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ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS OF SOME ELLIPTIC PROBLEMS

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In this paper, we discuss the asymptotic behavior of the positive solutions of the problem $-\Delta u = au - bu^p$, $u|_{\partial\Omega} = 0$ as $p \to 1+0$ and as $p \to \infty$. We show that, for each case, the behavior is determined by a limiting problem. Moreover, the limiting problem is of free boundary nature when $p \to \infty$.

1. Introduction and main results.

In this paper, we study the asymptotic behavior of positive solutions of the problem

$$-\Delta u = au - b(x)u^p,\ x \in \Omega;\ u = 0,\ x \in \partial\Omega, \tag{1.1}$$

for p near 1 and near ∞, respectively. Here Ω is a bounded smooth domain in R^N $(N \geq 1)$ and $b(x)$ is a nonnegative function in $C(\overline{\Omega})$, a and p are constants but the exponent p is always greater than 1.

Problem (1.1) arises from mathematical biology and Riemannian geometry, and has attracted considerable interests; see, for example, [**AT**], [**AM**], [**D**], [**dP**], [**DH**], [**FKLM**], [**KW**], [**M**] and [**O**]. The dependence of the positive solutions of (1.1) on the parameter a is well understood but little is known about the dependence on p.

When $b(x)$ is strictly positive on Ω, (1.1) is the steady-state logistic equation and it is well-known that for fixed $p > 1$ it has no positive solution if $a \leq \lambda_1^\Omega$ and there is a unique positive solution $u = u_a$ when $a > \lambda_1^\Omega$, where λ_1^Ω denotes the first eigenvalue of the problem

$$-\Delta u = \lambda u,\ u|_{\partial\Omega} = 0.$$

Moreover, $a \to u_a$ is continuous and strictly increasing as a function from $(\lambda_1^\Omega, \infty)$ to $C(\overline{\Omega})$ (with the natural order), and

$$\lim_{a \to \lambda_1^\Omega + 0} u_a(x) = 0 \text{ uniformly in } \overline{\Omega};$$

$$\lim_{a \to \infty} u_a(x) = \infty \text{ uniformly on any compact subset of } \Omega.$$

When $b^{-1}(0) := \{x \in \Omega : b(x) = 0\}$ is a proper subset of Ω, the behavior of (1.1) is more complicated. Assume for simplicity that $b^{-1}(0) = \overline{\Omega}_0 \subset\subset \Omega$, where Ω_0 is open, connected and with smooth boundary. Then it is well-known that (1.1) has no positive solution unless $a \in (\lambda_1^{\Omega}, \lambda_1^{\Omega_0})$, in which case there is a unique positive solution u_a which varies continuously with a and is strictly increasing in a. Moreover, $u_a \to 0$ uniformly on $\overline{\Omega}$ as $a \to \lambda_1^{\Omega} + 0$, but as $a \to \lambda_1^{\Omega_0}$, $u_a(x) \to \infty$ uniformly on $\overline{\Omega}_0$ and $u_a \to U$ uniformly on any compact subset of $\overline{\Omega} \setminus \overline{\Omega}_0$, where U is the unique minimal positive solution of the boundary blow-up problem

$$-\Delta u = au - b(x)u^p,\ x \in \Omega \setminus \overline{\Omega}_0;\ u|_{\partial\Omega} = 0,\ u|_{\partial\Omega_0} = \infty.$$

We refer to [**DH**] and the references therein for more details.

To understand the effect of the exponent p on the unique positive solution of (1.1), we fix a and consider the extreme cases, that is when $p \to 1+0$ and when $p \to \infty$. In each case, we obtain a limiting problem which determines the asymptotical behavior of (1.1).

To describe our results, we need to recall several simple properties of the first eigenvalue of the Laplacian operator. Let $\phi \in L^\infty(\Omega)$ and denote by $\lambda_1^{\Omega}(\phi)$ the first eigenvalue of the problem

$$-\Delta u + \phi u = \lambda u,\ u|_{\partial\Omega} = 0.$$

Clearly, $\lambda_1^{\Omega}(0) = \lambda_1^{\Omega}$. It is well-known that $\lambda_1^{\Omega}(\phi_n) \to \lambda_1^{\Omega}(\phi)$ whenever $\phi_n \to \phi$ in $L^\infty(\Omega)$, and when $\phi \le \psi$ but $\phi \not\equiv \psi$ in Ω, then $\lambda_1^{\Omega}(\phi) < \lambda_1^{\Omega}(\psi)$. It follows easily that, when $b(x) \ge \delta > 0$ on Ω, then $\lambda(\alpha) := \lambda_1^{\Omega}(\alpha b)$ is a strictly increasing function with $\lambda(0) = \lambda_1^{\Omega}$ and $\lambda(\alpha) \to \infty$ as $\alpha \to \infty$. Therefore, for any given $a > \lambda_1^{\Omega}$, there is a unique $\alpha > 0$ such that

$$a = \lambda_1^{\Omega}(\alpha b). \tag{1.2}$$

We denote by U_α the corresponding positive normalized eigenfunction:

$$-\Delta U_\alpha + \alpha b U_\alpha = aU_\alpha,\ U_\alpha > 0,\ U_\alpha|_{\partial\Omega} = 0,\ \|U_\alpha\|_\infty = 1. \tag{1.3}$$

Here and in what follows, we use the notation $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Omega)}$.

