in \mathfrak{s}E(\vec{p})$, and $Z^L_\Delta(\mathfrak{s}\vec{p}) - Z^W_\Delta(\mathfrak{s}\vec{p}) = Z^L_\Delta(\mathfrak{s}\vec{p}) - Z^W_\Delta(\mathfrak{s}\vec{p}) \in E(\mathfrak{s}\vec{p})$. The equivalence of (i)-(iii) follows. Clearly (i) holds when $\mathfrak{s} \in \mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$ and when $\mathfrak{s} \in \mathfrak{Aut}(W)$. □

Our problem is now reduced to showing that if $\mathfrak{s}Z(\vec{p}) = Z(\mathfrak{s}\vec{p})$ for some $\mathfrak{s} \in \mathfrak{S}_X$, then $\vec{p}$ and $\mathfrak{s}\vec{p}$ belong to the same $\mathfrak{Aut}(W)$-orbit of X^n. If $\mathfrak{s}$ is in either of the groups identified in Lemma 5.3, then $\vec{p}$ and $\mathfrak{s}\vec{p}$ belong to the same $\mathfrak{Aut}(W)$-orbit of X^n. We shall show that if the characteristic of $\mathbb{K}$ does not divide $|X|!$, then $\mathfrak{Aut}(W)\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p}) := \{\mathfrak{st} \mid \mathfrak{s} \in \mathfrak{Aut}(W),\ \mathfrak{t} \in \mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})\}$ is the complete set of permutations $\mathfrak{s}$ such that $\mathfrak{s}Z(\vec{p}) = Z(\mathfrak{s}\vec{p})$. We will then use this fact to complete our proof. Our goal now is to describe $Z(\vec{p})$ exactly. To do so, we use some facts about polynomial invariants of finite groups as applied to $\mathcal{L}$.

For all subgroups $\mathfrak{G} \subseteq \mathfrak{S}_X$, let $\mathcal{L}^{\mathfrak{G}}$ denote the ring of invariants of $\mathcal{L}$ under the action of $\mathfrak{G}$:

$$\mathcal{L}^{\mathfrak{G}} = \{f \in \mathcal{L} \mid f(M) = (\mathfrak{s}(f))(M) \text{ for all } \mathfrak{s} \in \mathfrak{G},\ M \in \mathcal{M}_{\overline{\mathbb{K}}}(X)\}.$$

See [21] and [22] for more on polynomial invariants of finite groups. Noether's original work on the subject can be found in [18] and [19].

We shall show that under suitable conditions, $E(\vec{p})$ actually spans the ring of invariants of $\mathcal{L}$ under the action of $\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$. We will then be able to appeal to the following result to describe $Z(\vec{p})$ exactly:

Lemma 5.4. *Pick $M \in \mathcal{M}_{\overline{\mathbb{K}}}(X)$. Then the set of common zeros of $\{f - f(M) \mid f \in \mathcal{L}^{\mathfrak{G}}\}$ is $\mathfrak{G} \cdot M := \{\mathfrak{g}M \mid \mathfrak{g} \in \mathfrak{G}\}$.*

Proof. Suppose $M' \notin \mathfrak{G} \cdot M$. Then $\mathfrak{G} \cdot M$ and $\mathfrak{G} \cdot M'$ are disjoint finite sets. Thus there exists a polynomial $h \in \mathcal{L}$ such that $h(\mathfrak{g}M') = 1$ and $h(\mathfrak{g}M) = 0$ for all $\mathfrak{g} \in \mathfrak{G}$. Now $f = \prod_{\mathfrak{g}\in\mathfrak{G}} \mathfrak{g}h \in \mathcal{L}^{\mathfrak{G}}$ has the property that $f(M) = 0$ and $f(M') = 1$. Thus every zero of $\{f - f(M) \mid f \in \mathcal{L}^{\mathfrak{G}}\}$ is in $\mathfrak{G} \cdot M$. The reverse containment is clear, so the result follows. $\square$

We now describe a simple criterion which ensures that we may apply the previous theorem. We deduced such a condition from Noether's work. Let $[\mathcal{L}^{\mathfrak{G}}] \subseteq \mathcal{L}^{\mathfrak{G}}$ denote the $\overline{\mathbb{K}}$-linear span of the polynomials of the form $\sum_{\mathfrak{g}\in\mathfrak{G}} \mathfrak{g}m$, where m runs over all monomials in the variables ℓ_{uv} $(u, v \in X)$. This sum is, up to a normalization constant, the so-called Reynolds operator of the group $\mathfrak{G}$ applied to m. We have the following result of Noether:

Theorem 5.5 ([19] (Noether))**.** *If* Char $\mathbb{K} \nmid |\mathfrak{G}|$, *then* $[\mathcal{L}^{\mathfrak{G}}] = \mathcal{L}^{\mathfrak{G}}$.

It is this criterion of Noether which leads to our condition that the characteristic of $\mathbb{K}$ does not divide $|X|!$. We now sandwich $\mathrm{span}(E(\vec{p}))$ between $[\mathcal{L}^{\mathfrak{G}}]$ and $\mathcal{L}^{\mathfrak{G}}$ for $\mathfrak{G} = \mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$. With the previous two results this gives an exact description of $Z(\vec{p})$ when the characteristic of $\mathbb{K}$ does not divide $|X|!$.

Lemma 5.6. *With the above notation,*

$$[\mathcal{L}^{\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})}] \subseteq \mathrm{span}(E(\vec{p})) \subseteq \mathcal{L}^{\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})}.$$

Proof. We first show that $[\mathcal{L}^{\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})}]$ is contained in the linear span of $E(\vec{p})$. Pick $f \in [\mathcal{L}^{\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})}]$, and let $m = \ell_{u_1v_1}^{n_1}\ell_{u_2v_2}^{n_2}\dots\ell_{u_jv_j}^{n_j}$ denote a monomial appearing in f (say with coefficient $\alpha \in \overline{\mathbb{K}}$) having the maximal number of distinct indices not in $\vec{p}$ appearing on the variables. Let $\Delta = (U, D, \vec{p})$ denote the open graph with U the set of spins which appear in $\vec{p}$ or as a subscript of some variable in m and D the multiset which contains n_i copies of (u_i, v_i) $(1 \le i \le j)$. We show that $f - \alpha(|X| - |U|)!Z^L_\Delta(\vec{p})$ has fewer monomials with as many distinct indices on the variables as m does. It will then follow from induction that $f \in \mathrm{span}(E(\vec{p}))$.

If every element of U appears in $\vec{p}$, then $Z^L_\Delta(\vec{p}) = m$ and $\sum_{\mathfrak{s}\in\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})} \mathfrak{s}m = |\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})|m$ since m is fixed by $\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$. This is the base case of the induction. Now suppose that not all indices of the variables in m are in $\vec{p}$, and consider the states σ over which the sum in (4) runs. Observe that σ is simply a map from U to X with the appropriate boundary condition, and it is either an injection or it is not. Suppose σ is an injection. Then there are $(|X|-|U|)!$ many ways to extend σ to a permutation of X. Any such permutation belongs to $\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$ by the boundary condition, and conversely any element of $\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$ restricts to a valid, injective state. In particular, if $(|X|-|U|)! = 0$, then m cannot appear in f with nonzero coefficient because this number is a factor of the number of repetitions of m. If σ is not an injection, then fewer indices of variables appear in the corresponding summand of $Z^L_\Delta(\vec{p})$ than in m because two or more have been identified by σ. Thus $\sum_{\mathfrak{s}\in\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})} \mathfrak{s}m - \alpha(|X|-|U|)!(Z^L_\Delta(\vec{p}) - Z^W_\Delta(\vec{p}))$ consists only of monomial terms with fewer distinct indices appearing on the variables than in m. By the definition of $[\mathcal{L}^{\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})}]$, every summand of $\sum_{\mathfrak{s}\in\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})} \mathfrak{s}m$ appears in f. It follows by induction that $f \in E(\vec{p})$, thus proving the containment $[\mathcal{L}^{\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})}] \subseteq \operatorname{span}(E(\vec{p}))$.