When $b^{-1}(0) = \overline{\Omega}_0$ is not empty, we assume as before that $\Omega_0 \subset\subset \Omega$ is open, connected and with smooth boundary. Then $\lambda(\alpha) = \lambda_1^{\Omega}(\alpha b)$ is still strictly increasing with $\lambda(0) = \lambda_1^{\Omega}$, but (see [**D**] and [**FKLM**])

$$\lim_{\alpha\to\infty} \lambda(\alpha) = \lambda_1^{\Omega_0}.$$

Thus for any given $a \in (\lambda_1^{\Omega}, \lambda_1^{\Omega_0})$, there is a unique $\alpha > 0$ satisfying (1.2) which determines a unique U_α through (1.3).

We are now ready to state our main results.

Theorem 1.1. *Suppose that $b(x) > 0$ on $\overline{\Omega}$ and $a > \lambda_1^{\Omega}$. Let u_p denote the unique positive solution of* (1.1)*. Then the following results hold:*

(i) *When* $a < \lambda_1^\Omega(b)$, *we have* $u_p \to 0$ *uniformly on* $\overline{\Omega}$ *as* $p \to 1+0$. *Moreover, as* $p \to 1+0$,

$$(p-1)\ln\|u_p\|_\infty \to \ln\alpha,\ u_p/\|u_p\|_\infty \to U_\alpha \text{ in } C^1(\overline{\Omega}), \tag{1.4}$$

where α *and* U_α *are determined by* (1.2) *and* (1.3), *respectively.*

(ii) *When* $a > \lambda_1^\Omega(b)$, *we have* $u_p \to \infty$ *uniformly on any compact subset of* Ω *as* $p \to 1+0$. *Moreover,* (1.4) *holds.*

(iii) *When* $a = \lambda_1(b,\Omega)$, *we have* $u_p \to cU_1$ *in* $C^1(\overline{\Omega})$ *as* $p \to 1+0$, *where* U_1 *is given by* (1.3) *with* $\alpha = 1$ *and*

$$c = \exp\left(\int_\Omega bU_1^2 \ln U_1 dx \Big/ \int_\Omega bU_1^2 dx\right).$$

To understand the case that $p \to \infty$, we need the following free boundary problem:

$$-\Delta w = a\chi_{\{w<1\}}w,\ w > 0,\ w|_{\partial\Omega} = 0,\ \|w\|_\infty = 1, \tag{1.5}$$

which also arises as a limiting problem for the degenerate predator-prey model (see [**DD2**]). The following result has been proved in [**DD2**]:

Proposition 1.2. *For any* $a \geq \lambda_1^\Omega$, (1.5) *has a unique weak solution, and when* $a < \lambda_1^\Omega$, (1.5) *has no solution.*

With the help of Proposition 1.2, we will prove the following:

Theorem 1.3. *Suppose that* $b(x) > 0$ *on* $\overline{\Omega}$ *and* $a > \lambda_1^\Omega$. *Let* u_p *denote the unique positive solution of* (1.1). *Then* $u_p \to v$ *in* $C^1(\overline{\Omega})$ *as* $p \to \infty$, *where* v *is the unique positive weak solution of* (1.5).

When $\overline{\Omega}_0 := b^{-1}(0)$ is a nontrivial subset of Ω, it turns out that the techniques in proving Theorems 1.1 and 1.3 are not enough. One new ingredient for dealing with this case is the following result obtained in [**DD1**, Lemma 2.2]:

Lemma 1.4. *Suppose that* $\{u_n\} \subset C^1(\overline{\Omega})$ *satisfies* (*in the weak sense*) *for some positive constant* λ,

$$-\Delta u_n \leq \lambda u_n,\ u_n \geq 0 \text{ in } \Omega;\ u_n|_{\partial\Omega} = 0,\ \|u_n\|_\infty = 1.$$

Then it has a subsequence converging weakly in $H_0^1(\Omega)$ *and strongly in* $L^q(\Omega)$ *for any* $q \geq 1$, *to some* u *with* $\|u\|_\infty = 1$.

Theorem 1.5. *Suppose that* $\overline{\Omega}_0 = b^{-1}(0)$ *has nonempty interior which is connected with smooth boundary and* $\Omega_0 \subset\subset \Omega$. *Let* $a \in (\lambda_1^\Omega, \lambda_1^{\Omega_0})$ *and denote by* u_p *the unique positive solution of* (1.1). *Then the conclusions* (i)-(iii) *in Theorem* 1.1 *hold.*

As Theorem 1.5 concludes that when $b^{-1}(0) \neq \varnothing$ and $p \to 1$, the behavior of u_p is the same as when $b^{-1}(0) = \varnothing$, it is tempting to think that this is also the case when $p \to \infty$. It turns out, however, this is not true.

Theorem 1.6. *Suppose that $\overline{\Omega}_0 = b^{-1}(0)$ has nonempty interior which is connected with smooth boundary and $\Omega_0 \subset\subset \Omega$. Let $a \in (\lambda_1^{\Omega}, \lambda_1^{\Omega_0})$ and denote by u_p the unique positive solution of (1.1). Suppose $p_n \to \infty$ and denote $u_n = u_{p_n}$. Then, subject to a subsequence, $u_n \to u$ in $L^q(\Omega)$ for all $q \geq 1$, where $u \in K$ is a nontrivial nonnegative solution of the following variational inequality:*

$$\int_\Omega \nabla u \cdot \nabla(v-u)dx - \int_\Omega au(v-u)dx \geq 0,\ \forall v \in K, \tag{1.6}$$

$$K := \{w \in H_0^1(\Omega) : w \leq 1 \quad a.e.\ \ in\ \Omega \setminus \Omega_0\}.$$

In a forthcoming paper ([**DD3**]), we will show that (1.6) has a unique positive solution for $a \in (\lambda_1^{\Omega}, \lambda_1^{\Omega_0})$, and hence $u_p \to u$ as $p \to \infty$ in $L^q(\Omega)$. Let us note that (1.6) is different from (1.5). In fact, it has been shown in [**DD2**] that (1.5) is equivalent to the variational inequality

$$\int_\Omega \nabla u \cdot \nabla(v-u)dx - \int_\Omega au(v-u)dx \geq 0,\ \forall v \in K_0,$$

$$K_0 := \{w \in H_0^1(\Omega) : w \leq 1 \ \text{ a.e. in } \ \Omega\}.$$

Moreover, it is possible to show that for any given compact subset D of Ω, there exists a large a_D such that the unique solution of (1.5) satisfies $w = 1$ on D when $a > a_D$. (More precise results are discussed briefly in [**DD2**].) It is easily seen that for such a, and for those $\Omega_0 \subset D$ satisfying $\lambda_1^{\Omega_0} > a$, if we let $u = w$ on $\overline{\Omega} \setminus \Omega_0$; and on Ω_0, let u equal the unique solution to $-\Delta u = au$, $u|_{\partial\Omega_0} = 1$, then u solves (1.6).

Remark 1.7. From our proofs, it is easy to see that our assumptions on the smoothness of $\partial\Omega_0$ can be considerably weakened. For example, all our main results hold if Ω_0 only has Lipschitz boundary.

Remark 1.8. If $b^{-1}(0)$ consists of a single point in Ω, then Theorems 1.5 and 1.6 reduce to Theorems 1.1 and 1.3, respectively. This follows easily by checking the proofs. We intend to further consider the case that $b^{-1}(0)$ has measure zero in [**DD3**].

The rest of the paper consists of the proofs of our results given above. Theorem 1.1 is proved in Section 2; Theorems 1.3 and 1.5 are proved in Sections 3 and 4, respectively; Section 5 gives the proof of Theorem 1.6. The main techniques involved are various elliptic estimates and comparison principles. Several results and techniques from [**DD2**] will be used, including fine properties of functions in Sobolev spaces and the use of variational inequalities.

2. Proof of Theorem 1.1.

Set $M_p = \|u_p\|_\infty = \max_{\overline{\Omega}} u_p$. Then it is clear that the maximum is achieved in the interior of the domain Ω, say at $x_p \in \Omega$. Using the equation for u_p at the maximum point $x = x_p$ we have

$$aM_p - b(x_p)M_p^p \geq 0.$$

Hence,

$$M_p^{p-1} \leq a/\min_{\overline{\Omega}} b. \tag{2.1}$$

To understand the asymptotic behaviour of u_p as $p \to 1+0$, we choose an arbitrary sequence $p_n \to 1+0$ and use the notation

$$u_n = u_{p_n},\ M_n = M_{p_n},\ \alpha_n = M_{p_n}^{p_n-1},\ w_n = u_n/M_n. \tag{2.2}$$

Clearly w_n satisfies

$$-\Delta w_n = aw_n - \alpha_n b w_n^{p_n},\ w_n|_{\partial\Omega} = 0. \tag{2.3}$$

From (2.1) one sees that the right-hand side of (2.3) has a bound in $L^\infty(\Omega)$ which is independent of n. Thus, by standard elliptic estimates, $\{w_n\}$ is bounded in $W^{2,q}(\Omega)$ for any $q > 1$. By the Sobolev imbedding theorem, this implies that this sequence is compact in $C^1(\overline{\Omega})$. Therefore, subject to a subsequence, $w_n \to w$ in $C^1(\overline{\Omega})$. We may also assume that $\alpha_n \to \alpha$. Then from (2.3) we obtain, in the weak sense,

$$-\Delta w = (a - \alpha b)w,\ w|_{\partial\Omega} = 0.$$

As w is nonnegative with $\|w\|_\infty = 1$, we necessarily have $a = \lambda_1^\Omega(\alpha b)$ and hence α is uniquely determined by (1.2) and $w = U_\alpha$ given by (1.3). This implies that $\alpha_n \to \alpha$ and $w_n \to U_\alpha$ hold for the entire original sequences. Therefore, we have proved that $M_p^{p-1} \to \alpha$ and $u_p/M_p \to U_\alpha$ in $C^1(\overline{\Omega})$ as $p \to 1+0$. This shows the validity of (1.4).

When $a < \lambda_1^\Omega(b)$, we must have $\alpha \in (0,1)$ and it follows from

$$\lim_{p\to 1+0}(p-1)\ln M_p = \ln\alpha \tag{2.4}$$

that $M_p \to 0$ as $p \to 1+0$. This proves Part (i) of Theorem 1.1.