We now prove the containment $\operatorname{span}(E(\vec{p})) \subseteq \mathcal{L}^{\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})}$. Pick an open graph $\Gamma = (V, E, \vec{b})$ of boundary size n and a permutation $\mathfrak{s} \in \mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$. Then applying $\mathfrak{s}$ to $Z^L_\Delta(\vec{p}) - Z^W_\Delta(\vec{p})$ has the same effect as applying $\mathfrak{s}$ to each state σ over which the sum defining $Z^L_\Delta(\vec{p})$ runs. Since $\mathfrak{s}$ fixes $\vec{p}$ pointwise, the map $\mathfrak{s}\sigma$ is also another state satisfying the boundary condition. Thus $Z^L_\Gamma(\vec{p}) - Z^W_\Gamma(\vec{p}) \in \mathcal{L}^{\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})}$. □

Suppose the characteristic of $\mathbb{K}$ does not divide $|X|!$. Then Theorem 5.5 and Lemma 5.6 imply that $\operatorname{span}(E(\vec{p})) = \mathcal{L}^{\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})}$, so $Z(\vec{p}) = \mathfrak{stab}_{\mathfrak{S}_X}(\vec{p}) \cdot W$ by Lemma 5.4. It is this fact about $Z(\vec{p})$ which we shall use to complete our proof. We note that the condition on the characteristic of the field is sufficient but it is not necessary. However, this condition always holds in characteristic zero, which we consider the most important case. For the moment, we leave the problem of improving this sufficient condition as an open problem, but proceed with this in mind. Let us say that $\vec{p} \in X^n$ is SSS if $Z(\vec{p}) = \mathfrak{stab}_{\mathfrak{S}_X}(\vec{p}) \cdot W$. The above discussion gives us the following:

Lemma 5.7. *Pick $\vec{p} \in X^n$. If* Char $\mathbb{K} \nmid |X|!$, *then $\vec{p}$ is* SSS.

Lemma 5.8. *Pick $\mathfrak{s} \in \mathfrak{S}_X$ and $\vec{p} \in X^n$. Suppose that $\vec{p}$ is* SSS. *Then the following are equivalent:*

(i) $\mathfrak{s} \in \mathfrak{Aut}(W)\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$.
(ii) $W \in \mathfrak{s}Z(\vec{p})$.

Proof. (i) ⇒ (ii): Since $\mathfrak{s} \in \mathfrak{Aut}(W)\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$, the equivalent conditions of Lemma 5.3 hold for $\mathfrak{s} \in \mathfrak{S}_X$ and $\vec{p} \in X^n$. In particular $\mathfrak{s}Z(\vec{p}) = Z(\mathfrak{s}\vec{p})$. Since $W \in Z(\mathfrak{s}\vec{p})$, (ii) follows.

(ii) $\Rightarrow$ (i): Since $\mathfrak{s}^{-1}W \in Z(\vec{p})$, SSS implies that there exists $\mathfrak{t} \in \mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$ such that $\mathfrak{s}^{-1}W = \mathfrak{t}W$. Thus, $\mathfrak{st}W = W$, so $\mathfrak{st} \in \mathfrak{Aut}(W)$ by definition. Now (i) follows. $\square$

Lemma 5.9. *Pick $\vec{p}$, $\vec{q} \in X^n$, and suppose that $\vec{p}$ and $\vec{q}$ are* SSS. *If $Z^W_\Delta(\vec{p}) = Z^W_\Delta(\vec{q})$ for all $\Delta \in \mathcal{O}_n$, then $\vec{p}$ and $\vec{q}$ belong to the same $\mathfrak{Aut}(W)$-orbit of X^n.*

Proof. By Lemma 5.2, there exists $\mathfrak{s} \in \mathfrak{S}_X$ with $\mathfrak{s}\vec{p} = \vec{q}$. Now $Z^W_\Delta(\vec{p}) = Z^W_\Delta(\mathfrak{s}\vec{p})$ for all $\Delta \in \mathcal{O}_n$, so $\mathfrak{s}Z(\vec{p}) = Z(\mathfrak{s}\vec{p})$ by Lemma 5.3. In particular, $W \in \mathfrak{s}Z(\vec{p})$ since $W \in Z(\mathfrak{s}\vec{p})$. Now Lemma 5.8 implies that $\mathfrak{s} \in \mathfrak{Aut}(W)\mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$. If $\mathfrak{s} \in \mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$, then $\vec{p} = \vec{q}$. Otherwise, $\mathfrak{s} \notin \mathfrak{stab}_{\mathfrak{S}_X}(\vec{p})$, so there must be an automorphism of W which maps $\vec{p}$ to $\vec{q}$. In either case, $\vec{p}$ and $\vec{q}$ belong to the same $\mathfrak{Aut}(W)$-orbit of X^n. $\square$