When $a > \lambda_1^\Omega(b)$, we must have $\alpha > 1$ and it follows from (2.4) that $M_p \to \infty$ as $p \to 1+0$. To prove Part (ii) of Theorem 1.1, it remains to show that as $p \to 1+0$, $u_p(x) \to \infty$ uniformly on any compact subset of Ω. To this end, for any given large number β, we define $V = \beta U_\alpha$ and obtain

$$\Delta V + aV - bV^p = b(\alpha V - V^p).$$

For those x where $V(x) \leq 1$, $\alpha V - V^p \geq (\alpha-1)V \geq 0$; on the set $\{x \in \Omega : V(x) \geq 1\}$, since $V^p \to V$ uniformly as $p \to 1$, and since $\alpha V - V \geq \alpha - 1 > 0$, we can find $\epsilon = \epsilon(\beta) > 0$ small enough such that $\alpha V - V^p > 0$ for all $p \in (1, 1+\epsilon)$. Thus, for $p \in (1, 1+\epsilon)$, V is a lower solution to (1.1). As any large positive constant is an upper solution of (1.1), its unique positive solution u_p must satisfy $u_p \geq V = \beta U_\alpha$. This implies that as $p \to 1+0$, $u_p \to \infty$ uniformly on any compact subset of Ω and Part (ii) of Theorem 1.1 is proved.

We consider now the case that $a = \lambda_1^\Omega(b)$. We have $\alpha = 1$ and hence cannot draw a conclusion for $\lim_{p\to 1+0} M_p$ from (2.4). Denote $w_p = u_p/M_p$. We have

$$-\Delta w_p = a w_p - b M_p^{p-1} w_p^p,\ w_p|_{\partial\Omega} = 0.$$

Multiply this equation by U_1, which is given by (1.3) with $\alpha = 1$, and integrate by parts. It results

$$\int_\Omega (a-b) U_1 w_p dx = \int_\Omega (a w_p - b M_p^{p-1} w_p^p) U_1 dx.$$

Hence

$$\int_\Omega b(w_p - M_p^{p-1} w_p^p) U_1 dx = 0,$$

and

$$\int_\Omega \frac{M_p^{p-1}-1}{p-1} b w_p^p U_1 dx = \int_\Omega \frac{1-w_p^{p-1}}{p-1} b w_p U_1 dx. \tag{2.5}$$

Since $w_p \to U_1$ as $p \to 1+0$ in $C^1(\overline{\Omega})$, and by the Hopf boundary lemma, $\partial U_1/\partial\nu < 0$ on $\partial\Omega$, we obtain $w_p/U_1 \to 1$ uniformly on $\overline{\Omega}$. It follows that

$$\|\ln w_p - \ln U_1\|_{L^\infty(\Omega)} = o(1)$$

as $p \to 1+0$. Therefore,

$$\frac{1-w_p^{p-1}}{p-1} w_p = \frac{1-e^{(p-1)(\ln U_1 + o(1))}}{p-1} w_p \to U_1 \ln U_1$$

uniformly on $\overline{\Omega}$ as $p \to 1+0$. From this, we see immediately that the right-hand side of (2.5) converges to

$$\int_\Omega b U_1^2 \ln U_1 dx.$$

Thus,

$$\lim_{p\to 1+0} \int_\Omega \frac{M_p^{p-1}-1}{p-1} b w_p^p U_1 dx = \int_\Omega b U_1^2 \ln U_1 dx,$$

and

$$\lim_{p\to 1+0} \frac{M_p^{p-1}-1}{p-1} = \int_\Omega b U_1^2 \ln U_1 dx \Big/ \int_\Omega b U_1^2 dx. \tag{2.6}$$

We show next that

$$c := \lim_{p\to 1+0} M_p$$

exists and is uniquely determined by

$$\ln c = \int_\Omega bU_1^2 \ln U_1 dx \Big/ \int_\Omega bU_1^2 dx.$$

We first claim that

$$M_* := \underline{\lim}_{p\to 1+0} M_p > 0, \ M^* := \overline{\lim}_{p\to 1+0} M_p < \infty.$$

Otherwise, we can find a sequence $p_n \to 1+0$ such that $M_n := M_{p_n} \to 0$ or $M_n \to \infty$. In the former case, we deduce, for all large n,

$$\frac{M_n^{p_n-1}-1}{p_n-1} \le \frac{\epsilon^{p_n-1}-1}{p_n-1} \to \ln\epsilon$$

as $n\to\infty$, for any given $\epsilon>0$. This leads to a contradiction to (2.6). In the latter case, we obtain, for all large n,

$$\frac{M_n^{p_n-1}-1}{p_n-1} \ge \frac{M^{p_n-1}-1}{p_n-1} \to \ln M$$

as $n\to\infty$, for any given $M>0$. This also leads to a contradiction to (2.6). Thus, $0 < M_* \le M^* < \infty$. For any given small $\epsilon>0$, a similar argument to the above leads to

$$\ln(M_*+\epsilon) \ge \int_\Omega bU_1^2 \ln U_1 dx \Big/ \int_\Omega bU_1^2 dx,$$
$$\ln(M^*-\epsilon) \le \int_\Omega bU_1^2 \ln U_1 dx \Big/ \int_\Omega bU_1^2 dx.$$

Thus we necessarily have

$$M_* = M^* = c = \exp\left(\int_\Omega bU_1^2 \ln U_1 dx \Big/ \int_\Omega bU_1^2 dx\right),$$

and $u_p \to cU_1$ as $p\to 1+0$ in $C^1(\overline{\Omega})$. This finishes the proof of Theorem 1.1.

3. Proof of Theorem 1.3.

We clearly still have (2.1). Let p_n be a sequence converging to ∞ and use the notations in (2.2). We find that w_n satisfies (2.3) whose right-hand side has a bound in $L^\infty(\Omega)$ which is independent of n. Thus, as in Section 2, subject to a subsequence, $w_n \to w$ in $C^1(\overline{\Omega})$.

The equation satisfied by w_n can also be written as

$$(3.1) \qquad -\Delta w_n = aw_n - bu_n^{p_n-1} w_n, \ w_n|_{\partial\Omega} = 0.$$

From (2.1) we deduce

$$0 \le u_n^{p_n-1} \le a/\min_{\overline{\Omega}} b.$$

Hence, by passing to a subsequence, we may assume that $bu_n^{p_n-1} \to \psi$ weakly in $L^2(\Omega)$. Clearly we must have $0 \le \psi \le \|b\|_\infty a/\min_{\overline{\Omega}} b$. Passing to the weak limit in (3.1) we find that w is a nontrivial weak solution to the problem

$$-\Delta w = (a-\psi)w, \; w|_{\partial\Omega} = 0. \tag{3.2}$$

As $a - \psi \in L^\infty(\Omega)$, it follows from the Harnack inequality that $w(x) > 0$ in Ω.

From (2.1) we obtain

$$M_n \le \left(a/\min_{\overline{\Omega}} b\right)^{1/(p_n-1)} \to 1 \text{ as } n \to \infty.$$

It follows that $\overline{\lim}_{n\to\infty} M_n \le 1$. If $\underline{\lim}_{n\to\infty} M_n < 1$, then by passing to a subsequence, we may assume that $M_n \le 1-\epsilon$ for all n and some $\epsilon > 0$. It follows then $u_n^{p_n-1} \le (1-\epsilon)^{p_n-1} \to 0$ as $n \to \infty$. Hence $\psi = 0$ and w is a positive solution to $-\Delta w = aw$, $w|_{\partial\Omega} = 0$. This implies that $a = \lambda_1^\Omega$, contradicting our assumption that $a > \lambda_1^\Omega$. Thus we have proved that $M_n \to 1$ as $n \to \infty$. It follows that $u_n \to w$ in $C^1(\overline{\Omega})$.

Let $\Omega_1 := \{x \in \Omega : w(x) < 1\}$. Then for any $x \in \Omega_1$, we can find $\delta > 0$ such that $u_n(x) < 1-\delta$ for all large n. It follows that $0 \le u_n(x)^{p_n-1} \le (1-\delta)^{p_n-1} \to 0$ as $n \to \infty$. Thus we must have $\psi = 0$ a.e. on Ω_1. On the rest of Ω, $w = 1$ and we necessarily have $\Delta w = 0$. (Here we regard w as a member of $W^{2,q}(\Omega)$, $q > 1$.) Thus from (3.2), we deduce $\psi = a$ a.e. on $\Omega \setminus \Omega_1$. Therefore, w satisfies

$$-\Delta w = a\chi_{\{w<1\}} w, \; w > 0, \; w|_{\partial\Omega} = 0, \; \|w\|_\infty = 1. \tag{3.3}$$

By Proposition 1.2, problem (3.3) has a unique solution v. Hence $u_n \to v$ in $C^1(\overline{\Omega})$ for the entire original sequence. This implies that $u_p \to v$ in $C^1(\overline{\Omega})$ as $p \to \infty$. The proof of Theorem 1.3 is complete.

4. Proof of Theorem 1.5.

We will mainly follow the lines of the proof of Theorem 1.1. The main difficulty is that the estimate (2.1) is of no use anymore and therefore it is unclear whether $\{\alpha_n\}$ is still bounded. We will use Lemma 1.4 to overcome this difficulty.

Let p_n be an arbitrary sequence of numbers converging to $1+0$. We employ the notations in (2.2) and find that w_n meets the conditions in Lemma 1.4. Hence, by passing to a subsequence, we may assume that $w_n \to w$ weakly in $H_0^1(\Omega)$, strongly in $L^q(\Omega)$ for any $q \ge 1$, and $\|w\|_\infty = 1$.

We claim that $\{\alpha_n\}$ is bounded. Otherwise, by passing to a subsequence, we may assume that $\alpha_n \to \infty$. Now we multiply (2.3), the equation satisfied by w_n, by ϕ/α_n with $\phi \in C_0^\infty(\Omega)$ and integrate by parts. We obtain

$$(\alpha_n)^{-1}\int_\Omega w_n(-\Delta\phi)dx = (\alpha_n)^{-1}\int_\Omega a w_n \phi dx - \int_\Omega b w_n^{p_n}\phi dx.$$

Letting $n \to \infty$, we deduce

$$\int_\Omega bw\phi dx = 0.$$

As ϕ is arbitrary, this implies that $bw = 0$ in Ω. Hence, $w = 0$ on $\Omega \setminus \Omega_0$. Since $w \in H_0^1(\Omega)$ and $\partial\Omega_0$ is smooth, this implies that $w|_{\Omega_0} \in H_0^1(\Omega_0)$. Multiplying the equation for w_n by an arbitrary $\phi \in C_0^\infty(\Omega_0)$ and integrating by parts, we obtain

$$\int_{\Omega_0} \nabla w_n \cdot \nabla \phi dx = \int_{\Omega_0} a w_n \phi dx.$$

Passing to $n \to \infty$ we obtain

$$\int_{\Omega_0} \nabla w \cdot \nabla \phi dx = \int_{\Omega_0} a w \phi dx.$$

Thus $w|_{\Omega_0}$ is a weak solution of the problem

$$-\Delta u = au,\ u|_{\partial\Omega_0} = 0.$$

As $w = 0$ on $\Omega \setminus \Omega_0$ and $\|w\|_\infty = 1$, $w|_{\Omega_0}$ is nonnegative and not identically zero. Hence we must have $a = \lambda_1^{\Omega_0}$, contradicting our assumption that $a < \lambda_1^{\Omega_0}$. This proves our claim that $\{\alpha_n\}$ is bounded.