Theorem 5.10. *Let $\mathbb{K}$ denote a field, let X denote a finite, nonempty set, and pick $W \in \mathcal{M}_{\mathbb{K}}(X)$. Suppose that* Char $\mathbb{K} \nmid |X|!$. *Then $\mathcal{O}^W = \mathcal{F}^W$.*

Proof. Immediate from Lemmas 5.1, 5.7 and 5.9. $\square$

This completes our main results. We now give an example which shows that $\mathcal{O}^W$ need not equal $\mathcal{F}^W$ if $|X|!$ is divisible by the characteristic of $\mathbb{K}$.

Example 5.11. Take as the ground field $\mathbb{F}_2$, the integers modulo 2. Let W denote the adjacency matrix of the complete bipartite graph $K_{1,3}$ on vertex set X. Each partite set is an orbit of $K_{1,3}$ under the action of its automorphism group, so $\dim \mathcal{F}_1(X, \mathfrak{Aut}(W)) = 2$. However, $\dim \mathcal{O}^W_1 = 1$ since the symmetry of $K_{1,3}$ implies that given an open graph $\Delta = (V, E, b)$ of boundary size 1, $Z^W_\Delta(p) \equiv Z^W_\Delta(q) \pmod 2$ for all vertices p, q of $K_{1,3}$. In particular, $\mathcal{O}^W_1 \neq \mathcal{F}^W_1$ over $\mathbb{F}_2$. Similar arguments show that when W is the adjacency matrix of a complete multipartite graph $K_{n_1,n_2,\dots,n_m}$ over a field $\mathbb{K}$ of characteristic $k > 0$, $\dim \mathcal{O}^W_1$ is equal to the number of congruence classes modulo k appearing among n_1, n_2, $\dots$, n_m while $\dim \mathcal{F}^W_1$ is equal to the number of distinct numbers among n_1, n_2, $\dots$, n_m.

The arguments used in this paper can be extended to planar subalgebras of $\mathcal{X}$ generated by finitely many functions $\Omega = \{f_i : X^{k_i} \to \mathbb{K}\}$. Here the elements of the planar algebra $\mathcal{P}^\Omega$ are the functions defined from the partition function (1) starting from planar tangles all of whose internal disks are labeled with compatible elements of Ω. (See [**15**] for more on labeled planar tangles.) The planar algebra $\mathcal{O}^\Omega$ can be defined using "open hypergraphs" in a fashion similar to the definition of $\mathcal{O}^W$ above. Then $\mathcal{P}^\Omega = \mathcal{O}^\Omega$ if and only if $\Phi^\Omega \in \mathcal{P}^\Omega_4$. Moreover, $\mathcal{O}^\Omega = \mathcal{F}(\mathfrak{Aut}(\Omega), X)$ as long as the characteristic of the ground field does not divide the order of $\mathfrak{Aut}(\Omega)$, where $\mathfrak{Aut}(\Omega) = \{\mathfrak{s} \in \mathfrak{S}_X \,|\, \mathfrak{s}f_i = f_i \text{ for all } f_i \in \Omega\}$.

References

[1] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London-New York, 1982, MR 86i:82002a, Zbl 0538.60093.

[2] N. Biggs, *Interaction Models*, London Mathematical Society Lecture Note Series, **30**, Cambridge University Press, Cambridge-New York-Melbourne, 1977, MR 58 #32647, Zbl 0375.05039.

[3] D. Bisch and V.F.R. Jones, *Singly generated planar algebras of small dimension*, Duke Math. J., **101** (2000), 41-75, MR 2002f:46118.

[4] J.H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra (Proc. Conf. Oxford, 1967), 1970, 329-358, MR 41 #2661, Zbl 0202.54703.

[5] H. Ehrig, G. Engels, H.-J. Kreowski and G. Rozenberg, eds., *Handbook of Graph Grammars and Computing by Graph Transformation. Vol.* 2. *Applications, Languages and Tools*, World Scientific Publishing Co., Inc., River Edge, NJ, 1999, MR 2001d:68012.