The rest of the proof follows that of Theorem 1.1 except that to prove $u_p \geq \beta U_\alpha$, we use Lemma 2.1 of [**DM**] (which holds for C^1 functions).

5. Proof of Theorem 1.6.

It turns out that Lemma 1.4 is not enough for our proof of Theorem 1.6. We will need some fine properties of the limiting function u in Lemma 1.4 and of functions in $H^1(R^N)$. These fine properties have already been used in [**DD2**] and we simply collect them in the following lemma:

Lemma 5.1. *Let u and u_n be as in Lemma 1.4. Then the following conclusions hold:*

(i) $\widetilde{u}(x) = \lim_{r\to 0} \int_{B_r(x)} u(y)dy/|B_r(x)|$ *exists for each $x \in \Omega$, where $B_r(x)$ denotes the ball with center x and radius r, and $|B_r(x)|$ stands for the volume of $B_r(x)$. Moreover, $u = \widetilde{u}$ a.e. in Ω.*

(ii) *$\widetilde{u}$ is upper semi-continuous (u.s.c. for short) on Ω, and for each $x_0 \in \Omega$ and any given $\epsilon > 0$, we can find a small ball $B_r(x_0) \subset \Omega$ such that for all large n,*

$$u_n(x) \leq \widetilde{u}(x_0) + \epsilon,\ \forall x \in B_r(x_0).$$

(iii) *If $v \in H^1(R^N)$, then $\widetilde{v}(x) = \lim_{r\to 0} \int_{B_r(x)} v(y)dy/|B_r(x)|$ exists for all $x \in R^N$ except possibly for a set of $(1,2)$-capacity 0. Moreover, $\widetilde{v} = v$ a.e. in R^N and if $\widetilde{v}$ vanishes on a closed set A in R^N (except for a subset of A of capacity zero), then there exists a sequence of functions*

$\phi_n \in H^1(R^N)$ such that each ϕ_n vanishes in a neighbourhood of A and $\phi_n \to \widetilde{v}$ in $H^1(R^N)$.

Let us now come back to the proof of Theorem 1.6. Let p_n be a sequence converging to ∞ and use the notations in (2.2). Then as before, by Lemma 1.4, subject to a subsequence, $w_n \to w$ weakly in $H_0^1(\Omega)$ and strongly in $L^q(\Omega)$ for any $q \geq 1$, and $\|w\|_\infty = 1$.

Claim 1. $\{M_n\}$ is bounded.

Proof. Since $a < \lambda_1^{\Omega_0}$, we can find a small δ-neighborhood Ω_δ of $\overline{\Omega}_0$ such that $a < \lambda_1^{\Omega_\delta}$. Let ϕ_δ denote the normalized positive eigenfunction corresponding to $\lambda_1^{\Omega_\delta}$:

$$-\Delta\phi_\delta = \lambda_1^{\Omega_\delta}\phi_\delta,\ \phi_\delta|_{\partial\Omega_\delta} = 0,\ \|\phi_\delta\|_\infty = 1,$$

and let $\psi \in C^2(\overline{\Omega})$ be an extension of $\phi_\delta|_{\Omega_{\delta/2}}$ to $\overline{\Omega}$ such that $\eta := \min_{\overline{\Omega}} \psi > 0$. We find, for any positive constant Q,

$$\begin{aligned}&\Delta(Q\psi) + a(Q\psi) - b(Q\psi)^p \leq (a - \lambda_1^{\Omega_\delta})Q\psi < 0,\ \forall x \in \Omega_{\delta/2},\\ &\Delta(Q\psi) + a(Q\psi) - b(Q\psi)^p = Q(\Delta\psi + a\psi) - bQ^p\psi^p,\ \forall x \in \Omega \setminus \Omega_{\delta/2}.\end{aligned}$$

Let $\xi = \inf_{\Omega\setminus\Omega_{\delta/2}} b$ and

$$Q_p := \left[\xi^{-1}\sup_{\Omega}(\Delta\psi + a\psi)\eta^{-p}\right]^{1/(p-1)}.$$

We easily see that for $Q = Q_p$,

$$\Delta(Q\psi) + a(Q\psi) - b(Q\psi)^p \leq 0,\ \forall x \in \Omega.$$

Therefore $Q_p\psi$ is an upper solution of (1.1). As (1.1) has arbitrarily small positive lower solutions, its unique positive solution u_p must satisfy $u_p \leq Q_p\psi$. Clearly $Q_p \to 1/\eta$ as $p \to \infty$. Thus, for any $p_0 > 1$, $\{M_p : p \geq p_0\}$ is bounded. In particular, $\{M_n\}$ is bounded. This proves Claim 1.

By passing to a subsequence, we may assume that $M_n \to c \in [0, \infty)$ as $n \to \infty$.

Claim 2. $c \geq 1$.

Proof. Let v_n be the unique solution of

$$-\Delta v = av - \|b\|_\infty v^{p_n},\ v|_{\partial\Omega} = 0.$$

By Theorem 1.3 we know $\|v_n\|_\infty \to 1$. On the other hand, a simple comparison argument shows $u_n \geq v_n$. Hence $c \geq 1$.

Claim 3. $w \leq 1/c$ a.e. in $\Omega \setminus \Omega_0$.

Proof. Otherwise the set $\{x \in \Omega \setminus \Omega_0 : w(x) > 1/c\}$ has positive measure and we can find some $c_1 > 1/c$ such that $\Omega_1 := \{x \in \Omega \setminus \Omega_0 : w(x) \geq c_1\}$ has positive measure. As $w_n \to w$ in $L^2(\Omega)$, by passing to a subsequence, $w_n \to w$ a.e. in Ω. Hence, by Egorov's theorem, we can find a subset of Ω_1, say Ω_2 which has positive measure and such that $w_n \to w$ uniformly on Ω_2. It follows that $u_n \to cw$ uniformly on Ω_2. Thus, there exists $\epsilon > 0$ such that for all large n, $u_n \geq 1 + \epsilon$ on Ω_2.

Let $\phi \in C_0^\infty(\Omega)$ be an arbitrary nonnegative function, and multiply the equation for w_n by ϕ and integrate over Ω. It results

$$\int_\Omega w_n(-\Delta\phi) = a\int_\Omega w_n\phi - \int_\Omega b u_n^{p_n-1} w_n\phi.$$

Hence, for all large n,

$$(1+\epsilon)^{p_n-1}\int_{\Omega_2} b w_n\phi \leq \int_{\Omega_2} b u_n^{p_n-1} w_n\phi \leq \int_\Omega w_n(\Delta\phi) + a\int_\Omega w_n\phi.$$

Dividing the above inequality by $(1+\epsilon)^{p_n-1}$ and letting $n \to \infty$, we deduce

$$\int_{\Omega_2} b w\phi = 0.$$

It follows that $w = 0$ a.e. in Ω_2, contradicting the assumption that $w \geq c_1$ there. This proves Claim 3.

Using $u_n = M_n w_n$ and denoting $\hat{u} = cw$, we see from Lemma 5.1 and Claims 1-3 above that the following result holds:

Lemma 5.2.

(i) $\{\|u_n\|_\infty\}$ *is bounded.*

(ii) *Subject to a subsequence,* $u_n \to \hat{u}$ *weakly in* $H_0^1(\Omega)$ *and strongly in* $L^q(\Omega)$, $\forall q \geq 1$.

(iii) $\hat{u} \leq 1$ *a.e. in* $\Omega \setminus \Omega_0$ *and* $\|\hat{u}\|_\infty \geq 1$.

(iv) $\widetilde{u}(x) := \lim_{r\to 0} \int_{B_r(x)} \hat{u}(y)dy/|B_r(x)|$ *exists for every* $x \in \Omega$.

(v) $\widetilde{u}(x)$ *is u.s.c. on* Ω *and* $\widetilde{u} = \hat{u}$ *a.e. in* Ω.

(vi) *For each* $x_0 \in \Omega$ *and any given* $\epsilon > 0$, *we can find a small ball* $B_r(x_0) \subset \Omega$ *such that for all large* n,
$$u_n(x) \leq \widetilde{u}(x_0) + \epsilon,\ \forall x \in B_r(x_0).$$

We are now ready to complete the proof of Theorem 1.6. Multiplying the equation for u_n by $\phi \in C_0^\infty(\Omega)$, we deduce

$$\int_\Omega \nabla u_n \cdot \nabla\phi dx = a\int_\Omega u_n\phi dx - \int_\Omega b(x)u_n^{p_n}\phi dx.$$

It follows that, subject to a subsequence,

$$\lim_{n\to\infty}\int_\Omega b(x)u_n^{p_n}\phi dx = -\int_\Omega \nabla\hat{u}\cdot\nabla\phi dx + a\int_\Omega \hat{u}\phi dx,\ \forall\phi \in C_0^\infty(\Omega). \tag{5.1}$$

Clearly the right-hand side of (5.1) defines a continuous linear functional on $H^1(\Omega)$:

$$T(\phi) = -\int_\Omega \nabla\hat{u}\cdot\nabla\phi dx + a\int_\Omega \hat{u}\phi dx.$$

Using the left-hand side of (5.1), and noticing that $b=0$ on Ω_0, we see that $T(\phi)\geq 0$ whenever $\phi\in C_0^\infty(\Omega)$ satisfies $\phi\geq 0$ on $\Omega\setminus\Omega_0$. Moreover, if $\operatorname{supp}(\phi)\subset\{\widetilde{u}<1\}\cup\Omega_0$, where $\{\widetilde{u}<1\}:=\{x\in\overline{\Omega}:\widetilde{u}(x)<1\}$, then by Lemma 5.2 (vi) and the fact that $\{\widetilde{u}<1\}$ is relatively open (due to $\widetilde{u}$ being u.s.c), we can find $\delta>0$ such that $u_n(x)\leq 1-\delta$ on the compact set $\operatorname{supp}(\phi)\setminus\Omega_0\subset\operatorname{supp}(\phi)\cap\{\widetilde{u}<1\}\subset\Omega$ for all large n. Therefore, since $b=0$ on Ω_0

$$0\leq\int_\Omega b(x)u_n^{p_n}\phi dx\leq\int_{\operatorname{supp}(\phi)\setminus\Omega_0} b(x)(1-\delta)^{p_n}\phi dx\to 0.$$