[6] H. Ehrig, H.-J. Kreowski, U. Montanari and G. Rozenberg, *Handbook of Graph Grammars and Computing by Graph Transformation. Vol.* 3. *Concurrency, Parallelism, and Distribution*, World Scientific Publishing Co., Inc., River Edge, NJ, 1999, MR 2001d:68013, Zbl 0951.68049.

[7] G. Hahn and C. Tardif, *Graph homomorphisms: Structure and symmetry*, in 'Graph Symmetry (Montreal, PQ, 1996),' 107-166, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **497**, Kluwer Acad. Publ., Dordrecht, 1997, MR 99c:05091, Zbl 0880.05079.

[8] P. de la Harpe and V.F.R. Jones, *Graph invariants related to statistical mechanical models: Examples and problems*, J. Combin. Theory Ser. B, **57**(**2**) (1993), 207-227, MR 94c:05033, Zbl 0729.57003.

[9] P. Hell and J. Nešetřil, *On the complexity of H-coloring*, J. Combin. Theory Ser. B, **48** (1990), 92-110, MR 91m:68082, Zbl 0639.05023.

[10] ——, *The existence problem for graph homomorphisms*, in 'Graph Theory in Memory of G.A. Dirac (Sandbjerg, 1985),' 255-265, Ann. Discrete Math., **41**, North-Holland, Amsterdam-New York, 1989, MR 90b:05051, Zbl 0673.05033.

[11] G. James and M. Liebeck, *Representations and Characters of Groups*, Cambridge Mathematical Textbooks, Cambridge University Press, Cambridge, 1993, MR 94h:20007, Zbl 0792.20006.

[12] V.F.R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math., **126** (1987), 335-388, MR 89c:46092, Zbl 0631.57005.

[13] ——, *On knot invariants related to some statistical mechanical models*, Pacific J. Math., **137** (1989), 311-224, MR 89m:57005, Zbl 0695.46029.

[14] ——, *Subfactors and Knots*, CBMS Regional Conference Series in Mathematics, **80**, Published for the Conference Board of the Mathematical Sciences, Washington, DC; American Mathematical Society, Providence, RI, 1991, MR 93b:57008, Zbl 0743.46058.

[15] ——, *Planar algebras*, I, NZ J. Math, to appear.

[16] ——, *The planar algebra of a bipartite graph*, in 'Knots in Hellas '98 (Delphi)', 94-117, Ser. Knots Everything, **24**, World Sci. Publishing, River Edge, NJ, 2000, CMP 1 865 703.

[17] U. Montanari and F. Rossi, *Graph rewriting, constraint solving and tiles for coordinating distributed systems*, Alg. Categorical Structures, **7** (1999), 333-370, MR 2001b:68070, Zbl 0949.68083.

[18] E. Noether, *Der Endlichkeitssatz der Invarianten endlicher Gruppen*, Math. Ann., **77** (1916), 89-92.

[19] ______, *Der Endlichkeitssatz der Invarianten endlicher linearer Gruppen der charakteristic p*, Abh. Akad. Wiss. Göttingen, 1926, 28-35.

[20] G. Rozenberg, ed., *Handbook of Graph Grammars and Computing by Graph Transformation. Vol.* 1*. Foundations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1997, MR 99b:68006, Zbl 0908.68095.

[21] L. Smith, *Polynomial Invariants of Finite Groups*, Research notes in mathematics, A.K. Peters, Ltd, Wellesley MA, 1995, MR 96f:13008, Zbl 0864.13002.

[22] B. Sturmfels, *Algorithms in Invariant Theory*, Springer-Verlag, Vienna, 1993, MR 94m:13004, Zbl 0802.13002.

[23] H.N.V. Temperley, *Graph Theory and Applications*, Ellis Horwood Ltd., Chichester; Halsted Press, New York, 1981, MR 83d:05001, Zbl 0481.05001.

[24] ______, *Lattice models in discrete statistical mechanics*, in 'Applications of Graph Theory,' R.J. Wilson and L.W. Beineke, Eds., Academic Press, London-New York, 1979, Zbl 0444.05047.

Received February 5, 2001 and revised June 27, 2002. Research conducted while the author was an NSF mathematical sciences postdoctoral research fellow at UC-Berkeley.

Department of Mathematics
University of South Florida
4202 E. Fowler Avenue, PHY114
Tampa, FL 33620
E-mail address: bcurtin@math.usf.edu