It follows that $T(\phi)=0$ if $\operatorname{supp}(\phi)\subset\{\widetilde{u}<1\}\cap\Omega_0$. Using the continuity of T on $H^1(\Omega)$ and the fact that functions in $H_0^1(\Omega)$ can be approximated in the $H^1(\Omega)$ norm by functions in $C_0^\infty(\Omega)$, we find that

$$T(\phi)\geq 0,\ \forall\phi\in H_0^1(\Omega)\text{ satisfying }\phi\geq 0\ \text{ a.e. on }\Omega\setminus\Omega_0, \tag{5.2}$$

$$T(\phi)=0,\ \forall\phi\in H_0^1(\Omega)\text{ satisfying }\operatorname{supp}(\phi)\subset\Omega_0\cup\{\hat{u}<1\}. \tag{5.3}$$

By Lemma 5.2 (iii), we easily see that $\widetilde{u}\leq 1$ on the open set $\Omega\setminus\overline{\Omega}_0$. We show next that $\widetilde{u}$ is close to 0 near $\partial\Omega$ and $\widetilde{u}\leq 1$ on $\partial\Omega_0$. By Lemma 5.2 (i), we can find $M>0$ such that $au_n<M$ on Ω for all $n\geq 1$. Therefore

$$-\Delta u_n = au_n - b(x)u_n^{p_n}\leq M\ \text{ on }\ \Omega.$$

If V is given by

$$-\Delta V = M\ \text{ in }\ \Omega,\ V|_{\partial\Omega}=0,$$

we obtain by the maximum principle that $u_n\leq V$. It follows that $\widetilde{u}\leq V$. Therefore $\widetilde{u}$ is close to 0 near $\partial\Omega$.

Since $\widetilde{u}\leq 1$ on $\Omega\setminus\overline{\Omega}_0$, we must have $\widetilde{u}\leq 1$ on $\partial\Omega_0$ except possibly for a set of capacity zero (see, e.g., [**Z**] pp. 190-191).

From the above analysis, we see that it is possible to choose $\phi\in C_0^\infty(\Omega)$ such that $0\leq\phi\leq 1$ on Ω and $\phi=1$ on a δ-neighborhood N_δ of $\{\hat{u}=1\}$. Let $v\in K$ be arbitrary and denote $\hat{v}=\max\{v,\phi\}$. Clearly $0\leq\hat{v}-v\in H_0^1(\Omega)$. Thus, by (5.2),

$$\begin{aligned}&\int_\Omega\nabla\hat{u}\cdot\nabla(v-\hat{u})dx - a\int_\Omega\hat{u}(v-\hat{u})dx = -T(v-\hat{u})\\ &= T(\hat{v}-v)+T(\hat{u}-\hat{v})\geq T(\hat{u}-\hat{v}).\end{aligned}$$

Denote $u^*=\hat{u}-\hat{v}$. Clearly $u^*\in H_0^1(\Omega)$. Now we choose $\psi\in C_0^\infty(\Omega)$ satisfying $0\leq\psi\leq 1$ on Ω, $\psi=0$ on $\Omega\setminus N_{(2/3)\delta}$, $\psi=1$ on $N_{(1/3)\delta}$. Then clearly

$$\operatorname{supp}((1-\psi)u^*) \subset \overline{\Omega} \setminus N_{(1/3)\delta} \subset \{\widetilde{u} < 1\} \cup \Omega_0.$$

Hence, by (5.3),

$$T(u^*) = T((1-\psi)u^*) + T(\psi u^*) = T(\psi u^*).$$

As $\psi = 0$ on $\Omega \setminus N_{(2/3)\delta}$, and $\hat{v} = \max\{v, \phi\} = 1$ a.e. on N_δ, we find that $\psi u^* = \psi(\hat{u}-1)$ a.e. on Ω. Since $\psi(\hat{u}-1)$ is zero outside $N_{(2/3)\delta}$ it can be regarded as a member of $H^1(R^N)$. It is easily seen that the representative of $\psi(\hat{u}-1)$ obtained through the limiting process in Lemma 5.1 (iii) is $\psi(\widetilde{u}-1)$. Thus we obtain

$$T(u^*) = T(\psi u^*) = T(\psi(\widetilde{u}-1)).$$

As $\widetilde{u} \le 1$ on $\overline{\Omega} \setminus \Omega_0$ and is u.s.c., we find that the set $A_1 := \{\widetilde{u} = 1\} \cap (\overline{\Omega} \setminus \Omega_0)$ is closed. Let $A_2 := R^N \setminus N_{(2/3)\delta}$ and $A = A_1 \cup A_2$. We know that $\psi(\widetilde{u}-1)$ vanishes on the closed set A (except possibly for a set of capacity zero) and so by Lemma 5.1 (iii), it can be approximated in the $H^1(R^N)$ norm by $\phi_n \in H^1(R^N)$ with each ϕ_n vanishing in a neighbourhood of A. Therefore, $\operatorname{supp}(\phi_n) \subset \{\widetilde{u} < 1\} \cup \Omega_0$, and by (5.3), $T(\phi_n) = 0$. It follows that

$$T(u^*) = T(\psi(\widetilde{u}-1)) = \lim_{n\to\infty} T(\phi_n) = 0.$$

We thus obtain

$$\int_\Omega \nabla\hat{u} \cdot \nabla(v-\hat{u})dx - a\int_\Omega \hat{u}(v-\hat{u})dx \ge 0, \ \forall v \in K.$$

That is to say that $\hat{u} \in K$ is a solution of (1.6). This finishes our proof of Theorem 1.6.

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